3 行列式的性质

记 $D=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$, $D^T=\begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix}$

行列式 D^T 称为行列式 D 的转置行列式(transpose).

若记 $D=\det(a_{ij})$, $D^T=\det(b_{ij})$，则 $b_{ij}=a_{ji}$.

性质1 行列式与它的转置行列式相等, 即 $D=D^T$.

性质1 行列式与它的转置行列式相等.

证明 若记 $D=\det(a_{ij}), D^T=\det(b_{ij})$ ，则 $a_{ij}=b_{ji}$

根据行列式的定义，有

$$\begin{aligned} D^T &= \sum_{p_1p_2\cdots p_n}(-1)^{t(p_1p_2\cdots p_n)} b_{1p_1} b_{2p_2}\cdots b_{np_n} \\ &\qquad\qquad\qquad\qquad\quad\ \Updownarrow\quad\Updownarrow\qquad\quad\Updownarrow \\ &= \sum_{p_1p_2\cdots p_n}(-1)^{t(p_1p_2\cdots p_n)} a_{p_1 1} a_{p_2 2}\cdots a_{p_n n} \\ &= D \end{aligned}$$

行列式中行与列具有同等的地位, 行列式的性质凡是对行成立的对列也同样成立.

性质2　互换行列式的两行（列），行列式变号.

备注：交换第i行（列）和第j行（列），记作$r_i \leftrightarrow r_j (c_i \leftrightarrow c_j)$.

验证

$$\begin{vmatrix} 1 & 7 & 5 \\ 6 & 6 & 2 \\ 3 & 5 & 8 \end{vmatrix} = -196 \qquad \begin{vmatrix} 1 & 7 & 5 \\ 3 & 5 & 8 \\ 6 & 6 & 2 \end{vmatrix} = 196$$

于是

$$\begin{vmatrix} 1 & 7 & 5 \\ 6 & 6 & 2 \\ 3 & 5 & 8 \end{vmatrix} = -\begin{vmatrix} 1 & 7 & 5 \\ 3 & 5 & 8 \\ 6 & 6 & 2 \end{vmatrix}$$

推论　如果行列式有两行（列）完全相同，则此行列式为零.

证明　互换相同的两行，有$D = -D$，所以$D = 0$.

性质3　行列式的某一行（列）中所有的元素都乘以同一个倍数 k，等于用数 k 乘以此行列式.

备注：第 i 行（列）乘以 k，记作 $r_i \times k(c_i \times k)$.

验证　我们以三阶行列式为例. 记

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \qquad D_1 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

根据三阶行列式的对角线法则，有

$$D_1=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$=a_{11}(ka_{22})a_{33}+a_{12}(ka_{23})a_{31}+a_{13}(ka_{21})a_{32}$$
$$-a_{13}(ka_{22})a_{31}-a_{12}(ka_{21})a_{33}-a_{11}(ka_{23})a_{32}$$

$$=k\begin{pmatrix} a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32} \\ -a_{13}a_{22}a_{31}-a_{12}a_{21}a_{33}-a_{11}a_{23}a_{32} \end{pmatrix}=kD$$

推论　行列式的某一行（列）中所有元素的公因子可以提到行列式符号的外面.

备注：第i行（列）提出公因子k，记作$r_i \div k(c_i \div k)$．

性质4　行列式中如果有两行（列）元素成比例，则此行列式为零.

验证　我们以4阶行列式为例.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ ka_{11} & ka_{12} & ka_{13} & ka_{14} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{11} & a_{12} & a_{13} & a_{14} \end{vmatrix} = k \cdot 0 = 0$$

性质5 若行列式的某一列（行）的元素都是两数之和，例如：

$$D=\begin{vmatrix} a_{11} & a_{12}+b_{12} & a_{13} \\ a_{21} & a_{22}+b_{22} & a_{23} \\ a_{31} & a_{32}+b_{32} & a_{33} \end{vmatrix}$$

则 $D=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}+\begin{vmatrix} a_{11} & b_{12} & a_{13} \\ a_{21} & b_{22} & a_{23} \\ a_{31} & b_{32} & a_{33} \end{vmatrix}$

