第四章　连续系统的频域分析（3-4）

周期信号的傅里叶级数

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(n\Omega t)+\sum_{n=1}^{\infty}b_n\sin(n\Omega t)\ =\sum_{n=-\infty}^{\infty}F_n e^{jn\Omega t}$$

$$F_n=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f(t)e^{-jn\Omega t}dt$$

非周期信号的傅里叶变换

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(j\omega)e^{j\omega t}d\omega$$

$$F(j\omega)=\int_{-\infty}^{\infty}f(t)e^{-j\omega t}dt$$

4.5 傅里叶变换的性质

■线性 $\sum_{i=1}^{n} a_i f_i(t) \longleftrightarrow \sum_{i=1}^{n} a_i F_i(\omega)$

■奇偶虚实性

实偶函数←-→实偶函数	实奇函数←-→虚奇函数
虚偶函数←-→虚偶函数	虚奇函数←-→实奇函数

■对称性 $F(jt) \longleftrightarrow 2\pi f(-\omega)$

■ 尺度变换特性 $f(at) \longleftrightarrow \frac{1}{|a|} F(\frac{\omega}{a})$

■ 时移特性 $f(t-t_0) \longleftrightarrow F(\omega) e^{-j\omega t_0}$

■ 频移特性 $f(t) e^{j\omega_0 t} \longleftrightarrow F(\omega - \omega_0)$

4.5 傅里叶变换的性质

- 时域微分特性 $\frac{df(t)}{dt} \longleftarrow - - \longrightarrow j\omega F(\omega)$ $\frac{d^n f(t)}{dt^n} \longleftarrow - - \longrightarrow (j\omega)^n F(\omega)$
- 时域积分特性 $\int_{-\infty}^{t} f(\tau)d\tau \longleftarrow - - \longrightarrow \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$
- 频域微分特性 $(-jt)^n f(t) \longleftarrow - - \longrightarrow F^n(j\omega)$
- 频域积分特性 $\pi f(0)\delta(t) + \frac{f(t)}{-jt} \longleftarrow - - \longrightarrow F^{(-1)}(j\omega)$
- 时域卷积定理 $f_1(t) * f_2(t) \longleftarrow - - \longrightarrow F_1(j\omega)F_2(j\omega)$
- 频域卷积定理 $f_1(t) \cdot f_2(t) \longleftarrow - - \longrightarrow \frac{1}{2\pi} F_1(j\omega) * F_2(j\omega)$
- Paseval定理 $E = \int_{-\infty}^{\infty} f^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega$ $P = \lim_{T\to\infty} \frac{|F(j\omega)|^2}{T}$

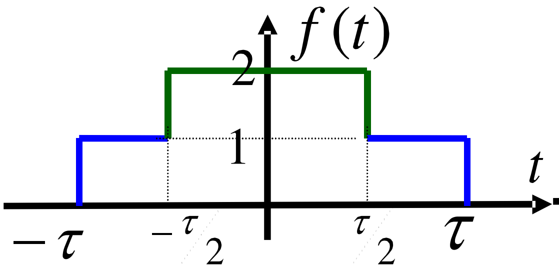
1. 线性

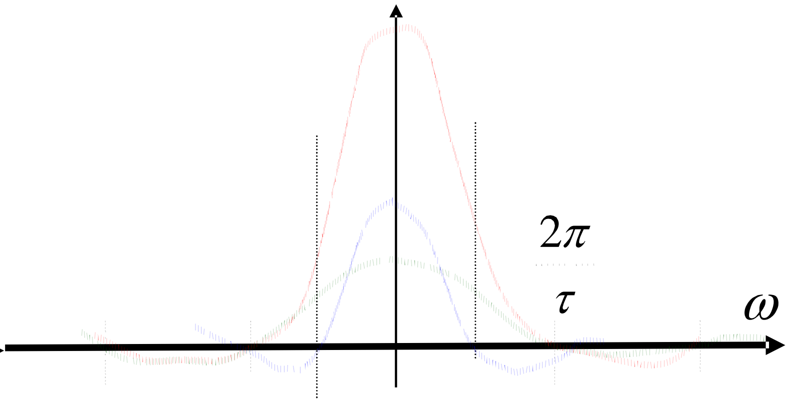
若 $f(t) \longleftrightarrow F(j\omega)$

则 $\sum_{i=1}^{n} a_i f_i(t) \longleftrightarrow \sum_{i=1}^{n} a_i F_i(\omega)$

$$g_\tau(t) \longleftrightarrow \tau Sa(\frac{\omega\tau}{2})$$

求f(t)的傅立叶变换：





$$f(t) = [\varepsilon(t+\tfrac{\tau}{2}) - \varepsilon(t-\tfrac{\tau}{2})] + [\varepsilon(t+\tau) - \varepsilon(t-\tau)]$$

$$F(\omega) = \tau[Sa(\frac{\omega\tau}{2}) + 2Sa(\omega\tau)]$$

2. 奇偶虚实性

若 $f(t) \longleftrightarrow F(j\omega)$

则 $f(-t) \longleftrightarrow F(-j\omega)$

$f^*(t) \longleftrightarrow F^*(-j\omega)$

$f^*(-t) \longleftrightarrow F^*(j\omega)$

时域反摺
频域也反摺

时域共轭
频域共轭
且反摺

$$FT[f(-t)] = \int_{-\infty}^{\infty} f(-t)e^{-j\omega t}dt$$

$$= -\int_{\infty}^{-\infty} f(-t)e^{-j(-\omega)(-t)}d(-t)$$

$$= \int_{-\infty}^{\infty} f(-t)e^{-j(-\omega)(-t)}d(-t) = F(-j\omega)$$

$$FT[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$$FT[f^*(t)] = \int_{-\infty}^{\infty} f^*(t)e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} f^*(t)[e^{-j(-\omega)t}]^* dt$$

$$= [\int_{\infty}^{-\infty} f(t)e^{-j(-\omega)t}dt]^* = F^*(-j\omega)$$

$$FT[f^*(-t)] = \int_{-\infty}^{\infty} f^*(-t)e^{-j\omega t}dt$$

$$= -\int_{\infty}^{-\infty} f^*(-t)[e^{-j\omega(-t)}]^* d(-t)$$

$$= [\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt]^* \quad = F^*(j\omega)$$

一、f(t)是实函数

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\cos\omega t dt - j\int_{-\infty}^{\infty} f(t)\sin\omega t dt$$

偶函数 $R(\omega)$

奇函数 $X(\omega)$

$R(\omega) = R(-\omega)$ $X(\omega) = -X(-\omega)$

幅度谱为偶函数，
而相位谱为奇函数

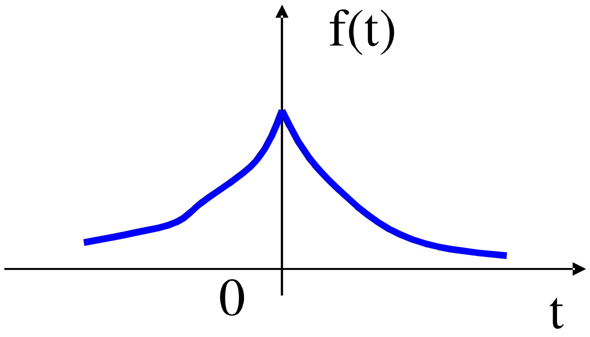
$$F(-\omega) = R(-\omega) + jX(-\omega)$$

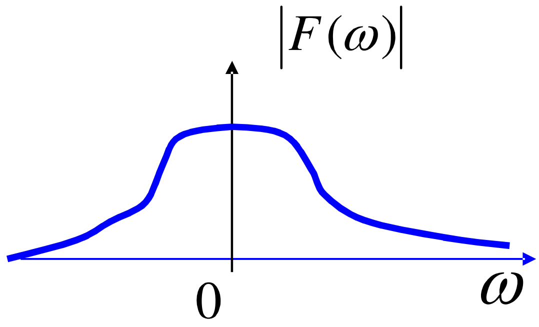
$$= R(\omega) - jX(\omega) = F^{*}(\omega)$$

f(t)是偶函数 $f(t)=f(-t)$

$$F(\omega)=\int_{-\infty}^{\infty} f(t)\cos\omega t\,dt$$

例： $f(t)=e^{-\alpha|t|} \qquad (-\infty<t<+\infty)$ $\qquad F(\omega)=\dfrac{2\alpha}{\alpha^2+\omega^2}$

