Limit Theorems for Some Critical Superprocesses

Yan-Xia Ren[*] **Renming Song**[†] and **Rui Zhang**[‡]

Abstract

In this paper we establish some conditional limit theorems for some critical superprocesses $X = \{X_t, t \geq 0\}$. First we identify the rate of non-extinction. Then we show that, for a large class of functions f, conditioned on non-extinction at time t, the limit, as $t \to \infty$, of $t^{-1}\langle f, X_t\rangle$ exists in distribution and we identify this limit. Finally, we also establish, under some conditions, a central limit theorem for $\langle f, X_t\rangle$ conditioned on non-extinction at time t.

AMS Subject Classifications (2000): Primary 60F05; 60J80; Secondary 60J25, 60J35

Keywords and Phrases: Superprocess, critical superprocess, non-extinction rate, central limit theorem

1 Introduction

1.1 Motivation

In 1966, Kesten, Ney and Spitzer [12] proved that if $\{Z_n, n \geq 0\}$ is a critical branching process with finite second moment, then

$$\lim_{n\to\infty} nP(Z_n > 0) = \frac{1}{\sigma^2} \tag{1.1}$$

and

$$\lim_{n\to\infty} P\left(\frac{1}{n}Z(n) > \frac{\sigma^2}{2}x|Z(n) > 0\right) = e^{-x}, \quad x \geq 0, \tag{1.2}$$

where σ^2 is the variance of the offspring distribution. The first result says that the non-extinction rate is of order $1/n$ as $n \to \infty$, and the second result says that, conditioned on non-extinction at time n, the total population size in generation n grows like n. For probabilistic proofs of these results, see Lyons, Pemantle and Peres [17]. For continuous time critical branching processes $\{Z_t, t \geq 0\}$, Athreya and Ney [4, Theorem 3 and Lemma 2 on page 113] proved the following limit theorem:

$$\lim_{t\to\infty} P\left(\frac{1}{t}Z(t) > \frac{\sigma^2}{2}x|Z(t) > 0\right) = e^{-x}, \quad x \geq 0, \tag{1.3}$$

where σ^2 is a positive constant determined by the branching rate and the variance of the offspring distribution.

[*]The research of this author is supported by NSFC (Grant No. 11271030 and 11128101) and Specialized Research Fund for the Doctoral Program of Higher Education.

[†]Research supported in part by a grant from the Simons Foundation (208236).

[‡]Supported by the China Scholarship Council.

For discrete time multi-type critical branching processes $\{\mathbf{Z}(n), n \geq 0\}$, Athreya and Ney [4] gave two limit theorems under the finite second moment condition, see [4, Section V.5]. Let $\mathbf{v}$ be a positive left eigenvector of the mean matrix associated with the eigenvalue 1. The first limit theorem says that if $\mathbf{w} \cdot \mathbf{v} > 0$, then

$$\lim_{n\to\infty} P\left(\frac{\mathbf{Z}(n)\cdot\mathbf{w}}{n} > x|\mathbf{Z}(n) > 0\right) = \int_x^\infty f(y)dy, \quad x \geq 0, \tag{1.4}$$

where

$$f(y) = \frac{1}{\gamma_1}e^{-y/\gamma_1}, \quad y \geq 0,$$

and γ_1 is a positive constant. The second limit theorem says that if $\mathbf{w} \cdot \mathbf{v} = 0$, then

$$\lim_{n\to\infty} P\left(\frac{\mathbf{Z}(n)\cdot\mathbf{w}}{\sqrt{n}} > x|\mathbf{Z}(n) > 0\right) = \int_x^\infty f_2(y)dy, \quad x \in \mathbb{R}, \tag{1.5}$$

where

$$f_2(y) = \frac{1}{2\gamma_2}e^{-|y|/\gamma_2}, \quad y \in \mathbb{R},$$

and γ_2 is a positive constant. The limit result (1.4) is a generalization of (1.2) from the single type case to the multi-type case, and was first proved by Joffe and Spitzer [11]. The limit result (1.5) was first proved in Ney [19].

For continuous time multi-type critical branching processes, Athreya and Ney [5] proved two limit theorems, similar to results (1.4) and (1.5) respectively, under the finite second moment condition, see [5, Theorems 1 and 2].

Asmussen and Hering [3] discussed similar questions for critical branching Markov processes $\{Y_t, t \geq 0\}$. In [3, Proposition 3.3 on page 201], Asmussen and Hering discussed the finite time extinction property of branching Markov processes. Under some conditions (see [3] for details), [3, Theorem 3.4 on page 202] provided the rate of non-extinction, more precisely, it was shown that

$$\lim_{t\to\infty} tP_\nu(\|Y_t\| \neq 0) = \mu^{-1}\phi_0(x)$$

uniformly in ν with ν satisfying $\mathrm{supp}(\nu) = n$ for any integer n, where μ is a positive constant and ϕ_0 is the first eigenfunction of the mean semigroup of $\{Y_t, t \geq 0\}$.

In [3, Theorem 3.8 on page 204], Asmussen and Hering gave a result similar to (1.4), under some condition which is satisfied by some critical multi-group branching diffusions. In [3, Theorem 3.3 on page 297], Asmussen and Hering gave a result similar to (1.5) for critical branching Markov processes under some condition of the mean matrix M at time $t = 1$ (see [3, (2.1) on page 293]). We also would like to mention that the conditions for the results of [3] mentioned in this paragraph are not very easy to check.

The main purpose of this paper is to consider similar types of limit theorems for critical superprocesses, under very general but easy to check conditions.

In our recent papers [20, 22], we established some spatial central limit theorems for supercritical superprocesses. See also [1, 18, 21] for related results. Our original motivation for the present paper is to establish spatial central limit theorems for critical superprocesses. In contrast with the papers mentioned above, the spatial process needs not be symmetric in this paper.

1.2 Superprocesses and assumptions

In this subsection, we describe the superprocesses we are going to work with and spell out our assumptions.

Suppose that E is a locally compact separable metric space and that m is a σ-finite Borel measure on E with full support. Suppose that ∂ is a separate point not contained in E. ∂ will be interpreted as the cemetery point. We will use E_∂ to denote $E \cup \{\partial\}$. Every function f on E is automatically extended to E_∂ by setting $f(\partial) = 0$. We will assume that $\xi = \{\xi_t, \Pi_x\}$ is a Hunt process on E and $\zeta := \inf\{t > 0 : \xi_t = \partial\}$ is the lifetime of ξ. We will use $\{P_t : t \geq 0\}$ to denote the semigroup of ξ. We will use $\mathcal{B}_b(E)$ ($\mathcal{B}_b^+(E)$) to denote the set of (positive) bounded Borel measurable functions on E.

The superprocess $X = \{X_t : t \geq 0\}$ we are going to work with is determined by three parameters: a spatial motion $\xi = \{\xi_t, \Pi_x\}$ on E which is a Hunt process, a branching rate function $\beta(x)$ on E which is a non-negative bounded measurable function and a branching mechanism φ of the form

$$\varphi(x,z) = -a(x)z + b(x)z^2 + \int_{(0,+\infty)} (e^{-zy} - 1 + zy) n(x, dy), \quad x \in E, \quad z > 0, \tag{1.6}$$

where $a \in \mathcal{B}_b(E)$, $b \in \mathcal{B}_b^+(E)$ and n is a kernel from E to $(0, \infty)$ satisfying

$$\sup_{x\in E} \int_{(0,+\infty)} y^2 n(x, dy) < \infty. \tag{1.7}$$

In our paper, we will not consider the special case that $b(\cdot) + n(\cdot, (0, \infty)) = 0$, a.e.-m.

The superprocess X is a Markov process taking values in $\mathcal{M}_F(E)$, the space of finite measures on E. The existence of such superprocesses is well-known, see, for instance, [6], [8] or [15]. For any $\mu \in \mathcal{M}_F(E)$, we denote the law of X with initial configuration μ by $\mathbb{P}_\mu$. As usual, $\langle f, \mu\rangle := \int_E f(x)\mu(dx)$ and $\|\mu\| := \langle 1, \mu\rangle$. Then for every $f \in \mathcal{B}_b^+(E)$ and $\mu \in \mathcal{M}_F(E)$,

$$-\log \mathbb{P}_\mu \left(e^{-\langle f, X_t\rangle}\right) = \langle u_f(t, \cdot), \mu\rangle, \tag{1.8}$$

where $u_f(t, x)$ is the unique positive solution to the equation

$$u_f(t, x) + \Pi_x \int_0^t \Psi(\xi_s, u_f(t - s, \xi_s)) ds = \Pi_x f(\xi_t), \tag{1.9}$$

where $\Psi(x, z) = \beta(x)\varphi(x, z)$, $x \in E$ and $z > 0$, while $\Psi(\partial, z) = 0, z > 0$. Define

$$\alpha(x) := \beta(x)a(x) \quad \text{and} \quad A(x) := \beta(x) \left(2b(x) + \int_0^\infty y^2 n(x, dy)\right). \tag{1.10}$$

Then, by our assumptions, $\alpha(x) \in \mathcal{B}_b(E)$ and $A(x) \in \mathcal{B}_b^+(E)$. Thus there exists $K > 0$ such that

$$\sup_{x\in E} (|\alpha(x)| + A(x)) \leq K. \tag{1.11}$$

For any $f \in \mathcal{B}_b(E)$ and $(t,x) \in (0,\infty) \times E$, define

$$T_t f(x) := \Pi_x \left[e^{\int_0^t \alpha(\xi_s)\, ds} f(\xi_t) \right]. \tag{1.12}$$

It is well-known that $T_t f(x) = \mathbb{P}_{\delta_x} \langle f, X_t \rangle$ for every $x \in E$.