验证 我们以三阶行列式为例.

$$D=\begin{vmatrix} a_{11} & a_{12}+b_{12} & a_{13} \\ a_{21} & a_{22}+b_{22} & a_{23} \\ a_{31} & a_{32}+b_{32} & a_{33} \end{vmatrix}$$

$$=\sum_{p_1p_2p_3}(-1)^{t(p_1p_2p_3)}a_{1p_1}(a_{2p_2}+b_{2p_2})a_{3p_3}$$

$$=\sum_{p_1p_2p_3}(-1)^{t(p_1p_2p_3)}a_{1p_1}a_{2p_2}a_{3p_3}+\sum_{p_1p_2p_3}(-1)^{t(p_1p_2p_3)}a_{1p_1}b_{2p_2}a_{3p_3}$$

$$=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}+\begin{vmatrix} a_{11} & b_{12} & a_{13} \\ a_{21} & b_{22} & a_{23} \\ a_{31} & b_{32} & a_{33} \end{vmatrix}$$

性质6 把行列式的某一列（行）的各元素乘以同一个倍数然后加到另一列(行)对应的元素上去，行列式不变.

备注： 以数 k 乘第j 行（列）加到第i 行（列）上，记作 $r_i + kr_j (c_i + kc_j)$.

验证 我们以三阶行列式为例. 记

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \quad D_1 = \begin{vmatrix} a_{11} & a_{12} + ka_{13} & a_{13} \\ a_{21} & a_{22} + ka_{23} & a_{23} \\ a_{31} & a_{32} + ka_{33} & a_{33} \end{vmatrix}$$

则 $D = D_1$.

二、应用举例

计算行列式常用方法：利用运算 $r_i + kr_j$ 把行列式化为上三角形行列式，从而算得行列式的值.

例 1

$$D = \left|\begin{array}{ccccc} 1 & -1 & 2 & -3 & 1 \\ -3 & 3 & -7 & 9 & -5 \\ 2 & 0 & 4 & -2 & 1 \\ 3 & -5 & 7 & -14 & 6 \\ 4 & -4 & 10 & -10 & 2 \end{array}\right| \begin{array}{l} \times 3 \\ \xleftarrow{\oplus} \\ \\ \\ \\ \end{array}$$

解

$$D = \begin{vmatrix} 1 & -1 & 2 & -3 & 1 \\ -3 & 3 & -7 & 9 & -5 \\ 2 & 0 & 4 & -2 & 1 \\ 3 & -5 & 7 & -14 & 6 \\ 4 & -4 & 10 & -10 & 2 \end{vmatrix} \begin{array}{l} \times 3 \\ \oplus \\ \\ \\ \\ \end{array}$$

$$\xlongequal{r_2 + 3r_1} \begin{vmatrix} 1 & -1 & 2 & -3 & 1 \\ 0 & 0 & -1 & 0 & -2 \\ 2 & 0 & 4 & -2 & 1 \\ 3 & -5 & 7 & -14 & 6 \\ 4 & -4 & 10 & -10 & 2 \end{vmatrix}$$

$$\overset{r_2+3r_1}{=} \begin{vmatrix} \boxed{1 \quad -1 \quad 2 \quad -3 \quad 1} \\ 0 \quad 0 \quad -1 \quad 0 \quad -2 \\ 2 \quad 0 \quad 4 \quad -2 \quad 1 \\ 3 \quad -5 \quad 7 \quad -14 \quad 6 \\ 4 \quad -4 \quad 10 \quad -10 \quad 2 \end{vmatrix} \begin{matrix} \times(-2) \\ \oplus \\ \leftarrow \\ \\ \\ \end{matrix}$$