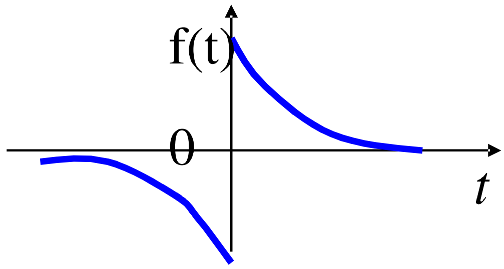




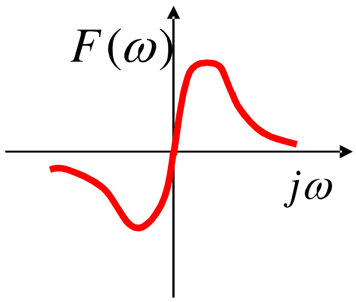
f(t)是奇函数 $f(t) = -f(-t)$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\cos\omega t dt - j\int_{-\infty}^{\infty} f(t)\sin\omega t dt$$

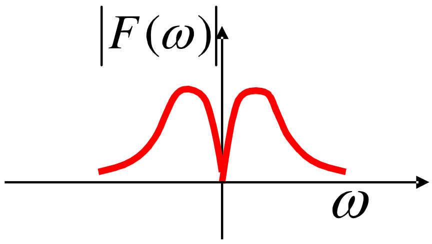
例 $f(t) = \begin{cases} e^{-at} & (t > 0) \\ -e^{-at} & (t < 0) \end{cases}$



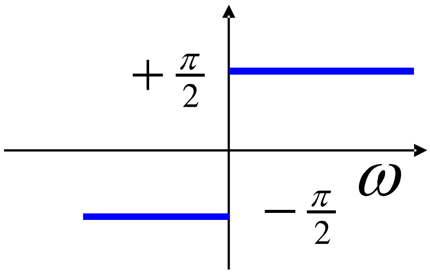
$$F(\omega) = \frac{-2j\omega}{\alpha^2 + \omega^2}$$



$$|F(\omega)| = \frac{2|\omega|}{\alpha^2 + \omega^2}$$



$$\varphi(\omega) = \begin{cases} -\frac{\pi}{2} & (\omega > 0) \\ +\frac{\pi}{2} & (\omega < 0) \end{cases}$$



二、f(t) = jg(t)是虚函数

$$F(\omega)=\int_{-\infty}^{\infty} g(t)\sin\omega t dt + j\int_{-\infty}^{\infty} g(t)\cos\omega t dt$$

奇函数 $R(\omega)$ 偶函数 $X(\omega)$

$$R(\omega)=-R(-\omega) \qquad X(\omega)=X(-\omega)$$

$$F(-\omega)=R(-\omega)+jX(-\omega)$$

$$=-R(\omega)+jX(\omega)=-F^{*}(\omega)$$

幅度谱为偶函数，
而相位谱为奇函数

若g(t)为偶函数,则其频谱为虚偶函数;
若g(t)为奇函数,则其频谱为实奇函数

$$F(j\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$ $$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega$$

3. 对称性

若 $f(t) \longleftarrow - - \longrightarrow F(j\omega)$

则 $F(t) \longleftarrow - - \longrightarrow 2\pi f(-\omega)$

证明：

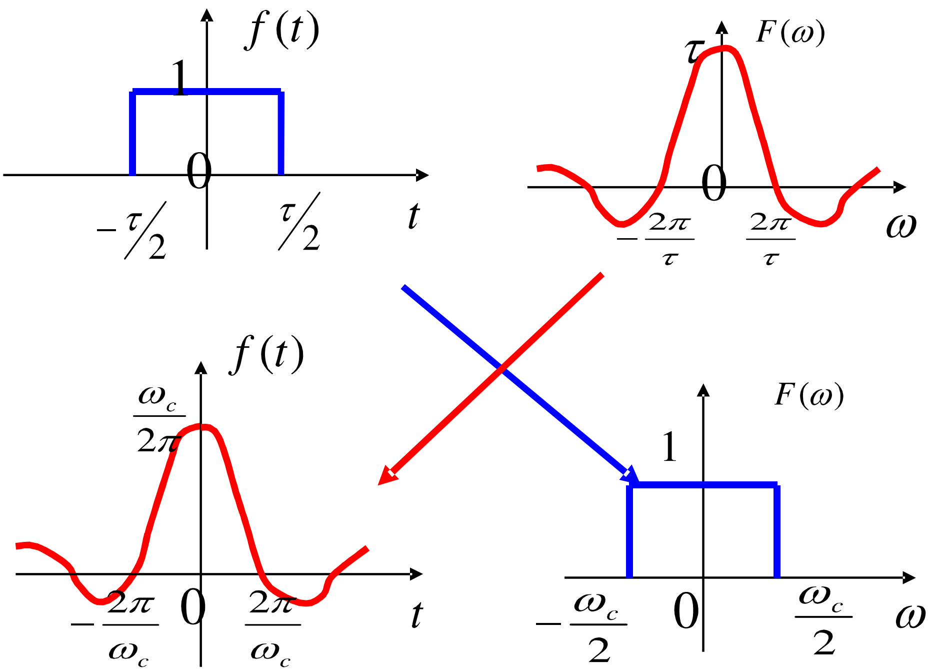
$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

$$f(-t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-j\omega t}d\omega,$$

$$f(-\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)e^{-j\omega t}dt$$

$$\boxed{F(t) \longleftarrow - - \longrightarrow 2\pi f(-\omega)}$$

若**f(t)**为偶函数，则时域和频域完全对称，直流和冲激函数的频谱的对称性是一例子



$$g_{\tau}(t) \leftarrow - - \rightarrow \tau\ Sa(\frac{\omega\tau}{2})$$

$$g_{\omega_c}(t) \leftarrow - - \rightarrow \omega_c\ Sa(\frac{\omega\omega_c}{2})$$

$$\omega_c\ Sa(\frac{\omega_c t}{2}) \leftarrow - - \rightarrow 2\pi g_{\omega_c}(\omega)$$

$$\frac{\omega_c}{2\pi}\ Sa(\frac{\omega_c t}{2}) \leftarrow - - \rightarrow g_{\omega_c}(\omega)$$

$$F(t) \longleftarrow - - \longrightarrow \quad 2\pi f(-\omega)$$

$a > 1, \quad t > 0$

$$f(t) = e^{-at}$$

FT

$$F(\omega) = \frac{1}{a + j\omega}$$

t 换成 -ω

$$F_1(\omega) = FT\left[\frac{1}{a + jt}\right] = ?$$

t 换成ω

对称性

$$F_1(\omega) = 2\pi f(-\omega) = 2\pi e^{+a\omega}$$

4. 尺度变换特性

若 $f(t) \longleftarrow\!-\!-\!\longrightarrow F(j\omega)$

则 $f(at) \longleftarrow\!-\!-\!\longrightarrow \frac{1}{|a|}F(\frac{\omega}{a})$

证明： $a>0 \quad FT[f(at)] = \frac{1}{a}\int_{-\infty}^{\infty} f(x)e^{-j\omega\frac{x}{a}}dx$

$$= \frac{1}{a}F(\frac{\omega}{a})$$

$$a<0 \quad FT[f(at)]=\frac{-1}{a}F(\frac{\omega}{a})$$

令$x=at$, 则$t=\frac{1}{a}x, \quad dt=\frac{1}{a}dx$

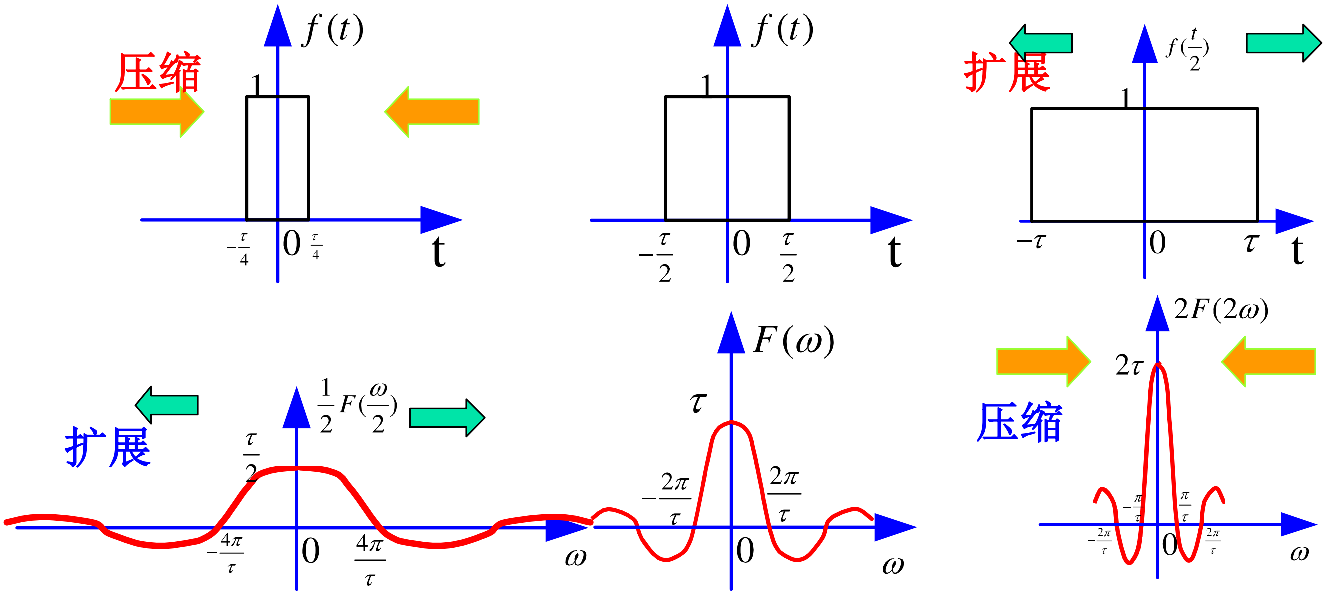
当$x\to\infty$时，$t\to-\infty$;　当$x\to-\infty$时，$t\to+\infty$;

