第四章　连续系统的频域分析(2)

第4章 连续系统的频域分析

4.2 傅里叶级数

周期信号的傅里叶级数:

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos(n\Omega t)+\sum_{n=1}^{\infty} b_n\sin(n\Omega t)$$

系数: $$a_n=\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos(n\Omega t)dt,\quad n=0,1,2,\ldots$$

$$b_n=\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin(n\Omega t)dt,\quad n=1,2,\ldots$$

$$f(t)=\frac{A_0}{2}+\sum_{n=1}^{\infty} A_n\cos(n\Omega t+\varphi_n)$$

系数：

$$\left.\begin{aligned} A_0 &= a_0 \\ A_n &= \sqrt{a_n^2+b_n^2} \qquad n=0,1,2,\ldots \\ \varphi_n &= \arctan(\frac{a_n}{b_n}) \end{aligned}\right\}$$

$$\left.\begin{aligned} a_0 &= A_0 \\ a_n &= \cos\varphi_n \qquad n=1,2,\ldots \\ b_n &= -\sin\varphi_n \end{aligned}\right\}$$

方波信号 $f(t)=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin(2n+1)\Omega t$

误差： $t_2=\frac{T}{2}，t_1=-\frac{T}{2}，K_i=\frac{T}{2}$

$$\overline{\varepsilon^2}=\frac{1}{t_2-t_1}\int_{t_1}^{t_2}[f(t)-\sum_{j=1}^{n}c_j\varphi_j(t)]^2dt$$

$$=\frac{1}{T}[\int_{-\frac{T}{2}}^{\frac{T}{2}}f^2(t)dt-\sum_{j=1}^{n}b_j^2\cdot\frac{T}{2}]$$

$$=\frac{1}{T}[\int_{-\frac{T}{2}}^{\frac{T}{2}}dt-\frac{T}{2}\sum_{j=1}^{n}b_j^2]$$

$$=1-\frac{1}{2}\sum_{j=1}^{n}b_j^2$$

只取基波：

$$\overline{\varepsilon^2}=1-\frac{1}{2}(\frac{4}{\pi})^2=0.189$$

只取基波和三次谐波：

$$\overline{\varepsilon^2}=1-\frac{1}{2}(\frac{4}{\pi})^2-\frac{1}{2}(\frac{4}{3\pi})^2=0.0994$$

取一三五次谐波：

$$\overline{\varepsilon^2}=1-\frac{1}{2}(\frac{4}{\pi})^2-\frac{1}{2}(\frac{4}{3\pi})^2-\frac{1}{2}(\frac{4}{5\pi})^2=0.0669$$

取一三五七次谐波：

$$\overline{\varepsilon^2}=1-\frac{1}{2}(\frac{4}{\pi})^2-\frac{1}{2}(\frac{4}{3\pi})^2-\frac{1}{2}(\frac{4}{5\pi})^2-\frac{1}{2}(\frac{4}{7\pi})^2=0.0504$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$$

信号: $f(t) = \sin 2\pi t$

$T = 1, t_2 = 0.5,\ t_1 = -0.5,\ K_i = 0.5,\ \Omega = 2\pi$

$$
\begin{aligned}
a_n &= \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos(n\Omega t)dt \\
&= 2\int_{-0.5}^{0.5} \sin 2\pi t\cos(n2\pi t)dt \\
&= \int_{-0.5}^{0.5} [\sin(n+1)2\pi t - \sin(n-1)2\pi t]dt \\
&= \frac{1}{(n+1)2\pi}[-\cos(n+1)2\pi t]\Big|_{-0.5}^{0.5} + \frac{1}{(n-1)2\pi}\cos(n-1)2\pi t\Big|_{-0.5}^{0.5} \\
&= 0
\end{aligned}
$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin(n\Omega t)dt$$

$$= 2\int_{-0.5}^{0.5} \sin 2\pi t \sin(n2\pi t)dt$$

$$= \begin{cases} \int_{-0.5}^{0.5} [1-\cos 4\pi t]dt & n=1 \\ \int_{-0.5}^{0.5} [\cos(n-1)2\pi t - \cos(n+1)2\pi t]dt & n\neq 1 \end{cases}$$

$$= \begin{cases} t\Big|_{-0.5}^{0.5} - \frac{1}{4\pi}[\sin 4\pi t]\Big|_{-0.5}^{0.5} & n=1 \\ \frac{1}{(n-1)2\pi}[\sin(n-1)2\pi t]\Big|_{-0.5}^{0.5} - \frac{1}{(n+1)2\pi}\sin(n+1)2\pi t\Big|_{-0.5}^{0.5} & n\neq 1 \end{cases}$$

$$= \begin{cases} 1 & n=1 \\ 0 & n\neq 1 \end{cases}$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$$

信号: $f(t)=\cos 2\pi t$

$T=1, t_2=0.5,\ t_1=-0.5,\ K_i=0.5,\ \Omega=2\pi$

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos(n\Omega t)dt$$

$$= 2\int_{-0.5}^{0.5} \cos 2\pi t\cos(n2\pi t)dt$$

$$= \begin{cases} \int_{-0.5}^{0.5} [1+\cos 4\pi t]dt & n=1 \\ \int_{-0.5}^{0.5} [\cos(n+1)2\pi t+\cos(n-1)2\pi t]dt & n\neq 1 \end{cases}$$

$$= \begin{cases} 1 & n=1 \\ 0 & n\neq 1 \end{cases}$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$$

信号: $f(t) = \cos 2\pi t$

$T = 1, t_2 = 0.5,\ t_1 = -0.5,\ K_i = 0.5,\ \Omega = 2\pi$

$$\begin{aligned}
b_n &= \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin(n\Omega t)dt \\
&= 2\int_{-0.5}^{0.5} \cos 2\pi t \sin(n2\pi t)dt \\
&= \int_{-0.5}^{0.5} [\sin(n+1)2\pi t - \sin(n-1)2\pi t]dt \\
&= \frac{1}{(n+1)2\pi}[-\cos(n+1)2\pi t]\Big|_{-0.5}^{0.5} + \frac{1}{(n-1)2\pi}\cos(n-1)2\pi t\Big|_{-0.5}^{0.5} \\
&= 0
\end{aligned}$$

