Design and Analysis of Algorithms

Median and Order statistics

Reference:
CLRS Chapter 9

Topics:

- **Order statistics**
- **Expected linear time selection**
- **Worst-case linear time selection**

Today

- Median and order statistics
- Two $O(n)$ time algorithms:
  - Randomized: similar to Quicksort
  - Deterministic: quite tricky
- Both are examples of divide and conquer

Order Statistics

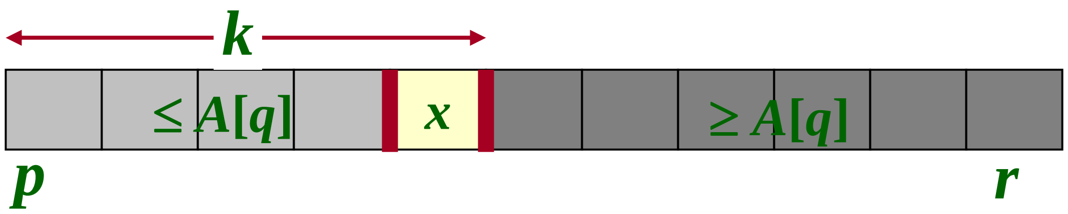
- Select the ith smallest of n elements (the element with rank i).
  - Minimum: $i = 1$
  - Maximum: $i = n$
  - Median: $i = \lfloor (n+1) / 2 \rfloor$ or $\lceil (n+1)/2 \rceil$

- How fast can we solve the problem?
  - $O(n)$ for min or max.
  - $O(n \lg n)$ by sorting for general i.
  - $O(n \lg i)$ with heaps.

- Next we will see how to do it in $O(n)$ time.

Randomized algorithm for finding the *i*th element

- Divide and conquer approach
- Main idea: PARTITION

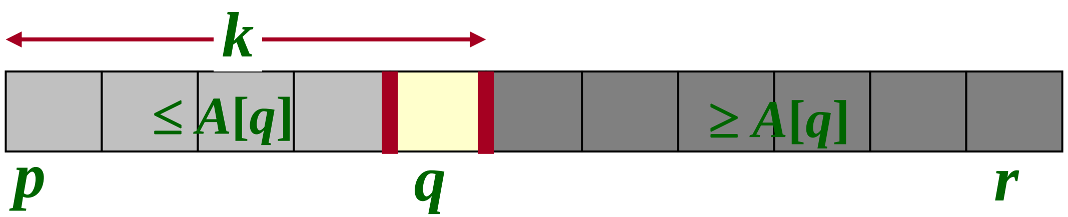


- If $i < k$, recurse on the left
- If $i > k$. recurse on the right
- Otherwise, output x

RANDOMIZD Divide-and-conquer

RANDOMIZED SELECT

```
RANDOMIZED-SELECT(A, p, r, i)
if p = r
    then return A[p]
q ← RANDOMIZED-PARTITION(A, p, r)    // as used in quicksort
k ← q–p+1                             // k = rank(A[q])
if i = k                              // the pivot value is the answer
    then return A[q]
if i < k
    then return RANDOMIZED-SELECT(A, p, q–1, i)
    else return RANDOMIZED-SELECT(A, q+1, r, i–k)
```



Example

- Select the *i* = 7th smallest:

6	10	13	5	8	3	2	11

pivot

$i = 7$

- Partition:

2	5	3	6	8	13	10	11

$k = 4$

Select the 7- 4 = 3rd smallest recursively

Analysis

- What is the worst-case running time?

  Unlucky:

  $$T(n) = T(n-1) + \Theta(n) = \Theta(n^2)$$

- Recall that a lucky partition splits into arrays with size ratio at most $9:1$
- What if all partitions are lucky?

  lucky:

  $$T(n) = T(9n/10) + \Theta(n) = \Theta(n)$$

  $$n^{\log_{10/9} 1} = n^0 = 1$$

  CASE 3

Expected running time

- The probability that a random pivot induces lucky partition is at least 8/10
- Let t_i be the number of partitions performed between the (i–1)-th and the i-th lucky partition
- The total time is at most…

$$T = t_1 n + t_2(9/10)n + t_3(9/10)^2 n + \ldots$$

- The total expected time is at most:
- $E[T] = E[t_1]\, n + E[t_2]\, (9/10)n + E[t_3]\, (9/10)^2 n + \ldots$

$= 10/8 * n * [1 + p + p^2 \ldots]$ Geometric series $p = (9/10) < 1$

$= 10/8 * n * [1/(1-p)]$

$= O(n)$

Digression: 9 to 1

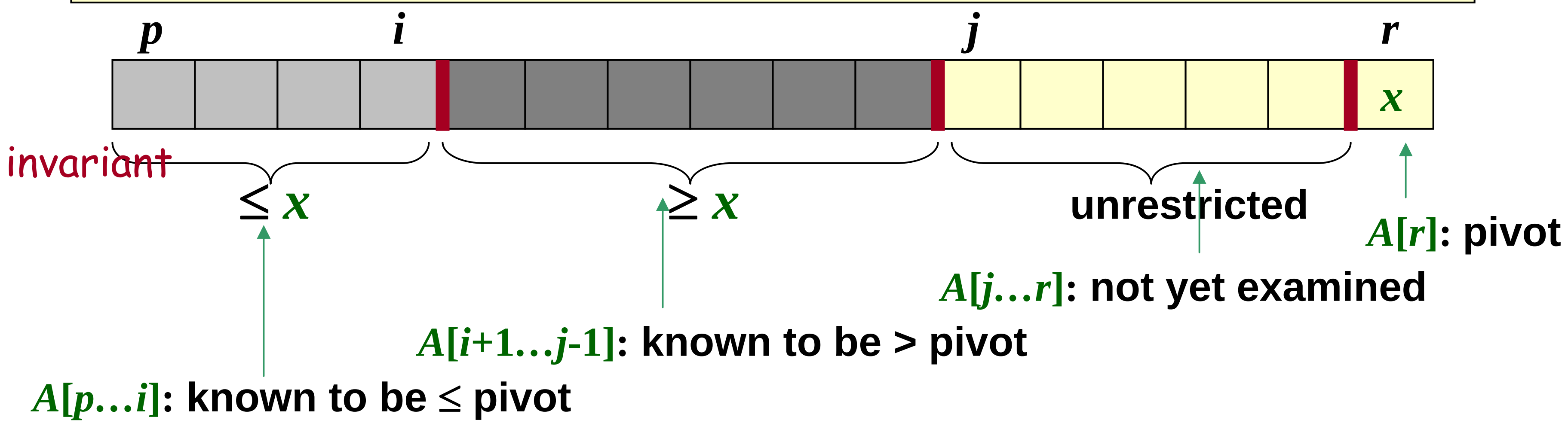
- Do we need to define the lucky partition as $9:1$ balanced?
- No. Suffices to say that both sides have size $\geq \alpha n$, for $0 < \alpha < ½$
- Need constant fraction of n, regardless of what the fraction is. But not constant number.
- Probability of getting a lucky partition is $1–2\alpha$