$$FT[f(at)]=\int_{-\infty}^{\infty}f(at)e^{-j\omega t}dt$$

$$=\frac{1}{a}\int_{\infty}^{-\infty}f(x)e^{-j\omega\frac{x}{a}}dx$$

$$=\frac{-1}{a}\int_{-\infty}^{\infty}f(x)e^{-j\omega\frac{x}{a}}dx=\frac{-1}{a}F(\frac{\omega}{a})$$

时域中的压缩（扩展）等于频域中的扩展（压缩）



$$g_{\frac{\tau}{2}}(t) \longleftrightarrow \frac{\tau}{2} Sa(\frac{\omega\tau}{4})$$

$$g_{\tau}(t) \longleftrightarrow \tau\ Sa(\frac{\omega\tau}{2})$$

$$g_{2\tau}(t) \longleftrightarrow 2\tau\ Sa(\omega\tau)$$

5. 时移特性

若 $f(t) \longleftrightarrow F(j\omega)$

则 $f(t-t_0) \longleftrightarrow F(\omega)e^{-j\omega t_0}$

证明： $x = t - t_0$

$$FT\left[f(x)\right] = \int_{-\infty}^{\infty} f(x) e^{-j\omega(x+t_0)} dx$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} f(x) e^{-j\omega x} dx = e^{-j\omega t_0} F(\omega)$$

$$\therefore \quad FT\left[f(t-t_0)\right] = e^{-j\omega t_0} F(\omega) \quad \#$$

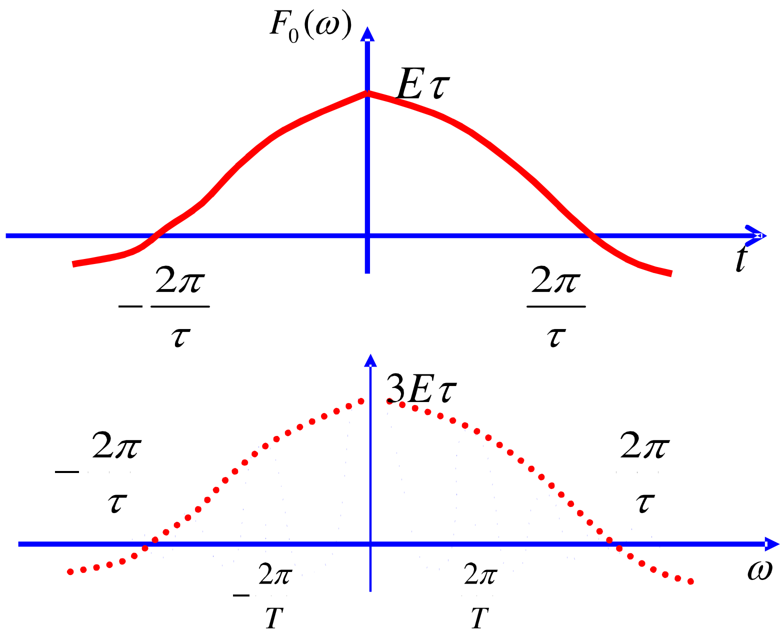
例：求三脉冲信号的频谱

单矩形脉冲 $f_0(t)$ 的频谱为 $F_0(\omega) = E\tau\, Sa(\frac{\omega\tau}{2})$

有如下三脉冲信号 $f(t) = f_0(t) + f_0(t+T) + f_0(t-T)$

其频谱为

$$\begin{aligned} F(\omega) &= F_0(\omega)(1 + e^{j\omega T} + e^{-j\omega T}) \\ &= F_0(\omega)(1 + 2\cos\omega T) \\ &= E\tau\, Sa(\frac{\omega\tau}{2})(1 + 2\cos\omega T) \end{aligned}$$



$F_0(\omega)$
$E\tau$
t
$-\frac{2\pi}{\tau}$
$\frac{2\pi}{\tau}$
$3E\tau$
$-\frac{2\pi}{\tau}$
$\frac{2\pi}{\tau}$
$-\frac{2\pi}{T}$
$\frac{2\pi}{T}$
ω

6. 频移特性

若 $f(t) \longleftarrow - - \longrightarrow F(j\omega)$

则 $f(t)e^{j\omega_0 t} \longleftarrow - - \longrightarrow F(\omega-\omega_0)$

证明: $FT[f(t)e^{j\omega_0 t}] = \int_{-\infty}^{\infty} f(t)e^{j\omega_0 t}e^{-j\omega t}dt$

$$= \int_{-\infty}^{\infty} f(t)e^{-j(\omega-\omega_0)t}dt = F(\omega-\omega_0)$$

同理: $FT[f(t)e^{-j\omega_0 t}] = F(\omega+\omega_0)$

调幅信号的频谱（载波技术）

求 $f(t)\cos\omega_0 t$ 的频谱

解: $$\cos\omega_0 t=\frac{1}{2}(e^{j\omega_0 t}+e^{-j\omega_0 t})$$

$$FT[f(t)\cos\omega_0 t]=\frac{1}{2}[F(\omega-\omega_0)+F(\omega+\omega_0)]$$

$$\sin\omega_0 t=\frac{1}{2j}(e^{j\omega_0 t}-e^{-j\omega_0 t})$$

$$FT[f(t)\sin\omega_0 t]=\frac{1}{2j}[F(\omega-\omega_0)-F(\omega+\omega_0)]$$

$$\cos\omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

载波频率ω_0

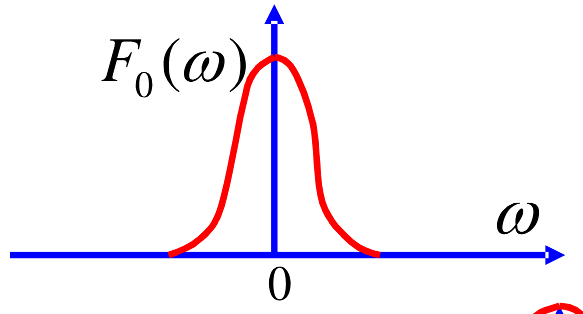
$f(t)$ 与 $\frac{1}{2}e^{j\omega_0 t}$ → $\frac{1}{2}F(\omega-\omega_0)$