Our standing assumption on ξ is that there exists a family of continuous strictly positive functions $\{p(t,x,y) : t > 0\}$ on $E \times E$ such that, for any $t > 0$ and nonnegative function f on E,

$$P_t f(x) = \int_E p(t,x,y) f(y) m(dy).$$

Define

$$a_t(x) := \int_E p(t,x,y)^2\, m(dy), \qquad \hat{a}_t(x) := \int_E p(t,y,x)^2\, m(dy).$$

In this paper, we assume that

Assumption 1.1 **(i)** *For any $t > 0$, $\int_E p(t,x,y)\, m(dx) \le 1$.*

(ii) *For any $t > 0$, we have*

$$e_t := \int_E a_t(x)\, m(dx) = \int_E \hat{a}_t(x)\, m(dx) = \int_E \int_E p(t,x,y)^2\, m(dy)\, m(dx) < \infty. \tag{1.13}$$

Moreover, the functions $x \to a_t(x)$ and $x \to \hat{a}_t(x)$ are continuous on E .

It is easy to see that

$$p(t+s,x,y) = \int_E p(t,x,z) p(s,z,y)\, m(dz) \le (a_t(x))^{1/2} (\hat{a}_s(y))^{1/2}, \tag{1.14}$$

which implies

$$a_{t+s}(x) \le \int_E \hat{a}_s(y)\, m(dy) a_t(x) \quad \text{and} \quad \hat{a}_{t+s}(x) \le \int_E a_s(y)\, m(dy) \hat{a}_t(x). \tag{1.15}$$

It is well known and easy to check that, $\{P_t : t \ge 0\}$ is a strongly continuous semigroup on $L^2(E,m)$. We claim that the function $t \to \int_E a_t(x)\, m(dx)$ is decreasing. In fact, by Fubini's theorem and Hölder's inequality, we get

$$\begin{aligned} a_{t+s}(x) &= \int_E p(t+s,x,y) \int_E p(t,x,z) p(s,z,y)\, m(dz)\, m(dy) \\ &= \int_E p(t,x,z) \int_E p(t+s,x,y) p(s,z,y)\, m(dy)\, m(dz) \\ &\le a_{t+s}(x)^{1/2} \int_E p(t,x,z) a_s(z)^{1/2}\, m(dz) \end{aligned}$$

which implies

$$a_{t+s}(x) \le \left(\int_E p(t,x,z) a_s(z)^{1/2}\, m(dz) \right)^2 \le \int_E p(t,x,z) a_s(z)\, m(dz). \tag{1.16}$$

Thus, by Fubini's theorem and Assumption 1.1(i), we get

$$\int_E a_{t+s}(x)\,m(dx) \le \int_E a_s(z) \int_E p(t,x,z)\,m(dx)\,m(dz) \le \int_E a_s(z)\,m(dz). \tag{1.17}$$

Therefore, the function $t \to \int_E a_t(x)\,m(dx)$ is decreasing.

We claim that (see Lemma 2.1 below) there exists a function $q(t,x,y)$ on $(0,\infty)\times E\times E$ which is continuous in (x,y) for each $t>0$ such that

$$e^{-Kt}p(t,x,y) \le q(t,x,y) \le e^{Kt}p(t,x,y), \quad (t,x,y)\in(0,\infty)\times E\times E \tag{1.18}$$

and that for any bounded Borel function f and any $(t,x)\in(0,\infty)\times E$,

$$T_tf(x) = \int_E q(t,x,y)f(y)\,m(dy).$$

It follows immediately that $\{T_t : t\ge 0\}$ is a strongly continuous semigroup on $L^2(E,m)$ and

$$\|T_tf\|_2^2 \le e^{2Kt}\|f\|_2^2. \tag{1.19}$$

Let $\{\widehat{T}_t, t>0\}$ be the adjoint operators on $L^2(E,m)$ of $\{T_t, t>0\}$, that is, for $f,g\in L^2(E,m)$,

$$\int_E f(x)T_tg(x)\,m(dx) = \int_E g(x)\widehat{T}_tf(x)\,m(dx)$$

and

$$\widehat{T}_tf(x) = \int_E q(t,y,x)f(y)\,m(dy).$$

It is well known that $\{\widehat{T}_t : t\ge 0\}$ is a strongly continuous semigroup on $L^2(E,m)$. For all $t>0$ and $f\in L^2(E,m)$, T_tf and $\widehat{T}_tf$ are continuous. In fact, since $q(t,x,y)$ is continuous in (x,y), by (1.14), (1.18) and Assumption 1.1(ii), using the dominated convergence theorem, we get that, for any $f\in L^2(E,m)$, T_tf and $\widehat{T}_tf$ are continuous.

Let L and $\widehat{L}$ be the infinitesimal generators of the semigroups $\{T_t\}$ and $\{\widehat{T}_t\}$ in $L^2(E,m)$ respectively. Define $\lambda_0 := \sup\Re(\sigma(L)) = \sup\Re(\sigma(\widehat{L}))$. By Jentzsch's theorem (Theorem V.6.6 on page 337 of [23]), λ_0 is an eigenvalue of multiplicity 1 for both L and $\widehat{L}$. Assume that ϕ_0 and ψ_0 are the eigenfunctions of L and $\widehat{L}$ respectively associated with λ_0. ψ_0 and ϕ_0 can be chosen to be continuous and strictly positive satisfying $\|\phi_0\|_2 = 1$ and $\langle\phi_0,\psi_0\rangle_m = 1$.

The main interest of this paper is critical superprocesses, so we assume that

Assumption 1.2 $\lambda_0 = 0$.

We also assume that

Assumption 1.3 **(i)** *ϕ_0 is bounded.*

(ii) *The semigroup $\{T_t, t>0\}$ is intrinsically ultracontractive, that is, there exists $c_t>0$ such that*

$$q(t,x,y) \le c_t\phi_0(x)\psi_0(y). \tag{1.20}$$

It is easy to get that, for any $t > 0$ and $x \in E$,

$$a_t(x) \le e^{2Kt} \int_E q(t,x,y)^2 \, m(dy) \le c_t^2 e^{2Kt} \int_E \psi_0(y)^2 \, m(dy) \phi_0(x)^2. \tag{1.21}$$

On the other hand, we have

$$\phi_0(x) = T_t(\phi_0)(x) \le e^{Kt}(e_{t/2})^{1/2} a_{t/2}(x)^{1/2}. \tag{1.22}$$

In [21] and [22], many examples of Markov processes satisfying the above Assumption 1.1 were given. In [16], quite a few examples of Hunt processes satisfying Assumptions 1.1 and 1.3 were given. If E consists of finitely many points, and $\xi = \{\xi_t : t \ge 0\}$ is a conservative irreducible Markov process on E, then ξ satisfies the Assumptions 1.1 and 1.3 for some finite measure m on E with full support. So, as special cases, our results give the analogs of the results of Athreya and Ney [5] for critical super-Markov chains.

Define $q_t(x) := \mathbb{P}_{\delta_x}(\|X_t\| = 0)$. Note that, since $\mathbb{P}_{\delta_x}\|X_t\| = T_t 1(x) > 0$, we have $\mathbb{P}_{\delta_x}(\|X_t\| = 0) < 1$. In this paper, we also assume that

Assumption 1.4 *There exists $t_0 > 0$ such that,*

$$\inf_{x \in E} q_{t_0}(x) > 0. \tag{1.23}$$

In Subsection 2.2, we will give sufficient conditions for Assumption 1.4. In Lemma 3.3, we will show that, under our assumptions, $\lim_{t\to\infty} q_t(x) = 1$, uniformly in $x \in E$.

