

# 第十六章

## 结构的稳定计算简介

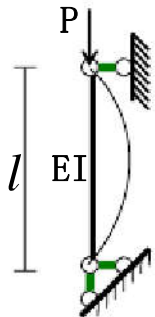
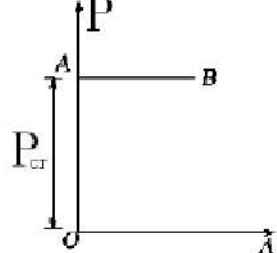
## 16.1 两类稳定问题概述

结构力学研究三项重要课题 强度、刚度、稳定性

材料力学讨论单根压杆的稳定问题

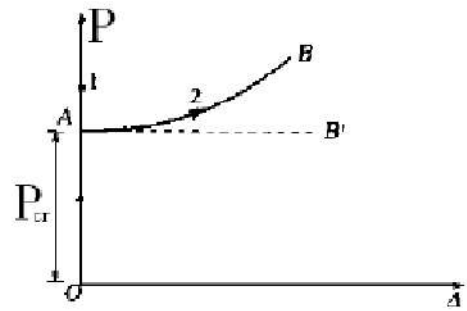
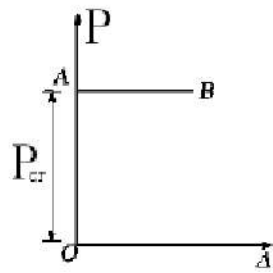
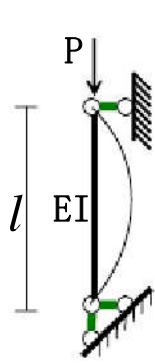
结构力学研究结构的稳定问题

### 1. 分支点失稳

|   |                                 |         |   |
|---|---------------------------------|---------|---|
|  | $P_{cr} = \frac{\pi^2 EI}{l^2}$ | ---临界荷载 |  |
|   | $P < P_{cr}$                    | 稳定平衡    |   |
|   | $P = P_{cr}$                    | 随遇平衡    |   |
|   | $P > P_{cr}$                    | 不稳定平衡   |   |

体系从稳定平衡状态转入不稳定平衡状态的破坏称为**失稳(屈曲)**。

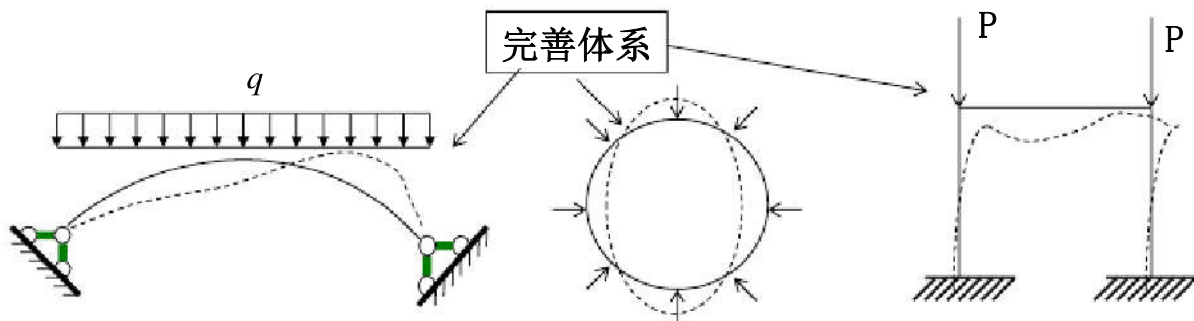
体系从稳定平衡状态转入不稳定平衡状态的破坏称为失稳。



分支点失稳的特点是：结构的变形在荷载达到临界值前后发生性质上的突变。

两种平衡状态：轴心受压和压弯。

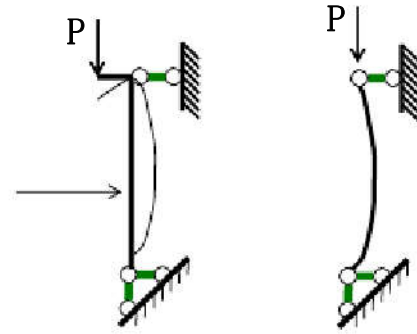
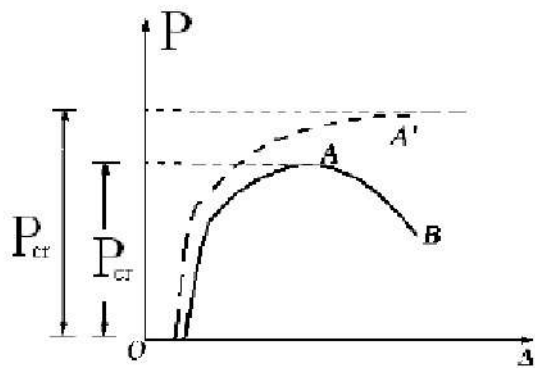
—— 第一类稳定问题



