

第七节

斯托克斯公式

*环流量与旋度

一、斯托克斯公式

*二、空间曲线积分与路径无关的条件

*三、环流量与旋度



一、斯托克斯公式

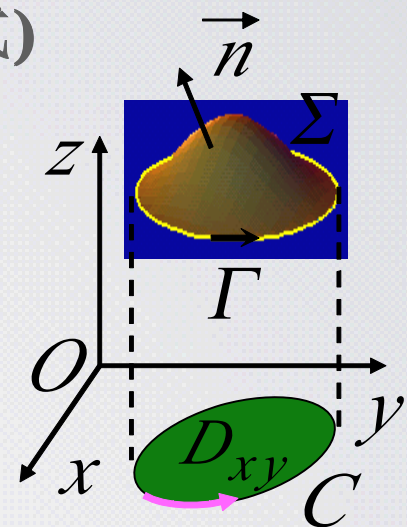
定理1. 设光滑曲面 Σ 的边界 Γ 是分段光滑曲线, Σ 的侧与 Γ 的正向符合右手法则, P, Q, R 在包含 Σ 在内的一个空间域内具有连续一阶偏导数,则有

$$\iint_{\Sigma} \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$
$$= \oint_{\Gamma} P dx + Q dy + R dz \quad (\text{斯托克斯公式})$$

证: 情形1. Σ 与平行 z 轴的直线只交于一点, 设其方程为

$$\Sigma : z = f(x, y), \quad (x, y) \in D_{xy}$$

为确定起见, 不妨设 Σ 取上侧 (如图).



$$\text{则 } \oint_{\Gamma} P dx = \oint_C P(x, y, z(x, y)) dx$$

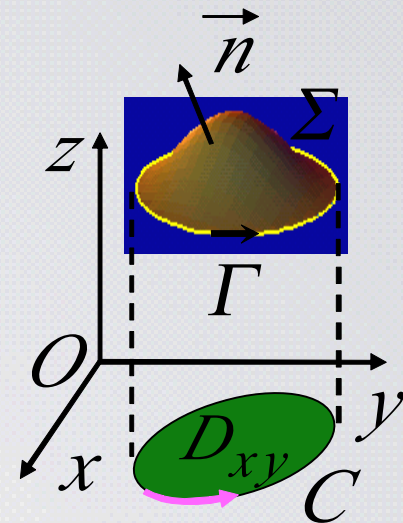
$$= - \iint_{D_{xy}} \frac{\partial}{\partial y} P(x, y, z(x, y)) dx dy \quad (\text{利用格林公式})$$

$$= - \iint_{D_{xy}} \left[\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z} \frac{\partial z}{\partial y} \right] dx dy$$

$$= - \iint_{\Sigma} \left[\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z} f_y \right] \cos \gamma dS$$

$$\therefore \cos \gamma = \frac{1}{\sqrt{1+f_x^2+f_y^2}}, \quad \cos \beta = \frac{-f_y}{\sqrt{1+f_x^2+f_y^2}},$$

$$\therefore f_y = - \frac{\cos \beta}{\cos \gamma}$$



$$\begin{aligned}
 \text{因此 } \oint_{\Gamma} P dx &= - \iint_{\Sigma} \left[\frac{\partial P}{\partial y} - \frac{\partial P \cos \beta}{\partial z \cos \gamma} \right] \cos \gamma dS \\
 &= \iint_{\Sigma} \left[\frac{\partial P}{\partial z} \cos \beta - \frac{\partial P}{\partial y} \cos \gamma \right] dS \\
 &= \iint_{\Sigma} \frac{\partial P}{\partial z} dz dx - \frac{\partial P}{\partial y} dx dy
 \end{aligned}$$

$$\text{同理可证 } \oint_{\Gamma} Q dy = \iint_{\Sigma} \frac{\partial Q}{\partial x} dx dy - \frac{\partial Q}{\partial z} dy dz$$

$$\oint_{\Gamma} R dz = \iint_{\Sigma} \frac{\partial R}{\partial y} dy dz - \frac{\partial R}{\partial x} dz dx$$

三式相加, 即得斯托克斯公式;



