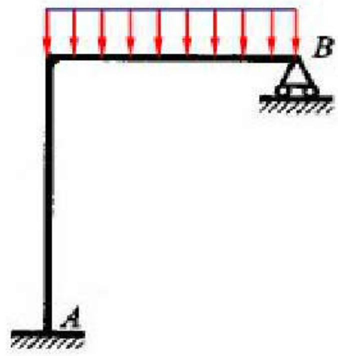

第十四章 静不定结构

§14.1静不定结构概述

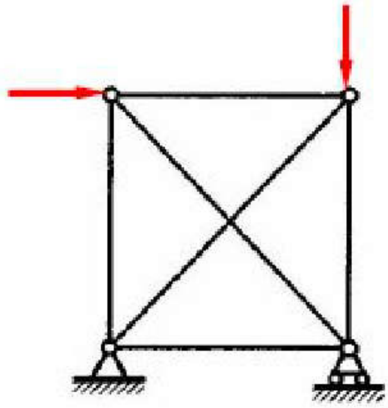
一、超静定结构

未知力不能由静力平衡方程完全求出的结构



外静不定：支反力不能由静力平衡方程完全求出的结构

内静不定：杆件内力不能由静力平衡方程完全求出的结构

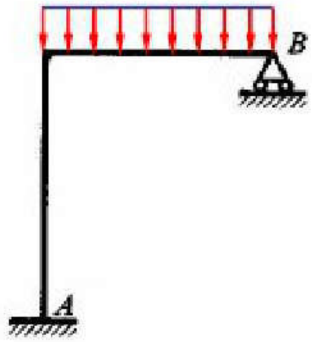


联合静不定结构：在结构外部和内部均存在多余约束，即支反力和内力是静不定的.

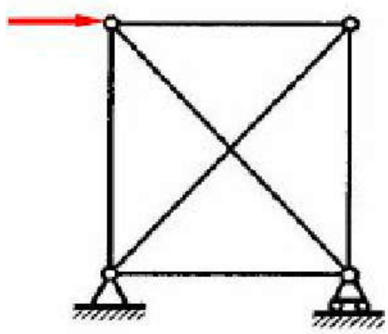
§14.1静不定结构概述

二、静不定问题分类

1.多余约束：对维持结构几何不变来说是多余的

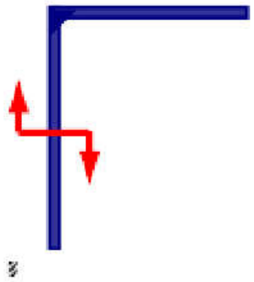


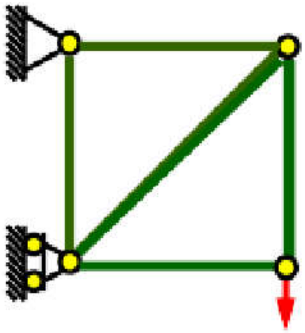
第一类:仅在结构外部存在多余约束,即支反力是静不定的,称为外力静不定系统;

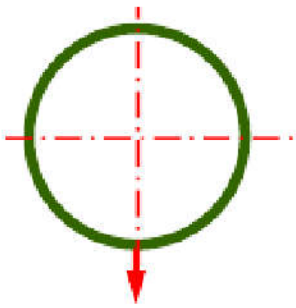


第二类:仅在结构内部存在多余约束,即内力是静不定的,称为内力静不定系统;

第三类:在结构外部和内部均存在多余约束,即支反力和内力是静不定的,称联合静不定结构.







2、基本静定系(静定基)

解除静不定结构的某些约束后得到的静定结构，称为原静定结构的基本静定系统或静定基

基本静定系可以有不同的选择，不是唯一的

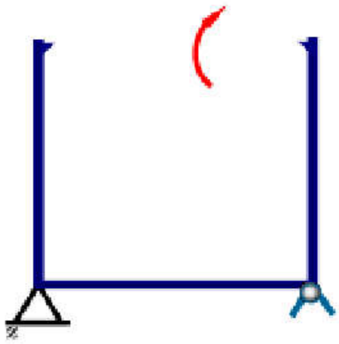
四、超静定次数的判定

1、外力超静定次数的判定 ：解除多余约束并使原结构几何不变。

解除多余约束的数量=静不定次数

2、内力超静定次数的判定:

（1）一个平面封闭框架为三次内力超静定;



（2）平面桁架的内力超静定次数等于未知力的个数减去二倍的节点数.

§ 14.2 用力法解静不定结构

一、力法的求解过程

力法: 以多余约束力为基本未知量，将变形或位移表示为未知力的函数，通过变形协调条件作为补充方程求来解未知约束力，这种方法称为力法

1.判定超静定次数

解除超静定结构的多余约束，用多余约束力X_1, X_2 , X_3 , … 代替多余约束,得到一个几何不变的静定系统,称为原静不定系统的“相当系统”；

2.在多余约束处满足“变形几何条件”，得到变形协调方程;

3.由补充方程求出多余约束力;

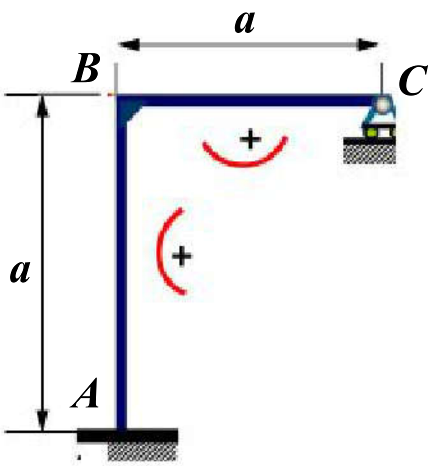
4.在相当系统上求解原超静定结构的内力和变形.

§ 14.2 用力法解静不定结构

安装尾顶针的工件，求支座反力

$$X_1 = \frac{Pa^2}{2l^3}\left(3l - a\right)$$

刚架的两杆抗弯刚度都是EI，作刚架弯矩图.

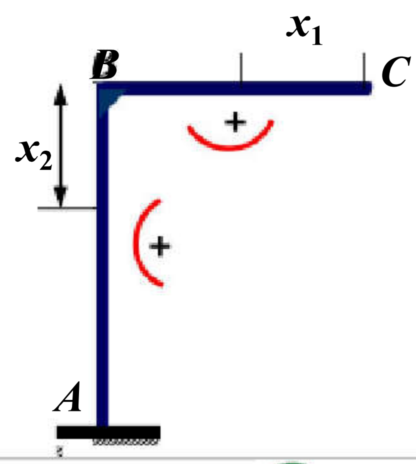


AB段 $M(x_2) = Fx_2$ $\bar{M}(x_2) = -a$

BC段 $M(x_1) = 0$ $\bar{M}(x_1) = x_1$

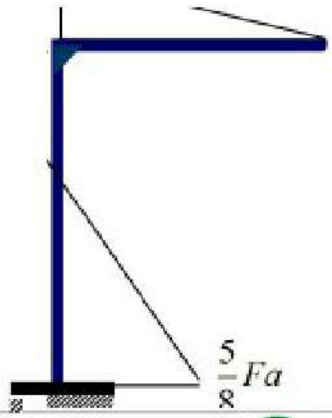
$$\delta_{11} = \int_0^a \frac{\bar{M}_1^2 \,\mathrm{d}\, x_1}{EI} + \int_0^a \frac{\bar{M}_2^2 \,\mathrm{d}\, x_2}{EI}$$

$$= \frac{1}{EI}\left[\int_0^a x_1^2 \,\mathrm{d}\, x_1 + \int_0^a a^2 \,\mathrm{d}\, x_2\right] = \frac{4a^3}{3EI}$$



代入协调方程

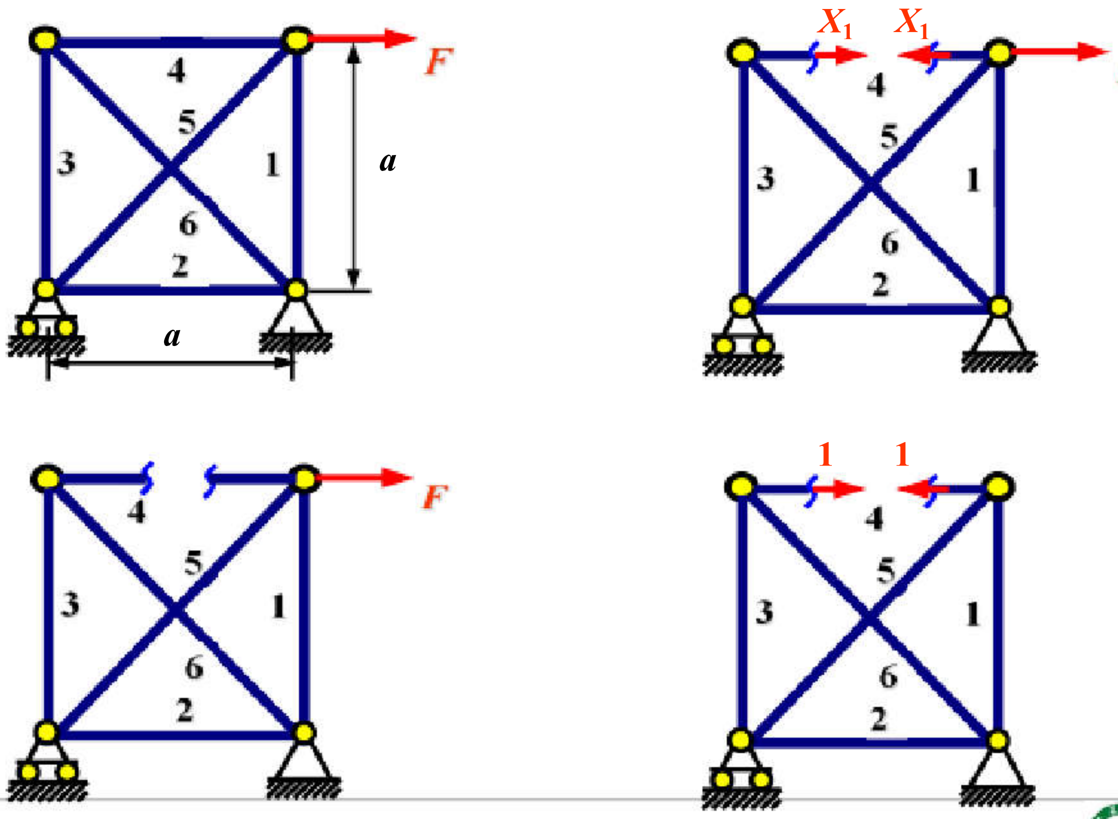
$$X_1 = \frac{-\Delta_{1F}}{\delta_{11}} = \frac{3}{8}F$$