Partitioning subroutine

PARTITION

```
PARTITION(A, p, r)        //A[p..r]
  x ← A[r]                //the rightmost element as pivot
  i ← p-1
for j ← p to r-1
      do if A[j] ≤ x
            then i ← i+1
                 exchange A[i]↔A[j]
exchange A[i+1]↔A[r]
return i+1
```

Running time = O(n) for n elements



Summary of randomized order-statistics selection

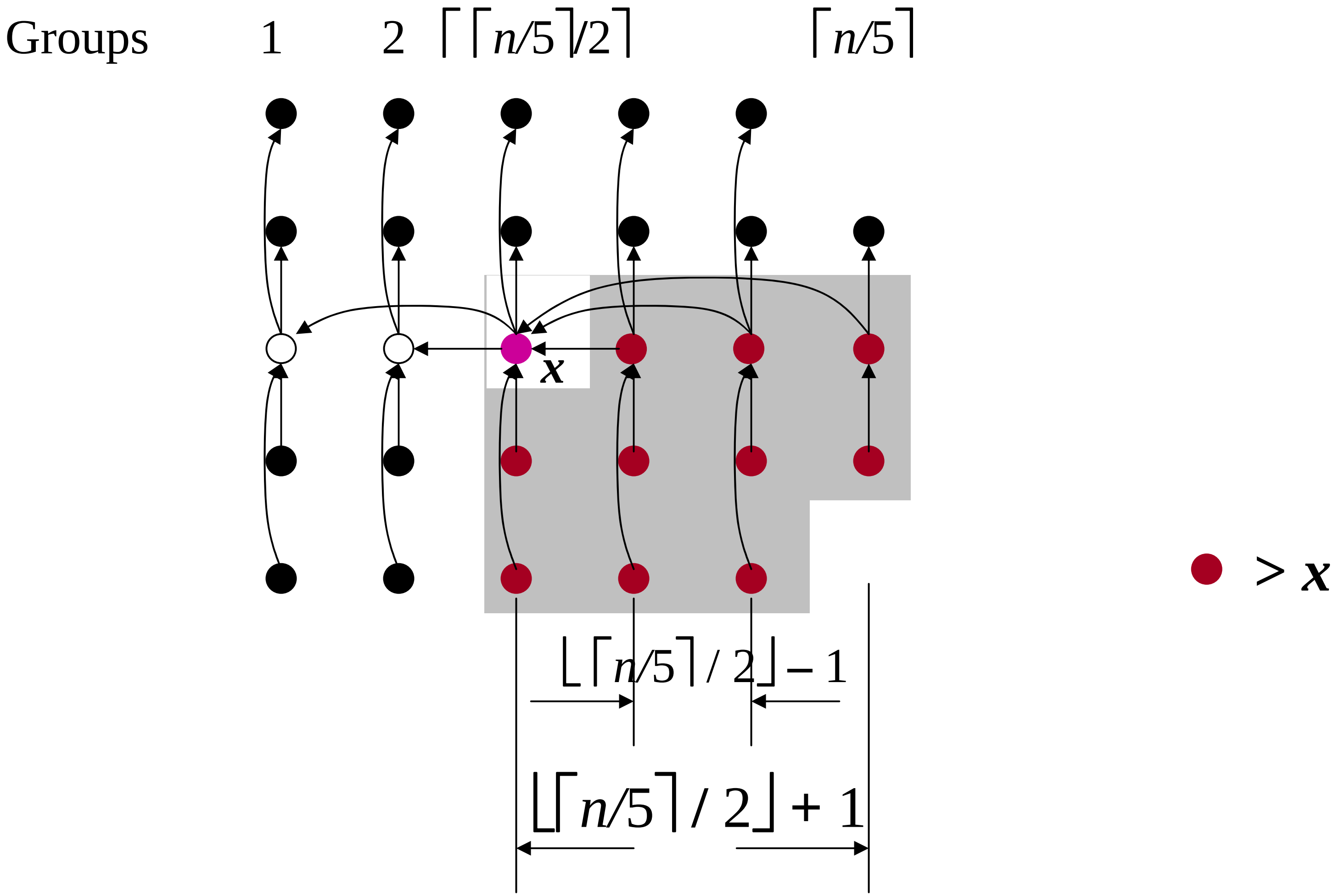
- Works fast: linear expected time.
- Excellent algorithm in practice.
- But, the worst case is very bad: $\Theta(n^2)$.

Q. Is there an algorithm that runs in linear time in the worst case?

A. Yes, due to [Blum-Floyd-Pratt-Rivest-Tarjan'1973]

- IDEA: Generate a good pivot recursively.

Pictorial Analysis of Select



Worst-Case Linear-Time Selection

SELECT(A, i)

1. Divide n elements into $\lfloor n/5 \rfloor$ groups of 5, plus extra
2. Find medians of each group by insertion sort.
3. Find median x of the $\lfloor n/5 \rfloor$ medians by calling SELECT() recursively
4. Partition the n elements around pivot x. Let k = rank(x)
5. **if** ($i = k$) **then** return x

   if ($i < k$) **then** call SELECT() recursively to find i-th smallest element in left partition

   else call SELECT() recursively to find (i-k)-th smallest element in right partition

Partition

- **`Partition()`.**
  - Divide n elements into $\lfloor n/5 \rfloor$ groups of 5 elements each, plus extra.