$\frac{1}{2}e^{-j\omega_0 t}$ 与 $f(t)$ → $\frac{1}{2}F(\omega+\omega_0)$

$$\frac{1}{2}[F(\omega-\omega_0)+F(\omega+\omega_0)]$$

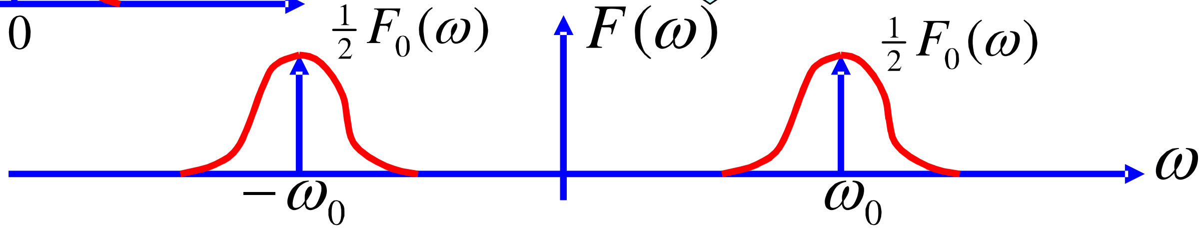
$$FT[f(t)\cos\omega_0 t]$$

$$FT[f(t)] = F_0(\omega)$$



$$\frac{1}{2} f(t)[e^{j\omega_0 t} + e^{-j\omega_0 t}]$$

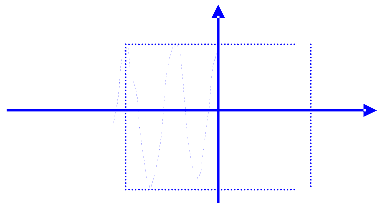
频移特性



$$\frac{1}{2}[F_0(\omega-\omega_0) + F_0(\omega+\omega_0)]$$

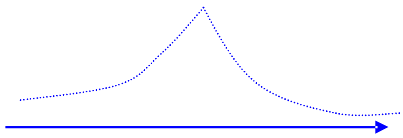
调幅信号都可看成乘积信号

矩形调幅



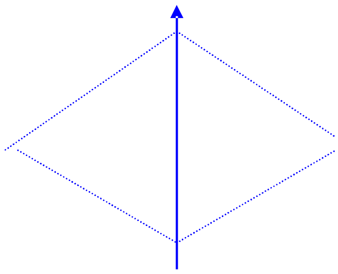
$G(t)\cos\omega_0 t$

指数衰减振荡



$e^{-at}\cos\omega_0 t$

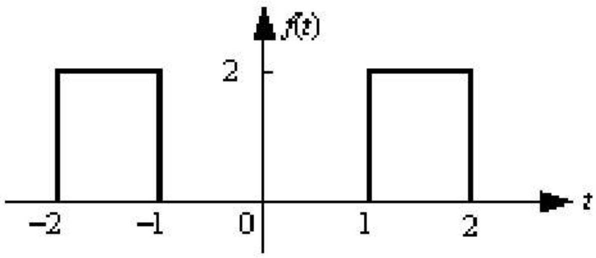
三角调幅



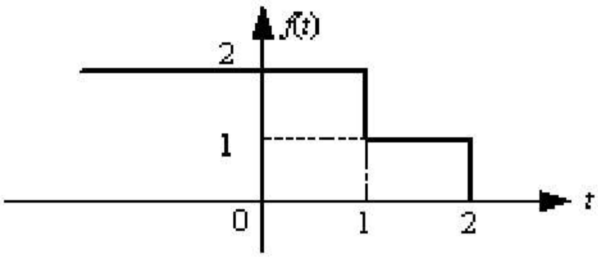
$\left(1-\frac{2t}{\tau}\right)\cos\omega_0 t$

求它们的频谱= ？（略）

试求下列非周期信号的频谱

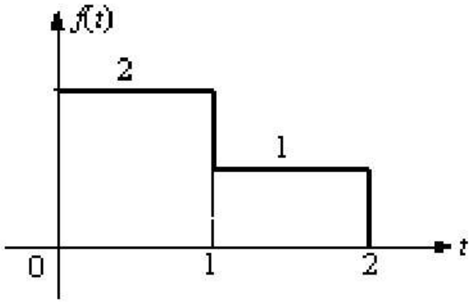


$$f(t)=2g_1(t+1.5)+2g_1(t-1.5)$$



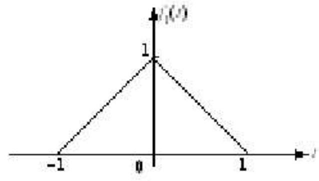
$$f(t)=2\varepsilon(-t)+2g_1(t-0.5)+g_1(t-1.5)$$

$$f(t)=\varepsilon(-t+1)+\varepsilon(-t+2)$$



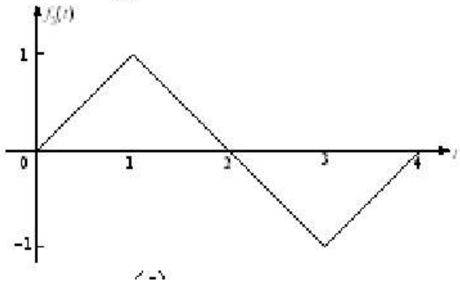
$$f(t)=2g_1(t-0.5)+g_1(t-1.5)$$

已知图1所示三角波信号f1(t)的频谱为$F_1(\omega)$试求下列非周期信号的频谱

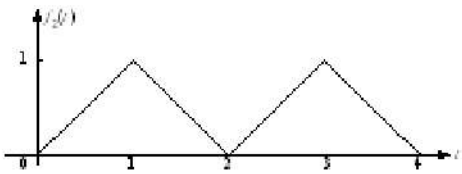


(a)

$$F_1(j\omega)=Sa^2\frac{\omega}{2}$$

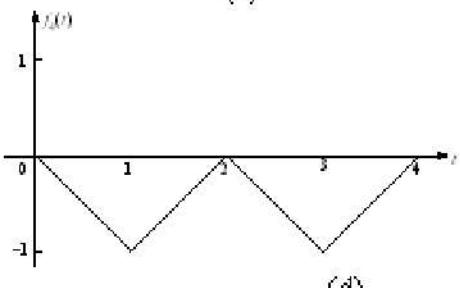


$$f_2(t)=f_1(t-1)-f_1(t-3)$$



(b)

$$f_3(t)=f_1(t-1)+f_1(t-3)$$



$$f_4(t)=-f_1(t-1)-f_1(t-3)$$

试求下列非周期信号的频谱

$$(1)\quad f(t)=\frac{\sin t}{t}$$

$$(2)\quad f(t)=\frac{1}{a^2+t^2}$$

$$(3)\quad f(t)=\frac{1}{a+jt}$$

$$(4)\quad f(t)=\delta(t-t_0)+\delta(t+t_0)$$

试求下列频谱对应的非周期信号

(1) $F(j\omega)=\dfrac{3}{j\omega+2}+\dfrac{4}{j\omega-4}$

(2) $F(j\omega)=\dfrac{3}{j(\omega+2)+4}+\dfrac{4}{j(\omega-2)+4}$

(3) $F(j\omega)=Sa(\omega\tau)$

(4) $F(j\omega)=\delta(\omega-\omega_0)$

(5) $F(j\omega)=g_{2\omega_c}(\omega)$

(6) $F(j\omega)=-\dfrac{2}{\omega^2}$

试求下列非周期信号的频谱

$$(1)\quad f(t-5)$$

$$(2)\quad f(5t)$$

$$(3)\quad e^{-jat}f(bt)$$

$$(4)\quad f(5-5t)$$

$$(5)\quad f(t)=\sin\omega_0 t+\cos\omega_0(t-t_0)$$

7. 时域卷积定理

若 $f_1(t) \longleftrightarrow F_1(j\omega), \quad f_2(t) \longleftrightarrow F_2(j\omega)$