1.3 Main results

In this subsection, we will state our main results. In the following, we use the notation

$$\mathbb{P}_{t,\mu}(\cdot) := \mathbb{P}_\mu\left(\cdot \mid \|X_t\| \neq 0\right).$$

Let $(\Omega, \mathcal{G})$ be the measurable space on which the process X is defined. Assume that $Y_t, t > 0$, and Y are random variables on $(\Omega, \mathcal{G})$. We write

$$Y_t|_{\mathbb{P}_{t,\mu}} \to Y \quad \text{in probability,}$$

if, for any $\epsilon > 0$,

$$\lim_{t\to\infty} \mathbb{P}_{t,\mu}(|Y_t - Y| \ge \epsilon) = 0.$$

Suppose that Z is a random variable on a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{G}}, P)$, we write

$$Y_t|_{\mathbb{P}_{t,\mu}} \stackrel{d}{\to} Z,$$

if, for all $a \in \mathbb{R}$ with $P(Z = a) = 0$,

$$\lim_{t\to\infty} \mathbb{P}_{t,\mu}(Y_t \le a) = P(Z \le a).$$

Define

$$\nu := \frac{1}{2}\langle A(\phi_0)^2, \psi_0\rangle_m. \tag{1.24}$$

It is easy to see that $0 < \nu < \infty$. Define

$$\mathcal{C}_p := \{f \in \mathcal{B}(E) : \langle |f|^p, \psi_0\rangle_m < \infty\}$$

and $\mathcal{C}_p^+ := \mathcal{C}_p \cap \mathcal{B}^+(E)$. By Assumption 1.2(ii) and the fact that $q(t, x, y)$ is continuous, using the dominated convergence theorem, we get, for $f \in \mathcal{C}_1$, $T_t f(x)$ is continuous.

Theorem 1.5 *For any non-zero $\mu \in \mathcal{M}_F(E)$,*

$$\lim_{t\to\infty} t\mathbb{P}_\mu(\|X_t\| \neq 0) = \nu^{-1}\langle \phi_0, \mu\rangle. \tag{1.25}$$

Theorem 1.6 *If $f \in \mathcal{C}_2$ then, for any non-zero $\mu \in \mathcal{M}_F(E)$, we have*

$$t^{-1}\langle f, X_t\rangle|_{\mathbb{P}_{t,\mu}} \stackrel{d}{\to} \langle f, \psi_0\rangle_m W, \tag{1.26}$$

where W is an exponential random variable with parameter $1/\nu$. In particular, we have

$$t^{-1}\langle \phi_0, X_t\rangle|_{\mathbb{P}_{t,\mu}} \stackrel{d}{\to} W. \tag{1.27}$$

Remark 1.7 *Our assumptions imply that $1 \in \mathcal{C}_2$, see Remark 2.6 below. Thus the limit result above implies that*

$$t^{-1}\langle 1, X_t\rangle|_{\mathbb{P}_{t,\mu}} \stackrel{d}{\to} \langle 1, \psi_0\rangle_m W,$$

which says that, conditioned on no-extinction at time t, the growth rate of the total mass $\langle 1, X_t\rangle$ is t as $t \to \infty$.

Note that, when $\langle f, \psi_0\rangle_m = 0$, $t^{-1}\langle f, X_t\rangle|_{\mathbb{P}_{t,\mu}} \to 0$ in probability. Therefore it is natural to consider central limit type theorems for $\langle f, X_t\rangle$.

Define

$$\sigma_f^2 = \int_0^\infty \langle A(T_s f)^2, \psi_0\rangle_m \, ds. \tag{1.28}$$

Theorem 1.8 *Suppose that $f \in \mathcal{C}_2$ and $\langle f, \psi_0\rangle_m = 0$, then we have, $\sigma_f^2 < \infty$ and for any non-zero $\mu \in \mathcal{M}_F(E)$,*

$$\left(t^{-1}\langle \phi_0, X_t\rangle, t^{-1/2}\langle f, X_t\rangle\right)|_{\mathbb{P}_{t,\mu}} \stackrel{d}{\to} \left(W, G(f)\sqrt{W}\right), \tag{1.29}$$

where $G(f) \sim \mathcal{N}(0, \sigma_f^2)$ is a normal random variable and W is the random variable defined in Theorem 1.6. Moreover, W and $G(f)$ are independent.

It follows from Theorem 1.6 that, when $\sigma_f^2 > 0$, the density of $G(f)\sqrt{W}$ is

$$d(x) = \frac{1}{\sqrt{2\nu\sigma_f^2}} \exp\left\{-\frac{2|x|}{\sqrt{2\nu\sigma_f^2}}\right\}, \qquad x \in \mathbb{R}.$$

As a consequence of Theorem 1.8, we immediately get the following central limit theorem.

Corollary 1.9 *Suppose that $f \in \mathcal{C}_2$ and $\langle f, \psi_0 \rangle_m = 0$, then we have, $\sigma_f^2 < \infty$ and for any non-zero $\mu \in \mathcal{M}_F(E)$,*

$$\left(t^{-1}\langle \phi_0, X_t \rangle, \frac{\langle f, X_t \rangle}{\sqrt{\langle \phi_0, X_t \rangle}} \right) |_{\mathbb{P}_{t,\mu}} \stackrel{d}{\to} (W, G(f)) \,, \tag{1.30}$$

where $G(f) \sim \mathcal{N}(0, \sigma_f^2)$ is a normal random variable and W is the random variable defined in Theorem 1.6. Moreover, W and $G(f)$ are independent.

2 Preliminaries

2.1 Density of $\{T_t : t \geq 0\}$

In this subsection, we show that, under Assumption 1.1, the semigroup $\{T_t : t \geq 0\}$ has a nice density $q(t, x, y)$.

Lemma 2.1 *Suppose that Assumption 1.1 holds. The semigroup $\{T_t : t \geq 0\}$ has a density $q(t, x, y)$ such that*

$$e^{-Kt}p(t, x, y) \leq q(t, x, y) \leq e^{Kt}p(t, x, y), \quad (t, x, y) \in (0, \infty) \times E \times E. \tag{2.1}$$

Furthermore, for any $t > 0$, $q(t, x, y)$ is a continuous function of (x, y) on $E \times E$.

Proof: For any $(t, x, y) \in (0, \infty) \times E \times E$, define

$$\begin{aligned} I_0(t, x, y) &:= p(t, x, y), \\ I_n(t, x, y) &:= \int_0^t \int_E p(s, x, z) I_{n-1}(t - s, z, y) \alpha(z) m(dz) ds, \qquad n \geq 1. \end{aligned}$$

Using arguments similar to those in Section 1.2 of [21], we easily get that the function

$$q(t, x, y) := \sum_{n=0}^{\infty} I_n(t, x, y), \qquad (t, x, y) \in (0, \infty) \times E \times E \tag{2.2}$$

is well defined and $q(t, x, y)$ is the transition density function of T_t satisfying (2.1). We omit the details.

We now prove the continuity of $q(t, x, y)$ in $(x, y) \in E \times E$ for each fixed $t > 0$. As in Section 1.2 of [21], it suffices to show that, for any $0 < \epsilon < t/2$,

$$\int_\epsilon^{t-\epsilon} \int_E p(s, x, z) p(t - s, z, y) \alpha(z) m(dz) ds$$

is continuous on $E \times E$. By (1.14), we get that

$$p(s, x, z) p(t - s, z, y) |\alpha(z)| \leq K a_{\epsilon/2}(x)^{1/2} \widehat{a}_{\epsilon/2}(y)^{1/2} \widehat{a}_{s-\epsilon/2}(z)^{1/2} a_{t-s-\epsilon/2}(z)^{1/2}.$$

By Hölder's inequality and (1.17), we get that

$$\int_\epsilon^{t-\epsilon} \int_E \widehat{a}_{s-\epsilon/2}(z)^{1/2} a_{t-s-\epsilon/2}(z)^{1/2} \, m(dz) \, ds$$

$$\leq \int_{\epsilon}^{t-\epsilon}\left(\int_E \widehat{a}_{s-\epsilon/2}(z)\,m(dz)\right)^{1/2}\left(\int_E a_{t-s-\epsilon/2}(z)\,m(dz)\right)^{1/2} ds$$
$$\leq \int_{\epsilon}^{t-\epsilon}\left(\int_E a_{s-\epsilon/2}(z)\,m(dz)\right)^{1/2}\left(\int_E a_{t-s-\epsilon/2}(z)\,m(dz)\right)^{1/2} ds \leq t\int_E a_{\epsilon/2}(z)\,m(dz).$$

The second inequality above follows from the fact $\int_E \widehat{a}_t(z)\,m(dz) = \int_E a_t(z)\,m(dz)$ and the last inequality above is a consequence of the fact that $t \to \int_E a_t(x)m(dx)$ is decreasing in t. Thus, by Assumption 1.1(ii) and the dominated convergence theorem, we get that $\int_{\epsilon}^{t-\epsilon}\int_E p(s,x,z)p(t-s,z,y)\alpha(z)m(dz)ds$ is continuous. □

2.2 Extinction and non-extinction of $\{X_t, t \geq 0\}$

In this subsection, we will give some sufficient conditions for Assumption 1.4, see Lemma 2.3 below. In the case when the function $a(x)$ in (1.6) is identically zero, this lemma follows from [6, Lemma 11.5.1]. Here we provide a proof for completeness.