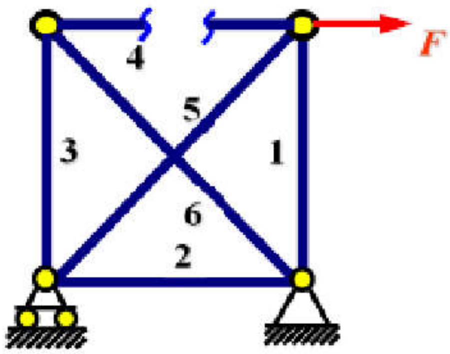
例题2

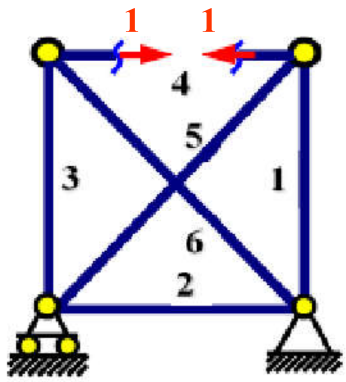
求如图桁架各杆的内力。各杆的材料相同，横截面面积相同。

解： 一次静不定 $\delta_{11}X_1+\Delta_{1F}=0$



例题2





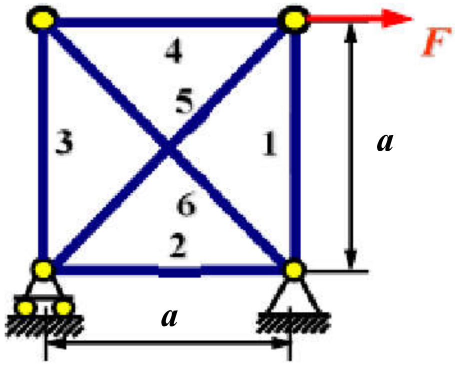
杆件编号	F_{Ni}	$\overline{F}_{Ni}$	l_i	$\overline{F}_{Ni}F_{Ni}l_i$	$\overline{F}_{Ni}\overline{F}_{Ni}l_i$	$F_{Ni}^F = F_{Ni} + \overline{F}_{Ni}X_1$
1	$-F$	1	a	$-Fa$	a	$-F/2$
2	$-F$	1	a	$-Fa$	a	$-F/2$
3	0	1	a	0	a	$F/2$
4	0	1	a	0	a	$F/2$
5	$\sqrt{2}F$	$-\sqrt{2}$	$\sqrt{2}a$	$-2\sqrt{2}Fa$	$2\sqrt{2}a$	$F/\sqrt{2}$
6	0	$-\sqrt{2}$	$\sqrt{2}a$	0	$2\sqrt{2}a$	$-F/\sqrt{2}$
				$\sum F_{Ni}\overline{F}_{Ni}l_i = -Fa(2+2\sqrt{2})$	$\sum \overline{F}_{Ni}\overline{F}_{Ni}l_i = 4(1+\sqrt{2})a$	

例题2

$$\Delta_{1F} = \sum \frac{F_i \overline{F}_i l_i}{EA_i} = -\frac{2(1+\sqrt{2})Fa}{EA}$$

$$\delta_{11} = \sum \frac{\overline{F}_i \cdot \overline{F}_i l_i}{EA_i} = -\frac{4(1+\sqrt{2})a}{EA}$$

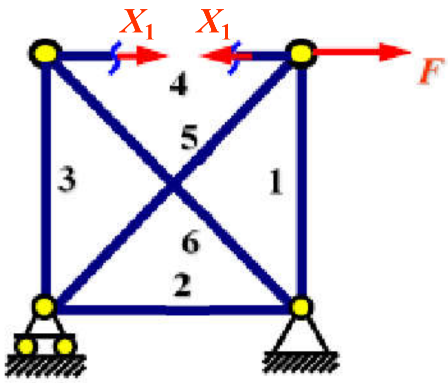
$$X_1 = -\frac{\Delta_{1F}}{\delta_{11}} = \frac{2(1+\sqrt{2})Fa}{4(1+\sqrt{2})a} = \frac{F}{2}$$



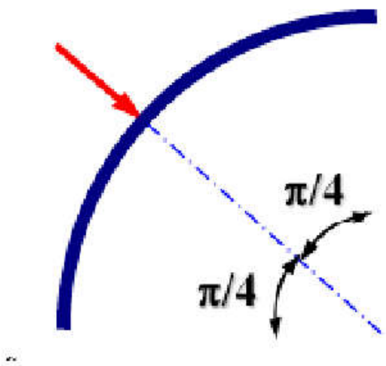
实际内力:

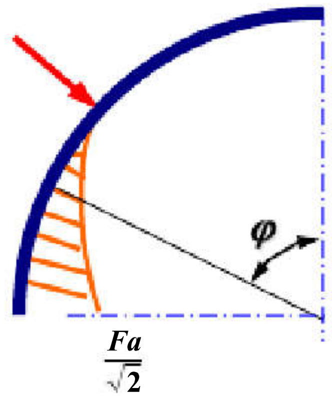
$$F_i^F = F_i + \overline{F}_i X_1$$

（见表）



轴线为1/4圆周的曲杆A端固定，B端铰支。在F作用下，试作曲杆的弯矩图。（设曲杆横截面尺寸远小于轴线半径）





只在F作用下:

$$M=0 \qquad (0\le\varphi\le\frac{\pi}{4})$$

$$M=Fa\sin(\varphi-\frac{\pi}{4})$$

单位力作用时:

$$\overline{M}=-a\sin\varphi$$

例题3

$$M=0 \qquad M=Fa\sin(\varphi-\frac{\pi}{4}) \qquad \overline{M}=-a\sin\varphi$$

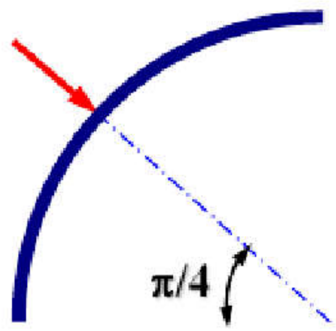
$$\Delta_{1F}=\int_s \frac{M\overline{M}ds}{EI}=\frac{1}{EI}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}[Fa\sin(\varphi-\frac{\pi}{4})](-\sin\varphi)ad\varphi=-\frac{Fa^3}{8\sqrt{2}EI}$$

$$\delta_{11}=\int_s \frac{\overline{M}\cdot\overline{M}ds}{EI}=\frac{1}{EI}\int_0^{\frac{\pi}{2}}(-a\sin\varphi)^2 ad\varphi=\frac{\pi a^3}{4EI}$$

将Δ_{1F}和δ_{11}代入 $\quad \delta_{11}X_1+\Delta_{1F}=0$

$$\frac{\pi a^3}{4EI}X_1-\frac{Pa^3\pi}{8\sqrt{2}EI}=0$$

解得 $\quad X_1=\frac{F}{2\sqrt{2}}$



例题3

曲杆在任意截面上的弯矩:

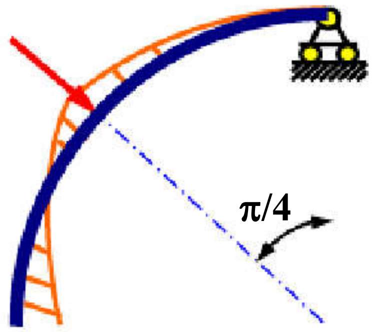
$$M = -X_1 a\sin\varphi = -\frac{Fa}{2\sqrt{2}}\sin\varphi$$

$$(0 \le \varphi \le \frac{\pi}{4})$$

$$M = Fa\sin(\varphi - \frac{\pi}{4}) - X_1 a\sin\varphi$$

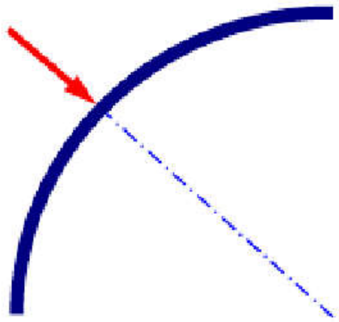
$$= Fa\left[\sin(\varphi - \frac{\pi}{4}) - \frac{1}{2\sqrt{2}}\sin\varphi\right]$$

$$(\frac{\pi}{4} \le \varphi \le \frac{\pi}{2})$$



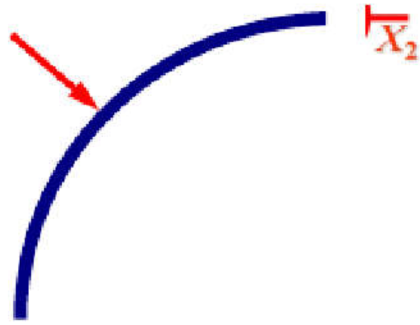
$$\frac{Fa}{2\sqrt{2}}$$

二、高次静不定、正则方程



***B*点的垂直位移(X_1方向)等于零**

$$\Delta_1 = \Delta_{1X_1} + \Delta_{1X_2} + \Delta_{1X_3} + \Delta_{1F} = 0$$



$$\Delta_{1X_1} = \delta_{11}X_1,\quad \Delta_{1X_2} = \delta_{12}X_2,\quad \boldsymbol{\Delta_{1X_3} = \delta_{13}X_3}$$

$$\boldsymbol{\delta_{11}X_1 + \delta_{12}X_2 + \delta_{13}X_3 + \Delta_{1F} = 0}$$

$$\begin{cases} \delta_{11}X_1 + \delta_{12}X_2 + \delta_{13}X_3 + \Delta_{1F} = 0 \\ \delta_{21}X_1 + \delta_{22}X_2 + \delta_{23}X_3 + \Delta_{2F} = 0 \\ \delta_{31}X_1 + \delta_{32}X_2 + \delta_{33}X_3 + \Delta_{3F} = 0 \end{cases}$$