则 $f_1(t) * f_2(t) \longleftrightarrow F_1(j\omega)F_2(j\omega)$

证明：$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau$

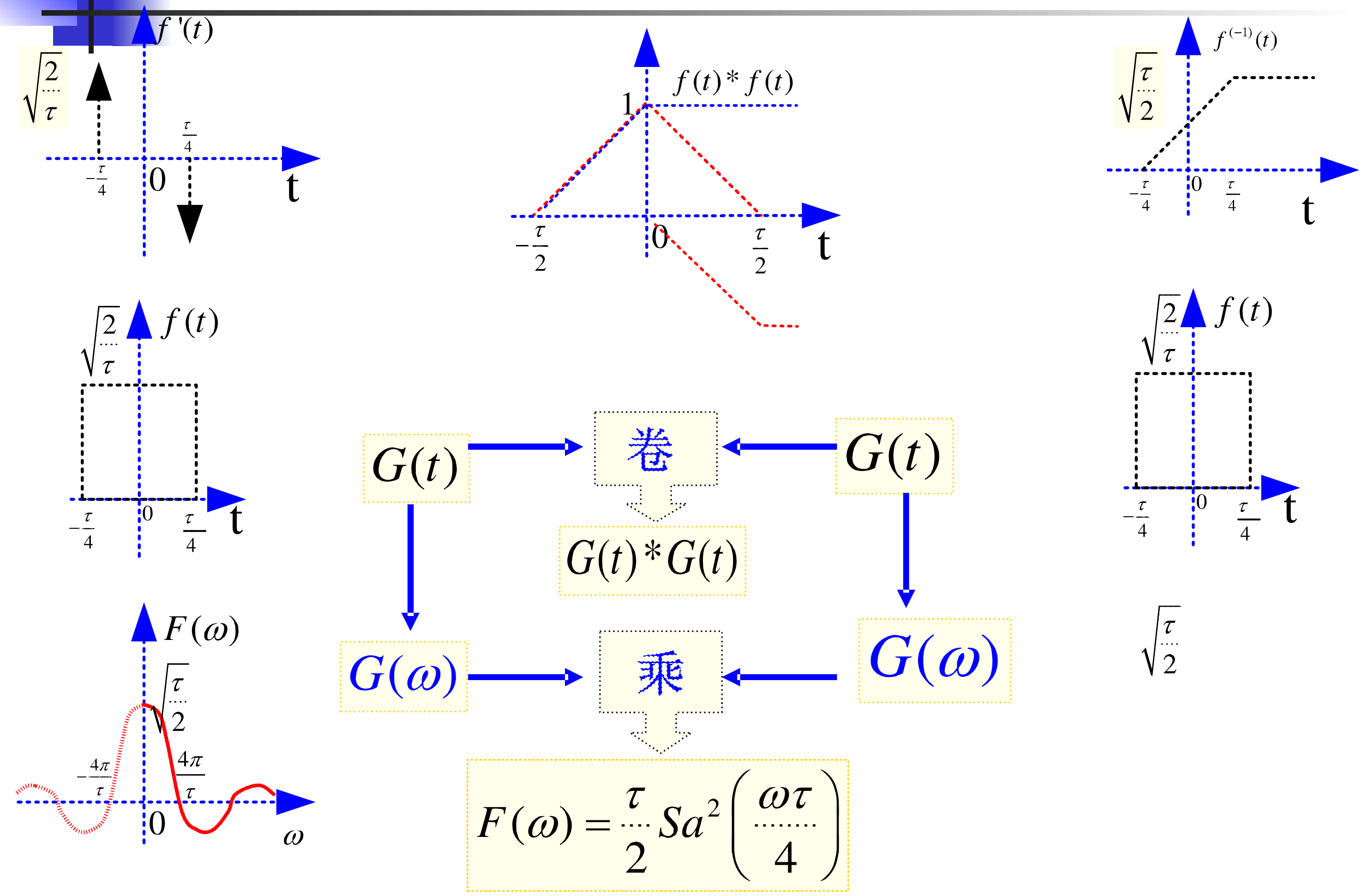
$$FT[f_1(t) * f_2(t)] = \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau] e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} f_1(\tau) [\int_{-\infty}^{\infty} f_2(t-\tau) e^{-j\omega(t-\tau)} d(t-\tau)] e^{-j\omega\tau} d\tau$$

$$= F_2(j\omega) \int_{-\infty}^{\infty} f_1(\tau) e^{-j\omega\tau} d\tau \quad = F_1(j\omega)F_2(j\omega)$$

例： 求三角脉冲的频谱

三角脉冲可看成两个同样矩形脉冲的卷积



8. 频域卷积定理

若 $f_1(t) \longleftrightarrow F_1(j\omega), \quad f_2(t) \longleftrightarrow F_2(j\omega)$

则 $f_1(t)\cdot f_2(t) \longleftrightarrow \frac{1}{2\pi}F_1(j\omega)*F_2(j\omega)$

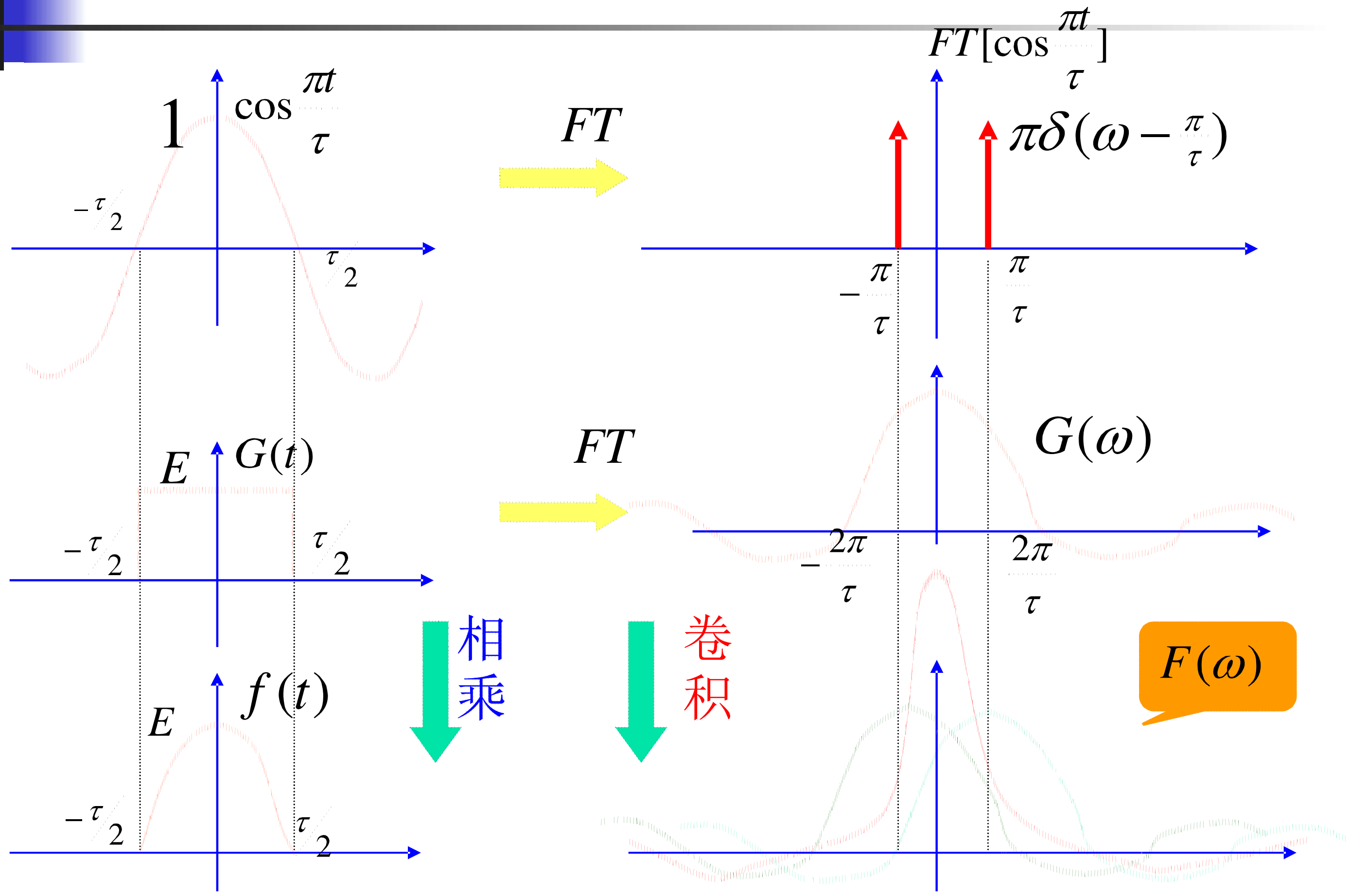
证明: $F_1(j\omega)*F_2(j\omega)=\int_{-\infty}^{\infty} F_1(j\eta)F_2(j\omega-j\eta)d\eta$

$$FT^{-1}[\frac{1}{2\pi}F_1(j\omega)*F_2(j\omega)]=\frac{1}{2\pi}\int_{-\infty}^{\infty}[\frac{1}{2\pi}\int_{-\infty}^{\infty}F_1(j\eta)F_2(j\omega-j\eta)d\eta]e^{j\omega t}d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}F_1(j\eta)[\frac{1}{2\pi}\int_{-\infty}^{\infty}F_2(j\omega-j\eta)e^{j(\omega-\eta)t}d(\omega-\eta)]e^{j\eta t}d\eta$$

$$=f_2(t)\frac{1}{2\pi}\int_{-\infty}^{\infty}F_1(j\eta)e^{j\eta t}d\eta = f_2(t)f_1(t)$$