Let $\widetilde{\Psi}(x,z)$ be a function on $E_\partial \times (0,\infty)$ with the form:

$$\widetilde{\Psi}(x,z) = -\tilde{a}(x)z + \tilde{b}(x)z^2 + \int_{(0,+\infty)}(e^{-zy} - 1 + zy)\tilde{n}(x,dy), \quad x \in E_\partial, \quad z > 0, \tag{2.3}$$

where $\tilde{a} \in \mathcal{B}_b(E_\partial)$, $\tilde{b} \in \mathcal{B}_b^+(E_\partial)$ and $\tilde{n}$ is a kernel from E_∂ to $(0,\infty)$ satisfying

$$\int_{(0,+\infty)} y \wedge y^2 \tilde{n}(x,dy) < \infty. \tag{2.4}$$

The following Lemma 2.2 is similar to [15, Corollary 5.18]. The proof of [15, Corollary 5.18] was based on the Laplace functional of the weighted occupation time of superprocesses. Below we give a proof without using the concept of the weighted occupation time.

Recall that, unless explicitly mentioned otherwise, every function f on E is automatically extended to E_∂ by setting $f(\partial) = 0$. The function g in the lemma below may not satisfy $g(\partial) = 0$.

Lemma 2.2 *Suppose that $\Psi(x,z) \geq \widetilde{\Psi}(x,z)$ for all $x \in E$ and $z > 0$. If f and g are bounded nonnegative measurable functions on E_∂ such that $f(\partial) = 0$ and $f(x) \leq g(x)$ for all $x \in E_\partial$. If $v_g(t,x)$ is the solution to the equation*

$$v_g(t,x) = -\Pi_x \int_0^t \widetilde{\Psi}(\xi_s, v_g(t-s,\xi_s))ds + \Pi_x g(\xi_t), \quad x \in E_\partial, t > 0,$$

then $v_g(t,x) \geq u_f(t,x)$ for all $t > 0$ and $x \in E$.

Proof: It is well known that u_f satisfies $u_f(t,\partial) = 0$ and

$$u_f(t,x) = -\Pi_x \int_0^t \Psi(\xi_s, u_f(t-s,\xi_s))ds + \Pi_x f(\xi_t), \quad x \in E_\partial.$$

Fix $T>0$ and, for any $r\in[0,T]$ and $x\in E_\partial$, define $G_1(r,x)=u_f(T-r,x)$ and $G_2(r,x)=v_g(T-r,x)$. For any $r\geq 0$ and $x\in E$, we will use $\Pi_{r,x}$ to denote the law of ξ with birth time r and starting point x. Then

$$G_1(r,x)=-\Pi_{r,x}\int_r^T\Psi(\xi_s,G_1(s,\xi_s))ds+\Pi_{r,x}f(\xi_T),\quad x\in E_\partial. \tag{2.5}$$

and

$$G_2(r,x)=-\Pi_{r,x}\int_r^T\widetilde{\Psi}(\xi_s,G_2(s,\xi_s))ds+\Pi_{r,x}g(\xi_T),\quad x\in E_\partial. \tag{2.6}$$

Put $G(r,x):=G_2(r,x)-G_1(r,x)$, $r\in[0,T]$, $x\in E_\partial$. It follows from [15, Proposition 2.14] that $G_1(r,x)$ and $G_2(r,x)$ are bounded, thus G is also bounded. For $s\in[0,T]$ and $x\in E_\partial$, define $w(s,x):=\Psi(x,G_1(s,x))-\widetilde{\Psi}(x,G_1(s,x))$ and

$$\rho(s,x):=\begin{cases}\frac{\widetilde{\Psi}(x,G_2(s,x))-\widetilde{\Psi}(x,G_1(s,x))}{G(s,x)} & \text{if } G(s,x)\neq 0;\\ 0, & \text{otherwise.}\end{cases}$$

Since G_1, G_2 are bounded, we can easily see that w is also bounded on $[0,T]\times E_\partial$. Since $\frac{\partial\widetilde{\Psi}(x,z)}{\partial z}$ is bounded on $E\times[0,S]$ for any $S>0$, we know that ρ is also bounded on $[0,T]\times E_\partial$. It follows from (2.5) and (2.6) that

$$\begin{aligned}G(r,x) &= \Pi_{r,x}\int_r^T\Psi(\xi_s,G_1(s,\xi_s))-\widetilde{\Psi}(\xi_s,G_2(s,\xi_s))ds+\Pi_{r,x}(g(\xi_T)-f(\xi_T))\\ &= \Pi_{r,x}\int_r^T\Psi(\xi_s,G_1(s,\xi_s))-\widetilde{\Psi}(\xi_s,G_1(s,\xi_s))ds\\ &\quad -\Pi_{r,x}\int_r^T\frac{\widetilde{\Psi}(\xi_s,G_2(s,\xi_s))-\widetilde{\Psi}(\xi_s,G_1(s,\xi_s))}{G(s,\xi_s)}G(s,\xi_s)ds+\Pi_{r,x}(g(\xi_T)-f(\xi_T))\\ &= \Pi_{r,x}\int_r^T w(s,\xi_s)\,ds-\Pi_{r,x}\int_r^T\rho(s,\xi_s)G(s,\xi_s)\,ds+\Pi_{r,x}(g(\xi_T)-f(\xi_T)).\end{aligned}$$

Applying [8, Lemma 1.5 in Appendix to Part I], we get that

$$G(r,x)=\Pi_{r,x}\int_r^T\exp\left\{-\int_r^s\rho(q,\xi_q)\,dq\right\}w(s,\xi_s)\,ds+\Pi_{r,x}\left(\exp\left\{-\int_r^T\rho(q,\xi_q)\,dq\right\}(g(\xi_T)-f(\xi_T))\right).$$

Since $G_1(r,\partial)=0$, $w(r,\partial)=\Psi(\partial,0)-\widetilde{\Psi}(\partial,0)=0$. Then, it follows from our assumptions that, for $r\in[0,T]$ and $x\in E_\partial$, $w\geq 0$ and $g-f\geq 0$, which implies $G(r,x)\geq 0$, $r\in[0,T]$, $x\in E$. Therefore, $v_g(t,x)\geq u_f(t,x)$ for all $t>0$ and $x\in E$. □

Lemma 2.3 *Suppose that $\widetilde{\Psi}(z)=\inf_{x\in E}\Psi(x,z)$ can be written in the form*

$$\widetilde{\Psi}(z)=\widetilde{a}z+\widetilde{b}z^2+\int_0^\infty(e^{-zy}-1+zy)\widetilde{n}(dy)$$

with $\widetilde{a}\in\mathbb{R}$, $\widetilde{b}\geq 0$ and $\widetilde{n}$ is a measure on $(0,\infty)$ satisfying $\int_0^\infty(y\wedge y^2)\widetilde{n}(dy)<\infty$. If $\widetilde{b}+\widetilde{n}(0,\infty)>0$ and $\widetilde{\Psi}(z)$ satisfies

$$\int^\infty\frac{1}{\widetilde{\Psi}(z)}dz<\infty, \tag{2.7}$$

then, for $t>0$, $\|-\log q_t\|_\infty<\infty$.

Proof: Let $\widetilde{X}$ be a continuous state branching processes with branching mechanism $\widetilde{\Psi}$. Let $\widetilde{\mathbb{P}}$ be the law of $\widetilde{X}$ with $\widetilde{X}_0 = 1$. Define

$$u_\theta(t,x) = -\log \mathbb{P}_{\delta_x} e^{-\theta \|X_t\|}, \qquad v_\theta(t) = -\log \widetilde{\mathbb{P}} e^{-\theta \widetilde{X}_t}.$$

It is easy to see that $u_\theta(t,\partial) = 0$ and, for $x \in E$ and $t > 0$,

$$u_\theta(t,x) = -\Pi_x \int_0^t \Psi(\xi_s, u_f(t-s,\xi_s))ds + \theta \Pi_x(t < \zeta)$$

and

$$v_\theta(t) = -\int_0^t \widetilde{\Psi}(v_\theta(s))ds + \theta.$$

Applying Lemma 2.2 with $\widetilde{\Psi}(x,z) = \widetilde{\Psi}(z),\ x \in E_\partial, z > 0$ and $g(x) = \theta,\ x \in E_\partial$. we get that, for all $t > 0$, $x \in E$ and $\theta > 0$, $u_\theta(t,x) \le v_\theta(t)$. Letting $\theta \to \infty$, we get $-\log \mathbb{P}_{\delta_x}(\|X_t\| = 0) \le -\log \widetilde{\mathbb{P}}(\tilde{X}_t = 0)$. It is well known that, under the conditions of this lemma, $\widetilde{\mathbb{P}}(\tilde{X}_t = 0) > 0$. Thus $\| -\log q_t\|_\infty = \| -\log \mathbb{P}_{\delta_\cdot}(\|X_t\| = 0)\|_\infty < \infty$. □

Note that, when Ψ does not depend on the spatial variable x and satisfies the integral condition of $\int^\infty \frac{1}{\Psi(\lambda)} d\lambda < \infty$, Assumption 1.4 is a consequence of Assumption 1.2.