情形2 曲面 Σ 与平行 z 轴的直线交点多于一个, 则可通过作辅助线把 Σ 分成与 z 轴只交于一点的几部分, 在每一部分上应用斯托克斯公式, 然后相加, 由于沿辅助曲线方向相反的两个曲线积分相加刚好抵消, 所以对这类曲面斯托克斯公式仍成立. 证毕

注意: 如果 Σ 是 xOy 面上的一块平面区域, 则斯托克斯公式就是格林公式, 故格林公式是斯托克斯公式的特例.

$$\iint_{\Sigma} \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

$$= \oint_{\Gamma} P dx + Q dy + R dz$$

为便于记忆, 斯托克斯公式还可写作:

$$\iint_{\Sigma} \begin{vmatrix} dydz & dzdx & dx dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \oint_{\Gamma} P dx + Q dy + R dz$$

或用第一类曲面积分表示:

$$\iint_{\Sigma} \begin{vmatrix} \cos \alpha & \cos \beta & \cos \gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS = \oint_{\Gamma} P dx + Q dy + R dz$$

例1. 利用斯托克斯公式计算积分 $\oint_{\Gamma} z dx + x dy + y dz$
 其中 Γ 为平面 $x + y + z = 1$ 被三坐标面所截三角形的整个边界, 方向如图所示.

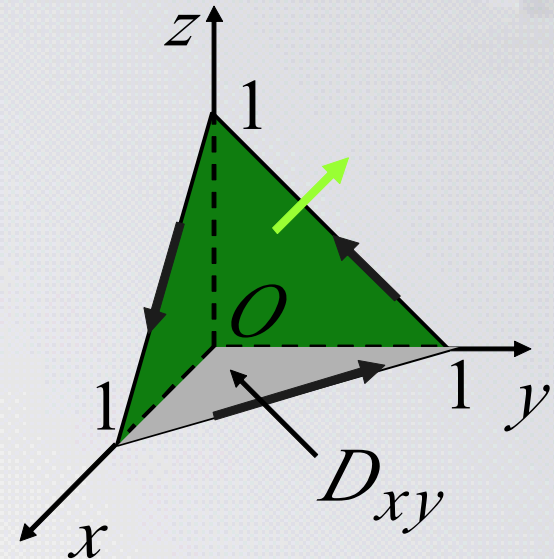
解: 记三角形域为 Σ , 取上侧, 则

$$\oint_{\Gamma} z dx + x dy + y dz$$

$$= \iint_{\Sigma} \begin{vmatrix} dy dz & dz dx & dx dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$$

$$= \iint_{\Sigma} dy dz + dz dx + dx dy = 3 \iint_{D_{xy}} dx dy = \frac{3}{2}$$

利用对称性



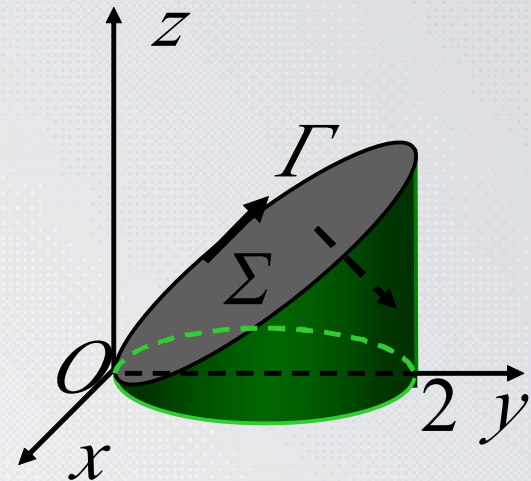
例2. Γ 为柱面 $x^2 + y^2 = 2y$ 与平面 $y = z$ 的交线, 从 z 轴正向看为顺时针, 计算 $I = \oint_{\Gamma} y^2 dx + xy dy + xz dz$.