例：求升余弦脉冲的频谱



$1 \quad \cos\frac{\pi t}{\tau}$
$-\tau/2$
$\tau/2$
FT
$FT[\cos\frac{\pi t}{\tau}]$
$\pi\delta(\omega-\frac{\pi}{\tau})$
$-\frac{\pi}{\tau}$
$\frac{\pi}{\tau}$
E
$G(t)$
$-\tau/2$
$\tau/2$
FT
$G(\omega)$
$-\frac{2\pi}{\tau}$
$\frac{2\pi}{\tau}$
相乘
卷积
E
$f(t)$
$-\tau/2$
$\tau/2$
$F(\omega)$

$$f(t) = G(t).\cos\frac{\pi t}{\tau}$$

乘

$$G(t)$$

FT

$$G(\omega) = E\tau Sa(\frac{\omega\tau}{2})$$

$$\cos\frac{\pi t}{\tau}$$

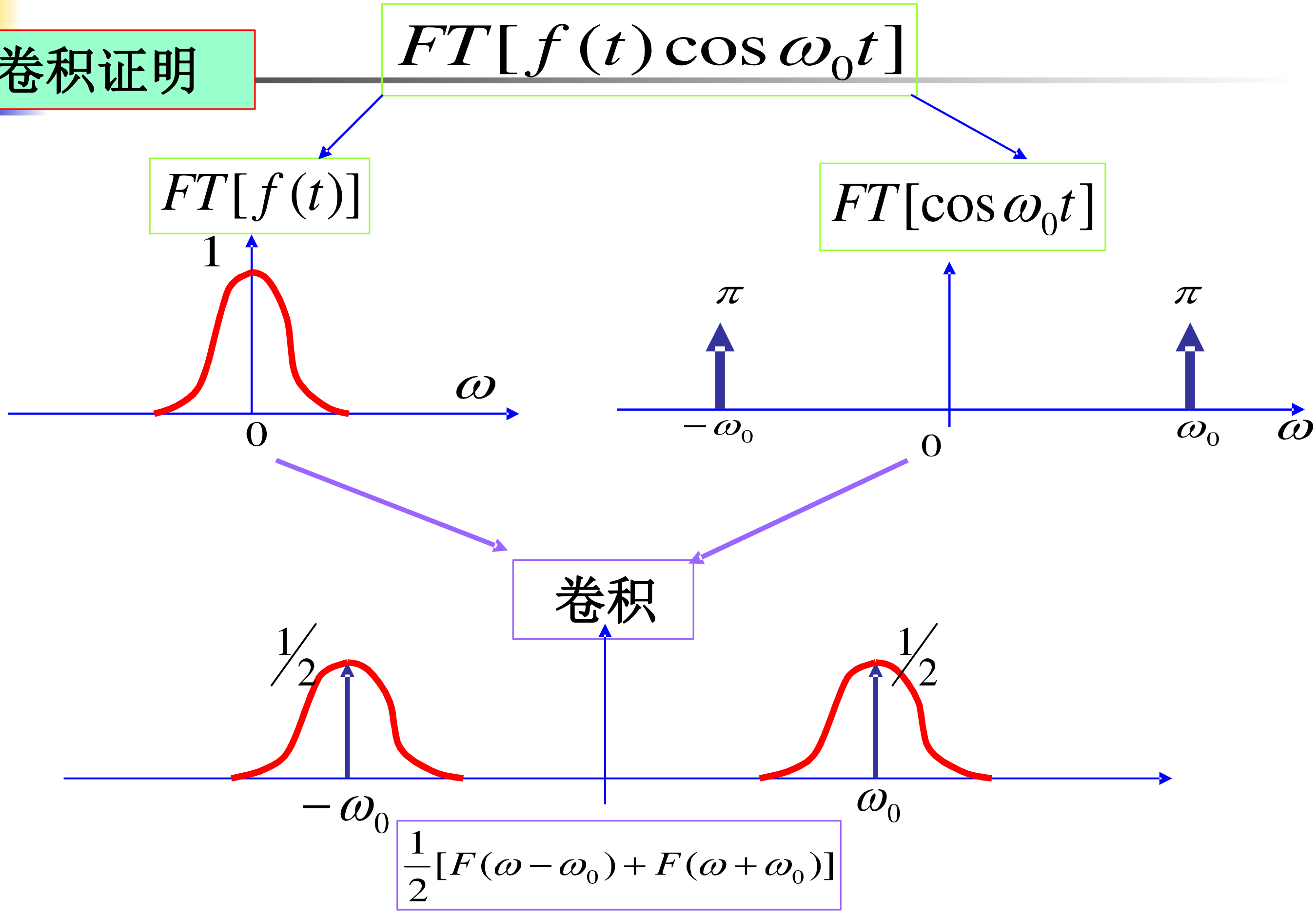
FT

$$\pi\delta(\omega + \frac{\pi}{\tau}) + \pi\delta(\omega - \frac{\pi}{\tau})$$

卷积

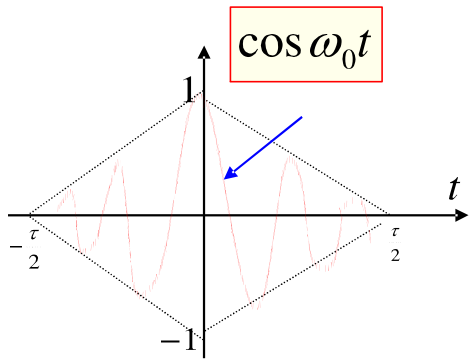
$$F(\omega) = \frac{2E\tau\cos(\frac{\omega\tau}{2})}{\pi\left[1 - (\frac{\omega\tau}{\pi})^2\right]}$$

利用卷积证明



$FT[f(t)\cos\omega_0 t]$
$FT[f(t)]$
1
0
ω
$FT[\cos\omega_0 t]$
π
π
$-\omega_0$
0
ω_0
ω
卷积
$1/2$
$1/2$
$-\omega_0$
ω_0
$\frac{1}{2}[F(\omega-\omega_0)+F(\omega+\omega_0)]$

求图中所示的三角调幅波信号的频谱



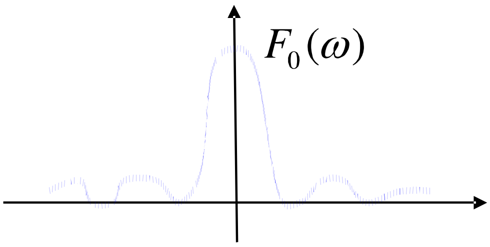
$$\cos\omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

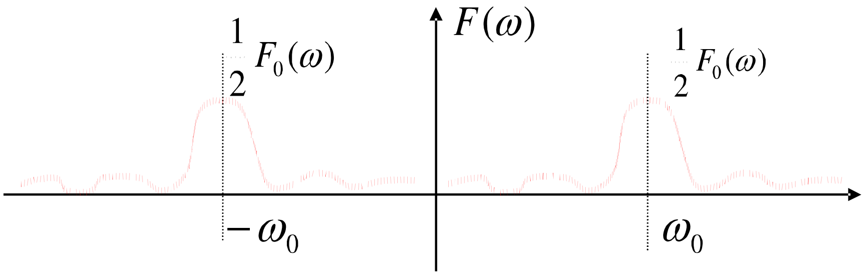
$$f_0(t) = 1 - \frac{2t}{\tau}$$

$$F_0(\omega) = \frac{E\tau}{2} Sa^2\left(\frac{\omega\tau}{4}\right)$$

三角波

$$F(\omega) = \frac{\tau}{4}\left\{Sa^2\left(\frac{(\omega-\omega_0)\tau}{4}\right) + Sa^2\left(\frac{(\omega+\omega_0)\tau}{4}\right)\right\}$$





$$E = 1$$

9.时域微分特性

若 $f(t) \longleftrightarrow F(j\omega)$

则 $\frac{df(t)}{dt} \longleftrightarrow j\omega F(\omega)$ $\frac{d^n f(t)}{dt^n} \longleftrightarrow (j\omega)^n F(\omega)$

证明:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

$$\frac{df(t)}{dt} = \frac{d}{dt}[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega]$$

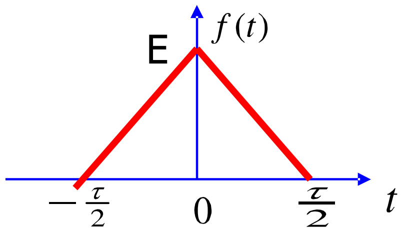
$$\frac{df(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} [j\omega F(\omega)]e^{j\omega t}d\omega$$

$$\frac{df(t)}{dt} \longleftrightarrow j\omega F(\omega)$$

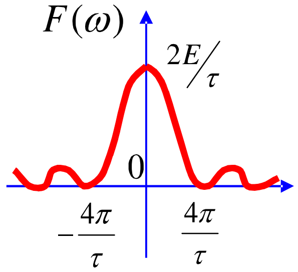
同理: $\frac{d^n f(t)}{dt^n} \longleftrightarrow (j\omega)^n F(\omega)$

$$F(\omega)=\frac{E\tau}{2}Sa^2(\frac{\omega\tau}{4})$$

三角脉冲



$$f(t)=\begin{cases}E(1-\frac{2}{\tau}|t|) & (|t|<\frac{\tau}{2})\\ 0 & (|t|>\frac{\tau}{2})\end{cases}$$