2.3 Excursion measures of $\{X_t, t \ge 0\}$

We use $\mathbb{D}$ to denote the space of $\mathcal{M}_F(E)$-valued cadlag functions on $[0,\infty)$. We use $\mathcal{F}$ to denote the σ-field generated by the sets $\{\omega \in \mathbb{D} : \omega_t(B) \le c\}$, where $B \in \mathcal{B}(E)$ and $c \in \mathbb{R}$. We assume X is canonical, that is, X is the coordinate map $X_t(\omega) = \omega_t$ on the measurable space $(\mathbb{D}, \mathcal{F})$.

It is known (see [15, Chapter 8]) that one can associate with $\{\mathbb{P}_{\delta_x} : x \in E\}$ a family of measures $\{\mathbb{N}_x : x \in E\}$, defined on the same measurable space as the probabilities $\{\mathbb{P}_{\delta_x} : x \in E\}$, such that

$$\mathbb{N}_x(1 - e^{-\langle f, X_t\rangle}) = -\log \mathbb{P}_{\delta_x}(e^{-\langle f, X_t\rangle}), \quad f \in \mathcal{B}_b^+(E),\ t \ge 0. \tag{2.8}$$

For earlier work on excursion measures of superprocesses, see [9, 10, 14].

Given X_t, let $N_t(d\omega, dx)$ be a Poisson random measure on $\mathbb{D} \times E$ with intensity $\mathbb{N}_x(d\omega)X_t(dx)$, in a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, P)$. Define

$$\Lambda_s^t := \int_E \int_{\mathbb{D}} \omega_s N_t(d\omega, dx).$$

Then, given X_t, the process $\{\Lambda_s^t, s \ge 0\}$ has the same law as $\{X_{t+s}, s \ge 0\}$. In fact, by (2.8) and the Markov property, we have, for $f \in \mathcal{B}_b^+(E)$,

$$\begin{aligned}
& \mathbb{P}_\mu \left[\exp\{-\langle f, X_{t+s}\rangle\} | X_t\right] = \mathbb{P}_{X_t}\left[\exp\{-\langle f, X_s\rangle\}\right] \\
= \; & \exp\left(\langle \log \mathbb{P}_{\delta_\cdot} \exp\{-\langle f, X_s\rangle\}, X_t\rangle\right) \\
= \; & \exp\left\{\int_E \int_{\mathbb{D}} (e^{-\langle f, \omega_s\rangle} - 1)\mathbb{N}_x(d\omega)X_t(dx)\right\}
\end{aligned}$$

$$
\begin{aligned}
&= P\left[\exp\left\{-\int_E\int_{\mathbb{D}}\langle f,\omega_s\rangle N_t(dx,d\omega)\right\}\right] \\
&= P\left[\exp\left\{-\langle f,\Lambda_s^t\rangle\right\}\right]. \qquad (2.9)
\end{aligned}
$$

Now we list some properties of $\mathbb{N}_x$. The proofs are similar to those of [9, Corollary 1.2, Proposition 1.1].

Proposition 2.4 *If* $\mathbb{P}_{\delta_x}|\langle f, X_t\rangle| < \infty$, *then*

$$\mathbb{N}_x\langle f, X_t\rangle = \mathbb{P}_{\delta_x}\langle f, X_t\rangle. \qquad (2.10)$$

If $\mathbb{P}_{\delta_x}\langle f, X_t\rangle^2 < \infty$, *then*

$$\mathbb{N}_x\langle f, X_t\rangle^2 = \mathbb{V}ar_{\delta_x}\langle f, X_t\rangle. \qquad (2.11)$$

2.4 Estimates for moments

In the remainder of this paper we will use the following notation: for two positive functions f and g on E, $f(x) \lesssim g(x)$ for $x \in E$ means that there exists a constant $c > 0$ such that $f(x) \le cg(x)$ for all $x \in E$. Throughout this paper, c is a constant whose value may varies from line to line.

In the following we will give an important lemma. The proof can be found in [13, Thorem 2.7].

Lemma 2.5 *There exist constants* $\gamma > 0$ *and* $c > 0$ *such that, for any* $(t, x, y) \in (1,\infty) \times E \times E$, *we have*

$$|q(t,x,y) - \phi_0(x)\psi_0(y)| \le ce^{-\gamma t}\phi_0(x)\psi_0(y). \qquad (2.12)$$

It follows that, if $f \in \mathcal{C}_1$, we have, for $(t,x) \in (1,\infty)\times E$,

$$|T_tf(x) - \langle f,\psi_0\rangle_m\phi_0(x)| \le ce^{-\gamma t}\langle |f|,\psi_0\rangle_m\phi_0(x) \qquad (2.13)$$

and

$$|T_tf(x)| \le (1+c)\langle |f|,\psi_0\rangle_m\phi_0(x). \qquad (2.14)$$

Hence, for $f \in \mathcal{C}_1$, T_tf is bounded and in $\mathcal{C}_1$. It follows from Proposition 2.4 that, for any $f \in \mathcal{C}_1$,

$$\int_E\int_D \langle |f|,\omega_s\rangle\mathbb{N}_x(d\omega)X_t(dx) < \infty, \quad \mathbb{P}_\mu\text{-a.s.}$$

Now applying (2.9), we get that for any $f \in \mathcal{C}_1$,

$$\mathbb{P}_\mu\left[\exp\left\{i\theta\langle f, X_{t+s}\rangle\right\}|X_t\right] = \exp\left\{\int_E\int_{\mathbb{D}}(e^{i\theta\langle f,\omega_s\rangle} - 1)\mathbb{N}_x(d\omega)X_t(dx)\right\}. \qquad (2.15)$$

Remark 2.6 *By Lemma 2.5, we get that*

$$q(t,x,y) \ge (1 - ce^{-\gamma t})\phi_0(x)\psi_0(y).$$

Since $q(t,x,\cdot) \in L^1(E,m)$, *we have* $\psi_0 \in L^1(E,m)$. *Thus* $\mathcal{B}_b(E) \subset \mathcal{C}_p$. *Moreover, by Hölder's inequality, we get* $\mathcal{C}_2 \subset \mathcal{C}_1$. □

Recall the second moments of the superprocess $\{X_t : t \geq 0\}$ (see, for example, [15, Corollary 2.39]): for $f \in \mathcal{B}_b(E)$, we have for any $t > 0$,

$$\mathbb{P}_\mu\langle f, X_t\rangle^2 = (\mathbb{P}_\mu\langle f, X_t\rangle)^2 + \int_E \int_0^t T_s[A(T_{t-s}f)^2](x)\, ds\mu(dx). \tag{2.16}$$

Thus,

$$\mathbb{V}\mathrm{ar}_\mu\langle f, X_t\rangle = \langle \mathbb{V}\mathrm{ar}_{\delta_\cdot}\langle f, X_t\rangle, \mu\rangle = \int_E \int_0^t T_s[A(T_{t-s}f)^2](x)\, ds\mu(dx), \tag{2.17}$$

where $\mathbb{V}\mathrm{ar}_\mu$ stands for the variance under $\mathbb{P}_\mu$. For any $f \in \mathcal{C}_2$ and $x \in E$, applying the Cauchy-Schwarz inequality, we have $(T_{t-s}f)^2(x) \leq e^{K(t-s)}T_{t-s}(f^2)(x)$, which implies that

$$\int_0^t T_s[A(T_{t-s}f)^2](x)\, ds \leq e^{Kt}T_t(f^2)(x) < \infty. \tag{2.18}$$

Thus, using a routine limit argument, one can easily check that (2.16) and (2.17) also hold for $f \in \mathcal{C}_2$.

Lemma 2.7 *Assume that $f \in \mathcal{C}_2$. If $\langle f, \psi_0\rangle_m = 0$, then, for $(t, x) \in (2, \infty) \times E$, we have*

$$\left|\mathbb{V}\mathrm{ar}_{\delta_x}\langle f, X_t\rangle - \sigma_f^2\phi_0(x)\right| \lesssim e^{-\gamma t}\phi_0(x), \tag{2.19}$$

where σ_f^2 is defined in (1.28)*. Therefore, for $(t, x) \in (2, \infty) \times E$, we have*

$$\mathbb{V}\mathrm{ar}_{\delta_x}\langle f, X_t\rangle \lesssim \phi_0(x). \tag{2.20}$$

Proof: First, we show that $\sigma_f^2 < \infty$. For $s \leq 1$, $|T_s f(x)|^2 \leq e^{Ks}T_s(f^2)(x)$. Hence, for $s \leq 1$,

$$\langle A(T_s f)^2, \psi_0\rangle \leq Ke^{sK}\langle T_s(f^2), \psi_0\rangle = Ke^{sK}\langle f^2, \psi_0\rangle. \tag{2.21}$$

For $s > 1$, by (2.13), $|T_s f(x)| \lesssim e^{-\gamma s}\langle |f|, \psi_0\rangle_m\phi_0(x)$. Hence, for $s > 1$,

$$\langle A(T_s f)^2, \psi_0\rangle \lesssim e^{-2\gamma s}. \tag{2.22}$$

Therefore,

$$\sigma_f^2 = \int_0^\infty \langle A(T_s f)^2, \psi_0\rangle_m\, ds \lesssim \int_0^1 e^{sK}\, ds + \int_1^\infty e^{-2\gamma s}\, ds < \infty.$$