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$n = 54$

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Partition

- **`Partition()`.**
  - **Divide n elements into $\lfloor n/5 \rfloor$ groups of 5 elements each, plus extra.**
  - **Find medians of each group by insertion sort.**
  - **Find x = "median of medians" by `SELECT()` on $\lfloor n/5 \rfloor$ medians.**

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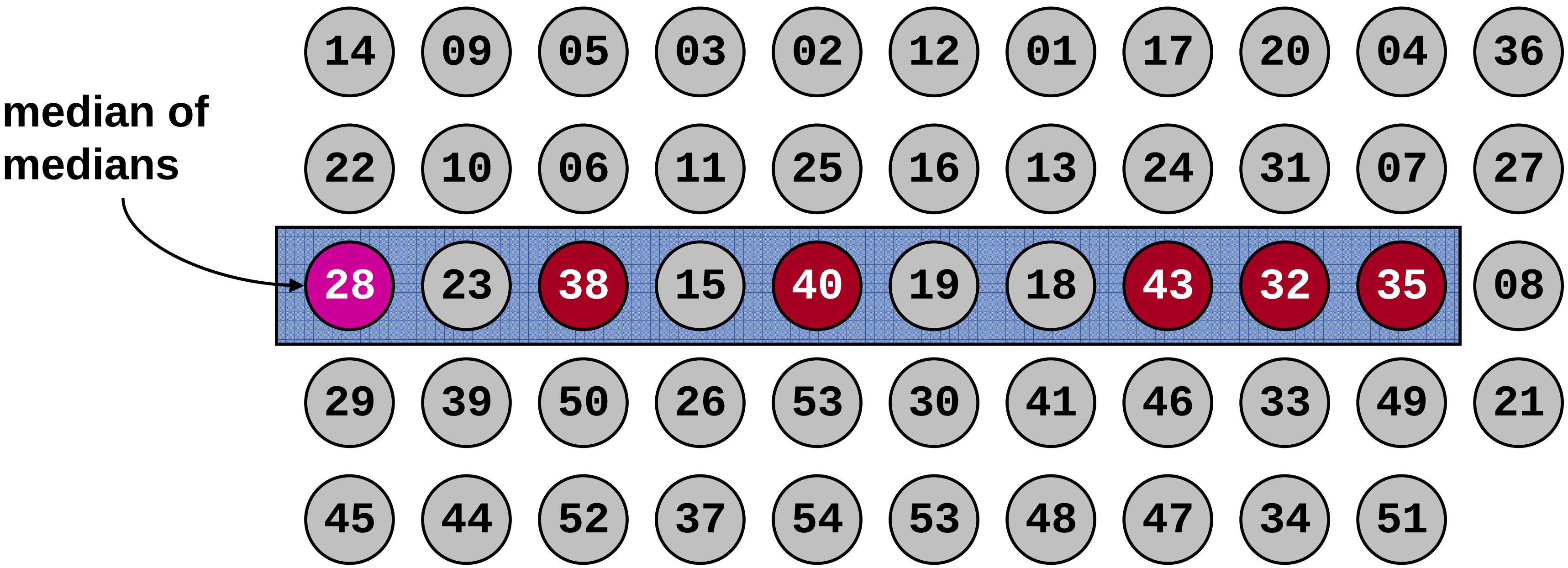
Selection Analysis

- **Crux of proof:**
  - At least half of medians found in step2 $> x$, except for…
    - » at least $\lceil \lceil n/5 \rceil /2 \rceil - 2 = \lceil n/10 \rceil -2$ medians $> x$



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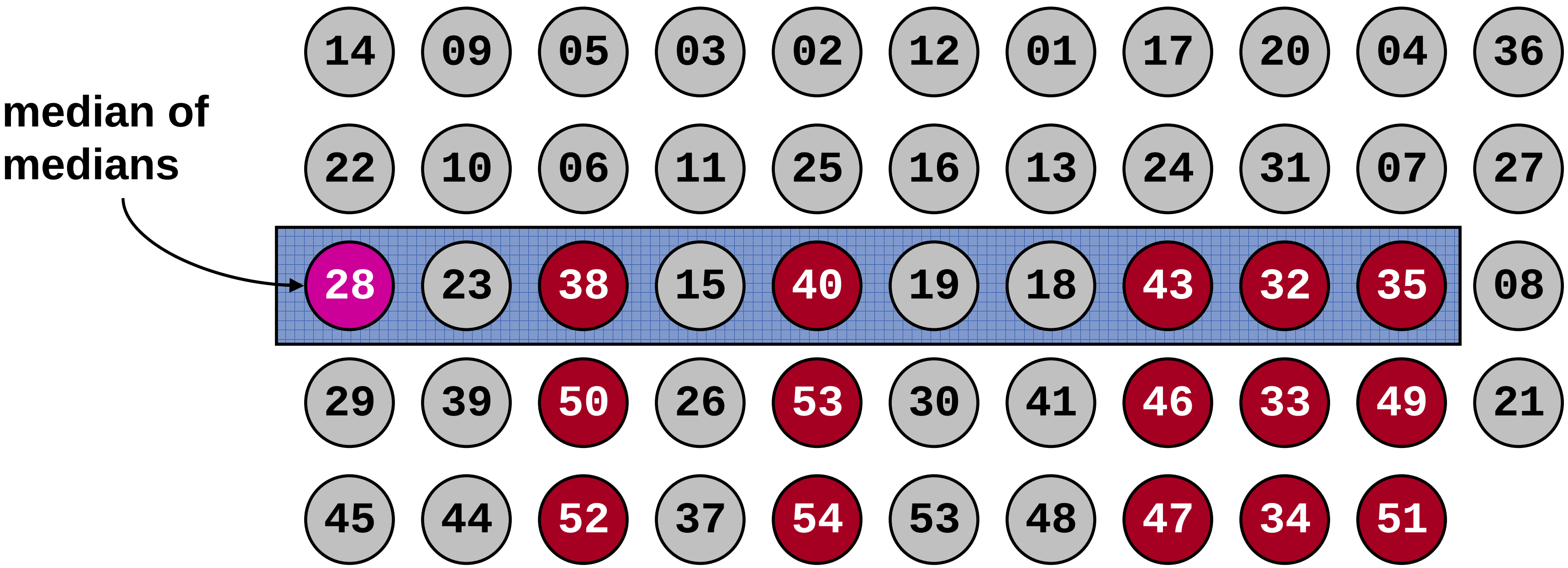
  - At least half of medians found in step2 $> x$, except for…
    - at least $\lceil \lceil n/5 \rceil /2 \rceil - 2 = \lceil n/10 \rceil - 2$ medians $> x$
  - At least $3(\lceil n/10 \rceil - 2)$ elements $> x$.