4.2 傅里叶级数

二、奇偶函数的傅里叶系数

1. f(t)为偶函数 $f(-t)=f(t)$

$$
\begin{aligned}
a_n &= \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos(n\Omega t)dt \\
&= \frac{2}{T}\int_{-\frac{T}{2}}^{0} f(t)\cos(n\Omega t)dt + \frac{2}{T}\int_{0}^{\frac{T}{2}} f(t)\cos(n\Omega t)dt \\
&= \frac{2}{T}\int_{\frac{T}{2}}^{0} f(-t)\cos(-n\Omega t)d(-t) + \frac{2}{T}\int_{0}^{\frac{T}{2}} f(t)\cos(n\Omega t)dt \\
&= \frac{4}{T}\int_{0}^{\frac{T}{2}} f(t)\cos(n\Omega t)dt
\end{aligned}
$$

4.2 傅里叶级数

$$
\begin{aligned}
b_n &= \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin(n\Omega t)dt \\
&= \frac{2}{T}\int_{-\frac{T}{2}}^{0} f(t)\sin(n\Omega t)dt + \frac{2}{T}\int_{0}^{\frac{T}{2}} f(t)\sin(n\Omega t)dt \\
&= \frac{2}{T}\int_{\frac{T}{2}}^{0} f(-t)\sin(-n\Omega t)d(-t) + \frac{2}{T}\int_{0}^{\frac{T}{2}} f(t)\sin(n\Omega t)dt \\
&= 0
\end{aligned}
$$

4.2 傅里叶级数

2. f(t)为奇函数 $f(-t)=-f(t)$

$$a_n=\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos(n\Omega t)dt$$

$$=\frac{2}{T}\int_{-\frac{T}{2}}^{0} f(t)\cos(n\Omega t)dt+\frac{2}{T}\int_{0}^{\frac{T}{2}} f(t)\cos(n\Omega t)dt$$

$$=\frac{2}{T}\int_{\frac{T}{2}}^{0} f(-t)\cos(-n\Omega t)d(-t)+\frac{2}{T}\int_{0}^{\frac{T}{2}} f(t)\cos(n\Omega t)dt$$

$$=0$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin(n\Omega t)dt$$

$$= \frac{2}{T}\int_{-\frac{T}{2}}^{0} f(t)\sin(n\Omega t)dt + \frac{2}{T}\int_{0}^{\frac{T}{2}} f(t)\sin(n\Omega t)dt$$

$$= \frac{2}{T}\int_{\frac{T}{2}}^{0} f(-t)\sin(-n\Omega t)d(-t) + \frac{2}{T}\int_{0}^{\frac{T}{2}} f(t)\sin(n\Omega t)dt$$

$$= \frac{4}{T}\int_{0}^{\frac{T}{2}} f(t)\sin(n\Omega t)dt$$

任意函数f(t)都可分解为奇函数和偶函数两部分

$$f(t) = f_{od}(t) + f_{ev}(t)$$

奇函数　　　偶函数

$$f(-t) = f_{od}(-t) + f_{ev}(-t) = -f_{od}(t) + f_{ev}(t)$$

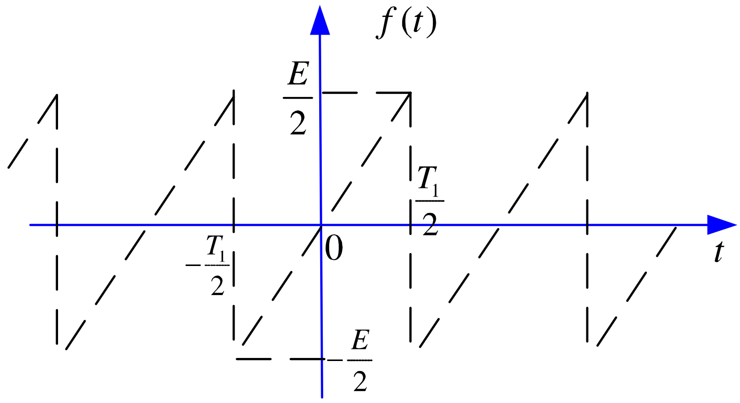
$$f_{ev}(t) = \frac{f(t) + f(-t)}{2}$$

$$f_{od}(t) = \frac{f(t) - f(-t)}{2}$$

3. f(t)为奇谐函数 $f(t) = -f(t \pm \frac{T}{2})$

$$a_0 = a_2 = a_4 = a_6 = \ldots = b_2 = b_4 = b_6 = \ldots = 0$$

例，周期锯齿脉冲信号



$$b_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\sin n\Omega t dt$$

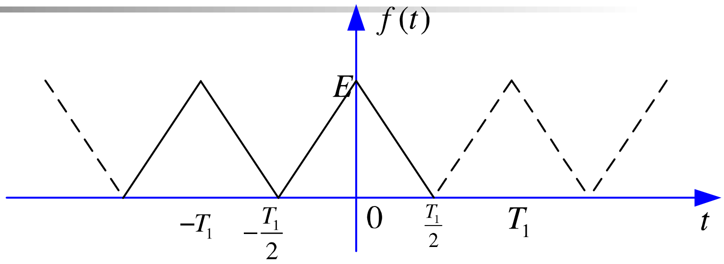
$$= \frac{4}{T}\frac{E}{T}\int_0^{\frac{T}{2}} t\sin n\Omega t dt$$

$$= \frac{4}{T}\frac{E}{T}[-\frac{t}{n\Omega}\cos n\Omega t|_0^{\frac{T}{2}} + \frac{1}{n\Omega}\int_0^{\frac{T}{2}}\cos n\Omega t dt]$$

$$= \frac{4}{T}\frac{E}{T}[-\frac{1}{n\Omega}\cdot\frac{T}{2}\cdot\cos n\pi + \frac{1}{(n\Omega)^2}\sin n\Omega t|_0^{\frac{T}{2}}] = -\frac{E}{n\pi}\cos n\pi$$