By (2.17), for $t > 2$, we have

$$\begin{aligned}
& \left|\mathbb{V}\mathrm{ar}_{\delta_x}\langle f, X_t\rangle - \sigma_f^2\phi_0(x)\right| \\
\leq\ & \int_0^{t-1} \left|T_{t-s}[A(T_s f)^2](x) - \langle A(T_s f)^2, \psi_0\rangle_m\phi_0(x)\right|\, ds \\
& + \int_{t-1}^t T_{t-s}[A(T_s f)^2](x)\, ds + \int_{t-1}^\infty \langle A(T_s f)^2, \psi_0\rangle_m\, ds\phi_0(x) \\
=:\ & V_1(t, x) + V_2(t, x) + V_3(t, x).
\end{aligned} \tag{2.23}$$

First, we consider $V_1(t,x)$. By (2.13), for $t-s>1$, we have

$$\left|T_{t-s}[A(T_sf)^2](x) - \langle A(T_sf)^2, \psi_0\rangle_m \phi_0(x)\right| \lesssim e^{-\gamma(t-s)}\langle A(T_sf)^2, \psi_0\rangle_m \phi_0(x).$$

Therefore, by (2.21) and (2.22), we have, for $(t,x)\in(2,\infty)\times E$,

$$V_1(t,x) \lesssim \int_1^t e^{-\gamma(t+s)}\,ds\,\phi_0(x) + \int_0^1 e^{-\gamma(t-s)}\,ds\,\phi_0(x) \lesssim e^{-\gamma t}\phi_0(x). \tag{2.24}$$

For $V_2(t,x)$, by (2.13), for $s>t-1>1$, $|T_sf(x)|\lesssim e^{-\gamma s}\phi_0(x)$. Thus,

$$V_2(t,x) \lesssim \int_{t-1}^t e^{-2\gamma s} T_{t-s}[\phi_0^2](x)\,ds = e^{-2\gamma t}\int_0^1 e^{2\gamma s} T_s[\phi_0^2](x)\,ds. \tag{2.25}$$

By Hölder's inequality, we have

$$\phi_0^2(x) = (T_1\phi_0(x))^2 \le e^K T_1(\phi_0^2)(x).$$

Thus by (2.25) and (2.14), for $(t,x)\in(2,\infty)\times E$, we have

$$V_2(t,x) \lesssim e^{-2\gamma t}\int_0^1 T_{s+1}(\phi_0^2)(x)\,ds \lesssim e^{-2\gamma t}\phi_0(x). \tag{2.26}$$

For $V_3(t,x)$, by (2.13), for $s>t-1>1$, $|T_sf(x)|\lesssim e^{-\gamma s}\phi_0(x)$. Thus,

$$V_3(t,x) \lesssim \int_{t-1}^\infty e^{-2\gamma s}\,ds\langle \phi_0^2, \psi_0\rangle_m \phi_0(x) \lesssim e^{-2\gamma t}\phi_0(x). \tag{2.27}$$

It follows from (2.24), (2.26) and (2.27) that, for $(t,x)\in(2,\infty)\times E$,

$$\left|\mathbb{V}\mathrm{ar}_{\delta_x}\langle f, X_t\rangle - \sigma_f^2\phi_0(x)\right| \lesssim e^{-\gamma t}\phi_0(x).$$

Now (2.20) follows immediately. □

3 Proofs of Main Results

In this section, we will prove our main theorems.

3.1 Proof of Theorem 1.6

For $x\in E$ and $z>0$, define

$$r(x,z) = \Psi(x,z) + \alpha(x)z \tag{3.1}$$

and

$$r^{(2)}(x,z) = \Psi(x,z) + \alpha(x)z - \frac{1}{2}A(x)z^2. \tag{3.2}$$

Lemma 3.1 *For any $x \in E$ and $z > 0$,*

$$0 \le r(x,z) \le Kz^2/2 \tag{3.3}$$

and

$$|r^{(2)}(x,z)| \le e(x,z)z^2, \tag{3.4}$$

where

$$e(x,z) = \beta(x) \int_0^\infty y^2 \left(1 \wedge \frac{1}{6} yz\right) n(x,dy). \tag{3.5}$$

Proof: It is easy to see that

$$r(x,z) = \beta(x)\left(b(x)z^2 + \int_0^\infty (e^{-zy} - 1 + zy)\, n(x,dy)\right) \tag{3.6}$$

and

$$r^{(2)}(x,z) = \beta(x)\int_0^\infty \left(e^{-zy} - 1 + zy - \frac{1}{2}y^2z^2\right) n(x,dy).$$

It follows easily from Taylor's expansion that, for $\theta > 0$,

$$0 < e^{-\theta} - 1 + \theta \le \frac{1}{2}\theta^2 \tag{3.7}$$

and

$$\left|e^{-\theta} - 1 + \theta - \frac{1}{2}\theta^2\right| \le \frac{1}{6}\theta^3. \tag{3.8}$$

By (3.7), we also have $\left|e^{-\theta} - 1 + \theta - \frac{1}{2}\theta^2\right| \le \theta^2$. Thus, we have

$$\left|e^{-\theta} - 1 + \theta - \frac{1}{2}\theta^2\right| \le \theta^2\left(1 \wedge \frac{1}{6}\theta\right). \tag{3.9}$$

Therefore, by (3.7) and (3.9), we have

$$0 < r(x,z) \le \beta(x)\left(b(x) + \frac{1}{2}\int_0^\infty y^2\, n(x,dy)\right) z^2 \le Kz^2/2$$

and

$$r^{(2)}(x,z) \le \beta(x)\int_0^\infty y^2 \left(1 \wedge \frac{1}{6}yz\right) n(x,dy) z^2.$$

The proof is now complete. □

Recall that

$$u_f(t,x) := -\log \mathbb{P}_{\delta_x} e^{-\langle f, X_t\rangle}.$$

Lemma 3.2 *If $f \in \mathcal{C}_1^+$, then $0 \le u_f(t,x) < \infty$ for all $t \ge 0, x \in E$, and the function R_f defined by*

$$R_f(t,x) \quad := \quad T_t f(x) - u_f(t,x) \tag{3.10}$$

satisfies

$$R_f(t,x) = \int_0^t T_s\left[r(\cdot, u_f(t-s,\cdot))\right](x)\, ds, \quad t \ge 0, x \in E. \tag{3.11}$$

Moreover,

$$0 \le R_f(t,x) \le e^{Kt} T_t(f^2)(x), \quad t \ge 0, x \in E. \tag{3.12}$$

Proof: First, we assume that $f \in \mathcal{B}_b^+$. Recall that $u_f(t,x) = -\log \mathbb{P}_{\delta_x} e^{-\langle f, X_t\rangle}$ satisfies

$$u_f(t,x) + \Pi_x \int_0^t \Psi(\xi_s, u_f(t-s,\xi_s))\, ds = \Pi_x(f(\xi_t)), \quad t \geq 0, x \in E. \tag{3.13}$$

It follows from the proof of [15, Theorem 2.23] that $u_f(t,x)$ also satisfies

$$u_f(t,x) = -\int_0^t T_s\left[r(\cdot, u_f(t-s,\cdot))\right](x)\, ds + T_t f(x), \quad t \geq 0, x \in E. \tag{3.14}$$

Thus,

$$R_f(t,x) = \int_0^t T_s\left[r(\cdot, u_f(t-s,\cdot))\right](x)\, ds, \quad t \geq 0, x \in E. \tag{3.15}$$

For general $f \in \mathcal{C}_1^+$, we have $T_t f(x) < \infty$. Let $f_n(x) = f(x) \wedge n \in \mathcal{B}_b^+$. Since (3.15) holds for f_n, applying the monotone convergence theorem, we get that (3.15) also holds for f. Therefore, by (3.3), $R_f(t,x) \geq 0$, which means $u_f(t,x) \leq T_t f(x) < \infty$. Recall that, as a consequence of the Cauchy-Schwarz inequality, we have $(T_{t-s}f)^2(y) \leq e^{K(t-s)} T_{t-s}(f^2)(y)$. Combining this with (3.3), we get

$$0 \leq R_f(t,x) \leq \frac{K}{2}\int_0^t T_s[(u_f(t-s))^2](x)\, ds \leq \frac{K}{2}\int_0^t T_s[(T_{t-s}f)^2](x) ds \leq e^{Kt} T_t(f^2)(x). \tag{3.16}$$

□

Recall that $q_t(x) = \mathbb{P}_{\delta_x}(\|X_t\| = 0)$.