$$\overset{r_2-2r_1}{=} \begin{matrix} (-4)\times \\ \\ \oplus \\ \\ \rightarrow \end{matrix} \begin{vmatrix} \boxed{1 \quad -1 \quad 2 \quad -3 \quad 1} \\ 0 \quad 0 \quad -1 \quad 0 \quad -2 \\ 0 \quad 2 \quad 0 \quad 4 \quad -1 \\ 3 \quad -5 \quad 7 \quad -14 \quad 6 \\ 4 \quad -4 \quad 10 \quad -10 \quad 2 \end{vmatrix} \begin{matrix} \times(-3) \\ \\ \oplus \\ \leftarrow \\ \\ \end{matrix}$$

$$\underset{r_4-4r_1}{\overset{r_3-3r_1}{=\!=\!=}}\begin{vmatrix} 1 & -1 & 2 & -3 & 1 \\ 0 & 0 & -1 & 0 & -2 \\ 0 & 2 & 0 & 4 & -1 \\ 0 & -2 & 1 & -5 & 3 \\ 0 & 0 & 2 & 2 & -2 \end{vmatrix}$$

$$\overset{r_2\leftrightarrow r_4}{=\!=\!=}-\begin{vmatrix} 1 & -1 & 2 & -3 & 1 \\ 0 & -2 & 1 & -5 & 3 \\ 0 & 2 & 0 & 4 & -1 \\ 0 & 0 & -1 & 0 & -2 \\ 0 & 0 & 2 & 2 & -2 \end{vmatrix} \oplus$$

$$\overset{r_3+r_2}{=\!=\!=} -\begin{vmatrix} 1 & -1 & 2 & -3 & 1 \\ 0 & -2 & 1 & -5 & 3 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & -1 & 0 & -2 \\ 0 & 0 & 2 & 2 & -2 \end{vmatrix} \begin{matrix} \\ \\ \\ \oplus \\ \\ \end{matrix}$$

$$\overset{r_4+r_3}{=\!=\!=} -\begin{vmatrix} 1 & -1 & 2 & -3 & 1 \\ 0 & -2 & 1 & -5 & 3 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 2 & -2 \end{vmatrix} \begin{matrix} \\ \\ \times(-2) \\ \oplus \\ \\ \end{matrix}$$

$$\xlongequal{r_5-2r_3} -\left|\begin{array}{ccccc} 1 & -1 & 2 & -3 & 1 \\ 0 & -2 & 1 & -5 & 3 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 4 & -6 \end{array}\right| \begin{array}{l} \\ \\ \\ \times 4 \\ \oplus \end{array}$$

$$\xlongequal{r_5+4r_4} -\begin{vmatrix} 1 & -1 & 2 & -3 & 1 \\ 0 & -2 & 1 & -5 & 3 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -6 \end{vmatrix} = -(-2)(-1)(-6) = 12.$$

例2 计算 n 阶行列式 $D=\begin{vmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b & b & b & \cdots & a \end{vmatrix}$

解 将第 $2,3,\cdots,n$ 列都加到第一列得

$$D=\begin{vmatrix} a+(n-1)b & b & b & \cdots & b \\ a+(n-1)b & a & b & \cdots & b \\ a+(n-1)b & b & a & \cdots & b \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a+(n-1)b & b & b & \cdots & a \end{vmatrix}$$

$$=\left[a+(n-1)b\right]\begin{vmatrix} 1 & b & b & \cdots & b \\ 1 & a & b & \cdots & b \\ 1 & b & a & \cdots & b \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & b & b & \cdots & a \end{vmatrix}$$

$$=\left[a+(n-1)b\right]\begin{vmatrix} 1 & b & b & \cdots & b \\ & a-b & & & \\ & & a-b & & \mathbf{0} \\ & \mathbf{0} & & \ddots & \\ & & & & a-b \end{vmatrix}=\left[a+(n-1)b\right](a-b)^{n-1}.$$

例3 设 $D=\begin{vmatrix} a_{11} & \cdots & a_{1k} & & & \\ \vdots & & \vdots & & 0 & \\ a_{k1} & \cdots & a_{kk} & & & \\ c_{11} & \cdots & c_{1k} & b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ c_{n1} & \cdots & c_{nk} & b_{n1} & \cdots & b_{nn} \end{vmatrix}$

$$D_1=\det(a_{ij})=\begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix},\ D_2=\det(b_{ij})=\begin{vmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{vmatrix},$$