Selection Analysis

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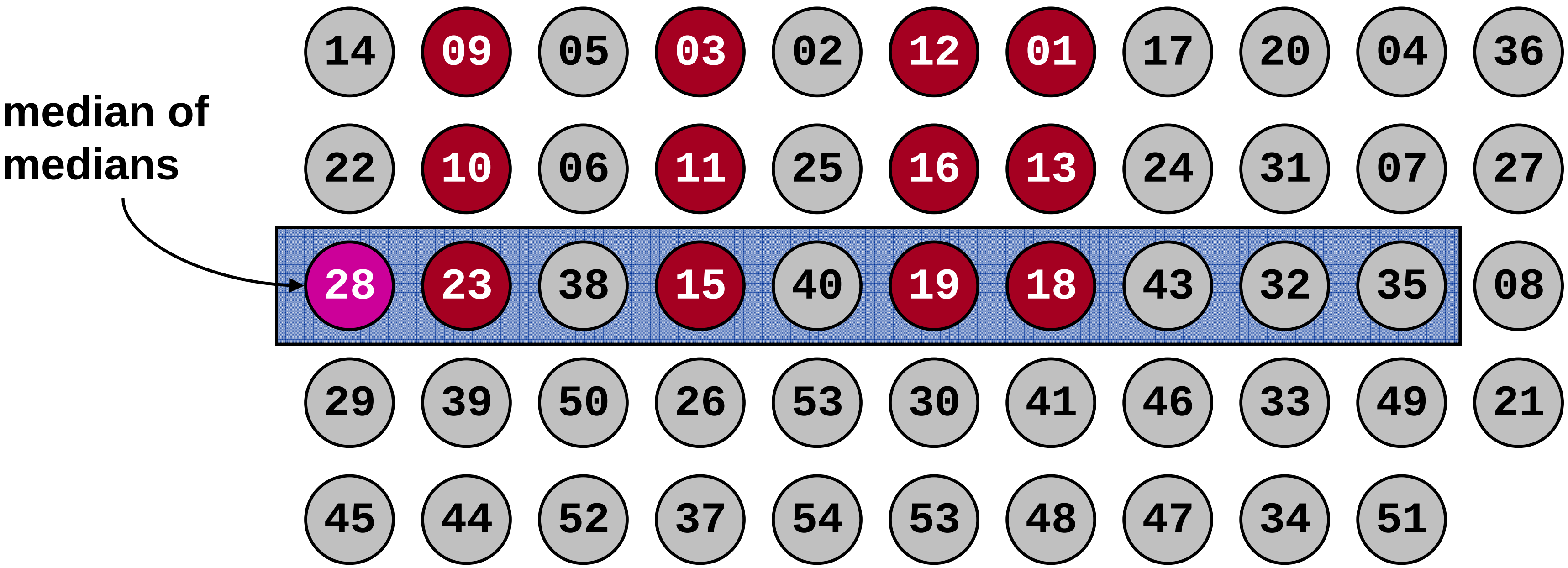
  - At least $3(\lceil n/10 \rceil - 2)$ elements $> x$.
  - At least $3(\lceil n/10 \rceil - 2)$ elements $< x$.



Selection Analysis

- **Crux of proof:**
  - At least half of medians found in step2 $> x$, except for…
    - » at least $\lceil \lceil n/5 \rceil /2 \rceil - 2 = \lceil n/10 \rceil - 2$ medians $> x$
  - At least $3(\lceil n/10 \rceil -2) \geq 3n/10 - 6$ elements $> x$.
  - At least $3n/10 - 6$ elements $< x$.

    $\Rightarrow$ SELECT( ) called recursively (step 5) with at most $n - (3n/10 - 6) = 7n/10 + 6$ elements in the worst case, regardless of which partition is used
- **Algorithm analysis**
  - Step 1,2, and 4 take $O(n)$ time,
  - Step 3 takes time $T(\lceil n/5 \rceil)$, and
  - Step 5 takes time at most $T(7n/10 + 6)$

Selection Analysis

- Claim: Thus after partitioning around x in step 4, step 5 will call SELECT() on at most 70% of the elements

$$T(n) \le \begin{cases} \Theta(1) & \text{if } n \le 140 \\ T(n/5) + T(7n/10 + 6) + O(n) & \text{if } n > 140 \end{cases}$$

- Claim: $T(n) \le cn$.
  - Base case: $n \le 140$.
  - Inductive hypothesis: assume true for $1, 2, \ldots, n\text{-}1$.
  - Induction step: for $n > 140$, we have:

$$\begin{aligned} T(n) &\le c\, n/5 + c\,(7n/10 + 6) + n \\ &\le cn/5 + c + 7cn/10 + 6c + n \\ &\le 9\,c\,n/10 + n + 7\,c \\ &\le c\,n \quad \Leftarrow \text{if we choose } c \text{ such that } 7c + n/10 \le cn/10 \\ &\qquad\qquad \text{e.g. true for } c = 20 \text{ if } n > 140 \end{aligned}$$

Linear-Time Median Selection: Applications

- Given a "black box" $O(n)$ algorithm that finds median, what can we do?
- i-th order statistic:
  - Find median x
  - Partition input around x
  - if ($i \le (n+1)/2$) recursively find i-th element of first half
  - else find $(i - (n+1)/2)$-th element in second half
  - $T(n) = T(n/2) + O(n) = O(n)$
- Worst-case $O(n \lg n)$ quicksort
  - Find median x and partition around it
  - Recursively quicksort two halves
  - $T(n) = 2\,T(n/2) + O(n) = O(n \lg n)$

Searching

Two fundamental problems in CS:

- Sorting
- Searching, defined as follows

(1)SELECTION PROBLEM

Input: Set *A* and index *i*

Output: *i*th smallest (or largest) element in *A*

(2)DICTIONARY PROBLEM

Input: Set *A* and key *k*

Output: element in *A* with the key if found,
and "not found" otherwise.