Lemma 3.3

$$\lim_{t\to\infty} \|-\log q_t\|_\infty = 0. \tag{3.17}$$

Proof: For $\theta > 0$, let

$$u_\theta(t,x) := -\log \mathbb{P}_{\delta_x} e^{-\langle \theta, X_t\rangle}.$$

By the Markov property of X,

$$q_{t+s}(x) = \lim_{\theta\to\infty} \mathbb{P}_{\delta_x}\left(e^{-\theta\|X_{t+s}\|}\right) = \lim_{\theta\to\infty} \mathbb{P}_{\delta_x}\left(e^{-\langle u_\theta(s)), X_t\rangle}\right) = \mathbb{P}_{\delta_x}\left(e^{-\langle -\log q_s, X_t\rangle}\right). \tag{3.18}$$

Since $q_t(x)$ is increasing in t, $q(x) := \lim_{t\to\infty} q_t(x)$ exists. Put $w(x) = -\log q(x)$. Letting $s \to \infty$ in (3.18), we get $q(x) = \mathbb{P}_{\delta_x}\left(e^{-\langle w, X_t\rangle}\right)$, which implies, for $t > 0$,

$$w(x) = u_w(t,x) \quad x \in E.$$

By Assumption 1.4, for $s > t_0$,

$$\|w\|_\infty \leq \|-\log q_s\|_\infty \leq \|-\log q_{t_0}\|_\infty = -\log\left(\inf_{x\in E} q_{t_0}(x)\right) < \infty,$$

which implies $w \in \mathcal{C}_1^+$, and $-\log q_s \in \mathcal{C}_1^+$. Thus, by Lemma 3.2, we have

$$w(x) = T_t(w)(x) - \int_0^t T_s(r(\cdot, w(\cdot)))(x)\, ds, \quad x \in E. \tag{3.19}$$

By (2.13), we have $\lim_{t\to\infty} T_t(w)(x) = \langle w, \psi_0\rangle_m \phi_0(x)$.

If $\langle r(\cdot, w(\cdot)), \psi_0\rangle_m > 0$, then

$$\lim_{t\to\infty} T_t(r(\cdot, w))(x) = \langle r(\cdot, w(\cdot)), \psi_0\rangle_m \phi_0(x) > 0, \quad \text{for any } x \in E,$$

which implies

$$\lim_{t\to\infty} \int_0^t T_s\left[r(\cdot, w(\cdot))\right](x)\, ds = \infty, \quad \text{for any } x \in E.$$

Thus, by (3.19), we get

$$0 \le w(x) = \lim_{t\to\infty} \left(T_t(w))(x) - \int_0^t T_s\left[r(\cdot, w(\cdot))\right](x)\, ds \right) = -\infty,$$

which is a contradiction. Therefore $r(x, w(x)) = 0$, a.e.-m. Then, by (3.19), we get, for all $x \in E$,

$$w(x) = \langle w, \psi_0\rangle_m \phi_0(x), \tag{3.20}$$

which implies that $w \equiv 0$ on E or $w(x) > 0$ for any $x \in E$. Since $r(x, w(x)) = 0$, a.e.-m., by (3.6), we obtain $w \equiv 0$ on E. For $s > t_0$, by (3.18) and Lemma 3.2, we get

$$-\log q_{2+s}(x) = u_{-\log q_s}(2, x) \le T_2(-\log q_s)(x) \le (1 + c)\langle -\log q_s, \psi_0\rangle_m \|\phi_0\|_\infty,$$

where in the last inequality we used (2.13). Since $-\log q_s(x) \to 0$, by the dominated convergence theorem, we get

$$\lim_{s\to\infty} \langle -\log q_s, \psi_0\rangle_m = 0.$$

Now (3.17) follows immediately. □

Lemma 3.4 *For any $f \in \mathcal{C}_1^+$, there exists a function $h_f(t, x)$ such that*

$$u_f(t, x) = (1 + h_f(t, x))\langle u_f(t, \cdot), \psi_0\rangle_m \phi_0(x). \tag{3.21}$$

Furthermore,

$$\lim_{t\to\infty} \|h_f(t)\|_\infty = 0 \quad \textit{uniformly in } f \in \mathcal{C}_1^+. \tag{3.22}$$

Proof: For any $f \in \mathcal{C}_1^+$, we have $u_f(t, x) \le T_t f(x) < \infty$ and $\langle u_f(t, \cdot), \psi_0\rangle_m \le \langle T_t f, \psi_0\rangle_m = \langle f, \psi_0\rangle_m < \infty$. So $u_f(t, x) \in \mathcal{C}_1^+$. If $m(f > 0) = 0$, then $T_t f(x) = 0$ for all $t > 0$ and $x \in E$, which implies $u_f(t, x) = 0$ and $\langle u_f(t, \cdot), \psi_0\rangle_m = 0$. In this case, we define $h_f(t, x) = 0$. If $m(f > 0) > 0$, then $T_t f(x) > 0$ for all $t > 0$ and $x \in E$, which implies $\mathbb{P}_{\delta_x}(\langle f, X_t\rangle = 0) < 1$. Thus we have $u_f(t, x) > 0$ and $\langle u_f(t, \cdot), \psi_0\rangle_m > 0$. Define

$$h_f(t, x) = \frac{u_f(t, x) - \langle u_f(t, \cdot), \psi_0\rangle_m \phi_0(x)}{\langle u_f(t, \cdot), \psi_0\rangle_m \phi_0(x)}.$$

We only need to prove that $\|h_f(t, \cdot)\|_\infty \to 0$ uniformly in $f \in \mathcal{C}_1^+ \setminus \{0\}$ as $t \to \infty$. It is easy to see that

$$u_f(t, x) \le -\log\left(\mathbb{P}_{\delta_x}(\|X_t\| = 0)\right) = -\log q_t(x),$$

which implies that

$$\|u_f(t,\cdot)\|_\infty \le \|-\log q_t\|_\infty \to 0 \quad \text{as } t\to\infty. \tag{3.23}$$

By the Markov property of X we have

$$u_f(t,x) = -\log \mathbb{P}_{\delta_x} e^{-\langle u_f(t-s,\cdot), X_s\rangle} = u_{u_f(t-s)}(s,x), \quad t\ge s>0, x\in E, \tag{3.24}$$

where in the subscript on the right-hand side, $u_f(t-s)$ stands for the function $x\to u_f(t-s,x)$. In the remainder of this proof, we keep this convention. By (3.10), we have

$$u_f(t,x) = T_s(u_f(t-s,\cdot))(x) - R_{u_f(t-s)}(s,x). \tag{3.25}$$

Thus,

$$\langle u_f(t,\cdot),\psi_0\rangle_m = \langle u_f(t-s,\cdot),\psi_0\rangle_m - \langle R_{u_f(t-s)}(s,\cdot),\psi_0\rangle_m. \tag{3.26}$$

Therefore, by (2.13) and (3.12), we have, for $1<s<t$ and $x\in E$,

$$\begin{aligned}
&|u_f(t,x) - \langle u_f(t,\cdot),\psi_0\rangle_m \phi_0(x)| \\
&\le |T_s(u_f(t-s,\cdot))(x) - \langle u_f(t-s,\cdot),\psi_0\rangle_m\phi_0(x)| + \left|R_{u_f(t-s)}(s,x)\right| + \left|\langle R_{u_f(t-s)}(s,\cdot),\psi_0\rangle_m\phi_0(x)\right| \\
&\lesssim e^{-\gamma s}\langle u_f(t-s,\cdot),\psi_0\rangle_m\phi_0(x) + e^{Ks}T_s(u_f(t-s,\cdot)^2)(x) + e^{Ks}\langle u_f(t-s,\cdot)^2,\psi_0\rangle_m\phi_0(x) \\
&\lesssim e^{-\gamma s}\langle u_f(t-s,\cdot),\psi_0\rangle_m\phi_0(x) + e^{Ks}\langle u_f(t-s,\cdot)^2,\psi_0\rangle_m\phi_0(x) \\
&\le \left[e^{-\gamma s} + e^{Ks}\|-\log q_{t-s}\|_\infty\right]\langle u_f(t-s,\cdot),\psi_0\rangle_m\phi_0(x),
\end{aligned}$$

where in the last inequality we used (3.23).

By Lemma 3.2 and (3.25), we get

$$\begin{aligned}
&T_s(u_f(t-s,\cdot))(x) \ge u_f(t,x) \ge T_s(u_f(t-s,\cdot))(x) - e^{Ks}T_s(u_f(t-s,\cdot)^2)(x) \\
&\ge T_s(u_f(t-s,\cdot))(x) - e^{Ks}\|-\log q_{t-s}\|_\infty T_s(u_f(t-s,\cdot))(x).
\end{aligned} \tag{3.27}$$

Thus, we have

$$\langle u_f(t-s,\cdot),\psi_0\rangle_m \ge \langle u_f(t,\cdot),\psi_0\rangle_m \ge (1-e^{Ks}\|-\log q_{t-s}\|_\infty)\langle u_f(t-s,\cdot),\psi_0\rangle_m. \tag{3.28}$$

For any $s>1$, $(1-e^{Ks}\|-\log q_{t-s}\|_\infty)>0$ when t is large enough. Therefore, as $t\to\infty$,

$$\|h_f(t,\cdot)\|_\infty \lesssim \frac{e^{-\gamma s}+e^{Ks}\|-\log q_{t-s}\|_\infty}{1-e^{Ks}\|-\log q_{t-s}\|_\infty} \to e^{-\gamma s}. \tag{3.29}$$

Now, letting $s\to\infty$, we get $\|h_f(t,\cdot)\|_\infty \to 0$ uniformly in $f\in\mathcal{C}_1^+\setminus\{0\}$ as $t\to\infty$. □

Lemma 3.5 *For any $\delta>0$,*

$$\lim_{n\to\infty}\frac{1}{n\delta}\left(\frac{1}{\langle u_f(n\delta),\psi_0\rangle_m} - \frac{1}{\langle f,\psi_0\rangle_m}\right) = \nu \tag{3.30}$$

uniformly in $f\in\mathcal{C}_1^+\setminus\{0\}$. Here ν is defined in (1.24).