解: 设 Σ 为平面 $z = y$ 上被 Γ 所围椭圆域, 且取下侧, 则其法线方向余弦

$$\cos \alpha = 0, \quad \cos \beta = \frac{1}{\sqrt{2}}, \quad \cos \gamma = -\frac{1}{\sqrt{2}}$$

利用斯托克斯公式得

$$I = \iint_{\Sigma} \begin{vmatrix} \cos \alpha & \cos \beta & \cos \gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & xy & xz \end{vmatrix} dS = \frac{1}{\sqrt{2}} \iint_{\Sigma} (y - z) dS = 0$$



*二、空间曲线积分与路径无关的条件

定理2. 设 G 是空间一维单连通域, 函数 P, Q, R 在 G 内具有连续一阶偏导数, 则下列四个条件相互等价:

(1) 对 G 内任一分段光滑闭曲线 Γ , 有

$$\oint_{\Gamma} P dx + Q dy + R dz = 0$$

(2) 对 G 内任一分段光滑曲线 Γ , $\int_{\Gamma} P dx + Q dy + R dz$ 与路径无关

(3) 在 G 内存在某一函数 u , 使 $du = P dx + Q dy + R dz$

(4) 在 G 内处处有

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

证: (4) \Rightarrow (1) 由斯托克斯公式可知结论成立;

(1) \Rightarrow (2) (自证)

(2) \Rightarrow (3) 设函数

$$u(x, y, z) = \int_{(x_0, y_0, z_0)}^{(x, y, z)} P dx + Q dy + R dz$$

则

$$\begin{aligned} \frac{\partial u}{\partial x} &= \lim_{\Delta x \rightarrow 0} \frac{u(x+\Delta x, y, z) - u(x, y, z)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \int_{(x, y, z)}^{(x+\Delta x, y, z)} P dx + Q dy + R dz \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \int_x^{x+\Delta x} P dx = \lim_{\Delta x \rightarrow 0} p(x+\Delta x, y, z) \\ &= P(x, y, z) \end{aligned}$$

同理可证 $\frac{\partial u}{\partial y} = Q(x, y, z), \quad \frac{\partial u}{\partial z} = R(x, y, z)$

故有 $du = Pdx + Qdy + Rdz$

(3) \Rightarrow (4) 若(3)成立, 则必有

$$\frac{\partial u}{\partial x} = P, \quad \frac{\partial u}{\partial y} = Q, \quad \frac{\partial u}{\partial z} = R$$

因 P, Q, R 一阶偏导数连续, 故有

$$\frac{\partial P}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$

同理 $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$

证毕

例3. 验证曲线积分 $\int_{\Gamma} (y+z)dx + (z+x)dy + (x+y)dz$ 与路径无关, 并求函数

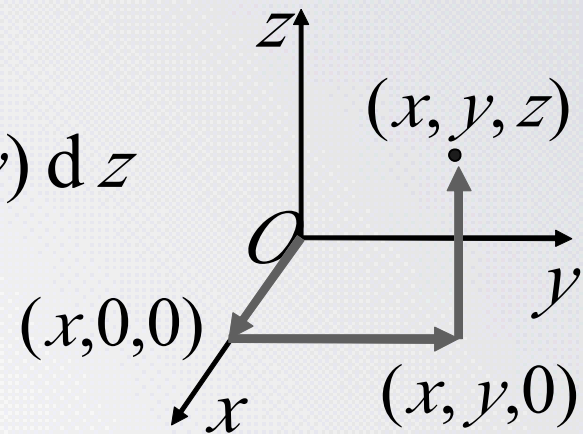
$$u(x, y, z) = \int_{(0,0,0)}^{(x,y,z)} (y+z)dx + (z+x)dy + (x+y)dz$$

解: 令 $P = y+z$, $Q = z+x$, $R = x+y$

$$\therefore \frac{\partial P}{\partial y} = 1 = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = 1 = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = 1 = \frac{\partial P}{\partial z}$$

\therefore 积分与路径无关, 因此

$$\begin{aligned} u(x, y, z) &= \int_0^x 0 dx + \int_0^y x dy + \int_0^z (x+y) dz \\ &= xy + (x+y)z \\ &= xy + yz + zx \end{aligned}$$



*三、环流量与旋度

斯托克斯公式

$$\begin{aligned} \iint_{\Sigma} \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \\ = \oint_{\Gamma} P dx + Q dy + R dz \end{aligned}$$