Min/Max problem

- **Find the minimum element in a set of *n* elements**

MINMUM(A)

```
MINIMUM(A)
1  min ← A[1]
2  for i ← 2 to length[A]
3       do if min > A[i]
4               then min ← A[i]
5  return min
```

- **Algorithm finds the maximum/ minimum in $n-1$ comparisons, which is optimal w.r.t. the number of comparisons**

Min-Max problem

- **Find the maximum and minimum elements in a set of *n* elements**

MAXIMUM-MINIMUM(A)

```
MAXIMUM-MINIMUM(A)
max ← min ← A[1]
for i ← 2 to length[A]
    do if A[i] > max
          then max ← A[i]
          else if A[i] < min
                  then min ← A[i]
return min & max
```

- **Algorithm finds the maximum and minimum in $n-1$ comparisons, if the array is already sorted in increasing order and in $2n$-2 comparisons in decreasing order.**

Min-Max problem

Search	Min	Max	Min & Max
#comparisons	$n-1$	$n-1$	$2n-2$

Algorithm1: Find maximum and then find minimum $2n-2$ **comparisons**

Algorithm2: a) Pair elements up $\lceil n/2 \rceil$ pairs

$$(a_1, a_2), (a_3, a_4), \ldots, (a_{n-1}, a_n)$$

b) Reorder: For $k = 1, \ldots, \lceil n/2 \rceil$ $\lceil n/2 \rceil$ comparisons

$$(a'_{2k-1}, a'_{2k}) = \begin{cases} (a_{2k-1}, a_{2k}) & \text{if } a_{2k-1} < a_{2k} \\ (a_{2k}, a_{2k-1}) & \text{otherwise} \end{cases}$$

c) For $k = 1, \ldots, \lceil n/2 \rceil$ $\lceil n/2 \rceil + \lceil n/2 \rceil$ comparisons

Compare a'_{2k-1} **with current min, and** a'_{2k} **with current max**

Algorithm2 finds the maximum and minimum in $3\lceil n/2 \rceil$

Min and Max Example

- Method 3: Divide and Conquer .
- Find the min and max of {3, 5, 6, 2, 4, 9, 3, 1}

- Example
  - $A = \{3,5,6,2\}$ and $B = \{4,9,3,1\}$
  - $\min(A) = 2$, $\min(B) = 1$
  - $\max(A) = 6$, $\max(B) = 9$

- $\min\{\min(A),\min(B)\} = 1$
- $\max\{\max(A), \max(B)\} = 9$

Min-Max problem

MAX & MIN

```
MAXMIN(i,j,fmax,fmin)
if (i=j)
    then fmax ← fmin ← a[i]
if (i=(j-1)) then
    if a[i]<a[j]
        then fmax ← a[j]
             fmin ← a[i]
        else fmax ← a[i]
             fmin ← a[j]
else
        mid ← ⌊(i+j)/2⌋
        MAXMIN(i,mid,gmax,gmin)
        MAXMIN(mid+1,j,hmax,hmin)
        fmax ← max{gmax,hmax}
        fmin ← min{gmin,hmin}
```

Time Complexity

- **Let $T(n)$ be the number of comparisons made when finding the min and max of n elements**

$$T(n) = \begin{cases} 0 & n = 1 \\ 1 & n = 2 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + 2 & n > 2 \end{cases}$$

- **To solve the recurrence, assume n is a power of 2 and use repeated substitution.**
- $T(n) = \lceil 3n/2 \rceil - 2$

Average-Case Analysis

- Define indicator random variable X_k

$$X_k = \mathrm{I}\{\text{the subarray } A[p..q] \text{ has exactly } k \text{ elements}\}$$

$$\mathrm{E}[X_k] = 1/n$$

- $T(n) \le 1/n(\ T(\max(1, n-1)) + \sum_{k=1 \text{ to } n-1} T(\max(k, n-k))\ ) + O(n)$

$$\le 1/n(\ T(n-1) + 2 \sum_{k=\lceil n/2 \rceil \text{ to } n-1} T(k)\ ) + O(n)$$

$$= 2/n \sum_{k=\lceil n/2 \rceil \text{ to } n-1} T(k) + O(n)$$

- Substitution Method: Guess $T(n) \le c\ n$

$$T(n) \le 2/n \sum_{k=\lceil n/2 \rceil \text{ to } n-1} ck + O(n)$$

$$\le 2c/n\ (\ \sum_{k=1 \text{ to } n-1} k - \sum_{k=1 \text{ to } \lceil n/2 \rceil -1} k\ ) + O(n)$$

$$= 2c/n\ ((n-1)n/2 - 1/2(\lceil n/2 \rceil -1)\lceil n/2 \rceil) + O(n)$$

$$\le c(n-1) - (c/n)(n/2 - 1)(n/2) + O(n)$$

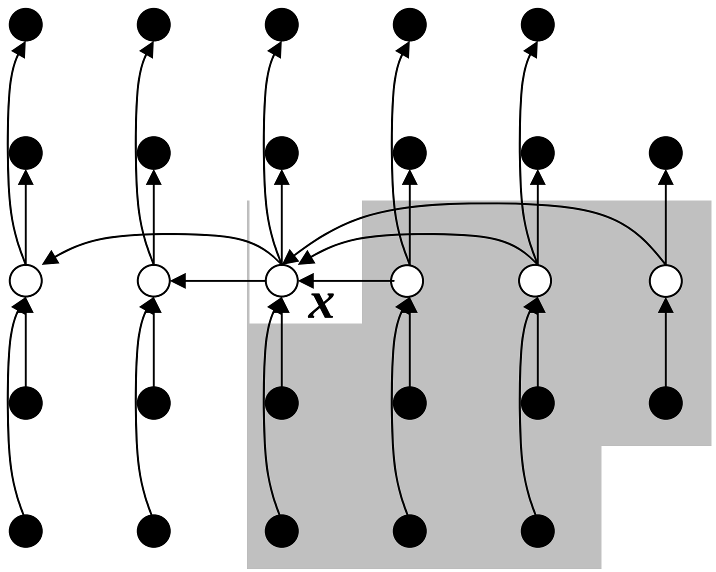
$$\le c(3n/4 - 1/2) + O(n)$$

$\le cn$ $\Leftarrow$ if we pick c large enough so that $c(n/4 + 1/2)$ dominates $O(n)$