$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\Omega t = \frac{E}{\pi}[\sin\Omega t - \frac{1}{2}\sin 2\Omega t + \frac{1}{3}\sin 3\Omega t + ...]$$

例，周期三角波脉冲信号



$$a_0 = \frac{2}{T}\int_0^{\frac{T}{2}} f(t)dt = \frac{2}{T}\int_0^{\frac{T}{2}} (E - \frac{2E}{T}t)dt$$

$$= \frac{2}{T}E t\Big|_0^{\frac{T}{2}} - \frac{2}{T}\cdot\frac{2E}{T}\cdot\frac{1}{2}t^2\Big|_0^{\frac{T}{2}} = E - \frac{E}{2} = \frac{E}{2}$$

$$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\cos n\Omega t dt \quad = \frac{4}{T}\int_0^{\frac{T}{2}} (E - \frac{2E}{T}t)\cos n\Omega t dt$$

$$= \frac{4}{T}\{\frac{E}{n\Omega}\sin n\Omega t\Big|_0^{\frac{T}{2}} - \frac{2E}{T}[\frac{t}{n\Omega}\sin n\Omega t\Big|_0^{\frac{T}{2}} + \frac{1}{n\Omega}\int_0^{\frac{T}{2}} \sin n\Omega t dt]$$

$$= \frac{4}{T}\frac{2E}{T}[-\frac{1}{(n\Omega)^2}\cos n\Omega t\Big|_0^{\frac{T}{2}}] = \frac{2E}{\pi^2}\bullet\frac{1-\cos n\pi}{n^2}$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\Omega t = \frac{E}{2} + \frac{4E}{\pi^2}[\cos\Omega t + \frac{1}{3^2}\cos 3\Omega t + \frac{1}{5^2}\cos 5\Omega t + \ldots]$$

4.2 傅里叶级数

三、傅里叶级数的指数形式

$$f(t)=a_0+\sum_{n=1}^{\infty}a_n\cos(n\Omega t)+\sum_{n=1}^{\infty}b_n\sin(n\Omega t)$$

根据欧拉公式

$$\begin{cases}\cos(n\Omega t)=\dfrac{1}{2}(e^{jn\Omega t}+e^{-jn\Omega t})\\ \sin(n\Omega t)=\dfrac{1}{2j}(e^{jn\Omega t}-e^{-jn\Omega t})\end{cases}$$

a_n是n的偶函数

b_n是n的奇函数

$$f(t)=a_0+\sum_{n=1}^{\infty}\frac{a_n-jb_n}{2}e^{jn\Omega t}+\sum_{n=1}^{\infty}\frac{a_n+jb_n}{2}e^{-jn\Omega t}$$

n用-n代替

$$\sum_{n=1}^{\infty}\frac{a_{-n}-jb_{-n}}{2}e^{-jn\Omega t}\qquad\sum_{n=-1}^{-\infty}\frac{a_n-jb_n}{2}e^{jn\Omega t}$$

如果设 $F_n=\dfrac{a_n-jb_n}{2}$ 且 $F_0=a_0$ ，则 $f(t)=\sum_{n=-\infty}^{\infty}F_ne^{jn\Omega t}$

4.2 傅里叶级数

$$a_n = \frac{2}{T}\int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} f(t)\cos(n\Omega t)dt$$

$$b_n = \frac{2}{T}\int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} f(t)\sin(n\Omega t)dt$$

$$F_n = \frac{1}{2}(a_n - jb_n)$$

$$= \frac{1}{2}[\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos(n\Omega t)dt - j\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin(n\Omega t)dt]$$

$$= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)[\cos(n\Omega t) - j\sin(n\Omega t)]dt$$

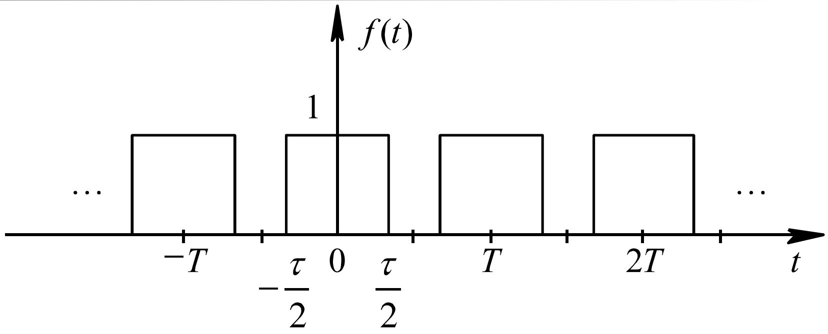
$$= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-jn\Omega t}dt$$

$$F_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-jn\Omega t}dt$$

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\Omega t}$$

4.2 傅里叶级数

例，周期矩形脉冲信号



取有限项逼近**f(t)**,则

$$F_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-jn\Omega t}dt = \frac{1}{T}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{-jn\Omega t}dt$$

$$= \frac{1}{T}\frac{e^{-jn\Omega t}}{-jn\Omega}\Bigg|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} = \frac{2}{Tn\Omega}\sin\frac{n\Omega\tau}{2} = \frac{\tau}{T}Sa\frac{n\Omega\tau}{2}$$

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\Omega t} = \frac{\tau}{T}\sum_{n=-\infty}^{\infty} Sa\frac{n\Omega\tau}{2}e^{jn\Omega t}$$