证明 $D=D_1D_2.$

证明

对 D_1 作运算 $r_i + kr_j$ ，把 D_1 化为下三角形行列式

设为 $$D_1 = \begin{vmatrix} p_{11} & & 0 \\ \vdots & \ddots & \\ p_{k1} & \cdots & p_{kk} \end{vmatrix} = p_{11} \cdots p_{kk};$$

对 D_2 作运算 $c_i + kc_j$，把 D_2 化为下三角形行列式

设为 $$D_2 = \begin{vmatrix} q_{11} & & 0 \\ \vdots & \ddots & \\ q_{n1} & \cdots & p_{nk} \end{vmatrix} = q_{11} \cdots q_{nn}.$$

对 D 的前 $\textcircled{k}$ 行作运算 $r_i + kr_j$，再对后 $\textcircled{n}$ 列作运算 $c_i + kc_j$，把 D 化为下三角形行列式

$$D = \begin{vmatrix} \boxed{\begin{matrix} p_{11} & & \\ \vdots & \ddots & \\ p_{k1} & \cdots & p_{kk} \end{matrix}} & \mathbf{0} \\ \begin{matrix} c_{11} & \cdots & c_{1k} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nk} \end{matrix} & \boxed{\begin{matrix} q_{11} & & \\ \vdots & \ddots & \\ q_{n1} & \cdots & q_{nn} \end{matrix}} \end{vmatrix},$$

故 $\quad D = p_{11} \cdots p_{kk} \cdot q_{11} \cdots q_{nn} = D_1 D_2 .$

三、小结

行列式的6个性质(行列式中行与列具有同等的地位，凡是对行成立的性质对列也同样成立).

计算行列式常用方法：(1)利用定义;(2)利用性质把行列式化为上三角形行列式，从而算得行列式的值.

思考题

计算4阶行列式

$$D = \begin{vmatrix} a^2+\frac{1}{a^2} & a & \frac{1}{a} & 1 \\ b^2+\frac{1}{b^2} & b & \frac{1}{b} & 1 \\ c^2+\frac{1}{c^2} & c & \frac{1}{c} & 1 \\ d^2+\frac{1}{d^2} & d & \frac{1}{d} & 1 \end{vmatrix} \qquad (\text{已知}\ abcd=1)$$

思考题解答

解

$$D=\begin{vmatrix} a^2 & a & \frac{1}{a} & 1 \\ b^2 & b & \frac{1}{b} & 1 \\ c^2 & c & \frac{1}{c} & 1 \\ d^2 & d & \frac{1}{d} & 1 \end{vmatrix} + \begin{vmatrix} \frac{1}{a^2} & a & \frac{1}{a} & 1 \\ \frac{1}{b^2} & b & \frac{1}{b} & 1 \\ \frac{1}{c^2} & c & \frac{1}{c} & 1 \\ \frac{1}{d^2} & d & \frac{1}{d} & 1 \end{vmatrix}$$

$$= abcd\begin{vmatrix} a & 1 & \frac{1}{a^2} & \frac{1}{a} \\ b & 1 & \frac{1}{b^2} & \frac{1}{b} \\ c & 1 & \frac{1}{c^2} & \frac{1}{c} \\ d & 1 & \frac{1}{d^2} & \frac{1}{d} \end{vmatrix} + (-1)^3\begin{vmatrix} a & 1 & \frac{1}{a^2} & \frac{1}{a} \\ b & 1 & \frac{1}{b^2} & \frac{1}{b} \\ c & 1 & \frac{1}{c^2} & \frac{1}{c} \\ d & 1 & \frac{1}{d^2} & \frac{1}{d} \end{vmatrix}$$

$$= 0.$$

5. 行列式按行（列）展开

一、引言

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

$$= a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) + a_{12}\left(a_{23}a_{31} - a_{21}a_{33}\right)$$

$$+ a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right)$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

结论 三阶行列式可以用二阶行列式表示.