## 2. 极值点失稳

第二类稳定问题

非完善体系

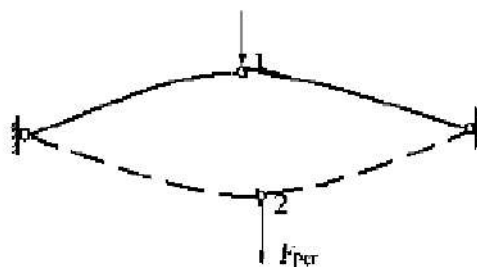
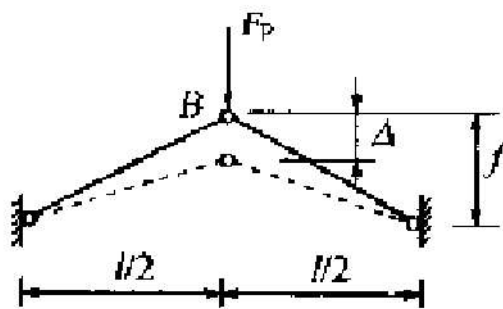
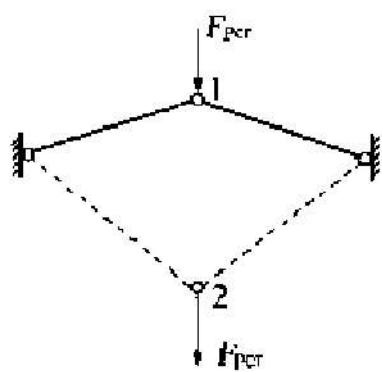


偏心受压

有初变形

极值点失稳的特点是：结构的变形在荷载达到临界值前后并不发生性质上的突变，只是原有变形的迅速增长。

# 跳跃失稳



--- 第三类稳定问题

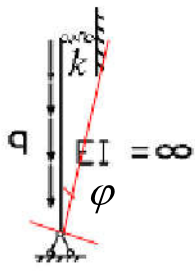
## 16.2 (3) 稳定问题的分析——静力法和能量法

静力法  
能量法

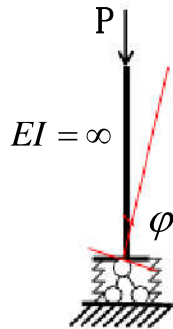
只讨论完善体系的分支点失稳问题

稳定自由度

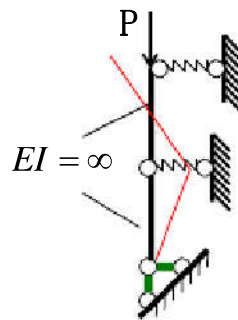
在稳定计算中，确定一个体系临界平衡的变形状态所需的独立几何参数的数目，称为稳定自由度。



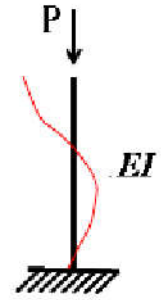
1个自由度



抗转动弹簧



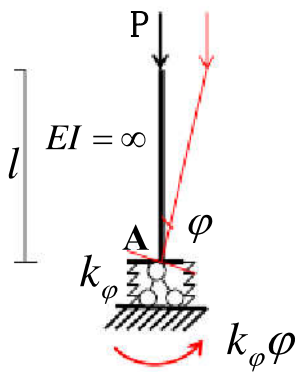
2个自由度



无限自由度

# 1. 静力法 求临界荷载

## 1) 一个自由度体系



$$\sum M_A = 0$$

$$k_\varphi \cdot \varphi - Pl \sin \varphi = 0$$

小挠度、小位移情况下:  $\sin \varphi \doteq \varphi$

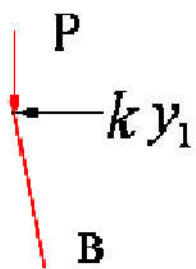
$$(k_\varphi - Pl)\varphi = 0$$

$$\varphi \neq 0 \quad k_\varphi - Pl = 0$$

----稳定方程 (特征方程)

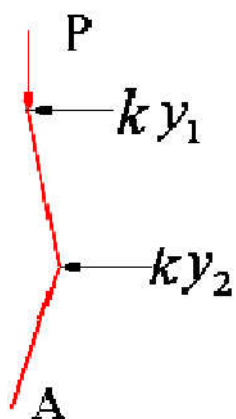
$$P_{cr} = k_\varphi / l \quad \text{---临界荷载}$$

## 2) N自由度体系



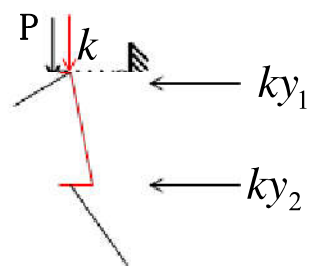
$$\sum M_B = 0$$

$$\sum M_A = 0$$



$$ky_1 \cdot l + P(y_2 - y_1) = 0$$

$$ky_2 \cdot l + ky_1 \cdot 2l - Py_1 = 0$$





$$ky_1 \cdot l + P(v - v) = 0$$

$$ky_2 \cdot l +$$

$$(kl$$

$$(2kl$$

$$\left| \begin{array}{l} kl - \\ 2kl - \end{array} \right.$$

$$kl(k$$

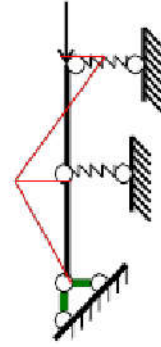
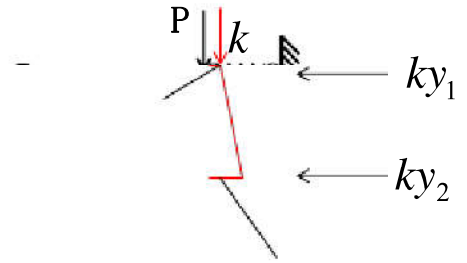
$$P^2 +$$

$$P = \frac{\sqrt{\quad}}{\quad}$$

(---)形式

$$\frac{\quad}{y_1} = 1.010$$

---大徳形式



# 16.4 弹性压杆 无限自由度体系

## 1. 等截面杆

挠曲线近似微分方程为  $EIy''(x) = M(x)$

$$M = -Py + Q(l - x)$$

$$EIy''(x) = -Py + Q(l - x)$$

或  $y''(x) + \frac{P}{EI}y = \frac{Q}{EI}(l - x)$  得

令  $n^2 = \frac{P}{EI}$

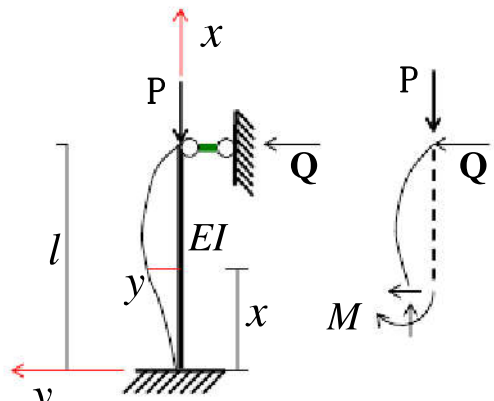
$$y''(x) + n^2y = n^2 \frac{Q}{P}(l - x)$$