Worst-Case Linear-Time Selection

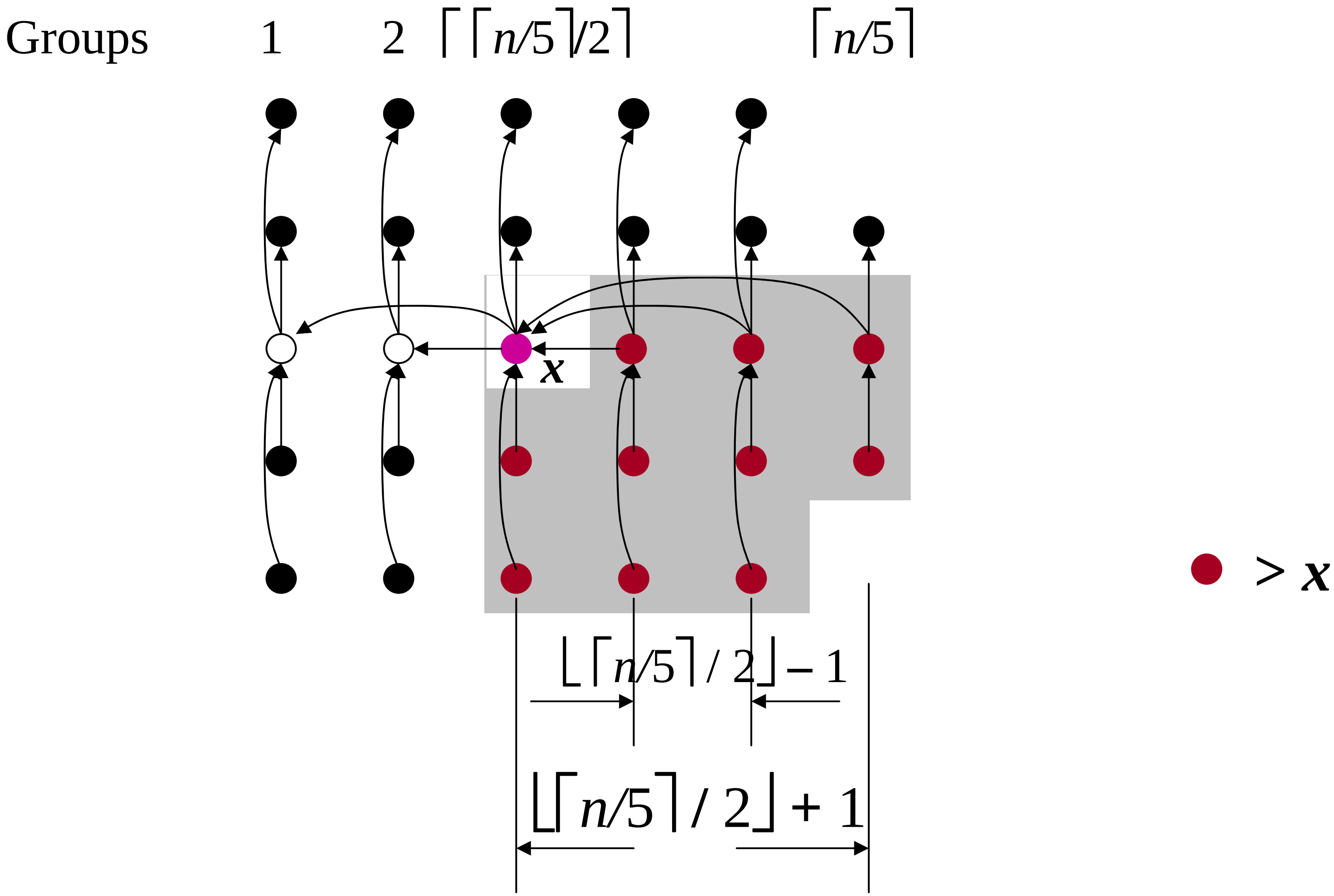
- Randomized algorithm works well in practice
- What follows is a worst-case linear time algorithm, really of theoretical interest only
- Basic idea:
  - Generate a good partitioning element
  - Call this element x

Pictorial Analysis of Select



- **Fig. 9.1 Analysis of the algorithm SELECT. The *n* elements are represented by small circles, and each group occupies a column. The medians of the group are whitened, and the median-of-medians *x* is labeled.**

Pictorial Analysis of Select



Worst-Case Linear-Time Selection

SELECT(i, n)

1. Divide the n elements into groups of 5. Find the median of each 5-element group by hand.
2. Recursively SELECT the median x of the $\lfloor n/5 \rfloor$ group medians to be the pivot.
3. Partition around pivot x. Let k = rank(x)
4. if $i = k$ then return x
   else if $i < k$
   then recursively SELECT the ith smallest element in lower part
   else recursively SELECT the $(i-k)$th smallest element in right part

Same as RAND-SELECT

Choosing the pivot

– Divide n elements into $\lfloor n/5 \rfloor$ groups of 5 elements each, plus extra.

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$n = 54$

Choosing the pivot

– Divide n elements into $\lfloor n/5 \rfloor$ groups of 5 elements each, plus extra.

– Find medians of each group by insertion sort.