思考题 任意一个行列式是否都可以用较低阶的行列式表示？

在n阶行列式中，把元素a_{ij}所在的第i行和第j列划后，留下来的$n-1$阶行列式叫做元素a_{ij}的余子式(complement minor)，记作M_{ij} .

把 $A_{ij}=(-1)^{i+j}M_{ij}$称为元素a_{ij}的代数余子式.

例如

$$D=\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$M_{23}=\begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}$$

$$A_{23}=(-1)^{2+3}M_{23}=-M_{23}$$

结论 因为行标和列标可唯一标识行列式的元素，所以行列式中每一个元素都分别对应着一个余子式和一个代数余子式.

引理　一个n阶行列式，如果其中第i行所有元素除a_{ij}外都为零，那么这行列式等于a_{ij}与它的代数余子式的乘积，即$D = a_{ij}A_{ij}$.

例如
$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{33}A_{33} = (-1)^{3+3}a_{33}M_{33}$$

$$= (-1)^{3+3}a_{33}\begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} = a_{33}\begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}$$

分析 当 a_{ij} 位于第1行第1列时,

$$D = \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

即有 $D = a_{11}M_{11}$.

又 $A_{11} = (-1)^{1+1} M_{11} = M_{11}$,

从而 $D = a_{11}A_{11}$.

下面再讨论一般情形.

我们以4阶行列式为例.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \overset{r_2 \leftrightarrow r_3}{=} (-1) \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 0 & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$\overset{r_1 \leftrightarrow r_2}{=} (-1)^2 \begin{vmatrix} 0 & 0 & 0 & a_{34} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = (-1)^{(3-1)} \begin{vmatrix} 0 & 0 & 0 & a_{34} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

思考题： 能否以$r_1 \leftrightarrow r_3$代替上述两次行变换？

思考题： 能否以$r_1 \leftrightarrow r_3$代替上述两次行变换?

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \overset{\substack{r_2 \leftrightarrow r_3 \\ r_1 \leftrightarrow r_2}}{=} (-1)^2 \begin{vmatrix} 0 & 0 & 0 & a_{34} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \overset{r_1 \leftrightarrow r_3}{=} (-1) \begin{vmatrix} 0 & 0 & 0 & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

答： 不能.

$$=(-1)^{(3-1)}\begin{vmatrix} 0 & 0 & 0 & a_{34} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \overset{\substack{c_3\leftrightarrow c_4 \\ c_2\leftrightarrow c_3 \\ c_1\leftrightarrow c_2}}{=} (-1)^{(3-1)}(-1)^{3}\begin{vmatrix} a_{34} & 0 & 0 & 0 \\ a_{14} & a_{11} & a_{12} & a_{13} \\ a_{24} & a_{21} & a_{22} & a_{23} \\ a_{44} & a_{41} & a_{42} & a_{43} \end{vmatrix}$$

$$=(-1)^{(3-1)}(-1)^{(4-1)}\begin{vmatrix} a_{34} & 0 & 0 & 0 \\ a_{14} & a_{11} & a_{12} & a_{13} \\ a_{24} & a_{21} & a_{22} & a_{23} \\ a_{44} & a_{41} & a_{42} & a_{43} \end{vmatrix}$$

a_{34} 被调换到第1行，第1列

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$=(-1)^{3+4-2}a_{34}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}=(-1)^{3+4}a_{34}M_{34}=a_{34}A_{34}$$

二、行列式按行（列）展开法则

定理3　行列式等于它的任一行（列）的各元素与其对应的代数余子式乘积之和，即

$$D = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \left(i = 1, 2, \cdots, n \right)$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11}+0+0 & 0+a_{12}+0 & 0+0+a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