设曲面 Σ 的法向量为 $\vec{n} = (\cos \alpha, \cos \beta, \cos \gamma)$

曲线 Γ 的单位切向量为 $\vec{\tau} = (\cos \lambda, \cos \mu, \cos \nu)$

则斯托克斯公式可写为

$$\begin{aligned} \iint_{\Sigma} \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \cos \alpha + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \cos \beta + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \cos \gamma \right] dS \\ = \oint_{\Gamma} (P \cos \lambda + Q \cos \mu + R \cos \nu) ds \end{aligned}$$



令 $\vec{A} = (P, Q, R)$, 引进一个向量

$$\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \nabla \times \vec{A} \quad \underline{\text{记作}} \quad \text{rot} \vec{A}$$

于是得斯托克斯公式的向量形式：

$$\iint_{\Sigma} \text{rot} \vec{A} \cdot \vec{n} dS = \oint_{\Gamma} \vec{A} \cdot \vec{\tau} ds$$

或
$$\iint_{\Sigma} (\text{rot} A)_n dS = \oint_{\Gamma} A_{\tau} ds \quad \textcircled{1}$$

定义： $\oint_{\Gamma} P dx + Q dy + R dz = \oint_{\Gamma} A_{\tau} ds$ 称为向量场 \vec{A} 沿有向闭曲线 Γ 的环流量. 向量 $\text{rot} \vec{A}$ 称为向量场 \vec{A} 的旋度.

旋度的力学意义:

设某刚体绕定轴 l 转动, 角速度为 $\vec{\omega}$, M 为刚体上任一点, 建立坐标系如图, 则

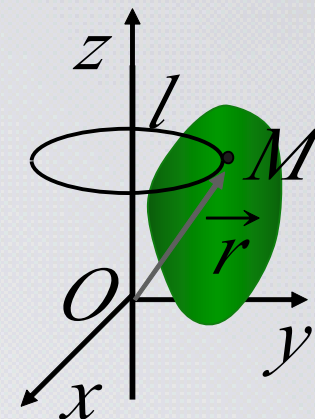
$$\vec{\omega} = (0, 0, \omega), \quad \vec{r} = (x, y, z)$$

点 M 的线速度为

$$\vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = (-\omega y, \omega x, 0)$$

$$\text{rot } \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = (0, 0, 2\omega) = 2\vec{\omega}$$

(此即“旋度”一词的来源)

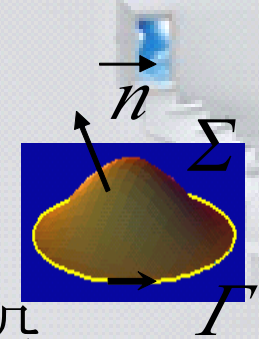


斯托克斯公式①的物理意义:

$$\iint_{\Sigma} (\text{rot } A)_n dS = \oint_{\Gamma} A_{\tau} ds$$

向量场 \vec{A} 产生的旋度场
穿过 Σ 的通量

向量场 \vec{A} 沿
 Γ 的环流量



注意 Σ 与 Γ 的方向形成右手系!

例4. 求电场强度 $\vec{E} = \frac{q}{r^3} \vec{r}$ 的旋度.

解:
$$\text{rot } \vec{E} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{qx}{r^3} & \frac{qy}{r^3} & \frac{qz}{r^3} \end{vmatrix} = (0, 0, 0) \quad (\text{除原点外})$$

这说明, 在除点电荷所在原点外, 整个电场无旋.

例5. 设 $\vec{A} = (2y, 3x, z^2)$, $\Sigma : x^2 + y^2 + z^2 = 4$, \vec{n} 为 Σ 的外法向量, 计算 $I = \oiint_{\Sigma} \text{rot } \vec{A} \cdot \vec{n} dS$.

解: $\text{rot } \vec{A} = \nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 3x & z^2 \end{vmatrix} = (0, 0, 1)$

$$\vec{n} = (\cos \alpha, \cos \beta, \cos \gamma)$$

$$\begin{aligned} \therefore I &= \oiint_{\Sigma} \cos \gamma dS = \iint_{\Sigma_{\text{上}}} dx dy + \iint_{\Sigma_{\text{下}}} dx dy \\ &= \iint_{D_{xy}} dx dy - \iint_{D_{xy}} dx dy = 0 \end{aligned}$$