4.3 周期信号的频谱

$$f(t)=\sum_{n=-\infty}^{\infty}F_n e^{jn\Omega t}$$

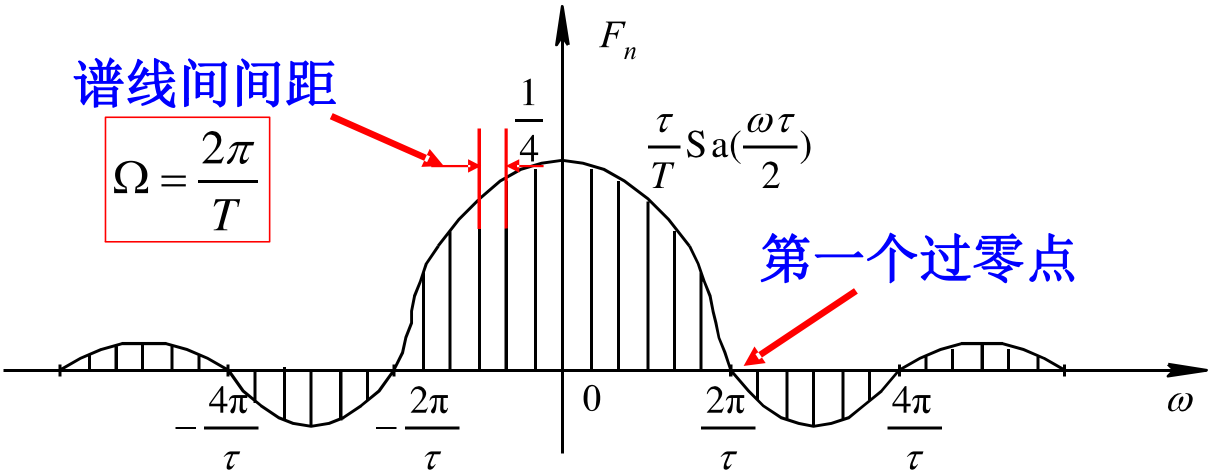
$$F_n=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f(t)e^{jn\Omega t}dt$$

一、周期信号的频谱

$$f(t)=\sum_{n=-\infty}^{\infty}F_n e^{jn\Omega t}=\frac{\tau}{T}\sum_{n=-\infty}^{\infty}Sa\frac{n\Omega\tau}{2}e^{jn\Omega t}$$

谱线间间距

$$\Omega=\frac{2\pi}{T}$$



第一个过零点

4.3 周期信号的频谱

周期信号频谱的特点:

1. 任何信号均由多次谐波叠加而成;

2. 周期信号的频谱是线谱，谱线间的间距为 $\Omega = \frac{2\pi}{T}$

3. 第一个过零点在 $\frac{2\pi}{\tau}$

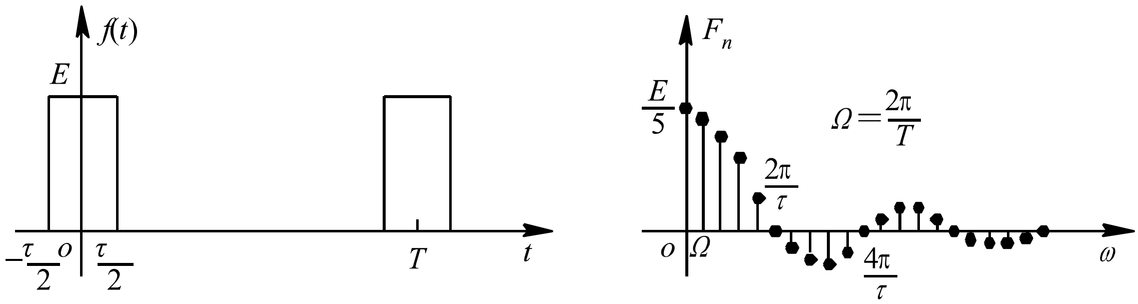
4. 信号的带宽 $B_\omega = \frac{2\pi}{\tau}(rad/s)$ 或 $B_f = \frac{1}{\tau}(Hz)$

5. 信号的功率 $P = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(t)dt = \sum_{k=-\infty}^{\infty} |F_k|^2$

$$P = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)[\sum_{k=-\infty}^{\infty} F_k e^{jk\Omega t}]dt = \sum_{k=-\infty}^{\infty} F_k[\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{jk\Omega t}dt = \sum_{k=-\infty}^{\infty} F_k F_{-k} = \sum_{k=-\infty}^{\infty} F_k F_k^* = \sum_{k=-\infty}^{\infty} |F_k|^2$$

4.3 周期信号的频谱

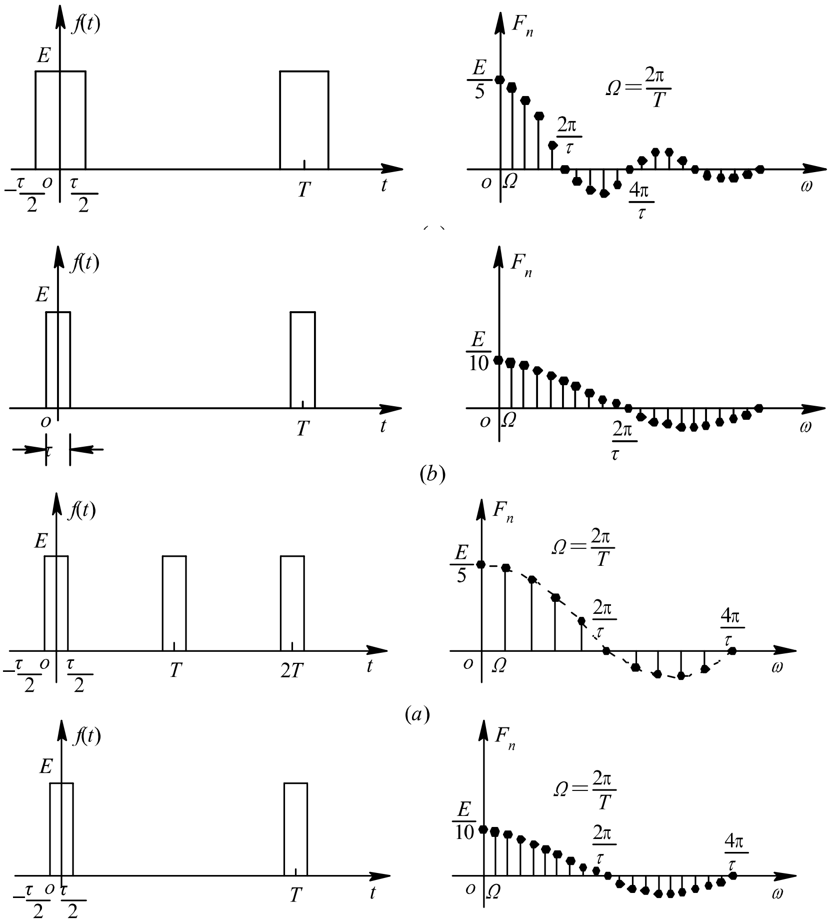
周期矩形脉冲信号的频率谱 $F_n = \dfrac{\tau}{T} Sa \dfrac{n\Omega\tau}{2}$



