Chapter 6 The Stability of Linear Feedback Systems

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- The stability of a feedback system is directly related to the location of the roots of the characteristic equation of the system transfer function. The Routh-Hurwitz method is introduced as a useful tool for assessing system stability. The technique allows us to compute the number of roots of the characteristic equation in the right half plane without actually computing the values of the roots.

- Thus, we can determine stability without the added computational burden of determining characteristic root locations. This gives us a design method for determining values of certain system parameters that will lead to closed-loop stability. For stable systems, we will introduce the notion of relative stability, which allows us to characterize the degree of stability.

6.1The Concept of Stability

- When considering the design and analysis of feedback control systems, stability is of the utmost importance. We can say that a closed-loop feedback system is either stable or it is not stable. This type of stable/not stable characterization is referred to as absolute stability.

- A system possessing absolute stability is called a stable system. Given that a closed-loop system is stable, we can further characterize the degree of stability. This is referred to as relative stability. The pioneers of aircraft design were familiar with the notion of relative stability—the more stable an aircraft was, the more difficult it was to maneuver (that is ,to turn).One outcome of the relative instability of modern fighter aircraft is high maneuverability.

- A fighter aircraft is less stable than a commercial transport, hence it can maneuver more quickly. We can determine that a system is stable (in absolute sense) by determining that all the transfer function poles lie in the left-half s-plane, or equivalently, that all the roots of the characteristic equation lie in the left-half s-plane. Given that all the poles (or roots) are in the left-half s-plane, we investigate relative-stability by examining the relative locations of the poses (or roots).

- A stable system is defined as a system with a bounded (limited) system response. That is : A stable system is a dynamic system with a bounded response to a bounded input.

In terms of linear systems, we recognize that the stability requirement may be defined in terms of the location of the poles of the closed-loop transfer function. The closed-loop system transfer function is written as

$$T(s) = \frac{p(s)}{q(s)} = \frac{K\prod_{i=1}^{M}(s+z_i)}{s^N\prod_{k=1}^{Q}(s+\sigma_k)\prod_{m=1}^{R}[s^2+2\alpha_m s+(\alpha_m^2+\omega_m^2)]} \tag{6.1}$$

- Where q(s)=0 is the characteristic equation whose roots are the poles of the closed-loop system. The output response for an impulse input is then
- Where A_k and B_m are constants that depend on σ_k, z_i, α_m, K, and ω_m.

$$y(t) = \sum_{k=1}^{Q} A_k e^{-\sigma_k t} + \sum_{m=1}^{R} B_m \left(\frac{1}{\omega_m}\right) e^{-\sigma_m t} \sin(\omega_m t + \theta_m) \qquad (6.2)$$

- To obtain a bounded response, the poles of the closed-loop system must be in the left-hand portion of the s-plane. Thus, a necessary and sufficient condition for a feedback system to be stable is that all the poles of the system transfer function have negative real parts.

- 1. A system is stable If all the poles of the transfer function are in the left-hand s-plane.
- 2. A system is not stable if not all the roots are in the left-hand s-plane.

- For an unstable system, the characteristic equation has at least one root in the right half of the s-plane or repeated jw roots, the output will become unbounded for any input.

3. A system is called marginally stable

- If the characteristic equation has simple roots on the imaginary axis (jw axis) with all other roots in the left-half s-plane, the steady-state output will be sustained oscillations for a bounded input.

6.2 THE ROUTH-HURWITZ STABILITY CRITERION

- In the late 1800s, A.Hurwitz and E.J.Routh independently published a method of investigating the stability of a linear system. The Routh-Hurwitz stability method provides an answer to the question of stability by considering the characteristic equation of the system.

1.Routh-Hurwitz Array

- The characteristic equation is written as

$$\Delta(s) = q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0 \qquad (6.3)$$

- Ordering the coefficients of the characteristic equation into an array or schedule as follows:

$$\begin{array}{c|cccc} s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \end{array}$$

- Further rows of the schedule are then completed as

$$
\begin{array}{c|ccc}
s^{n} & a_{n} & a_{n-2} & a_{n-4} \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} \\
s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} \\
s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} \\
\vdots & \vdots & \vdots & \vdots \\
s^{0} & h_{n-1} & &
\end{array}
$$

- Where

$$b_{n-1} = \frac{(a_{n-1})(a_{n-2}) - a_n(a_{n-3})}{a_{n-1}} = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix},$$

$$b_{n-3} = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix},$$

$$c_{n-1} = \frac{-1}{b_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix},$$

- And so on.