通解为

$$y(x) = A \cos nx + B \sin nx + \frac{Q}{P}(l - x)$$

由边界条件

$$y(0) = 0, y'(0) = 0, y(l) = 0$$

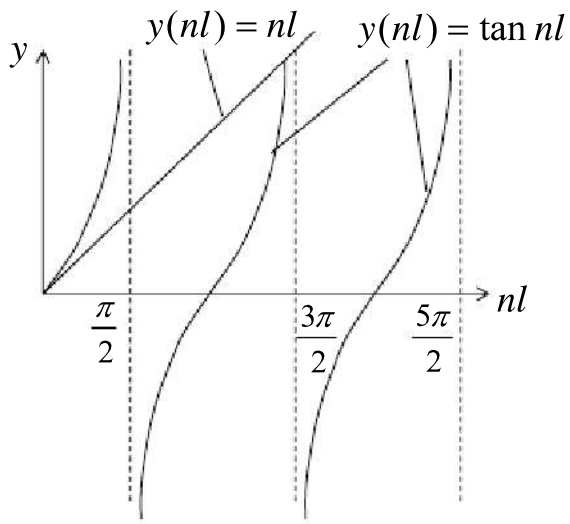


$$\begin{cases} A + \frac{Q}{P}l = 0 \\ Bn - \frac{Q}{P} = 0 \\ A \cos nl + B \sin nl = 0 \end{cases} \quad \text{以A、B、} \\ \text{Q/P为变量}$$

$$\begin{vmatrix} 1 & 0 & l \\ 0 & n & -1 \\ \cos nl & \sin nl & 0 \end{vmatrix} = 0 \quad \text{稳定方程}$$

$$-nl \cos nl + \sin nl = 0$$

$$\tan nl = nl$$

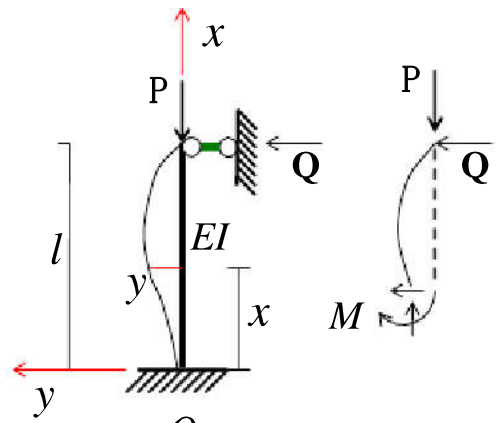


经试算

$$nl \approx 4.493$$

$$P_{cr} = n^2 EI = \left(\frac{4.493}{l}\right)^2 EI$$

$$= 20.19 EI / l^2 = \frac{\pi^2 EI}{(0.7l)^2}$$



得

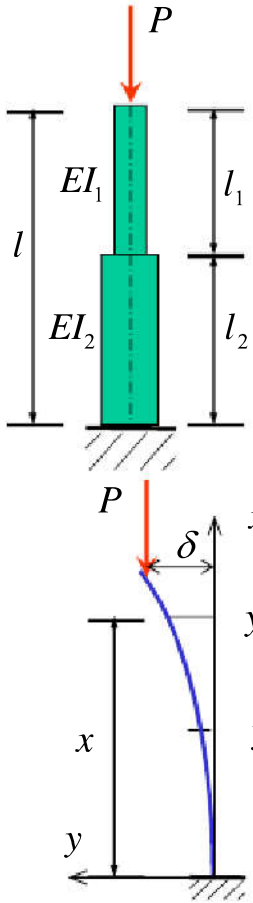
$$\begin{cases} A + \frac{Q}{P}l = 0 \\ Bn - \frac{Q}{P} = 0 \\ A \cos nl + B \sin nl = 0 \end{cases}$$

$$\begin{vmatrix} 1 & 0 & l \\ 0 & n & -1 \\ \cos nl & \sin nl & 0 \end{vmatrix} = 0 \quad \text{稳定方程}$$

$$-nl \cos nl + \sin nl = 0$$

$$\tan nl = nl$$

## 2.变截面压杆



平衡微分方程为:  $EI_1 y_1'' = P(\delta - y_1)$

$$EI_2 y_2'' = P(\delta - y_2)$$

$$\text{令 } n_1^2 = \frac{P}{EI_1}, \quad n_2^2 = \frac{P}{EI_2}$$

通解分别为:

$$y_1 = A_1 \cos n_1 x + B_1 \sin n_1 x + \delta$$

$$y_2 = A_2 \cos n_2 x + B_2 \sin n_2 x + \delta$$

边界条件为:

当  $x=0$  时,  $y_2 = 0, y_2' = 0$

$\rightarrow A_2 = -\delta, B_2 = 0$

当  $x=l_2$  时,  $y_1 = y_2, y_1' = y_2'$

当  $x=l$  时,  $y_1 = \delta$

得齐次方程为:

$$\begin{cases} A_1 \cos n_1 l + B_1 \sin n_1 l = 0 \\ A_1 \cos n_1 l_2 + B_1 \sin n_1 l_2 + \delta \cos n_2 l_2 = 0 \\ A_1 n_1 \sin n_1 l_2 - B_1 n_1 \cos n_1 l_2 + \delta n_2 \sin n_2 l_2 = 0 \end{cases}$$

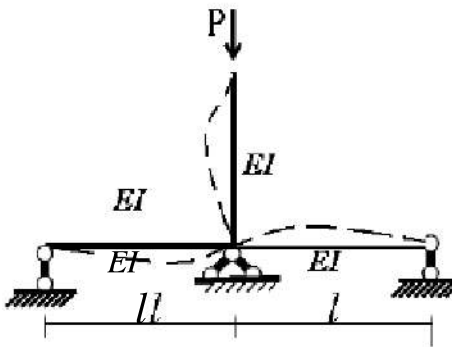
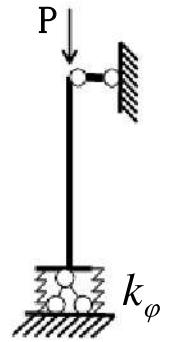
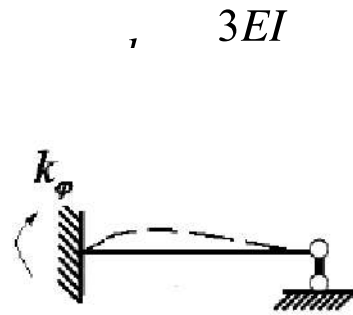
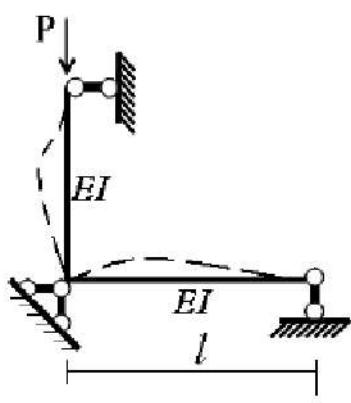
特征方程

$$\begin{vmatrix} \cos n_1 l & \sin n_1 l & 0 \\ \cos n_1 l_2 & \sin n_1 l_2 & \cos n_2 l_2 \\ \sin n_1 l_2 & -\cos n_1 l_2 & \frac{n_2}{n_1} \sin n_2 l_2 \end{vmatrix} = 0$$

超越方程为:

$$\operatorname{tgn} n_1 l_1 \cdot \operatorname{tgn} n_2 l_2 = \frac{n_1}{n_2}$$

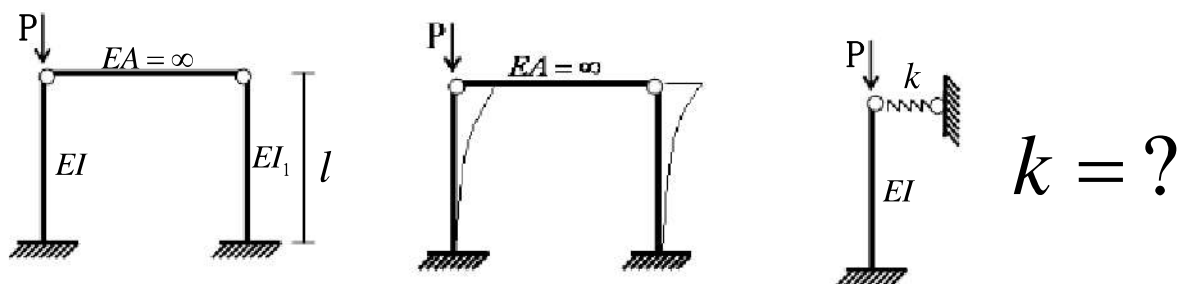
### 3. 具有弹性支座压杆的稳定



$$k_\varphi = \frac{6EI}{l}$$

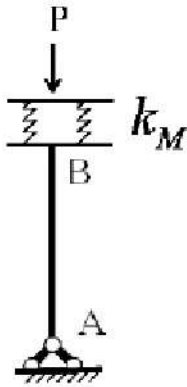
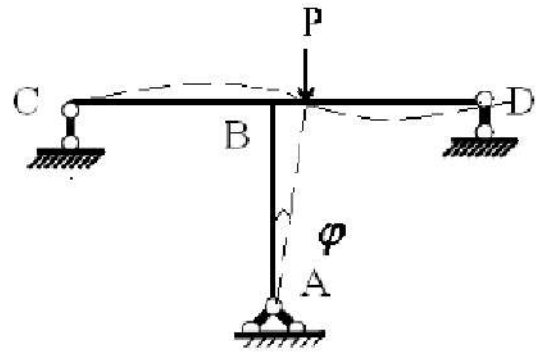
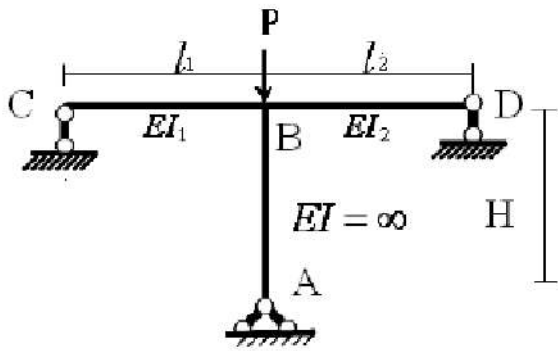
### 3.具有弹性支座压杆的稳定