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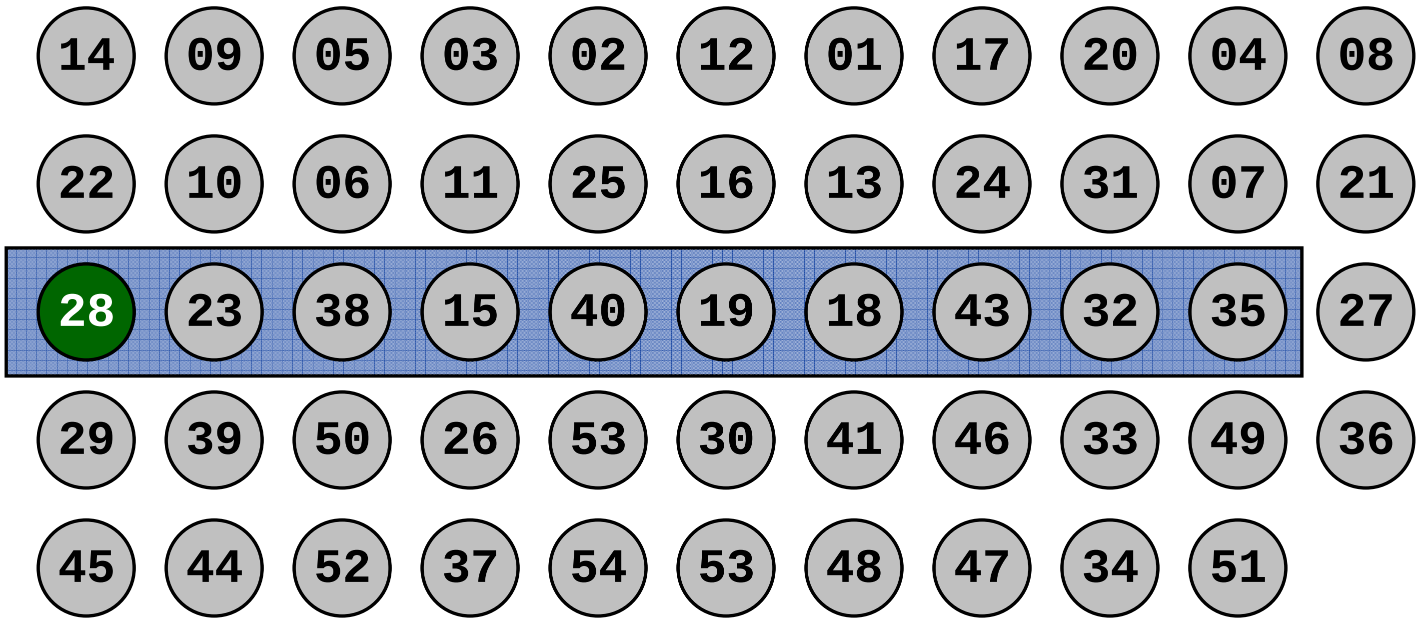
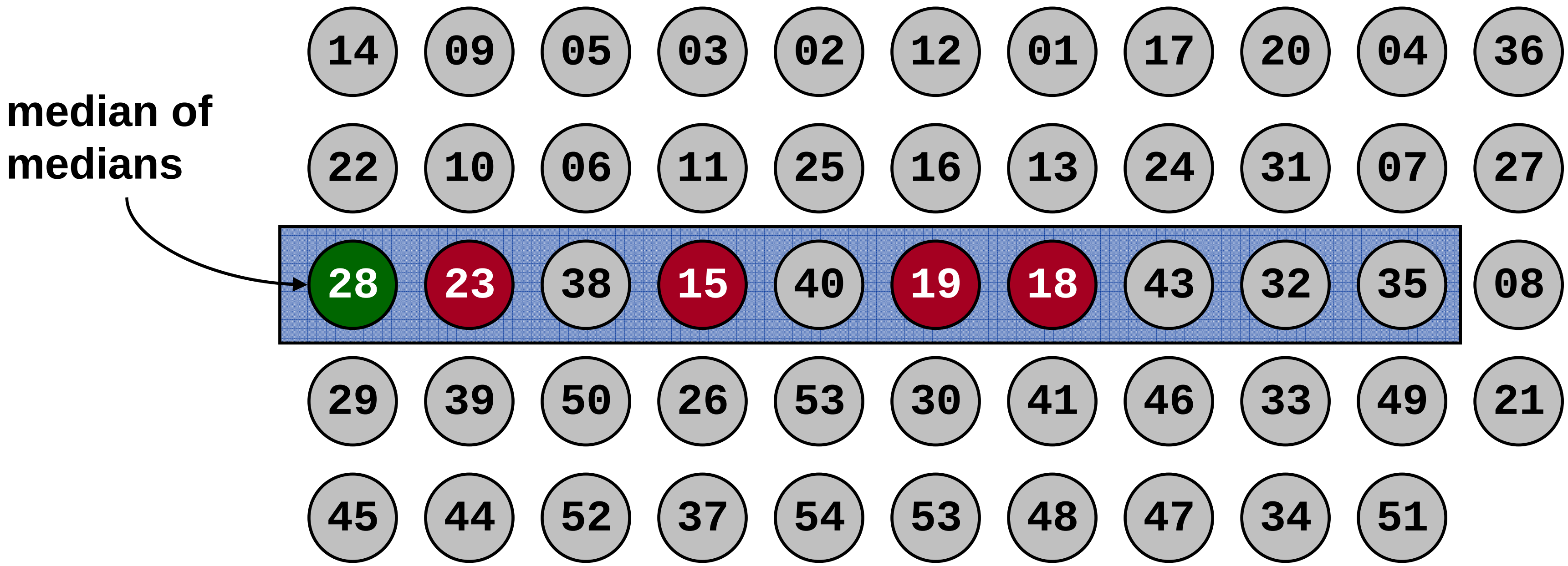
Choosing the pivot

- Divide n elements into $\lfloor n/5 \rfloor$ groups of 5 elements each, plus extra.
- Find medians of each group by insertion sort.
- Find x = "median of medians" by `SELECT()` on $\lfloor n/5 \rfloor$ medians.