2. The Routh-Hurwitz Criterion

- The Routh-Hurwitz criterion states that **the number of roots of q(s) with positive real parts is equal to the number of changes in sign of the first column of the Routh array.**

Example 6.2 $\Delta(s) = q(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$

- The Routh-array is

$$\begin{array}{c|cc} s^3 & a_3 & a_1 \\ s^2 & a_2 & a_0 \\ s^1 & b1 & 0 \\ s^0 & c1 & 0 \end{array}$$

- Where $b_1 = \dfrac{a_2 a_1 - a_0 a_3}{a_2}$ *and* $c_1 = \dfrac{b_1 a_0}{b_1} = a_0$

- For the system to be stable, it is necessary and sufficient that the coefficients be positive.

$$a_2 a_1 > a_0 a_3$$

❖ Example(p318)

$$\Delta(s) = q(s) = s^3 + s^2 + 2s + 24 = 0$$

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 1 & 24 \\ s^1 & -22 & 0 \\ s^0 & 24 & 0 \end{array}$$

- Because tow changes in sign appear in the first column, the system is unstable. Answer is

$$q(s) = s^3 + s^2 + 2s + 24 = (s - 1 + j\sqrt{7})(s - 1 - j\sqrt{7})(s + 3) = 0$$

- *Case 2*. There is a zero in the first column, but some other elements of the row containing the zero in the first column are nonzero.
- If only one element in the array is zero, it may be replaced with a small positive number, ε, that is allowed to approach zero after completing the array.

- Example(p319)

$$q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10 = 0$$

- The Routh-array is

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 11 \\ s^4 & 2 & 4 & 10 \\ s^3 & \varepsilon & 6 & 0 \\ s^2 & c_1 & 10 & 0 \\ s^1 & d_1 & 0 & 0 \\ s^0 & 10 & 0 & 0 \end{array}$$

Where

$$c_1 = \frac{4\varepsilon - 12}{\varepsilon} = \frac{-12}{\varepsilon} \quad and \quad d_1 = \frac{6c_1 - 10\varepsilon}{c_1} \to 6$$

There are two sign changes due to the large negative number in the first column,

$$c_1 = \frac{-12}{\varepsilon}$$

Therefore, the system is unstable, and two roots lie in the right half of the plane.

Case 3. There is a zero in the first column, and the other elements of the row containing the zero are also zero.

Case3 occurs when all the elements in one row are zero or when the row consists of a single element that is zero.The polynomial contains singularities that are symmetrically located about the origin of the s-plane:

Such as $(s+\sigma)(s-\sigma) \quad or \quad (s+j\omega)(s-j\omega)$

Solve: utilizing the auxiliary polynomial U(s).For example

$$q(s) = s^3 + 2s^2 + 4s + K = 0$$ (p320)

Where K is an adjustable loop gain. Array is then

$$\begin{array}{c|cc} s^3 & 1 & 4 \\ s^2 & 2 & K \\ s^1 & \dfrac{8-K}{2} & 0 \\ s^0 & K & 0 \end{array}$$

(1) For the stable system, we require that
0<K<8.

(2) When K=8, two roots is on the jw-axis and that is a marginal stable case.

U(s) is the equation of the row preceding the row of zeros:

$$U(s) = 2s^2 + Ks^0 = 2s^2 + 8 = 2(s^2 + 4) = 2(s + j2)(s - j2)$$

(3)When K=8, all the roots of the characteristic equation q(s) :

We divide q(s) by U(s) to obtain

$$
\begin{array}{r}
(1/2)s + 1 \\
2s^2 + 8 \overline{\big)\, s^3 + 2s^2 + 4s + 8} \\
\underline{s^3 \qquad\qquad + 4s} \\
2s^2 \qquad\qquad + 8 \\
2s^2 \qquad\qquad + 8
\end{array}
$$

when K=8, the factors of the q(s) are

$$q(s)=(s+2)(s+j2)(s-j2)$$

The marginal case response is an unacceptable scillation.

- *Case 4*. Repeated roots of the characteristic equation on the jw-axis.

If the jw-axis roots of the characteristic equation are simple, the system is neither stable nor unstable; it is instead called marginally stable, since it has undamped sinusoidal mode.

If the jw-axis roots are repeated, the system response will be unstable with a form

$$t[\sin(\omega t + \phi)]$$

For example (p321)

$$q(s) = (s+1)(s+j)(s-j)(s+j)(s-j) = s^5 + s^4 + 2s^3 + 2s^2 + s + 1 = 0$$

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 1 \\ s^4 & 1 & 2 & 1 \\ s^3 & \varepsilon & \varepsilon & 0 \\ s^2 & & & \\ s^1 & & & \\ s^0 & & & \end{array} \Rightarrow \begin{array}{c|ccc} s^5 & 1 & 2 & 1 \\ s^4 & 1 & 2 & 1 \\ s^3 & 4 & 4 & 0 \\ s^2 & 1 & 1 & 0 \\ s^1 & 2 & 0 & 0 \\ s^0 & 1 & 0 & 0 \end{array}$$