练习:简化成具有弹簧支座的压杆

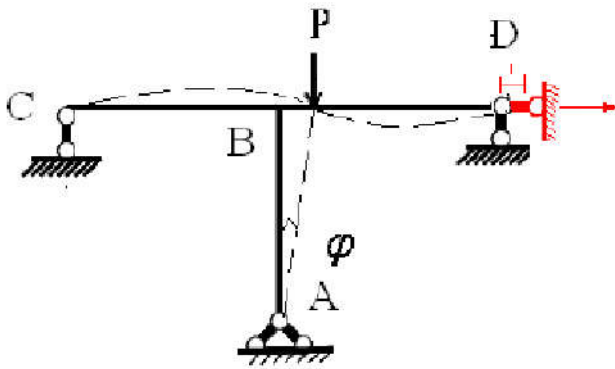
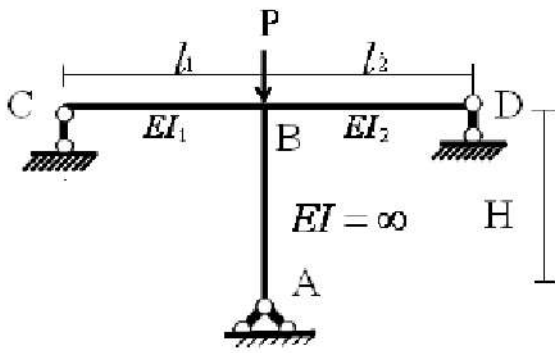


$k = ?$

$$k = \frac{3EI_1}{l^3}$$



$$k_M = (3i_1 + 3i_2)$$



$$k_N = (3i_1 + 3i_2) \frac{1}{H^2}$$



挠曲线近似微分方程为

## 弹性杆

$$EIy''(x) = M(x)$$

$$M = -Py + Q(l - x)$$

$$EIy''(x) = -Py + Q(l - x)$$

$$\sum M_A = 0 \quad Ql = k_\varphi \varphi$$

$$\text{令 } n^2 = \frac{P}{EI}$$

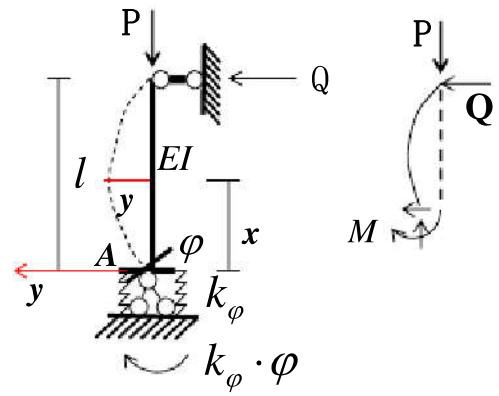
$$y''(x) + n^2 y = \frac{k_\varphi \varphi}{EI \cdot l} (l - x)$$

通解为

$$y(x) = A \cos nx + B \sin nx + \frac{k_\varphi \varphi}{Pl} (l - x)$$

$$\text{边界条件 } y(0) = 0, y'(0) = \varphi, y(l) = 0$$

$$\begin{cases} A + \frac{k_\varphi}{P} \varphi = 0 \\ Bn - \left(\frac{k_\varphi}{Pl} + 1\right) \varphi = 0 \\ A \cos nl + B \sin nl = 0 \end{cases}$$

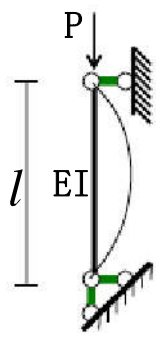


稳定方程

$$\begin{vmatrix} 1 & 0 & k_\varphi / P \\ 0 & n & -(k_\varphi / Pl + 1) \\ \cos nl & \sin nl & 0 \end{vmatrix} = 0$$

$$\tan nl = \frac{nl}{1 + \frac{EI}{k_\varphi l} (nl)^2}$$

解方程可得  $nl$  的最小正根  $P_{cr} = n^2 EI$



若  $k_{\varphi} = 0$

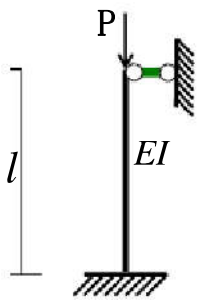
$$\tan nl = 0$$

$$\sin nl = 0$$

$$nl = \pi$$

$$P_{cr} = \frac{\pi^2 EI}{l^2}$$

为两端铰支的压杆

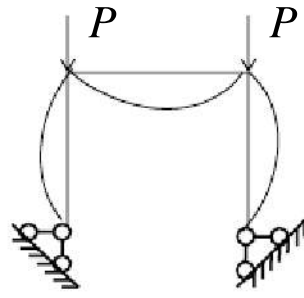
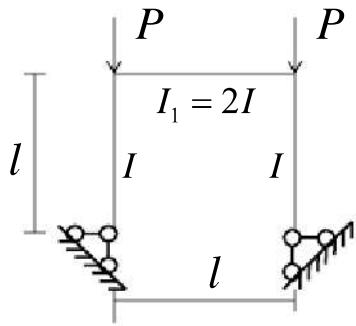