谱线间间距 $\Omega = \dfrac{2\pi}{T}$

第一个过零点 $\dfrac{2\pi}{\tau}$

4.3 周期信号的频谱



当周期T一定时，谱线间间距不变，门宽τ减小，带宽变宽

当门宽τ一定时，信号的带宽一定. 周期T变大时，谱线间距变小.

4.4 非周期信号的频谱

一、傅里叶变换的推导

$$F_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{jn\Omega t}dt \rightarrow 0$$

$$T \rightarrow \infty \qquad F_n T \rightarrow F(j\omega)$$

$$F_n T = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{jn\Omega t}dt$$

$$\Omega \underset{T\Omega=2\pi}{\rightarrow} d\omega \qquad n\Omega \rightarrow \omega$$

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \quad \text{———— ①}$$

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\Omega t}$$

$$F_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{jn\Omega t}dt$$

频谱不能用幅度来表示,而是用密度函数表示

4.4 非周期信号的频谱

$$f(t)=\sum_{n=-\infty}^{\infty}F_n e^{jn\Omega t} \qquad =\sum_{n=-\infty}^{\infty}\frac{F_n T}{T}e^{jn\Omega t}$$

$$T\to\infty \qquad F_n T\to F(j\omega) \qquad \sum_{n=-\infty}^{\infty}\to\int$$

$$\frac{1}{T}=\frac{1}{2\pi}\Omega\to\frac{1}{2\pi}d\omega \qquad n\Omega\to\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(j\omega)e^{j\omega t}d\omega \quad \text{——— ②}$$

$$F(j\omega)=\int_{-\infty}^{\infty}f(t)e^{-j\omega t}dt \quad \text{——— ①}$$

①②称为一对傅里叶变换.

4.4 非周期信号的频谱

傅里叶变换的模与相位

$$F(j\omega)=|F(j\omega)|e^{j\varphi(\omega)} \quad =\int_{-\infty}^{\infty}f(t)e^{-j\omega t}dt$$

$$=\int_{-\infty}^{\infty}f(t)\cos\omega t dt - j\int_{-\infty}^{\infty}f(t)\sin\omega t dt \quad = R(\omega)+jX(\omega)$$

$$R(\omega)=\int_{-\infty}^{\infty}f(t)\cos\omega t dt \qquad X(\omega)=-\int_{-\infty}^{\infty}f(t)\sin\omega t dt$$

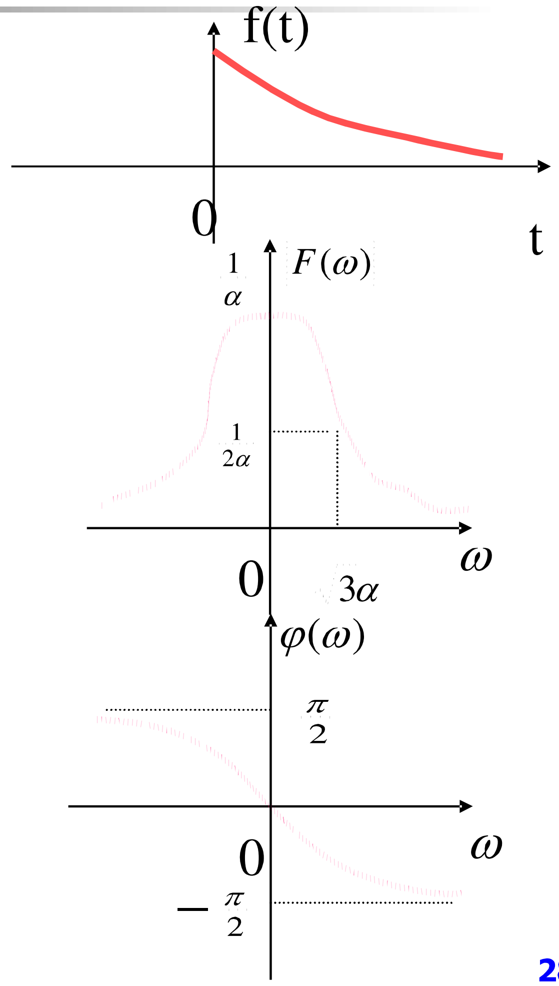
$$R(\omega)=R(-\omega) \qquad X(\omega)=-X(-\omega)$$

$$|F(j\omega)|=\sqrt{R^2(\omega)+X^2(\omega)} \qquad \varphi(\omega)=-\arctan\frac{X(\omega)}{R(\omega)}$$

ω的偶函数　　ω的奇函数

二、典型非周期信号的傅里叶变换

(1) 单边指数信号



信号表达式 $f(t)=\begin{cases} e^{-\alpha t} & (t \geq 0) \\ 0 & (t<0) \end{cases}$

$$F(\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$$=\int_{0}^{\infty} e^{-\alpha t}e^{-j\omega t}dt$$

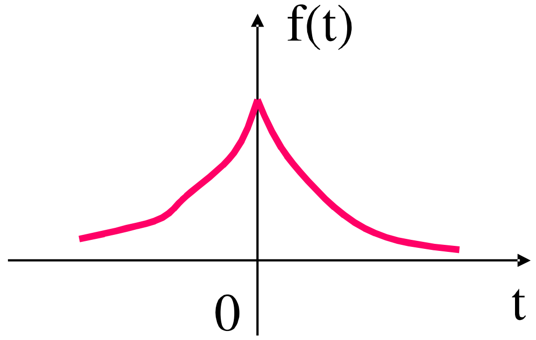
$$=\frac{1}{\alpha+j\omega} \qquad (\alpha>0)$$