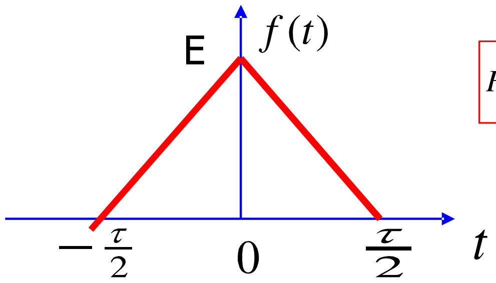


方法一：代入定义计算

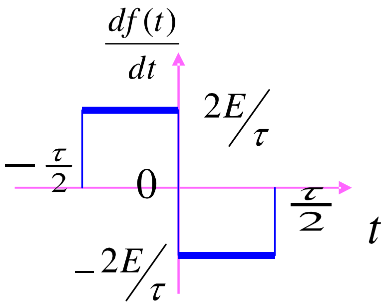
$$F(j\omega)=\int_{-\infty}^{\infty}f(t)e^{-j\omega t}dt$$

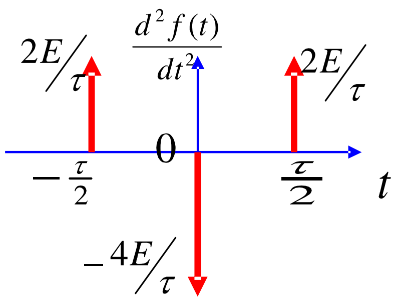
$$=\int_{-\frac{\tau}{2}}^{+\frac{\tau}{2}}E(1-\frac{2}{\tau}|t|)e^{-j\omega t}dt$$

■方法二：利用二阶导数的FT



$$F(\omega)=\frac{E\tau}{2}Sa^2\left(\frac{\omega\tau}{4}\right)$$





$$\frac{d^2 f(t)}{dt^2}=\frac{2E}{\tau}\left[\delta(t+\tfrac{\tau}{2})+\delta(t-\tfrac{\tau}{2})-2\delta(t)\right]$$

$$(j\omega)^2F(\omega)=\frac{2E}{\tau}\left(e^{j\omega\frac{\tau}{2}}+e^{-j\omega\frac{\tau}{2}}-2\right)$$

$$=-\frac{8E}{\tau}\sin^2\left(\frac{\omega\tau}{4}\right)=-\frac{\omega^2E\tau}{2}Sa^2\left(\frac{\omega\tau}{4}\right)$$

10.时域积分特性

若 $f(t) \longleftrightarrow F(j\omega)$

则 $\int_{-\infty}^{t} f(\tau)d\tau \longleftrightarrow \dfrac{F(\omega)}{j\omega}$

$$\omega = 0, \quad \left|\frac{F(\omega)}{\omega}\right| < \infty \quad or\ F(0) = 0$$

$$\int_{-\infty}^{t} f(\tau)d\tau \longleftrightarrow \frac{F(\omega)}{j\omega} + \boxed{\pi F(0)\delta(\omega)} \qquad F(0) \neq 0$$

证明: $$\int_{-\infty}^{t} f(\tau)d\tau = \int_{-\infty}^{t} f(\tau)d\tau * \delta(t) = f(t) * u(t)$$

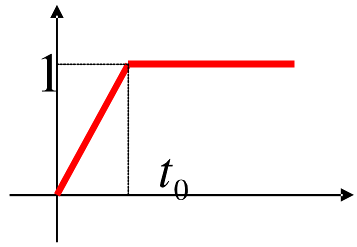
$$\int_{-\infty}^{t} f(\tau)d\tau \longleftarrow - - \longrightarrow FT[f(t)]\cdot FT[u(t)]$$

$$\int_{-\infty}^{t} f(\tau)d\tau \longleftarrow - - \longrightarrow F(j\omega)\cdot[\pi\delta(\omega) + \frac{1}{j\omega}]$$

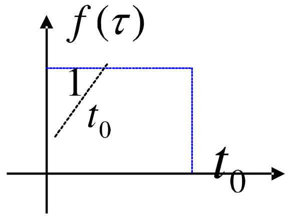
$$\int_{-\infty}^{t} f(\tau)d\tau \longleftarrow - - \longrightarrow \pi F(0)\delta(\omega) + \frac{1}{j\omega}F(\omega)$$

若F(0)=0, $$\int_{-\infty}^{t} f(\tau)d\tau \longleftarrow - - \longrightarrow \frac{1}{j\omega}F(\omega)$$

斜平信号 $y(t)=\begin{cases}0 & (\tau<0)\\ t/t_0 & (0<\tau<t_0)\\ 1 & (\tau>t_0)\end{cases}$ 的频谱



看成高 $1/t_0$,宽 t_0 的矩形脉冲 $f(\tau)$ 的积分



$$f(\tau)=\begin{cases}0 & (\tau<0)\\ 1/t_0 & (0<\tau<t_0)\\ 0 & (\tau>t_0)\end{cases}$$

$$y(t)=\int_{-\infty}^{t} f(\tau)d\tau$$

$$=f(t)*u(t)$$

$$Y(\omega)=FT[y(t)]$$

$$=\frac{1}{j\omega}F(\omega)+\pi F(0)\delta(\omega)$$

F(0)不为0

$$=\frac{1}{j\omega}Sa(\frac{\omega t_0}{2})e^{-j\frac{\omega t_0}{2}}+\pi\delta(\omega)$$

11. 频域微分特性

设 $F^{(n)}(j\omega)=\dfrac{d^n F(j\omega)}{d\omega^n}$

若 $f(t)\leftarrow--\rightarrow \quad F(j\omega)$

则 $(-jt)^{(n)}f(t)\leftarrow--\rightarrow \quad F^{(n)}(j\omega)$

证明： $F(j\omega)=\int_{-\infty}^{\infty}f(t)e^{-j\omega t}dt$

$$\frac{dF(j\omega)}{d\omega}=\int_{-\infty}^{\infty}f(t)\frac{d}{d\omega}[e^{-j\omega t}]dt$$

$$=\int_{-\infty}^{\infty}(-jt)f(t)e^{-j\omega t}dt$$

12. 频域积分特性

设 $F^{(-1)}(j\omega)=\int_{-\infty}^{\omega}F(j\eta)d\eta$

若 $f(t)\longleftarrow\!-\!-\!\longrightarrow F(j\omega)$

则 $\pi f(0)\delta(t)+\dfrac{1}{-jt}f(t)\longleftarrow\!-\!-\!\longrightarrow F^{(-1)}(j\omega)$

证明 $F^{(-1)}(j\omega)=\int_{-\infty}^{\omega}F(j\eta)d\eta$

$$=\int_{-\infty}^{\omega}F(j\eta)d\eta*\delta(\omega)$$

$$=F(j\eta)*\varepsilon(\omega)$$

$$y(t) = 2\pi f(t) \cdot FT^{(-1)}[\varepsilon(\omega)]$$

$$= 2\pi \cdot f(t) \cdot \frac{1}{2\pi}[\pi\delta(t) + \frac{1}{-jt}]$$

$$= \pi f(0)\delta(t) + \frac{f(t)}{-jt}$$

$$\varepsilon(t) \longleftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

$$\varepsilon(-t) \longleftrightarrow \pi\delta(-\omega) + \frac{1}{-j\omega}$$

$$\pi\delta(t) + \frac{1}{-jt} \longleftrightarrow 2\pi\varepsilon(\omega)$$

13. 相关定理

相关函数:某信号与其另一时延τ的信号之间的相似程度,定义为:

$$R_{12}(\tau)=\int_{-\infty}^{\infty} f_1(t)f_2(t-\tau)dt=\int_{-\infty}^{\infty} f_1(t+\tau)f_2(t)dt=f_1(t)*f_2(-t)$$

$$R_{21}(\tau)=\int_{-\infty}^{\infty} f_2(t)f_1(t-\tau)dt=\int_{-\infty}^{\infty} f_2(t+\tau)f_1(t)dt=f_1(-t)*f_2(t)$$

可见: $R_{12}(\tau)=R_{21}(-\tau)$

若$f_1(t)$和$f_2(t)$是同一函数,则称为自相关函数

$$R(\tau)=\int_{-\infty}^{\infty} f(t)f(t-\tau)dt=f(t)*f(-t)$$

$$R(\tau)=R(-\tau)$$

13. 相关定理

若 $f_1(t) \longleftrightarrow F_1(j\omega) \quad f_2(t) \longleftrightarrow F_2(j\omega)$