若  $k_{\varphi} = \infty$

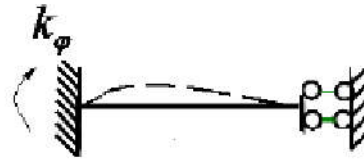
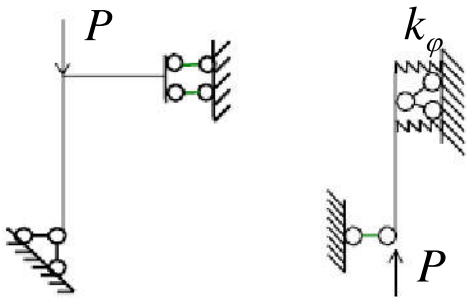
$$\tan nl = nl$$

$$P_{cr} = 20.19EI / l^2$$

一端固定，一端铰支的压杆

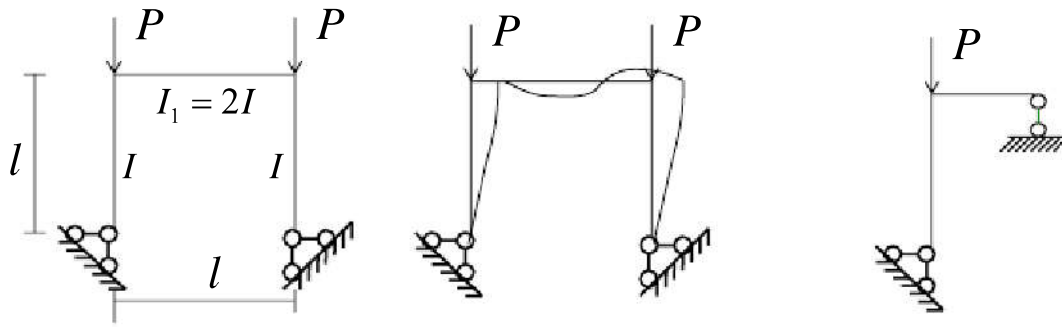


正对称失稳

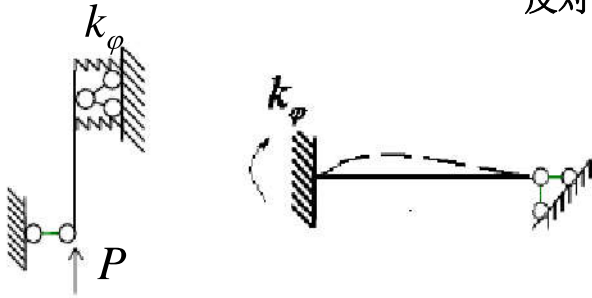


$$k_{\varphi} = \frac{2EI}{l/2} = 4EI/l$$

$$P_{cr\text{正}} = 14.67EI/l^2$$



反对称失稳



$$k_{\phi} = 3 \cdot \frac{2EI}{l/2} = 12EI / l$$

$$P_{cr\text{反}} = 2.10EI / l^2$$

$$P_{cr\text{正}} = 14.67EI / l^2$$

1. 应变能

弯曲应变能  $V_e = P \Delta / 2 = \frac{1}{2} \int_0^l M \kappa dx$

拉压应变能  $V_e = P \Delta / 2 = \frac{1}{2} \int_0^l N \varepsilon dx$

剪切应变能  $V_e = P \Delta / 2 = -$

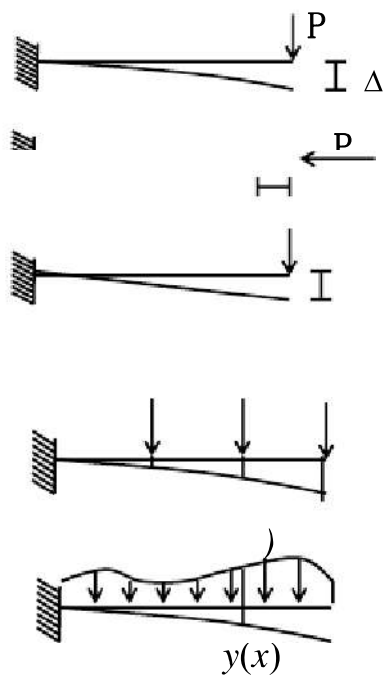
2. 外力势能

外力从变形状态退回至原始状态中所作的功。

$V_P^* = -\sum P_i \Delta_i$        $V_P^* = -\int_0^l q(x) y(x) dx$

3. 结构势能

$E_P = V_e + V_P^*$



#### 4. 势能驻值原理

在弹性结构的一切可能位移中，真实位移使结构势能取驻值。



对于稳定平衡状态, 真实位移使结构势能取极小值.

例一:求图示结构的临界荷载.  
有限自由度

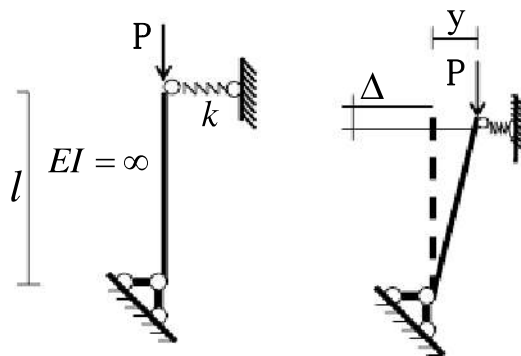
解: 应变能

$$V_e = \frac{1}{2} \cdot ky \cdot y$$

外力势能

$$V_P^* = -\sum P_i \Delta_i = -P\Delta$$

$$= -\frac{Py^2}{2l}$$



$$\Delta = l - l \cos \varphi = 2l \sin^2 \frac{\varphi}{2}$$

$$\doteq 2l \left(\frac{\varphi}{2}\right)^2 = \frac{1}{2} l \left(\frac{y}{l}\right)^2 = \frac{y^2}{2l}$$

结构势能  $E_P = V_e + V_P^* = \frac{lk - P}{2l} y^2$

由势能驻值原理  $\frac{dE_P}{dy} = \frac{lk - P}{l} y = 0$

得临界荷载  $P_{cr} = lk$

例二:求图示结构的临界荷载.