幅频: $|F(\omega)|=\frac{1}{\sqrt{\alpha^2+\omega^2}}$

相频: $\varphi(\omega)=-arctg(\frac{\omega}{\alpha})$

(2) 双边指数信号

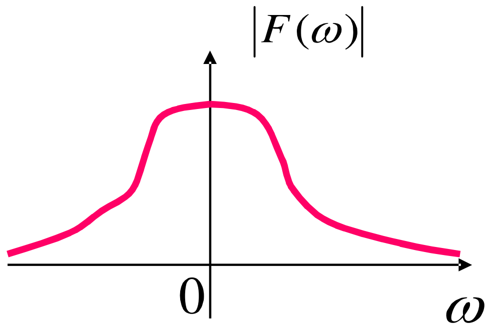
信号表达式 $f(t)=e^{-\alpha|t|} \quad (-\infty<t<+\infty)$



$$F(\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$$=\int_{-\infty}^{0} e^{\alpha t}e^{-j\omega t}dt+\int_{0}^{\infty} e^{-\alpha t}e^{-j\omega t}dt$$

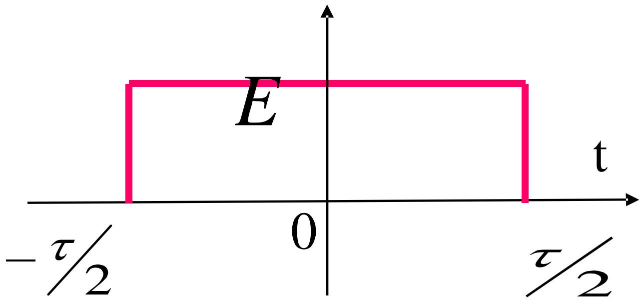
$$=\frac{1}{\alpha-j\omega}+\frac{1}{\alpha+j\omega}=\frac{2\alpha}{\alpha^2+\omega^2} \qquad (\alpha>0)$$



幅频: $F(\omega)=\dfrac{2\alpha}{\alpha^2+\omega^2}$

相频: $\varphi(\omega)=0$

(3) 矩形脉冲信号

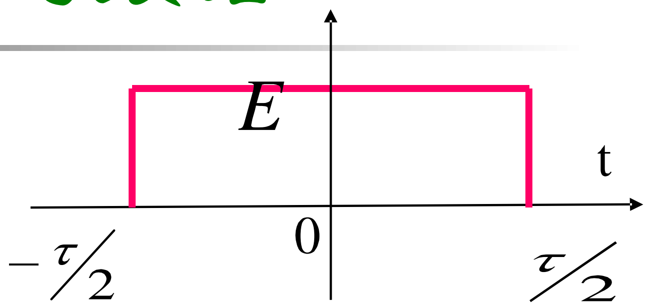


信号表达式

$$f(t)=\begin{cases}E & (|t|\le \frac{\tau}{2})\\ 0 & (|t|>\frac{\tau}{2})\end{cases}$$

$$F(\omega)=\int_{-\tau/2}^{\tau/2}Ee^{-j\omega t}dt=\frac{2E}{\omega}\sin(\frac{\omega\tau}{2})$$

$$=E\tau\left(\frac{\sin(\frac{\omega\tau}{2})}{\frac{\omega\tau}{2}}\right)=E\tau\ Sa(\frac{\omega\tau}{2})$$

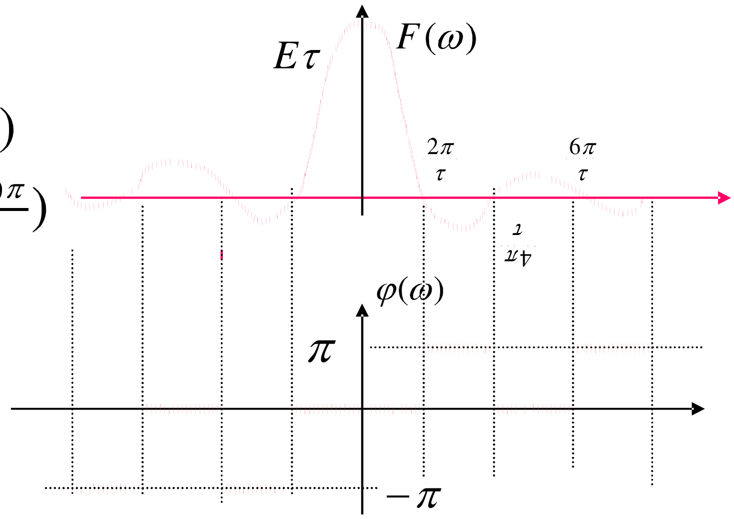


(3) 矩形脉冲信号

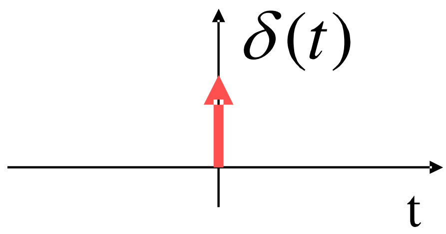
幅频: $|F(\omega)| = E\tau \left|Sa(\frac{\omega\tau}{2})\right|$

相频:

$$\varphi(\omega)=\begin{cases}0 & (\frac{4n\pi}{\tau}<|\omega|<\frac{2(2n+1)\pi}{\tau})\\ \pi & (\frac{2(2n+1)\pi}{\tau}<|\omega|<\frac{4(n+1)\pi}{\tau})\end{cases}$$