方程系数和常数项的计算:

$$\Delta_{1F} = \int_l \frac{M\bar{M}_1 ds}{EI} \qquad \Delta_{2F} = \int_l \frac{M\bar{M}_2 ds}{EI} \qquad \Delta_{3F} = \int_l \frac{M\bar{M}_3 ds}{EI}$$

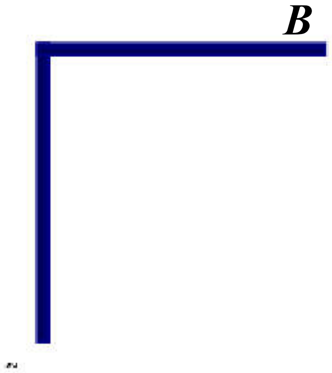
$$\delta_{12} = \int_l \frac{\bar{M}_1 \cdot \bar{M}_2 ds}{EI} = \int_l \frac{\bar{M}_2 \cdot \bar{M}_1 ds}{EI} = \delta_{21}$$

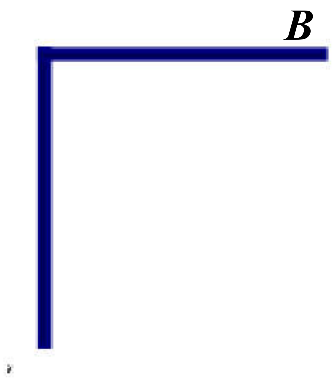
$$\delta_{23} = \delta_{32} \qquad \delta_{13} = \delta_{31}$$

方程有6个独立的系数

求解如图静不定刚架。设两杆的EI相等。

解:　　1. 计算系数:


B

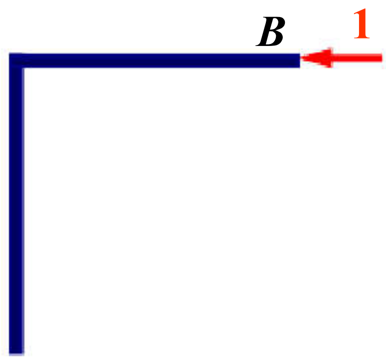


$$\Delta_{1F} = -\frac{1}{EI}\int_0^a \frac{qx_2^2}{2}\cdot a\cdot dx_2 \quad = -\frac{qa^4}{6EI}$$

$$\Delta_{2F} = -\frac{1}{EI}\int_0^a \frac{qx_2^2}{2}\cdot x_2\cdot dx_2 = -\frac{qa^4}{8EI}$$

$$\Delta_{3F} = -\frac{1}{EI}\int_0^a \frac{qx_2^2}{2}\cdot 1\cdot dx_2 = -\frac{qa^3}{6EI}$$

例题4

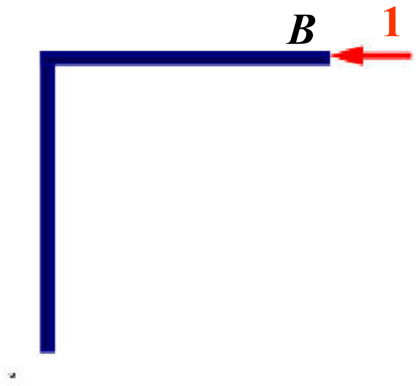


$$\delta_{11} = \frac{1}{EI}\int_0^a x_1 \cdot x_1 \cdot dx_1 + \frac{1}{EI}\int_0^a a \cdot a \cdot dx_2 = \frac{4a^3}{3EI}$$

$$\delta_{22} = \frac{1}{EI}\int_0^a x_2 \cdot x_2 \cdot dx_2 = \frac{a^3}{3EI}$$

$$\delta_{33} = \frac{1}{EI}\int_0^a 1 \cdot 1 \cdot dx_1 + \frac{1}{EI}\int_0^a 1 \cdot 1 \cdot dx_2 = \frac{2a}{EI}$$

例题4

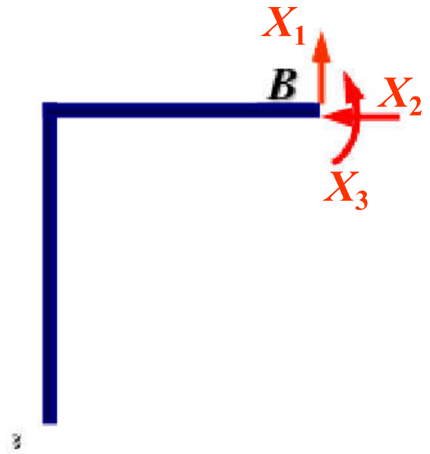


（3）求 δ_{ij}

$$\delta_{12}=\delta_{21}=\frac{1}{EI}\int_0^a x_2\cdot a\cdot dx_2=\frac{a^3}{2EI}$$

$$\delta_{13}=\delta_{31}=\frac{1}{EI}\int_0^a x_1\cdot 1\cdot dx_1+\frac{1}{EI}\int_0^a a\cdot 1\cdot dx_2=\frac{3a^2}{2EI}$$

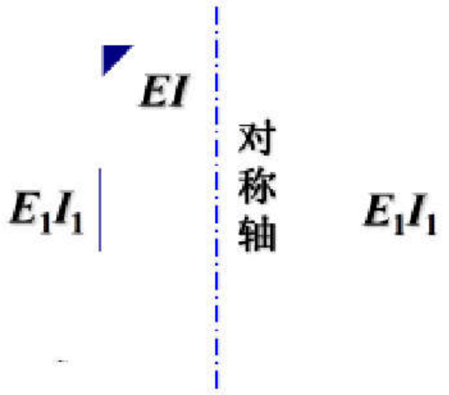
$$\delta_{23}=\delta_{32}=\frac{1}{EI}\int_0^a x_2\cdot 1\cdot dx_2=\frac{a^2}{2EI}$$

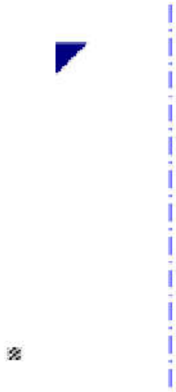


求解得: $X_1=-\dfrac{qa}{16}$ $X_2=-\dfrac{7qa}{16}$ $X_3=-\dfrac{qa^2}{48}$

§ 14.3　对称及反对称性质的利用

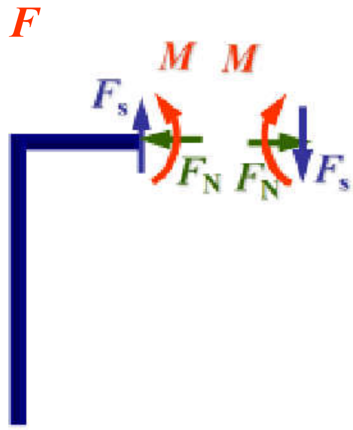
一、概念：





2.反对称载荷：对称结构，若载荷作用的位置、大小对称，方向反对称

3、内力分成：对称内力、反对称内力

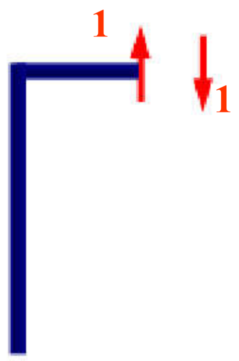


对称的内力：弯矩M、轴力F_N

反对称的内力：剪力F_S

对称与反对称性质的利用：用于超静定问题的降次即方程的减少

二、对称性质

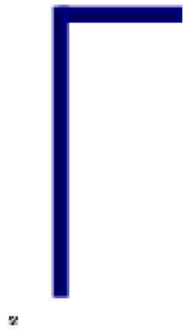


$$\Delta_{2F} = \int_l \frac{M_F \bar{M}_2}{EI} dx = 0 \qquad \delta_{12} = \delta_{21} = \int_l \frac{\bar{M}_1 \bar{M}_2}{EI} dx = 0 \qquad \delta_{23} = \delta_{32} = 0$$

$$\delta_{11} X_1 + \delta_{13} X_3 + \Delta_{1F} = 0$$
$$\delta_{31} X_1 + \delta_{33} X_3 + \Delta_{3F} = 0$$
$$\delta_{22} X_2 = 0$$

结论：受对称载荷，对称截面上反对称内力为零

三、反对称性质



$$\Delta_{1F} = \int_l \frac{M_F \bar{M}_1}{EI} dx = 0 \qquad \Delta_{3F} = 0 \qquad \delta_{12} = \delta_{21} = \delta_{23} = \delta_{32} = 0$$