解: 应变能  $V_e = \frac{1}{2} ky_1^2 + \frac{1}{2} ky_2^2$

外力势能  $V_p^* = -\sum P_i \Delta_i = -P[\frac{y_2^2}{2l} + \frac{(y_2 - y_1)^2}{2l}]$

结构势能  $E_p = V_e + V_p^*$

$$= \frac{1}{2} ky_1^2 + \frac{1}{2} ky_2^2 - P[\frac{y_2^2}{2l} + \frac{(y_2 - y_1)^2}{2l}]$$

$$= \frac{1}{2l} [(kl - P)y_1^2 + 2Py_1y_2 + (kl - 2P)y_2^2]$$

$$\delta E_p = \frac{\partial E_p}{\partial y_1} \delta y_1 + \frac{\partial E_p}{\partial y_2} \delta y_2 = 0$$

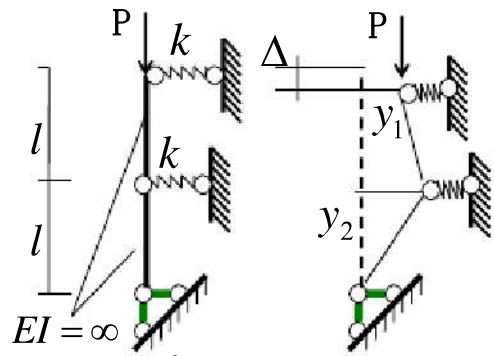
$$\begin{vmatrix} kl - P & P \\ P & kl - 2P \end{vmatrix} = 0$$

$$\frac{\partial E_p}{\partial y_1} = 0 \quad \frac{\partial E_p}{\partial y_1} = \frac{1}{l} [(kl - P)y_1 + Py_2] = 0 \quad P^2 - 3klP + k^2l^2 = 0$$

$$\frac{\partial E_p}{\partial y_2} = 0 \quad \frac{\partial E_p}{\partial y_2} = \frac{1}{l} [Py_1 + (kl - 2P)y_2] = 0 \quad P = \frac{3 \pm \sqrt{5}}{2} kl = \begin{cases} 2.618kl \\ 0.382kl \end{cases}$$

• 同静力法计算结果一样

$$P_{cr} = 0.382kl$$





# 用能量法求无限自由度体系的临界荷载

## 瑞利里兹法

应变能  $V_e = \frac{1}{2} \int_0^l \frac{M^2(x)}{EI} dx$

$$M = EIy''(x)$$

$$V_e = \frac{1}{2} \int_0^l EI [y''(x)]^2 dx$$

$$ds - dx = dx\sqrt{1+(y')^2} - dx$$

$$= dx[(1+(y')^2)^{1/2} - 1] \doteq dx[1 + \frac{1}{2}(y')^2 - 1]$$

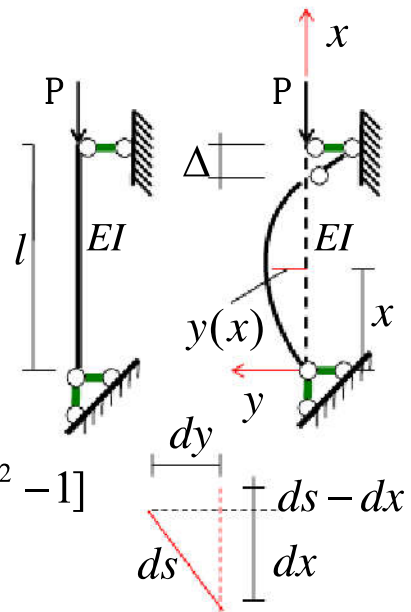
$$= \frac{1}{2}(y')^2 dx$$

$$\Delta = \int_0^l (ds - dx) = \int_0^l \frac{1}{2}(y')^2 dx$$

外力势能  $V_p^* = -P\Delta = -\frac{P}{2} \int_0^l (y')^2 dx$

结构势能  $E_p = V_e + V_p^*$

$$= \frac{1}{2} \int_0^l EI(y'')^2 dx - \frac{P}{2} \int_0^l (y')^2 dx$$



设  $y(x) = a_1\varphi_1(x) + a_2\varphi_2(x) + \dots + a_n\varphi_n(x)$   
 $= \sum_{i=1}^n a_i\varphi_i(x)$

将无限自由度化为有限自由度。结构势能则为  $a_1, a_2, \dots, a_n$  的多元函数, 求其极值即可求出临界荷载。

例:求图示体系的临界荷载.

解: 1. 设  $y(x) = a \sin \frac{\pi x}{l}$

$$V_e = \frac{1}{2} \int_0^l EI [y''(x)]^2 dx = \frac{\pi^4 EI}{4l^3} a^2$$

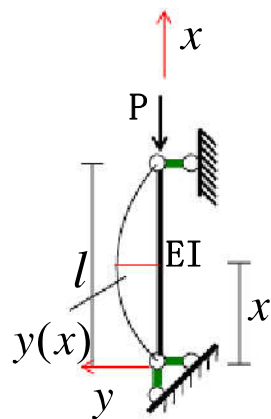
$$V_e^* = -\frac{P}{2} \int_0^l (y')^2 dx = \frac{\pi^2}{4l} Pa^2$$

$$E_P = \left( \frac{\pi^4 EI}{4l^3} - \frac{\pi^2}{4l} P \right) a^2$$

$$\frac{dE_P}{da} = \left( \frac{\pi^4 EI}{2l^3} - \frac{\pi^2}{2l} P \right) a = 0$$

$$\frac{\pi^4 EI}{2l^3} - \frac{\pi^2}{2l} P = 0$$

$$P_{cr} = \frac{\pi^2 EI}{l^2}$$



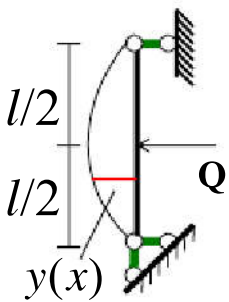
精确解:  $P_{cr} = \frac{\pi^2 EI}{l^2}$

2. 设  $y(x) = \frac{4a}{l^2}(lx - x^2)$

$$P_{cr} = \frac{12EI}{l^2} \quad \text{误差: +21.6\%}$$

3. 设杆中作用集中荷载所引起的位移作为失稳时的位移.