$$U_1(s) = s^4 + 2s^2 + 1 = (s^2+1)^2$$

$$dU_1/ds = 4s^3 + 4s$$

$$U_2(s) = s^2 + 1 = 0 \Rightarrow s = \pm j$$

$$dU_2/ds = 2$$

Example 6.4 Robot control

$$q(s) = s^5 + s^4 + 4s^3 + 24s^2 + 3s + 63 = 0$$

$$\begin{array}{c|ccc} s^5 & 1 & 4 & 3 \\ s^4 & 1 & 24 & 63 \\ s^3 & -20 & -60 & 0 \\ s^2 & 21 & 63 & 0 \\ s^1 & 0 & 0 & 0 \\ s^0 & & & \end{array} \Rightarrow \begin{array}{c|ccc} s^5 & 1 & 4 & 3 \\ s^4 & 1 & 24 & 63 \\ s^3 & -20 & -60 & 0 \\ s^2 & 21 & 63 & 0 \\ s^1 & 42 & 0 & 0 \\ s^0 & 63 & 0 & 0 \end{array}$$

$$U(s) = 21s^2 + 63 = 0 \Rightarrow s = \pm j\sqrt{3}$$

$$dU/ds = 42s$$

- The two changes in sign in the first column indicate the presence of two roots in the right-hand plane, and the system is unstable. The roots in the right-hand plane are

$$s = +1 \pm j\sqrt{6}$$

6.3 THE RELATIVE STABILITY OF FEEDBACK CONTROL SYSTEMS

- The verification of stability using the Routh-Hurwitz criterion provides only a partial answer to the question of stability.

- 1.The Routh-Hurwitz criterion ascertains the absolute stability of a system by determining whether any of the roots of the characteristic equation lie in the right half of the s-plane.

- 2.If the system satisfies the Routh-Hurwitz criterion and is absolutely stable, it is desirable to determine the relative stability; that is, it is necessary to investigate the relative damping of each root of the characteristic equation. The relative stability of a system can be defined as the property that is measured by the relative real part of each root or pair of roots.

EXAMPLE 6.6 Axis shift

$$q(s) = s^3 + 4s^2 + 6s + 4 = 0$$

(1)There is not sign changes in the first column, the system is absolute stable.

(2)Let

$$s_n = s + 2, \Rightarrow q(s) = (s_n - 2)^3 + 4(s_n - 2)^2 + 6(s_n - 2) + 4 = s_n^3 - 2s_n^2 + 2s_n$$

The system is unstable and has not relative stability.

(3)Let

$$s_n = s + 1, \Rightarrow q(s) = (s_n - 1)^3 + 4(s_n - 1)^2 + 6(s_n - 1) + 4 = s_n^3 + s_n^2 + s_n + 1$$

$$\begin{array}{c|cc} s_n^3 & 1 & 1 \\ s_n^2 & 1 & 1 \\ s_n^1 & 0 & 0 \\ s_n^0 & 1 & 0 \end{array}$$

There are roots on the shifted imaginary axis that can be obtained from the auxiliary polynomial

$$U(s) = s_n^2 + 1 = (s_n + j)(s_n - j) = (s + 1 + j)(s + 1 - j)$$

The shifting of the s-plane axis to ascertain the relative stability of a system is a very useful approach, particularly for higher-order systems with several pairs of closed-loop complex conjugate roots.

EXERCISES

- E6.1 A system has a characteristic equation

$$s^3 + 3Ks^2 + (2 + K)s + 5 = 0.$$

Determine the range of K for a stable system.

E6.2 A system has a characteristic equation

$$s^3 + 9s^2 + 26s + 24 = 0.$$

Using the Routh-Hurwitz criterion, show that the system is stable.

E6.3
E6.4
E6.5
E6.7
E6.9
E6.19

CHAPTER 7
THE ROOT LOCUS METHOD

- The locus of roots in the s-plane can be determined by a graphical method. A graph of the locus of roots as one system parameter varies is known as a root locus plot.

- The relative stability and the transient performance of a closed-loop control system are directly related to the location of the closed-loop roots of the characteristic equation in the s-plane. It is frequently necessary to adjust one or more system parameters in order to obtain suitable root locations. Therefore, it is worthwhile to determine how the roots of the characteristic equation of a given system migrate about the s-plane as the parameters are varied; that, it is useful to determine the locus of roots in the s-plane as a parameter is varied.

- The root locus method was introduced by Evans in 1948 and has been developed and utilized extensively in control engineering practice. The root locus technique is a graphical method for sketching the locus of roots in the s-plane as a parameter is varied. In fact, the root locus method provides the engineer with a measure of the sensitivity of the roots of the system to a variation in the parameter being considered. The root locus technique may be used to great advantage in conjunction with the Routh-Hurwitz criterion.

- The root locus method provides graphical information, and therefore an approximate sketch can be used to obtain qualitative information concerning the stability and performance of the system. Furthermore, the locus of roots of the characteristic equation of a multi-loop system may be investigated as readily as for a single-loop system. If the root locations are not satisfactory, the necessary parameter adjustments often can be readily ascertained from the root locus.