正则方程为：

$$\delta_{11} X_1 + \delta_{13} X_3 = 0$$

$$\delta_{31} X_1 + \delta_{33} X_3 = 0$$

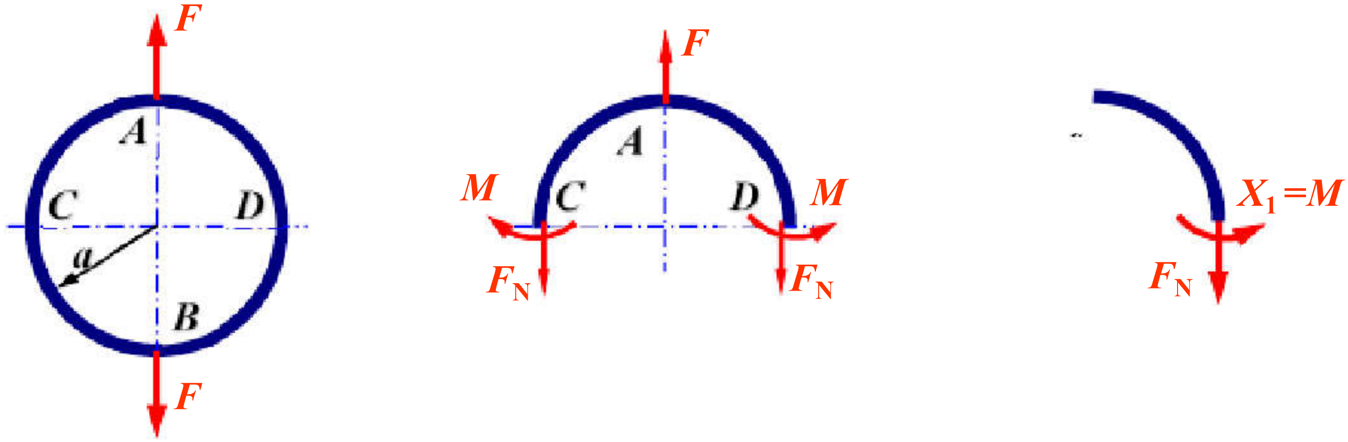
$$\delta_{22} X_2 = -\Delta_{2F}$$

$$X_1 = X_3 = 0$$

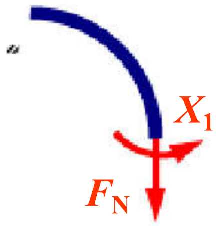
结论：受反对称载荷，对称截面上对称的内力为零

在等截面圆环直径AB的两端，沿直径作用方向相反的一对F力。试求AB直径的长度变化。

解：利用对称性选择静定基 $\delta_{11}X_1+\Delta_{1F}=0$



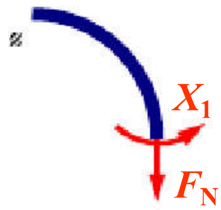
$$M = \frac{Fa}{2}(1 - \cos\varphi) \qquad \overline{M} = -1$$



$$\Delta_{1F} = \int_0^{\frac{\pi}{2}} \frac{M\overline{M}ad\varphi}{EI} = \frac{Fa^2}{2EI}\int_0^{\frac{\pi}{2}}(1 - \cos\varphi)(-1)d\varphi$$

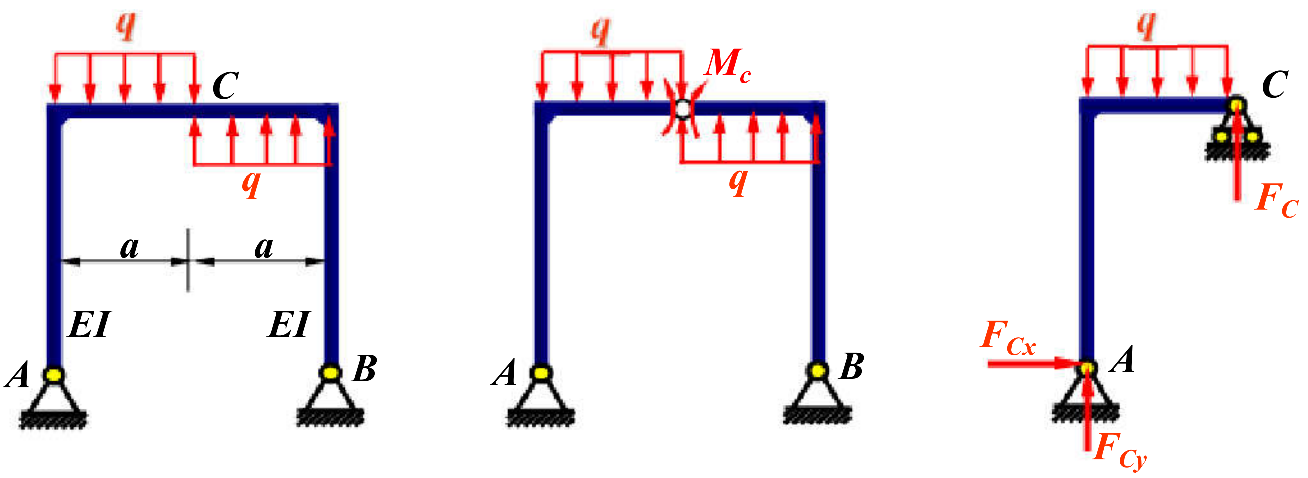
任意截面上的弯矩为：

$$M(\varphi)=\frac{Fa}{2}(1-\cos\varphi)-X_1=Fa(\frac{1}{\pi}-\frac{\cos\varphi}{2})$$



例题5

已知如图，求刚架的反力。



解：　一次静不定　解除C截面的转动约束，代之以M_C

∵载荷反对称，

∴ $M_C=0$, $F_C=0$,

$Q_C \neq 0$

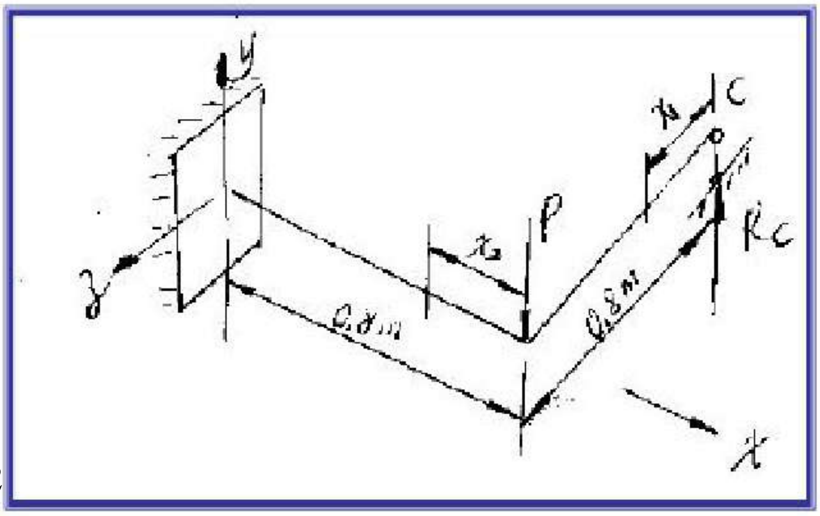
$$\sum M_A = 0 \qquad F_C = \frac{1}{2}qa$$

$$\sum F_y = 0 \qquad F_{Ay} = \frac{1}{2}qa$$

$$\sum F_x = 0 \qquad F_{Ax} = 0$$

例：折杆ABC位于同一平面内,P=10kN,E=200Gpa, G=80Gpa,I=400×10^3mm^4,求C处支反力。

解：一次静不定。解除C处约束，代之以R_C



(1)真实力作用下：

BC段：$M(x_1)=R_C x_1$

AB段：

$$M(x_2)=(R_C-P)x_2$$

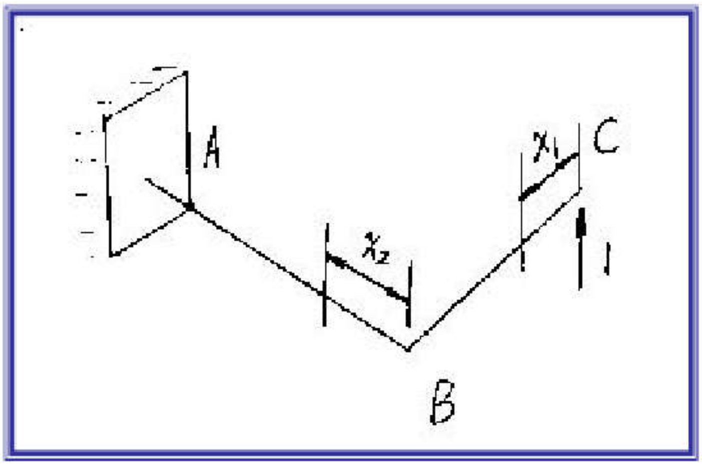
$$T(x_2)=0.8R_C$$

(2)在C处加单位力

BC: $\overline{M}(x_1) = x_1$

AB: $\overline{M}(x_2) = x_2$

$\overline{T}(x_2) = 0.8$



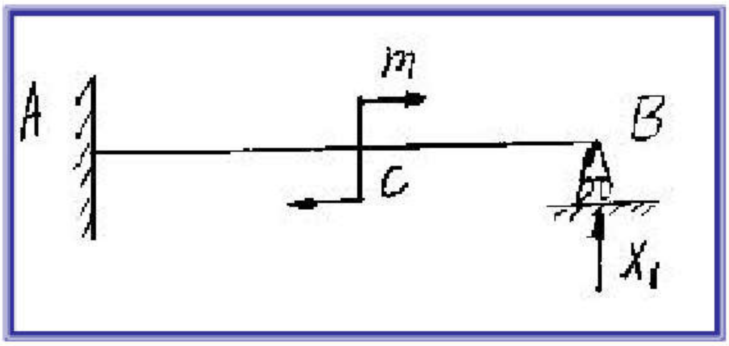
(3)
$$\delta_C = \frac{1}{EI}[\int_0^{0.8} R_C x_1 \times x_1 dx_1$$

$$+ \int_0^{0.8} (R_C - P)x_2 \times x_2 dx_2]$$

$$+ \frac{1}{GI_P}\int_0^{0.8} 0.8R_C \times 0.8 dx_2 = 0 \qquad R_C = \frac{4}{23}P = 1.74KN$$