同理可得 $= a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$

$$= a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$$

例

$$D=\begin{vmatrix} 3 & 1 & -1 & 2 \\ -5 & 1 & 3 & -4 \\ 2 & 0 & 1 & -1 \\ 1 & -5 & 3 & -3 \end{vmatrix} \xlongequal[c_4+c_3]{c_1+(-2)c_3} \begin{vmatrix} 5 & 1 & -1 & 1 \\ -11 & 1 & 3 & -1 \\ 0 & 0 & 1 & 0 \\ -5 & -5 & 3 & 0 \end{vmatrix}$$

$$=(-1)^{3+3}\begin{vmatrix} 5 & 1 & 1 \\ -11 & 1 & -1 \\ -5 & -5 & 0 \end{vmatrix} \xlongequal{r_2+r_1} \begin{vmatrix} 5 & 1 & 1 \\ -6 & 2 & 0 \\ -5 & -5 & 0 \end{vmatrix}$$

$$=(-1)^{1+3}\begin{vmatrix} -6 & 2 \\ -5 & -5 \end{vmatrix}=\begin{vmatrix} -8 & 2 \\ 0 & -5 \end{vmatrix}=40.$$

例 证明范德蒙德(Vandermonde)行列式

$$D_n = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{n \ge i > j \ge 1} (x_i - x_j). \quad (1)$$

证明 用数学归纳法

$$D_2 = \begin{vmatrix} 1 & 1 \\ x_1 & x_2 \end{vmatrix} = x_2 - x_1 = \prod_{2 \ge i > j \ge 1} (x_i - x_j)$$

所以n=2时(1)式成立.

假设(1)对于$n-1$阶范德蒙行列式成立，从第n行开始，后行减去前行的x_1倍：

$$D_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & x_2 - x_1 & x_3 - x_1 & \cdots & x_n - x_1 \\ 0 & x_2(x_2 - x_1) & x_3(x_3 - x_1) & \cdots & x_n(x_n - x_1) \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & x_2^{n-2}(x_2 - x_1) & x_3^{n-2}(x_3 - x_1) & \cdots & x_n^{n-2}(x_n - x_1) \end{vmatrix}$$

按照第1列展开，并提出每列的公因子$(x_i - x_1)$，就有

$$= (x_2 - x_1)(x_3 - x_1)\cdots(x_n - x_1)\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_2 & x_3 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_2^{n-2} & x_3^{n-2} & \cdots & x_n^{n-2} \end{vmatrix}$$

$n-1$阶范德蒙德行列式

$$\therefore \quad D_n = (x_2 - x_1)(x_3 - x_1)\cdots(x_n - x_1)\prod_{n \geq i > j \geq 2}(x_i - x_j)$$

$$= \prod_{n \geq i > j \geq 1}(x_i - x_j).$$

推论　行列式任一行（列）的元素与另一行（列）的对应元素的代数余子式乘积之和等于零，即

$$a_{i1}A_{j1}+a_{i2}A_{j2}+\cdots+a_{in}A_{jn}=0,\quad i\neq j.$$

分析　我们以3阶行列式为例.

$$a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13}=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

把第1行的元素换成第2行的对应元素，则

$$a_{21}A_{11}+a_{22}A_{12}+a_{23}A_{13}=\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=0.$$

定理3　行列式等于它的任一行（列）的各元素与其对应的代数余子式乘积之和，即

$$a_{i1}A_{i1}+a_{i2}A_{i2}+\cdots+a_{in}A_{in}=D\left(i=1,2,\cdots,n\right)$$

推论　行列式任一行（列）的元素与另一行（列）的对应元素的代数余子式乘积之和等于零，即

$$a_{i1}A_{j1}+a_{i2}A_{j2}+\cdots+a_{in}A_{jn}=0,\quad i\neq j.$$

综上所述，有

$$a_{i1}A_{j1}+a_{i2}A_{j2}+\cdots+a_{in}A_{jn}=\begin{cases}D, & i=j\\ 0, & i\neq j\end{cases}$$

同理可得

$$a_{1i}A_{1j}+a_{2i}A_{2j}+\cdots+a_{ni}A_{nj}=\begin{cases}D, & i=j\\ 0, & i\neq j\end{cases}$$

例　计算行列式 $D=\begin{vmatrix} 5 & 3 & -1 & 2 & 0 \\ 1 & 7 & 2 & 5 & 2 \\ 0 & -2 & 3 & 1 & 0 \\ 0 & -4 & -1 & 4 & 0 \\ 0 & 2 & 3 & 5 & 0 \end{vmatrix}$