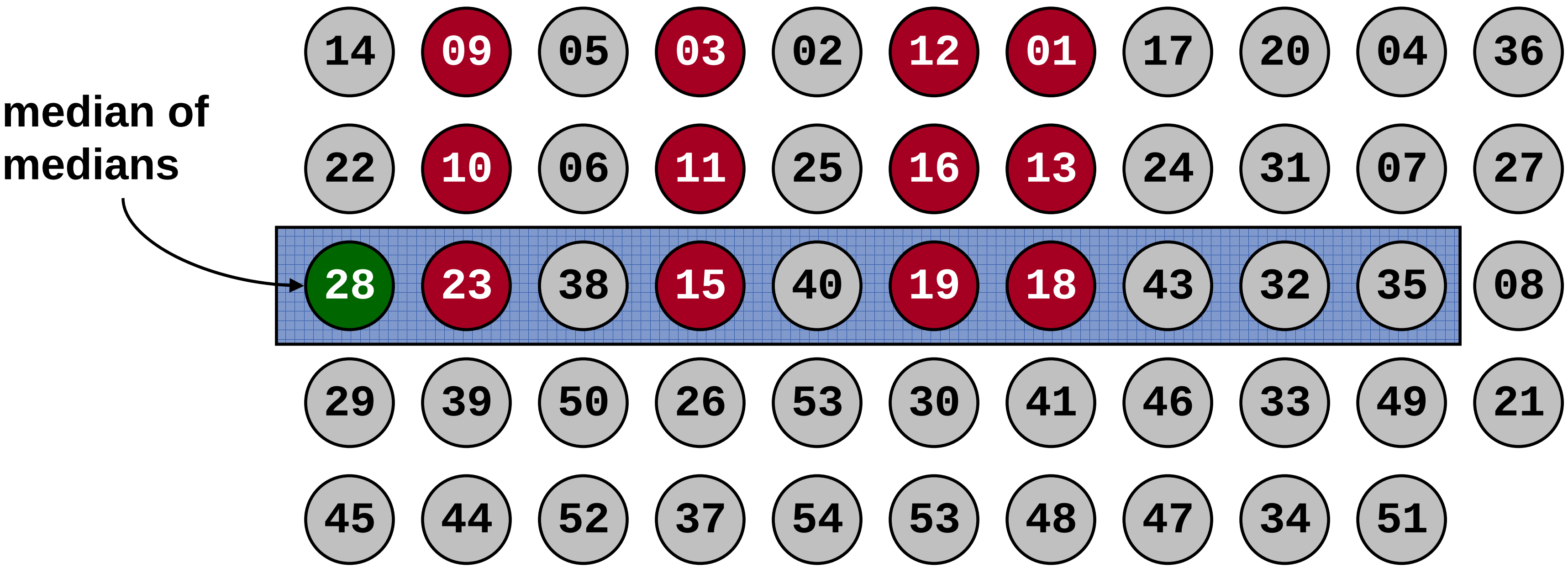
Selection Analysis

- **Crux of proof:**
  - At least **half** the group medians are $\le x$, which is at least $\lfloor \lfloor n/5 \rfloor / 2 \rfloor = \lfloor n/10 \rfloor$ group medians



Selection Analysis

- Crux of proof:
  - At least **half** the group medians are $\leq x$, which is at least $\lfloor \lfloor n/5 \rfloor / 2 \rfloor = \lfloor n/10 \rfloor$ group medians
  - Therefore, at least $3\lfloor n/10 \rfloor$ elements $\leq x$.



Selection Analysis

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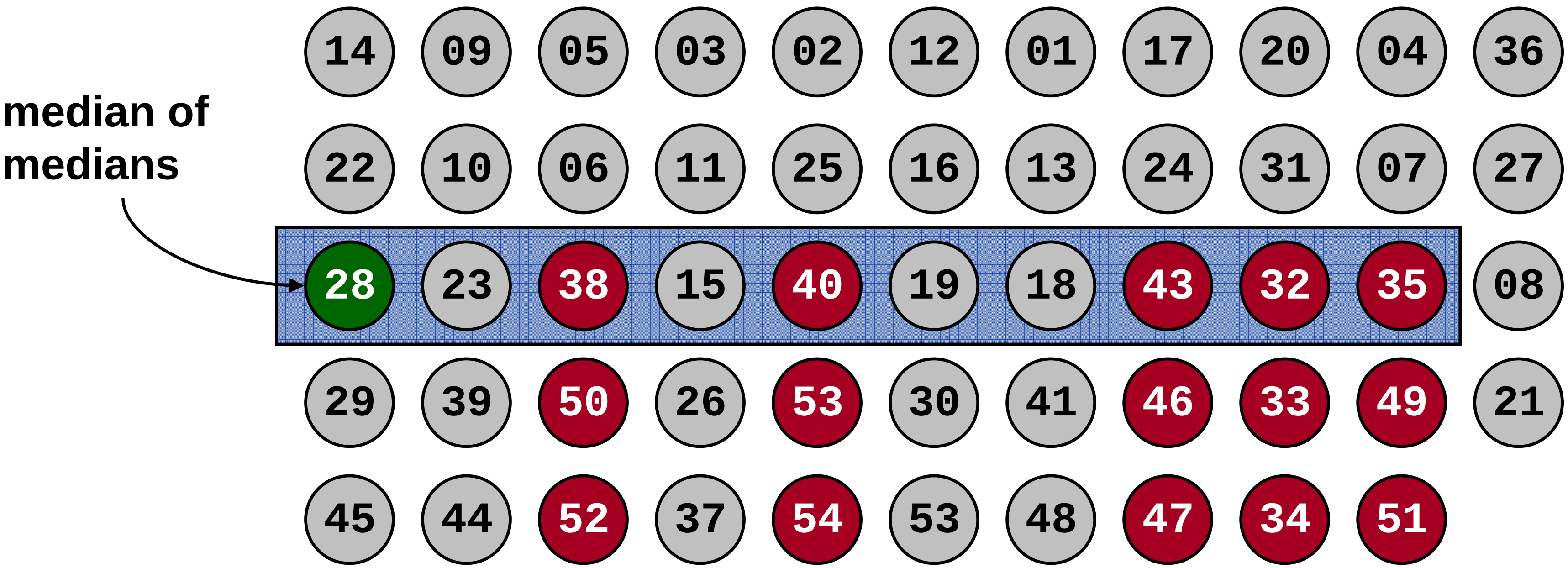
  - At least **half** the group medians are $\leq x$, which is at least $\lfloor \lfloor n/5 \rfloor / 2 \rfloor = \lfloor n/10 \rfloor$ group medians
  - Therefore, at least $3\lfloor n/10 \rfloor$ elements $\leq x$.
  - Similarly, at least $3\lfloor n/10 \rfloor$ elements $\geq x$.



Worst-Case Linear-Time Selection

$T(n)$ SELECT(i, n)

$\Theta(n)$ — 1. Divide the n elements into groups of 5. Find the median of each 5-element group by hand.

$T(n/5)$ — 2. Recursively SELECT the median x of the $\lfloor n/5 \rfloor$ group medians to be the pivot.

$\Theta(n)$ — 3. Partition around pivot x. Let k = rank(x)

$T(7n/10)$ — 4. **if** $i = k$ **then** return x

else if $i < k$

then recursively SELECT the ith smallest element in lower part

else recursively SELECT the $(i-k)$th smallest element in right part

Solving the recurrence

$$T(n) = T(1/5\, n) + T(7/10\, n) + \Theta(n)$$

Substitution:

$T(n) \le cn$

$$T(n) \le 1/5\, cn + 7/10\, cn + \Theta(n)$$
$$= 18/20\, cn + \Theta(n)$$
$$= cn - (2/20\, cn - \Theta(n))$$
$$\le cn$$

If c is chosen large enough to handle the $\Theta(n)$.