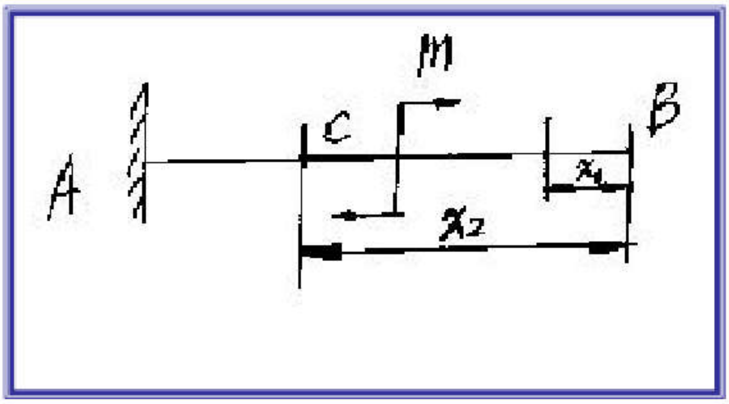
例:已知如图，EI，求约束反力。

解：　　一次静不定，解除B处约束代之以X_1



$$\therefore \Delta_1 = \Delta_{1P} + \delta_{11} X_1 = 0$$

外力作用下：



BC: $M(x_1) = 0$

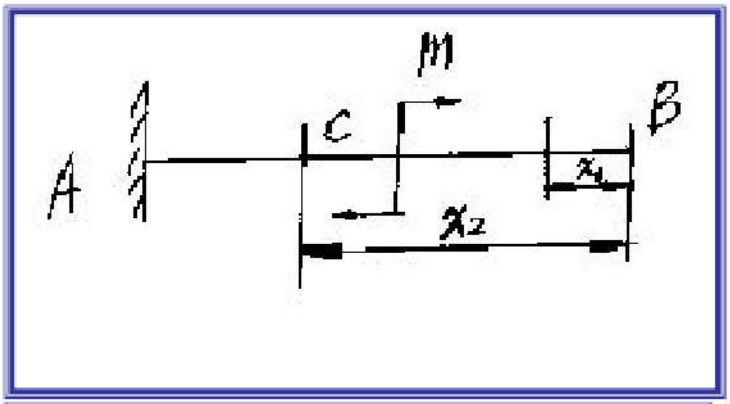
AC: $M(x_2) = -m$

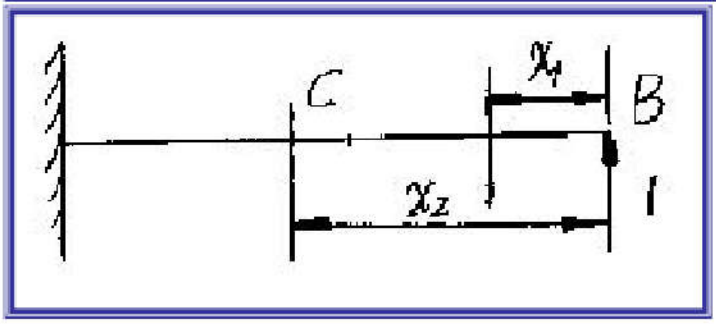
加单位力：

$$\overline{M}(x_1) = x_1 \quad \overline{M}(x_2) = x_2$$

$$\Delta_{1P} = \int_l \frac{M(x)\overline{M}(x)}{EI} dx$$

$$= \int_{\frac{l}{2}}^{l} \frac{(-m)x_2}{EI} dx_2 = -\frac{3}{8}\frac{ml^2}{EI}$$





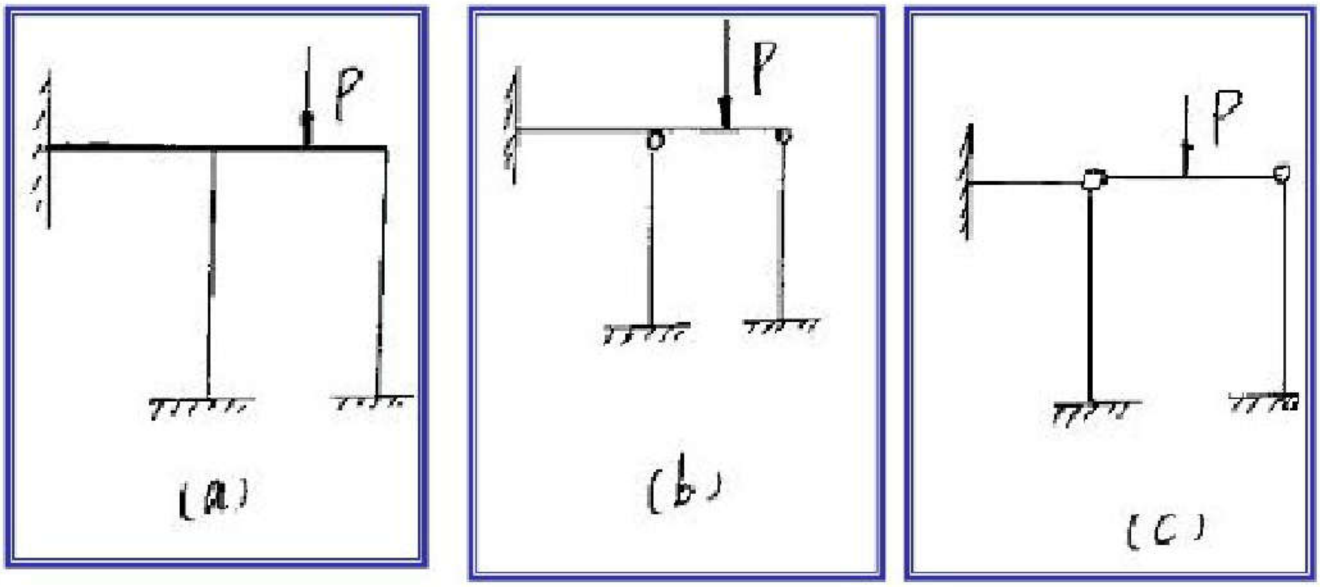
$$\delta_{11} = \int_l \frac{\overline{M}(x)\overline{M}(x)}{EI} dx$$

$$= \int_0^l \frac{x^2}{EI} dx = \frac{l^3}{3EI} \qquad X_1 = \frac{9}{8} ml(\uparrow)$$