解

$$D=\begin{vmatrix} 5 & 3 & -1 & 2 & 0 \\ 1 & 7 & 2 & 5 & 2 \\ 0 & -2 & 3 & 1 & 0 \\ 0 & -4 & -1 & 4 & 0 \\ 0 & 2 & 3 & 5 & 0 \end{vmatrix}=\left(-1\right)^{2+5}2\begin{vmatrix} 5 & 3 & -1 & 2 \\ 0 & -2 & 3 & 1 \\ 0 & -4 & -1 & 4 \\ 0 & 2 & 3 & 5 \end{vmatrix}$$

$$= (-1)^{2+5} 2 \begin{vmatrix} 5 & 3 & -1 & 2 \\ 0 & -2 & 3 & 1 \\ 0 & -4 & -1 & 4 \\ 0 & 2 & 3 & 5 \end{vmatrix} = -2 \cdot 5 \begin{vmatrix} -2 & 3 & 1 \\ -4 & -1 & 4 \\ 2 & 3 & 5 \end{vmatrix}$$

$$\xrightarrow[r_3 + r_1]{r_2 + (-2)r_1} -10 \begin{vmatrix} -2 & 3 & 1 \\ 0 & -7 & 2 \\ 0 & 6 & 6 \end{vmatrix} = -10 \cdot (-2) \begin{vmatrix} -7 & 2 \\ 6 & 6 \end{vmatrix}$$

$$= 20 \cdot (-42 - 12) = -1080.$$

例　设 $D=\begin{vmatrix} 3 & -5 & 2 & 1 \\ 1 & 1 & 0 & -5 \\ -1 & 3 & 1 & 3 \\ 2 & -4 & -1 & -3 \end{vmatrix}$，$D$的$(i,j)$元的余子式和代数余子式依次记作$M_{ij}$和$A_{ij}$，求

$$A_{11}+A_{12}+A_{13}+A_{14} \text{ 及 } M_{11}+M_{21}+M_{31}+M_{41}.$$

分析　利用

$$a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13}+a_{14}A_{14}=\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

解 $A_{11}+A_{12}+A_{13}+A_{14}=\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & -5 \\ -1 & 3 & 1 & 3 \\ 2 & -4 & -1 & -3 \end{vmatrix}$

$$\xlongequal[r_3-r_1]{r_4+r_3}\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & -5 \\ -2 & 2 & 0 & 2 \\ 1 & -1 & 0 & 0 \end{vmatrix}=\begin{vmatrix} 1 & 1 & -5 \\ -2 & 2 & 2 \\ 1 & -1 & 0 \end{vmatrix}$$

$$\xlongequal{c_2+c_1}\begin{vmatrix} 1 & 2 & -5 \\ -2 & 0 & 2 \\ 1 & 0 & 0 \end{vmatrix}=\begin{vmatrix} 2 & -5 \\ 0 & 2 \end{vmatrix}=4.$$

$$M_{11}+M_{21}+M_{34}+M_{41}=A_{11}-A_{21}+A_{31}-A_{41}$$

$$=\begin{vmatrix} 1 & -5 & 2 & 1 \\ -1 & 1 & 0 & -5 \\ 1 & 3 & 1 & 3 \\ -1 & -4 & -1 & -3 \end{vmatrix} \xlongequal{r_4+r_3} \begin{vmatrix} 1 & -5 & 2 & 1 \\ -1 & 1 & 0 & -5 \\ 1 & 3 & 1 & 3 \\ 0 & -1 & 0 & 0 \end{vmatrix}$$

$$=-\begin{vmatrix} 1 & 2 & 1 \\ -1 & 0 & -5 \\ 1 & 1 & 3 \end{vmatrix} \xlongequal{r_1-2r_3} -\begin{vmatrix} -1 & 0 & -5 \\ -1 & 0 & -5 \\ 1 & 1 & 3 \end{vmatrix}=0.$$