三、奇异函数的傅里叶变换



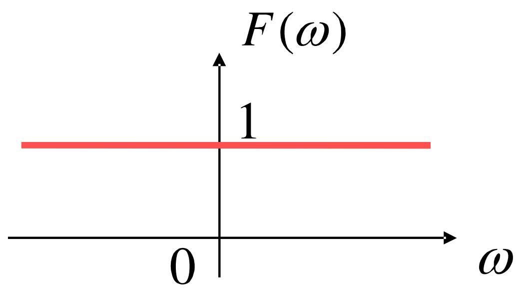
(1) 冲激函数的傅里叶变换和逆变换

信号表达式 $\delta(t)=\begin{cases}1 & t=0\\ 0 & t\neq 0\end{cases}$

$$F(\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$$=\int_{-\infty}^{\infty}\delta(t)e^{-j\omega t}dt=1$$

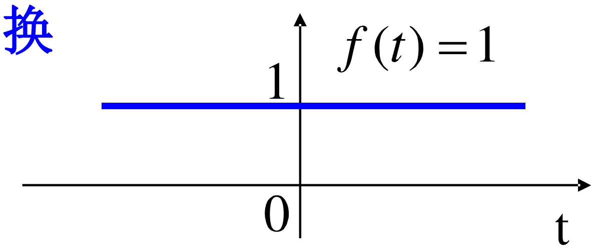
$$\delta(t)\longleftarrow - - \longrightarrow \quad 1$$



4.4 非周期信号的频谱

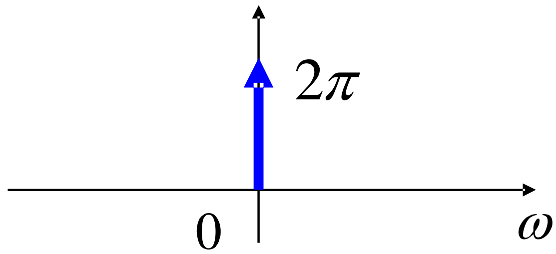
(1) 冲激函数的傅里叶变换和逆变换

$$\delta(\omega)=\begin{cases}1 & \omega=0\\ 0 & \omega\neq 0\end{cases}$$



$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{j\omega t}dt$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega)e^{j\omega t}dt=\frac{1}{2\pi}$$



$$1\longleftarrow - -\longrightarrow \quad 2\pi\delta(\omega)$$

(2) 冲激函数的导数的频谱

$$FT[f(t)] = F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$$FT[\delta'(t)] = \int_{-\infty}^{\infty} \delta'(t)e^{-j\omega t}dt$$

$$= -(e^{-j\omega t})'\Big|_{t=0} = j\omega$$

$$\boxed{FT[\delta'(t)] = j\omega}$$

$$\boxed{FT[\delta^{(n)}(t)] = (j\omega)^n}$$

(3) 符号函数

信号表达式 $\operatorname{sgn}(t)=\begin{cases}+1 & (t>0)\\ -1 & (t<0)\end{cases}$

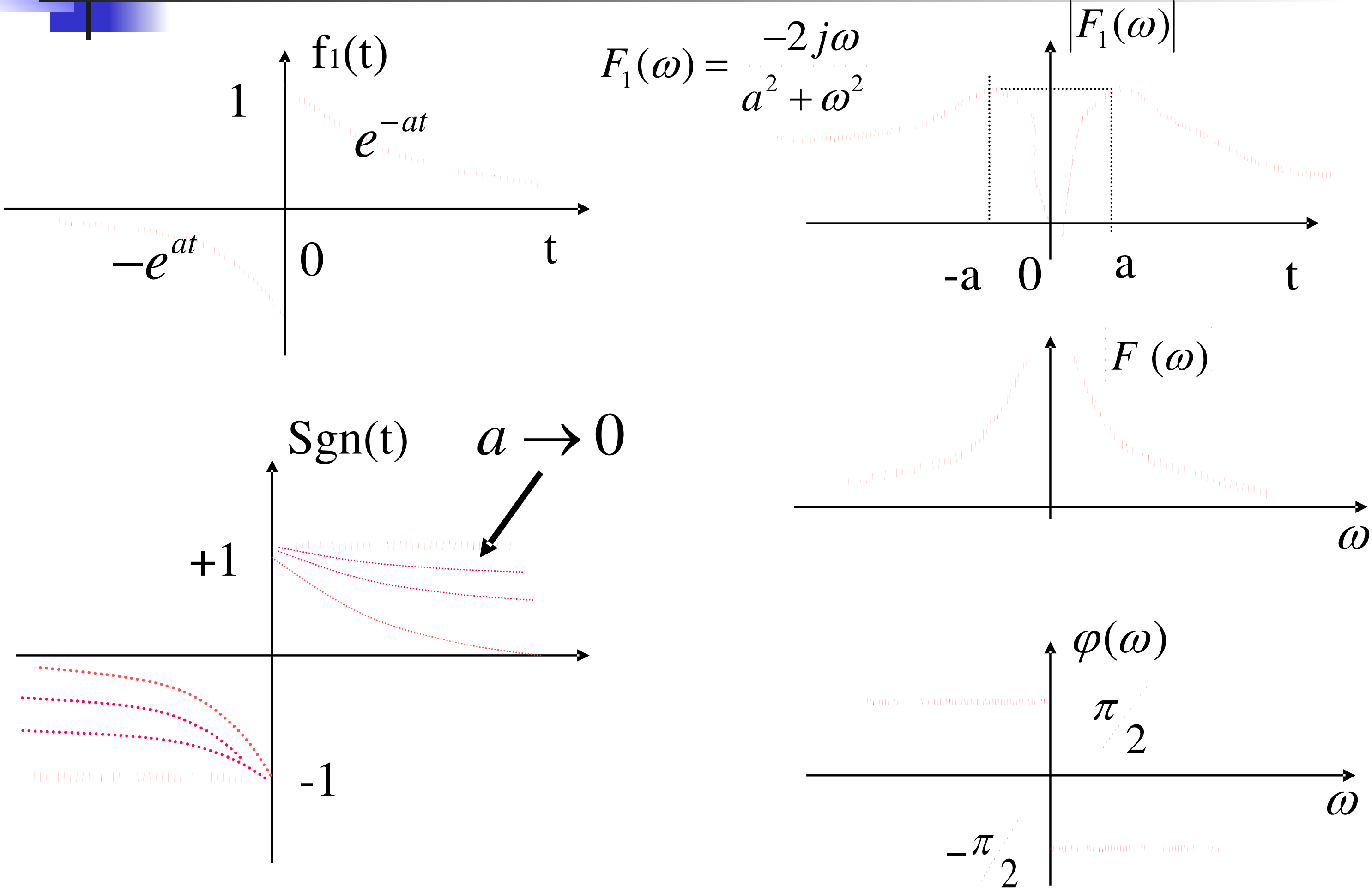
$$f(t)=\lim_{a\to 0} f_1(t)=\lim_{a\to 0}[\operatorname{sgn}(t).e^{-a|t|}]$$