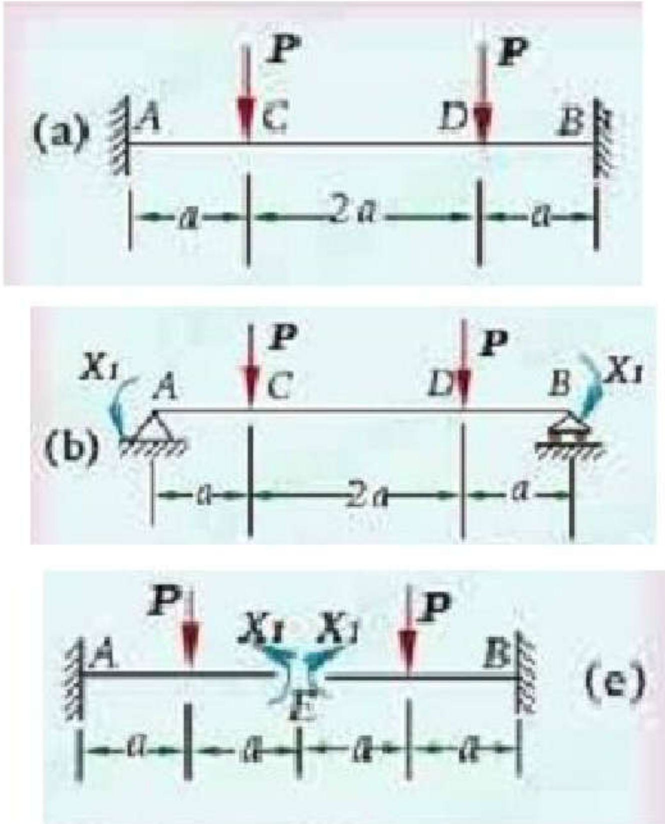
考虑整体平衡，可求出A处反力

例:判断图示结构静不定次数

解： 图(a):9-3=6 图(b):9-3-2=4 图(c):9-3-2-1=3



例： 两端固支梁受力如图所示，其抗弯刚度为EI，试求约束反力。（不计轴力）



解： 选择静定基如图所示

$$\delta_{11}X_1+\Delta_{1P}=0$$

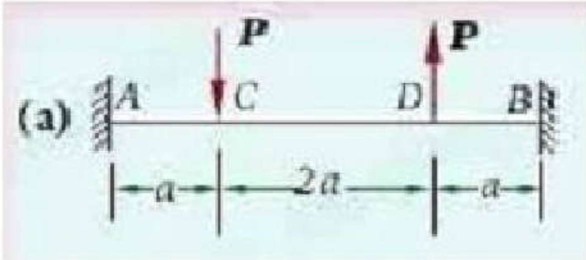
1．计算系数：

$$\delta_{11}=\frac{4a}{EI}\quad \Delta_{1P}=-\frac{3Pa^2}{EI}$$

2．解方程：

$$\frac{4a}{EI}X_1-\frac{3Pa^2}{EI}=0\quad X_1=\frac{3}{4}Pa$$

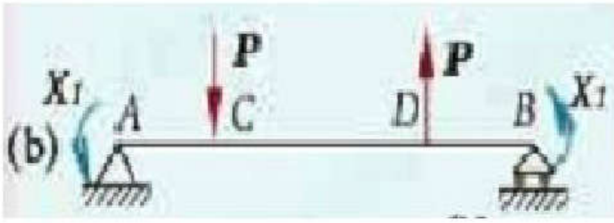
例:抗弯刚度为EI的两端固支梁受力如图。



解：选择静定基如图(b)所示

$$\delta_{11}X_1+\Delta_{1P}=0$$

1．计算系数：

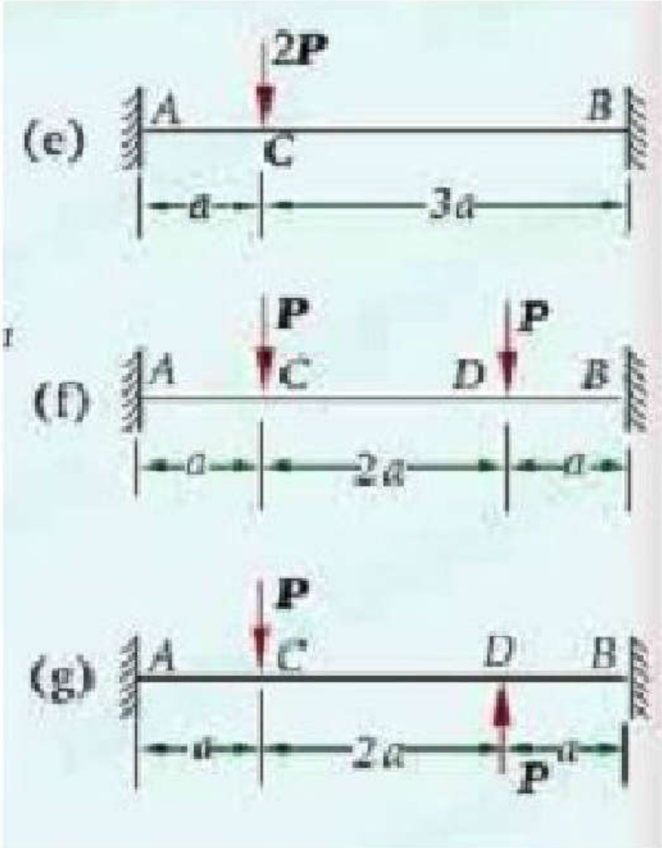


$$\delta_{11}=\frac{4a}{3EI}\qquad \Delta_{1P}=-\frac{Pa^2}{2EI}$$

2．解方程：

$$\frac{4a}{3EI}X_1-\frac{Pa^2}{2EI}=0\qquad X_1=\frac{3}{8}Pa$$

（方向如图（b）所示）



如果结构受力如图（e），则属一般受力情况,但可分解为对称（图（f））与反对称（图（g））两种情况的叠加

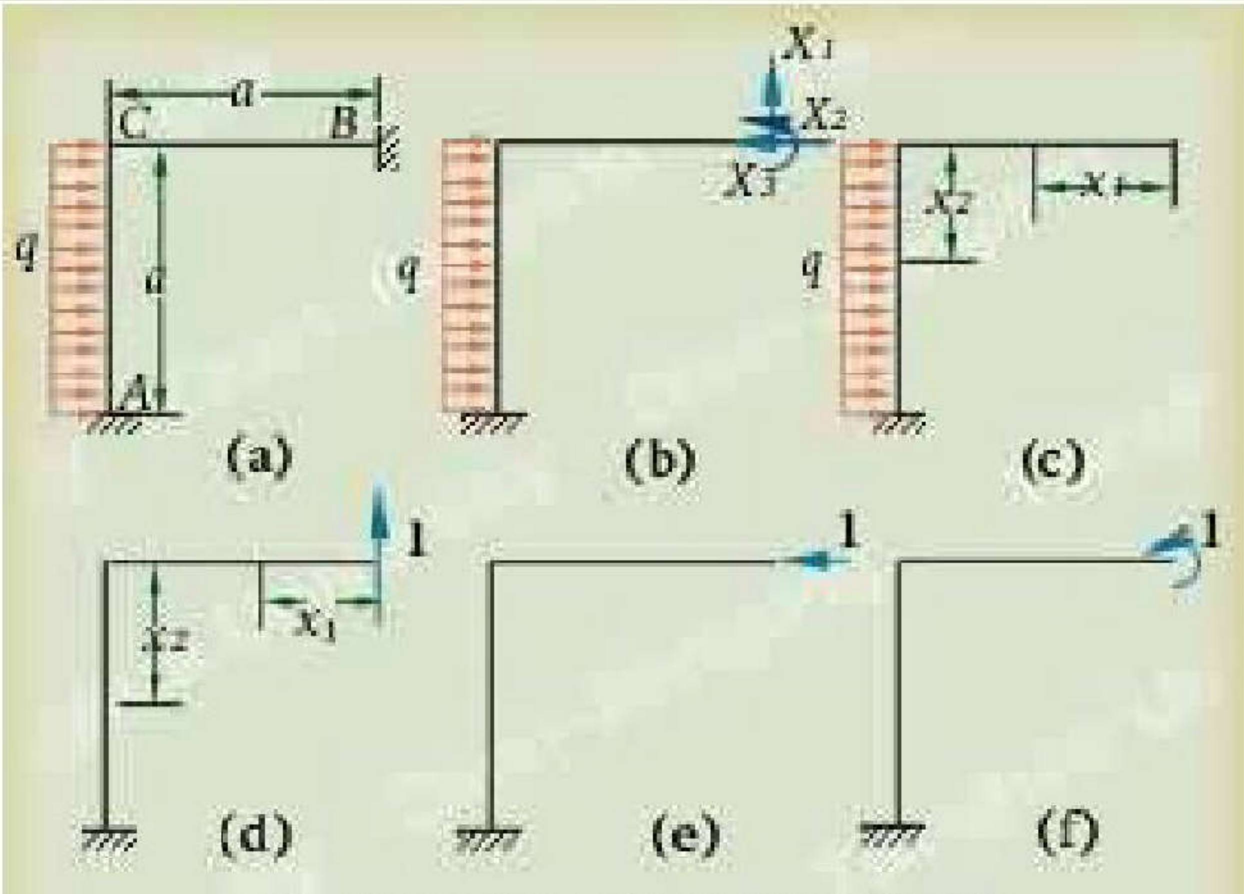
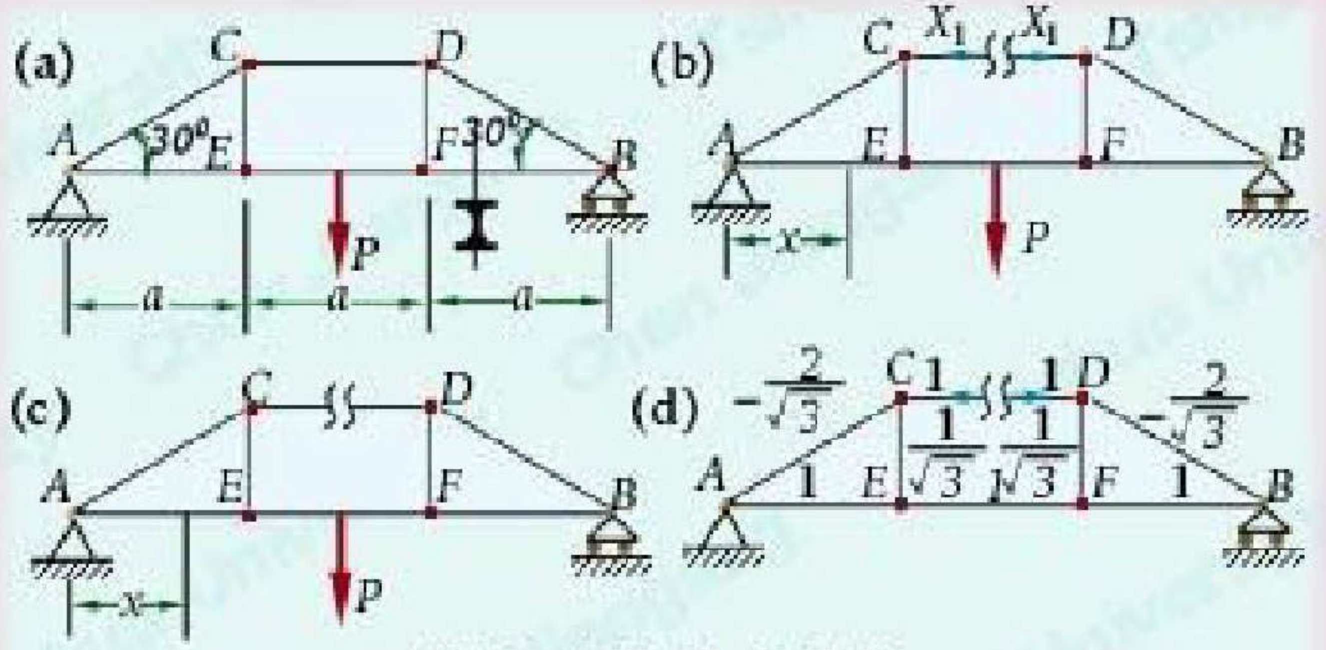


图 12-7 静不定刚架的求解


(a)
C
D
A
30°
E
F
30°
B
P
a
a
a
(b)
C
X_1
X_1
D
A
E
F
B
x
P
(c)
C
D
A
E
F
B
x
P
(d)
$-\frac{2}{\sqrt{3}}$
C
1
1
D
$-\frac{2}{\sqrt{3}}$
$\frac{1}{\sqrt{3}}$
$\frac{1}{\sqrt{3}}$
A
1
E
F
1
B

例 12-1图吊车桁架

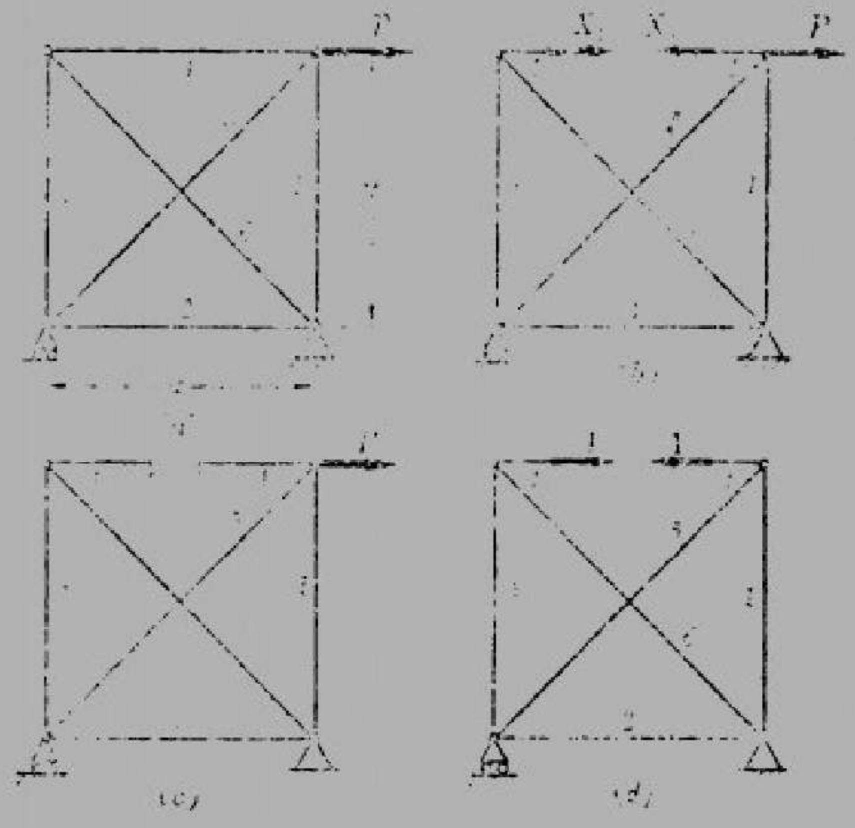
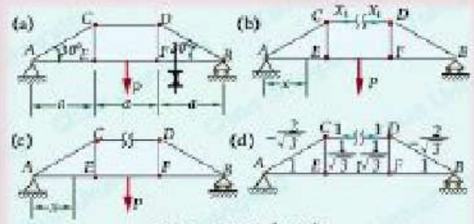