则 $R_{12}(\tau) \longleftrightarrow F_1(j\omega)F_2{}^*(j\omega)$

$$R_{21}(\tau) \longleftrightarrow F_1(j\omega)F_2{}^*(j\omega)$$

证明: $R_{12}(\tau) = f_1(t) * f_2(-t)$

$$FT[R_{12}(\tau)] = FT[f_1(t)]FT[f_2(-t)]$$

$$= F_1(j\omega)F_2{}^*(j\omega)$$

同理: $R_{21}(\tau) \longleftrightarrow F_1(j\omega)F_2{}^*(j\omega)$

$$R(\tau) \longleftrightarrow |F(j\omega)|^2$$

14. Paseval定理

信号的能量定义为在时间(-∞,+∞) 区间上信号的能量,用字母E表示

$$E \overset{def}{=} \lim_{T\to\infty}\int_{-T}^{T}|f(t)|^2\,dt \quad = \int_{-\infty}^{\infty} f^2(t)dt$$

$$E = \int_{-\infty}^{\infty} f^2(t)dt = \int_{-\infty}^{\infty} f(t)\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} F(j\omega)e^{j\omega t}d\omega\right]dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\left[\int_{-\infty}^{+\infty} f(t)e^{j\omega t}dt\right]d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)F^*(j\omega)d\omega \quad = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|F(j\omega)\right|^2 d\omega \quad = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{E}(j\omega)d\omega$$

$$R(\tau) \longleftrightarrow \quad \mathcal{E}(j\omega)$$

14. Paseval定理

信号的功率定义为在时间(-∞,+∞) 区间上信号的平均功率,用字母P表示

$$P \overset{def}{=} \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}|f(t)|^2\,dt \quad = \frac{1}{T}\int_{-\infty}^{\infty} f^2(t)dt$$

$$E = \int_{-\infty}^{\infty} f^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|F(j\omega)|^2\,d\omega$$

$$P = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|F(j\omega)|^2}{T}d\omega \qquad = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{P}(j\omega)d\omega$$

14. Paseval定理

若f (t) 是功率有限信号,则称为自相关函数

$$R(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)f(t-\tau)dt \quad =\lim_{T\to\infty}\frac{1}{T}\left[f_T(\tau)*f_T(-\tau)\right]$$

$$FT[R(\tau)]=FT\left\{\lim_{T\to\infty}\frac{1}{T}\left[f_T(\tau)*f_T(-\tau)\right]\right\}$$

$$FT[R(\tau)]=\lim_{T\to\infty}\frac{1}{T}\left|F_T(j\omega)\right|^2$$

$$\mathcal{FT}[R(\tau)]=\mathcal{P}(j\omega)$$

$$R(\tau)=\mathcal{FT}^{-1}[\mathcal{P}(j\omega)]$$

试求下列非周期信号的频谱

$$f(t)=e^{-2t}\cos\omega_0 t\varepsilon(t)$$

$$f(t)=e^{-2|t|}\cos\omega_0 t\varepsilon(t)$$

$$f(t)=\sin^2\omega_0 t\cdot\varepsilon(t)$$

(5) $f(t)\delta(t-a)$

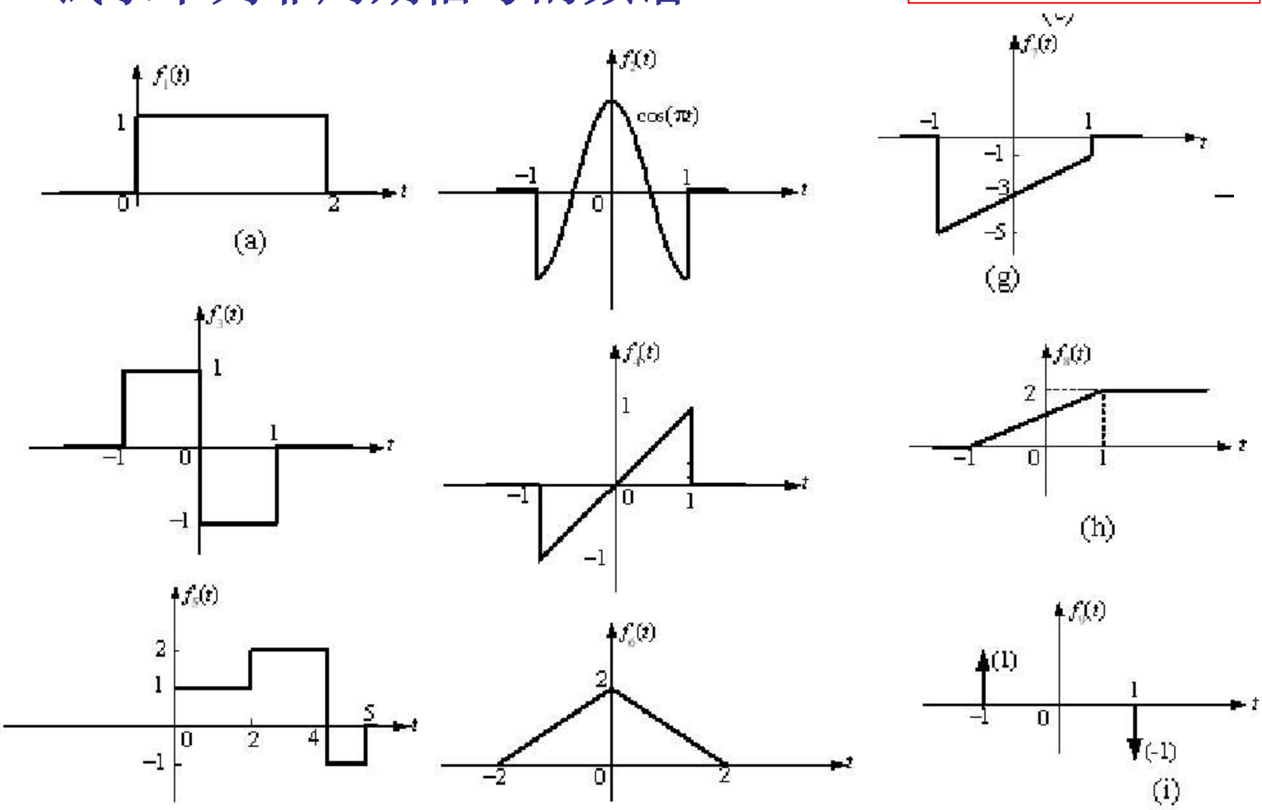
(6) $e^{-at}\varepsilon(-t)$

(7) $f(t)*\delta(\frac{t}{a}-b)$

(8) $(t-2)f(t)$

$$FT[g_1(t)] = Sa\frac{\omega}{2}$$

试求下列非周期信号的频谱



试求下列频谱对应的非周期信号

(1) $F(j\omega)=\dfrac{j\omega}{(j\omega+2)^2}$

(2) $F(j\omega)=\dfrac{4\sin(2\omega-2)}{2\omega-2}+\dfrac{4\sin(2\omega+2)}{2\omega+2}$

(3) $F(j\omega)=\dfrac{1}{j\omega(j\omega+2)}+2\pi\delta(\omega)$

(4) $F(j\omega)=\dfrac{d}{d\omega}[4\cos(3\omega)\dfrac{\sin 2\omega}{\omega}]$

(5) $F(j\omega)=\dfrac{2\sin\omega}{\omega(j\omega+1)}$

(6) $F(j\omega)=\dfrac{4\sin^2\omega}{\omega^2}$

试求下列非周期信号的频谱

$$f(t) = e^{-2t}\varepsilon(t)\cdot\sin\pi t$$

$$f(t) = \frac{2\sin\pi t}{\pi t}\cdot\frac{2\sin\pi t}{\pi t}$$

$$f(t) = \int_{-\infty}^{t}\frac{\sin\pi x}{\pi x}dx$$

$$f(t) = \frac{d}{dt}[\frac{\sin\pi t}{\pi t}\cdot\frac{\sin 2\pi t}{\pi t}]$$

$$f(t) = e^{-3|t-2|}$$

$$f(t) = \frac{d}{dt}[te^{-2t}\sin t\varepsilon(t)]$$

$$f(t) = e^{-2t+1}\varepsilon(\frac{t-4}{2})$$

常用信号的傅里叶变换

$$\delta(t) \longleftrightarrow 1$$

$$\delta(t-t_0) \longleftrightarrow e^{-j\omega t_0}$$

$$e^{-\alpha t}u(t) \longleftrightarrow \frac{1}{\alpha + j\omega}$$

$$\delta'(t) \longleftrightarrow j\omega$$

$$e^{-\alpha|t|} \longleftrightarrow \frac{2\alpha}{\alpha^2+\omega^2}$$

$$1 \longleftrightarrow 2\pi\delta(\omega)$$

$$u(t) \longleftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

$$e^{-j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega-\omega_0)$$

$$\mathrm{sgn}(t) \longleftrightarrow \frac{2}{j\omega}$$

$$g_\tau(t) \longleftrightarrow \tau\, Sa(\frac{\omega\tau}{2})$$

$$\cos\omega_0 t \longleftrightarrow \pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$$

$$\sin\omega_0 t \longleftrightarrow j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$$