$$F_1(\omega)=\frac{-2j\omega}{a^2+\omega^2}$$

$$\lim_{a\to 0} F_1(\omega)=\lim_{a\to 0}\frac{-2j\omega}{a^2+\omega^2}=\frac{2}{j\omega}$$

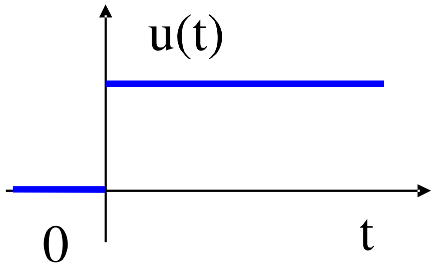
幅频: $|F(\omega)|=\dfrac{2}{\omega}$ 相频: $\varphi(\omega)=\begin{cases}-\frac{\pi}{2} & (\omega>0)\\ +\frac{\pi}{2} & (\omega<0)\end{cases}$

4.4 非周期信号的频谱



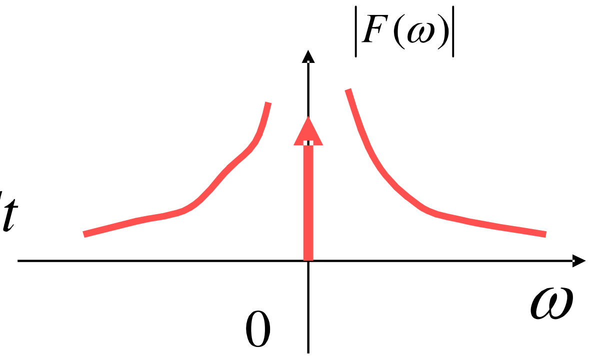
f1(t)
1
e^{-at}
$-e^{at}$
0
t
Sgn(t)
$a \to 0$
+1
-1
$F_1(\omega) = \frac{-2j\omega}{a^2+\omega^2}$
$|F_1(\omega)|$
-a
0
a
t
$F(\omega)$
ω
$\varphi(\omega)$
$\pi/2$
$-\pi/2$
ω

4.4 非周期信号的频谱



(4) 阶跃函数的傅里叶变换

信号表达式 $\varepsilon(t)=\begin{cases}1 & t>0\\ 0 & t<0\end{cases}=\frac{1}{2}+\frac{1}{2}\operatorname{sgn}(t)$



$$F(\omega)=\int_{-\infty}^{\infty}f(t)e^{-j\omega t}dt$$

$$=\int_{-\infty}^{\infty}(\tfrac{1}{2}+\tfrac{1}{2}\operatorname{sgn}(t))e^{-j\omega t}dt$$

$$=\pi\delta(\omega)+\frac{1}{j\omega}$$

$$\varepsilon(t)\longleftarrow - - \longrightarrow \quad \pi\delta(\omega)+\frac{1}{j\omega}$$

4.1 信号的正交分解

最小误差值 $\overline{\varepsilon^2} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2}[f(t) - \sum_{j=1}^{n} c_j\varphi_j(t)]^2 dt$

$$= \frac{1}{t_2 - t_1}[\int_{t_1}^{t_2} f^2(t)dt + \sum_{j=1}^{n} c_j^2 \int_{t_1}^{t_2} \varphi_j^2(t)dt - 2\sum_{j=1}^{n} c_j \int_{t_1}^{t_2} f(t)\varphi_j(t)dt]$$

$$= \frac{1}{t_2 - t_1}[\int_{t_1}^{t_2} f^2(t)dt + \sum_{j=1}^{n} c_j^2 K_i - 2\sum_{j=1}^{n} c_j^2 K_j]$$

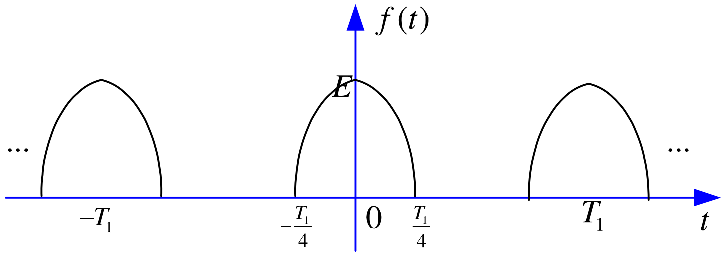
$$= \frac{1}{t_2 - t_1}[\int_{t_1}^{t_2} f^2(t)dt - \sum_{j=1}^{n} c_j^2 K_j]$$

当$n \to \infty$, $\overline{\varepsilon^2} = 0$ $\boxed{\int_{t_1}^{t_2} f^2(t)dt = \sum_{j=1}^{n} c_j^2 K_j}$ 帕斯瓦尔方程

$$\boxed{f(t) = \sum_{j=1}^{\infty} c_j\varphi_j(t)}$$

Thank you !

例，周期半波余弦信号



$$a_0 = \frac{2}{T}\int_0^{\frac{T}{2}} f(t)dt = \frac{2}{T}\int_0^{\frac{T}{4}} E\cos\Omega t dt$$

$$= \frac{2}{T}E\frac{1}{\Omega}\sin\Omega t\Big|_0^{\frac{T}{4}} = \frac{E}{\pi}\sin(\frac{2\pi}{T}\frac{T}{4} - 0) = \frac{E}{\pi}$$