图 11-10

$$\Delta_{1P} = \int_l \frac{M\overline{M}dx}{EI} = \frac{2}{EI}\left[\int_0^a \frac{Px}{2}\left(-\frac{x}{\sqrt{3}}\right)dx + \int_a^{\frac{3}{2}a} \frac{Px}{2}\left(-\frac{a}{\sqrt{3}}\right)dx\right]$$

$$= -\frac{23Pa^3}{24\sqrt{3}EI}$$



例 12-1图 吊车桁架

$$\delta_{11} = \int_l \frac{\overline{M}\cdot\overline{M}dx}{EI} + \sum\frac{\overline{N}\cdot\overline{N}a}{EA_1} + \sum\frac{\overline{N}\cdot\overline{N}l}{EA}$$

$$= \frac{2}{EI}\left[\int_0^a \left(-\frac{x}{\sqrt{3}}\right)^2 dx + \int_a^{\frac{3}{2}a}\left(-\frac{x}{\sqrt{3}}\right)^2 dx\right] + \frac{1}{EA_1}(a+a+a)$$

$$+\frac{1}{EA}\left[2\times\left(-\frac{2}{\sqrt{3}}\right)^2\cdot\frac{2a}{\sqrt{3}} + 2\times\left(\frac{1}{\sqrt{3}}\right)^2\cdot\frac{a}{\sqrt{3}} + (-1)^2\cdot a\right]$$

$$= \frac{5a^3}{9EI} + \frac{3a}{EA_1} + \frac{a}{EA}(2\sqrt{3}+1)$$

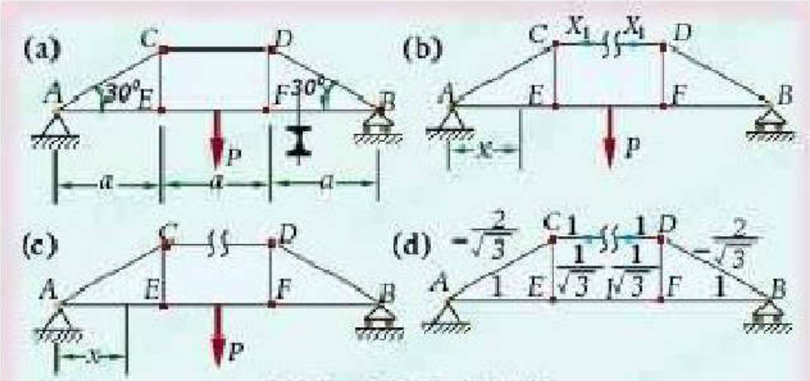
$$\delta_{11}=\frac{5a^3}{9EI}+\frac{3a}{EA_1}+\frac{a}{EA}(2\sqrt{3}+1)$$

因$A_1>>A$

$$\boldsymbol{\delta_{11}}=\frac{5a^3}{9EI}+\frac{a}{EA}(2\sqrt{3}+1)$$

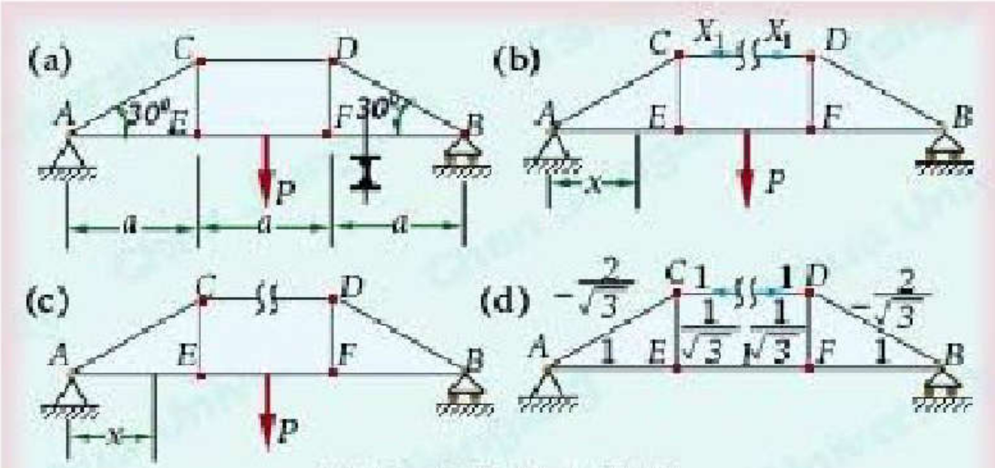
$$X_1=\frac{23Pa^2}{24\sqrt{3}\left[\frac{5}{9}a^2+\frac{I}{A}(2\sqrt{3}+1)\right]}$$

最大弯矩: $\boldsymbol{M_{max}}$**=?**



例 12-1图吊车桁架

例：以工字梁AB为大梁的桥式起重机,除工字梁外,其他各杆只承受拉或压,各杆横截面积皆为A。材料都为A3钢。若吊重P作用于跨度中点,试求工字梁的最大弯矩。



例 12-1图 吊车桁架

绘图:受力图、静定基图、外力图、单位力图

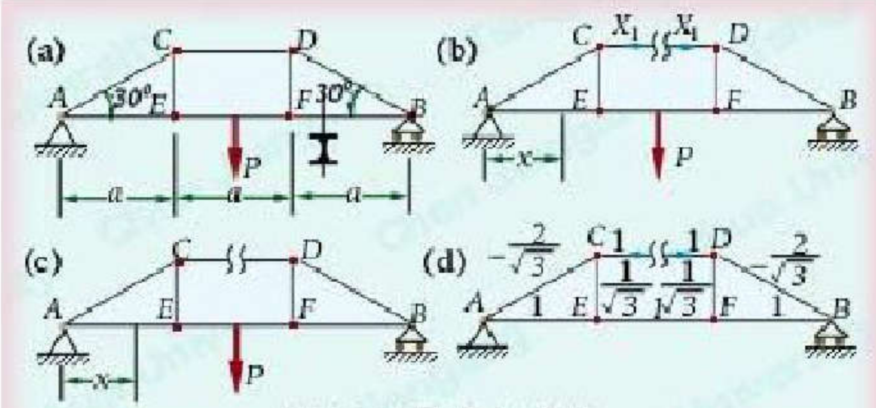
杆切开，截面之间不分离。即$\Delta_1 = 0$，但并不意味着此杆无弹性变形

解： 1.求反力

$$R_A = R_B = \frac{P}{2}$$

2.选择图示静定基

$$\delta_{11}X_1 + \Delta_{1P} = 0$$



例12-1图 吊车桁架

3.计算系数

工字梁AB: $M = \frac{P}{2}x$ $\quad 0 \le x \le \frac{3}{2}a$

AE: $\overline{M} = -\frac{x}{\sqrt{3}}$ $\quad$ EF: $\overline{M} = -\frac{a}{\sqrt{3}}$

单位力作用在各杆的轴力如图（d）所示。

解题步骤：

1．绘图：受力图、静定基图、外力图、单位力图

2．列出平衡方程（对静定基而言）

3．列出正则方程（几何方程）

4．求 δ_{11}、Δ_{1P}

5．解方程求出未知力

6．再求超静定系统的内力、位移

二、力法正则方程

(Generalized equations in the force method)

上例中以多余力为未知量的变形协调方程可改写成下式

$$\delta_{11}X_1 + \Delta_{1F} = 0$$

变形协调方程的标准形式,即所谓的力法正则方程.

X_1— 多余未知量;

δ_{11}— 在基本静定系上, X_1取单位值时引起的在X_1作用点X_1方向的位移;

Δ_{1F}—在基本静定系上,由原载荷引起的在X_1作用点沿X_1方向的位移;