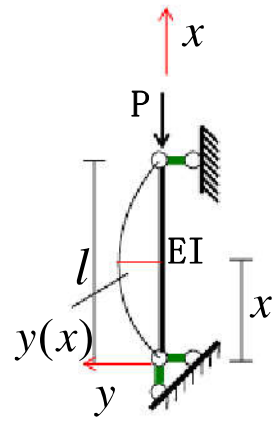
$$y(x) = \frac{Q}{EI} \left( \frac{l^2 x}{16} - \frac{x^3}{12} \right) \quad (0 \leq x \leq \frac{l}{2})$$



$$\text{令 } a = \frac{48l^3 Q}{EI}$$

$$y(x) = a \left( \frac{3x}{l} - \frac{4x^3}{l^3} \right)$$

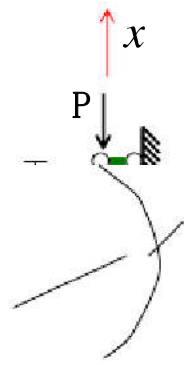
$$P_{cr} = \frac{10EI}{l^2} \quad \text{误差: +1.3\%}$$



精确解:  $P_{cr} = \frac{\pi^2 EI}{l^2}$

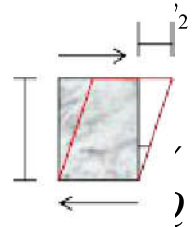
## 16.6 剪力对临界力的影响

同时考虑弯矩和剪力对变形的影响时



自  
下  
而  
上

$$\frac{d^2 y_2(x)}{dx^2} = -\frac{\kappa}{GA} \frac{d^2 M}{dx^2}$$



截面形状系数  
矩形截面为1.2  
圆形截面为1.11

挠曲线微分方程为

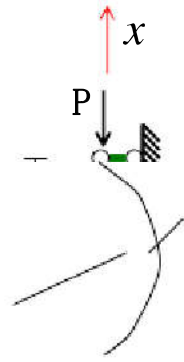
$$\frac{d^2 y(x)}{dx^2} = \frac{M}{EI} - \frac{\kappa}{GA} \frac{d^2 M}{dx^2}$$

$$\frac{dx}{GA} - \frac{dx}{GA}$$

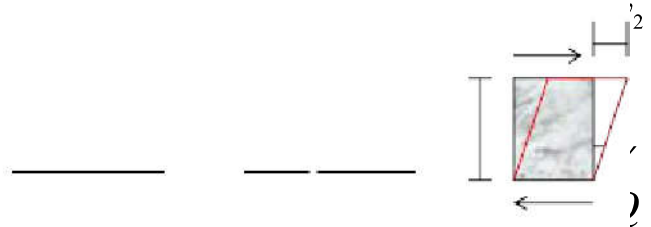
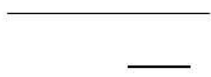
对于图示两端铰支的等截面杆,有

$$M = -Py, M'' = -Py''$$

$$d^2 y(x) = -Py, \quad \kappa P d^2 y$$



令



方

边界条件

$$y(0) = 0$$

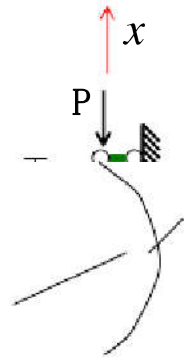
$$y(l) = 0$$

$$\frac{d^2 y}{dx^2} = -\frac{P}{EI} y = \frac{P}{GA} \frac{d^2 y}{dx^2}$$

对于图示两端铰支的等截面杆,有

$$M = -Py, M'' = -Py''$$

$$\frac{d^2 y(x)}{dx^2} + \frac{Py}{EI} = 0$$



令

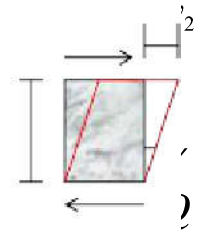
$$k = \frac{P}{EI}$$

方

边

$$y(l) = 0$$

$$y'' + ky = 0$$



不计剪力对临界荷载的影响  
所得到的临界荷载是大还是小?

$$P_k = \text{---}$$

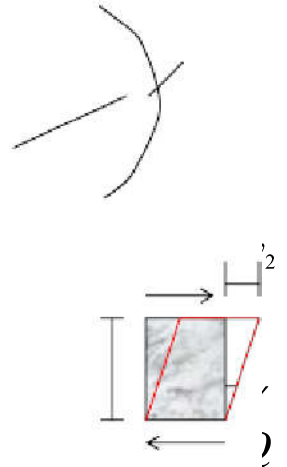
$$\alpha = \text{---}$$

$$= \text{---}$$

$$= \text{---}$$

$$= \text{---}$$

欧拉临界应力



对于Q235钢,比例极限约为200MPa.

若取  $\sigma_k = 200\text{MPa}$ ,  $G = 80\text{GPa}$ ,  $\kappa = 1.2$

$$\alpha = \frac{1}{1.003} \doteq 1$$

结论: **实体**杆件中,剪力对临界荷载的影响很小,可略去不计.

$$\frac{1}{GA} \frac{d^2 P}{dx^2} = 0$$