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结论：

结构对称，受对称载荷，对称截面上反对称的内力、反对称的位移为零

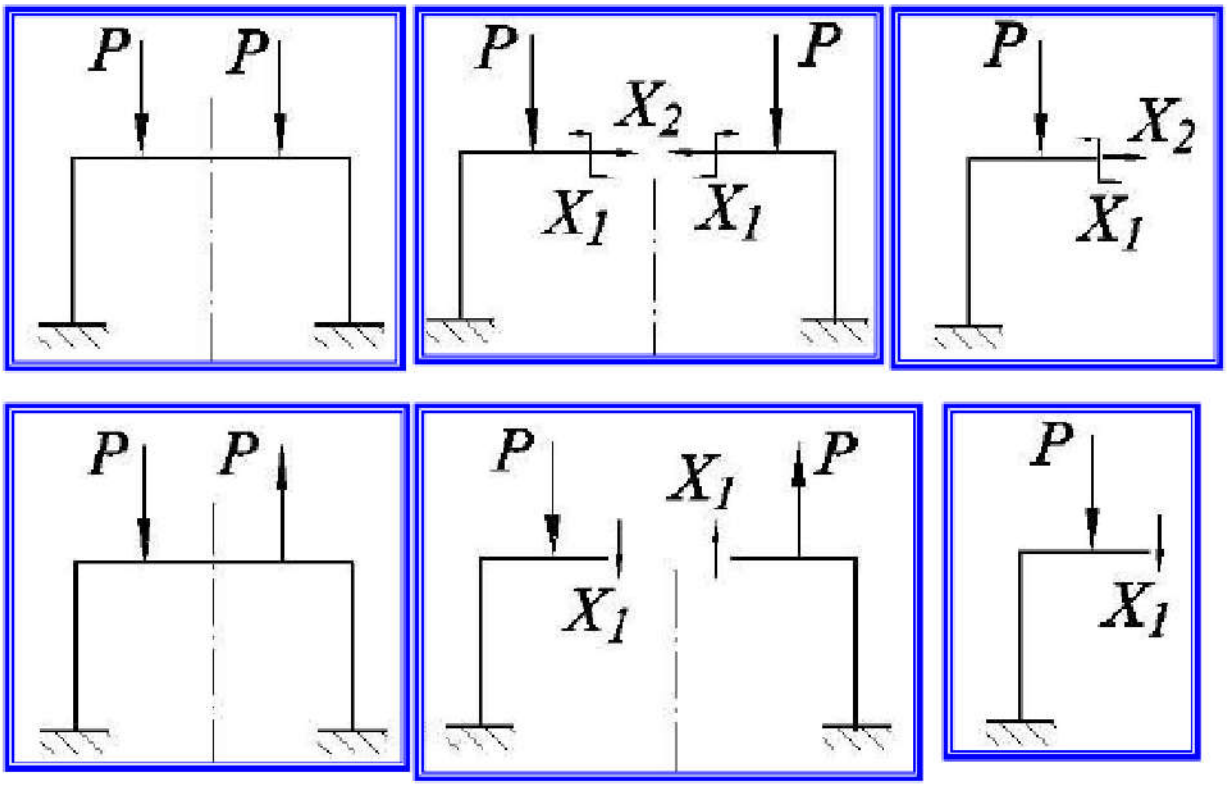
受反对称载荷，对称截面上对称的内力、对称的位移为零

在对称面上内力：轴力、弯矩为对称的内力

剪力、扭矩为反对称的内力

平面问题中：在对称面上，转角、扭转角、垂直于对称轴的线位移为反对称位移

㈣关于对称与反对称问题静定基的多种选取方法



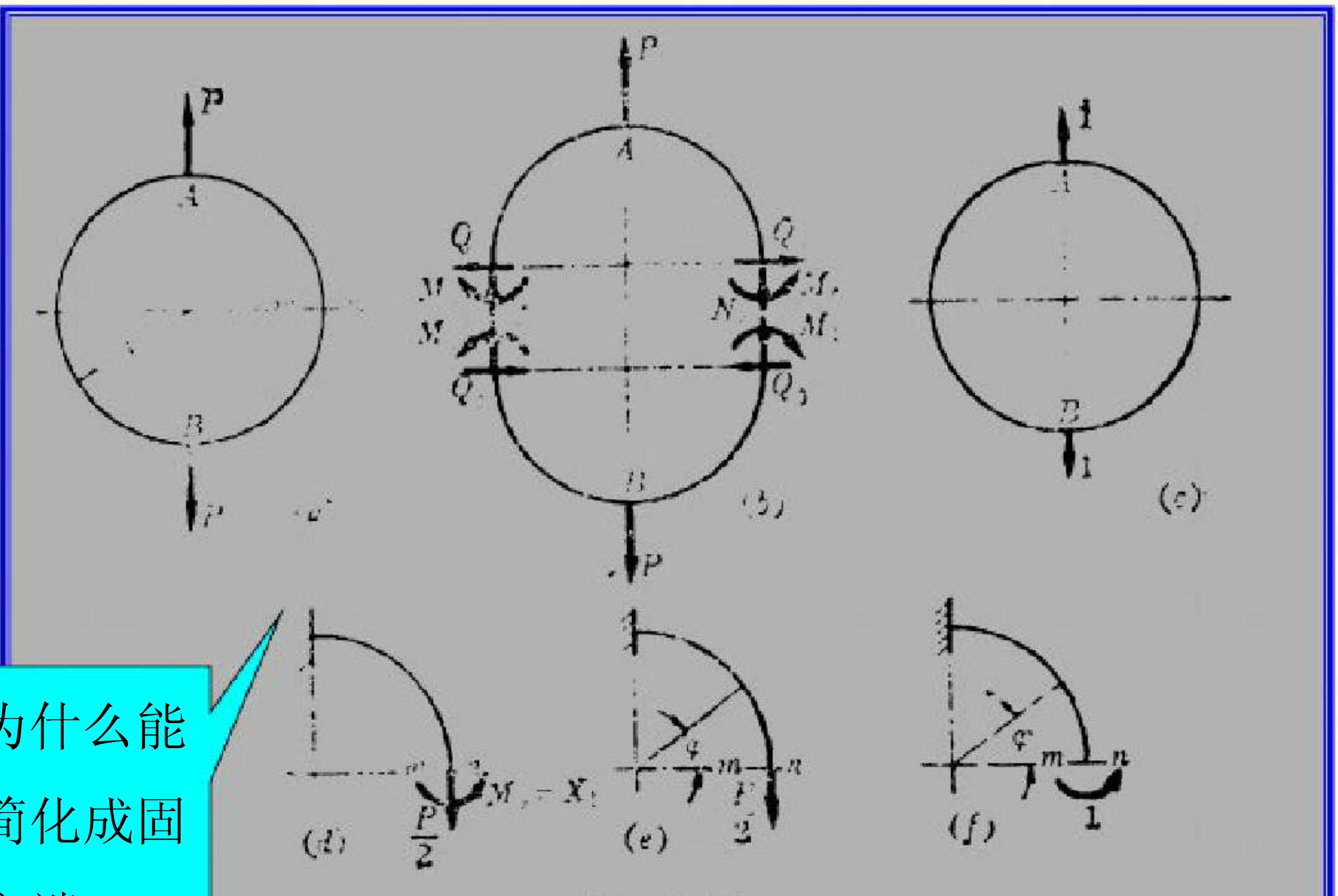
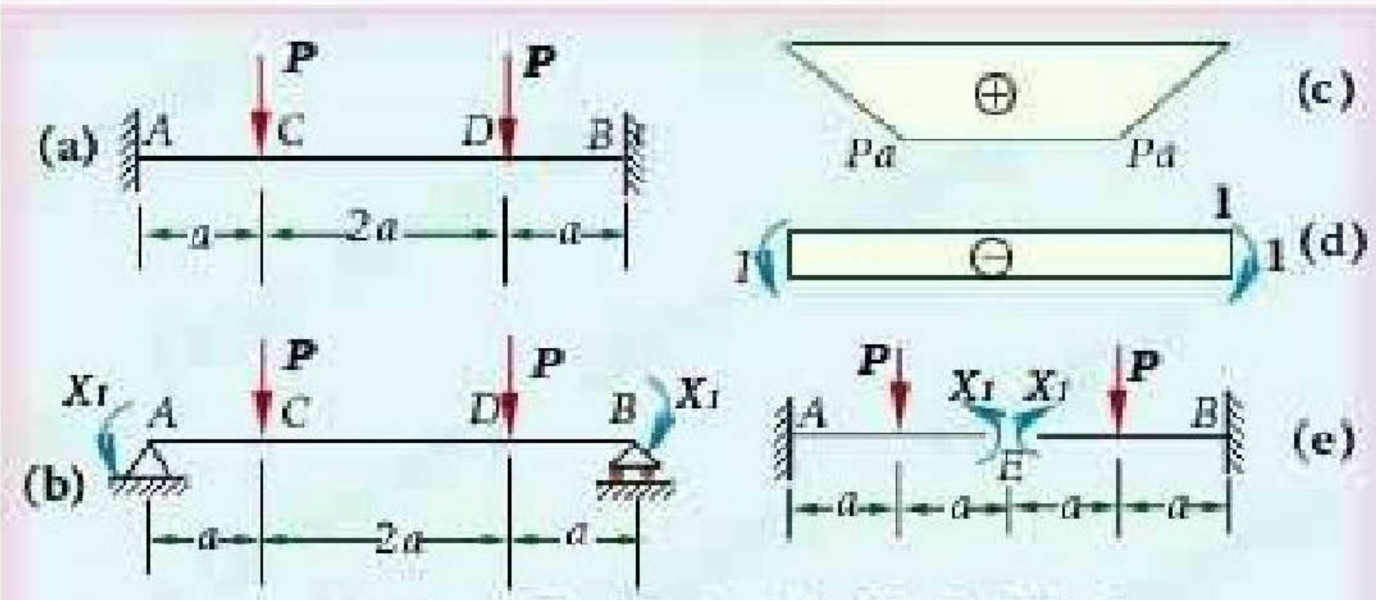


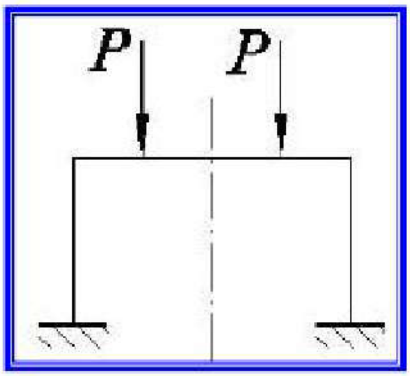
图 11-15

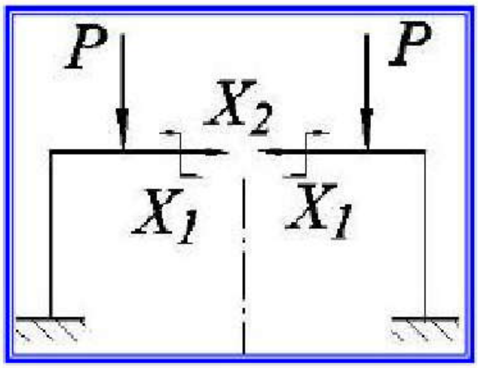
为什么能简化成固定端？

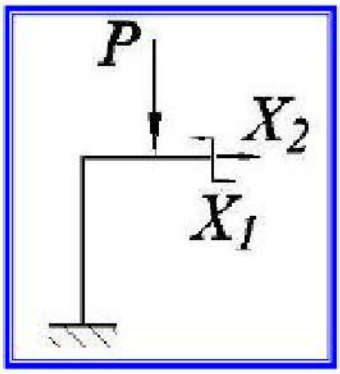


例 12-3图　两端固支梁对称受力

说明：







$$\delta_{11}X_1+\delta_{12}X_2+\Delta_{1P}=0$$

$$\delta_{21}X_1+\delta_{22}X_2+\Delta_{2P}=0$$

不同的静定基，等式右侧 **0**表示什么物理意义？

例题