Proof: We write $u_f(t,x)$ as $u_t(x)$, $x \in E$. Since f is non-negative and $m(f>0)>0$, we have $u_f(t,x)>0$ for all $t>0$ and $x\in E$. Consequently, we have $\langle u_f(t,\cdot),\psi_0\rangle_m>0$. It is clear that $u_0=f$. First note that

$$
\begin{aligned}
&\frac{1}{n\delta}\left(\frac{1}{\langle u_{n\delta},\psi_0\rangle_m}-\frac{1}{\langle f,\psi_0\rangle_m}\right)\\
=\;&\frac{1}{n\delta}\sum_{k=0}^{n-1}\left(\frac{1}{\langle u_{(k+1)\delta},\psi_0\rangle_m}-\frac{1}{\langle u_{k\delta},\psi_0\rangle_m}\right)\\
=\;&\frac{1}{n\delta}\sum_{k=0}^{n-1}\left(\frac{\langle u_{k\delta},\psi_0\rangle_m-\langle u_{(k+1)\delta},\psi_0\rangle_m}{\langle u_{(k+1)\delta},\psi_0\rangle_m\langle u_{k\delta},\psi_0\rangle_m}\right).
\end{aligned}
$$

Recall the identity (3.24) and the definition of $r^{(2)}(x,z)$ given in (3.2). Using (3.26) with $t=(k+1)\delta$ and $s=\delta$, we get

$$
\begin{aligned}
&\langle u_{k\delta},\psi_0\rangle_m-\langle u_{(k+1)\delta},\psi_0\rangle_m=\langle R_{u_{k\delta}}(\delta,\cdot),\psi_0\rangle_m\\
&=\int_0^\delta\langle r(\cdot,u_f(k\delta+s,\cdot)),\psi_0\rangle_m\,ds\\
&=\frac12\int_0^\delta\langle A(u_{k\delta+s})^2,\psi_0\rangle_m\,ds+\int_0^\delta\langle r^{(2)}(\cdot,u_{k\delta+s}(\cdot)),\psi_0\rangle_m\,ds\\
&=: I_1+I_2.
\end{aligned}
$$

By (3.21) and (3.28), we have, for $s\in[0,\delta]$,

$$
\begin{aligned}
|u_{t+s}(x)-\langle u_t,\psi_0\rangle_m\phi_0(x)| &\le |u_{t+s}(x)-\langle u_{t+s},\psi_0\rangle_m\phi_0(x)|+|\langle u_t,\psi_0\rangle_m-\langle u_{t+s},\psi_0\rangle_m|\phi_0(x)\\
&\le \|h_f(t+s)\|_\infty\langle u_{t+s},\psi_0\rangle_m\phi_0(x)+e^{Ks}\|-\log q_t\|_\infty\langle u_t,\psi_0\rangle_m\phi_0(x)\\
&\le \left(\|h_f(t+s)\|_\infty+e^{Ks}\|-q_t(x)\|_\infty\right)\langle u_t,\psi_0\rangle_m\phi_0(x)\\
&\le c_f(t)\langle u_t,\psi_0\rangle_m\phi_0(x), \qquad (3.31)
\end{aligned}
$$

where $c_f(t)=\sup_{0\le s\le\delta}\left(\|h_f(t+s)\|_\infty+e^{Ks}\|-q_t(x)\|_\infty\right)$. It is easy to see that $c_f(t)\to0$, as $t\to\infty$, uniformly in $f\in\mathcal{C}_1^+$. Thus, by (3.21) we have for $s\in[0,\delta]$,

$$
\frac{|u_{t+s}(x)^2-\langle u_t,\psi_0\rangle_m^2(\phi_0(x))^2|}{\langle u_t,\psi_0\rangle_m^2}\le(2+c_f(t))\,c_f(t)(\phi_0(x))^2. \qquad (3.32)
$$

Therefore, we have,

$$
\begin{aligned}
\left|\frac{I_1}{\langle u_{k\delta},\psi_0\rangle_m^2}-\delta\nu\right| &= \frac{\left|\int_0^\delta\langle A\left((u_{k\delta+s})^2-\langle u_{k\delta},\psi_0\rangle_m^2\phi_0^2\right),\psi_0\rangle_m\,ds\right|}{2\langle u_{k\delta},\psi_0\rangle_m^2}\\
&\le \frac12\langle A\phi_0^2,\psi_0\rangle_m\delta\,(2+c_f(k\delta))\,c_f(k\delta)\to0,\quad\text{as}\quad k\to\infty,
\end{aligned}
$$

uniformly in $f\in\mathcal{C}_1^+\setminus\{0\}$. By (3.28), we have

$$
0\le 1-\frac{\langle u_{(k+1)\delta},\psi_0\rangle_m}{\langle u_{k\delta},\psi_0\rangle_m}\le e^{K\delta}\|-\log q_{k\delta}\|_\infty, \qquad (3.33)
$$

which implies that

$$\frac{\langle u_{k\delta}, \psi_0\rangle_m}{\langle u_{(k+1)\delta}, \psi_0\rangle_m} \to 1, \quad \text{as } k \to \infty, \tag{3.34}$$

uniformly in $f \in \mathcal{C}_1^+ \setminus \{0\}$. It follows that

$$\lim_{k\to\infty} \frac{I_1}{\langle u_{k\delta}, \psi_0\rangle_m \langle u_{(k+1)\delta}, \psi_0\rangle_m} = \delta\nu \tag{3.35}$$

uniformly in $f \in \mathcal{C}_1^+ \setminus \{0\}$.

For I_2, by (3.4) and (3.31), we have

$$\begin{aligned}
\frac{\langle r^{(2)}(\cdot, u_{k\delta+s}(\cdot)), \psi_0\rangle_m}{\langle u_{k\delta}, \psi_0\rangle_m^2} &\le \frac{\langle e(\cdot, u_{k\delta+s}(\cdot))u_{k\delta+u}^2, \psi_0\rangle_m}{\langle u_{k\delta}, \psi_0\rangle_m^2} \\
&\le (1 + c_f(k\delta))^2 \langle e(\cdot, u_{k\delta+s}(\cdot))\phi_0^2, \psi_0\rangle_m \\
&\le (1 + c_f(k\delta))^2 \langle e(\cdot, \| - \log q_{k\delta}\|_\infty)\phi_0^2, \psi_0\rangle_m,
\end{aligned}$$

here the last inequality follows from $\|u_{k\delta+u}\|_\infty \le \| - \log q_{k\delta+u}\|_\infty \le \| - \log q_{k\delta}\|_\infty$ and the fact $z \to e(x, z)$ is increasing. It is easy to see that the function $e(x, z) \downarrow 0$ as $z \downarrow 0$. Thus, as $k \to \infty$,

$$\frac{I_2}{\langle u_{k\delta}, \psi_0\rangle_m^2} \le \delta(1 + c_f(k\delta))^2 \langle e(\cdot, \| - \log q_{k\delta}\|_\infty)\phi_0^2, \psi_0\rangle_m \to 0$$

uniformly in $f \in \mathcal{C}_1^+ \setminus \{0\}$. By (3.34), we have

$$\lim_{k\to\infty} \frac{I_2}{\langle u_{k\delta}, \psi_0\rangle_m \langle u_{(k+1)\delta}, \psi_0\rangle_m} = 0 \tag{3.36}$$

uniformly in $f \in \mathcal{C}_1^+ \setminus \{0\}$. Using (3.35) and (3.36), we get,

$$\lim_{k\to\infty} \frac{\langle u_{k\delta}, \psi_0\rangle_m - \langle u_{(k+1)\delta}, \psi_0\rangle_m}{\langle u_{(k+1)\delta}, \psi_0\rangle_m \langle u_{k\delta}, \psi_0\rangle_m} = \delta\nu$$

uniformly in $f \in \mathcal{C}_1^+ \setminus \{0\}$. Now, (3.30) follows immediately. □

Proof of Theorem 1.5: For $t > 0$, we have

$$\mathbb{P}_\mu(\|X_t\| \ne 0) = \lim_{\theta\to\infty} 1 - \exp\{-\langle u_\theta(t), \mu\rangle\}. \tag{3.37}$$

By Lemma 3.5, we have

$$\lim_{n\to\infty} \frac{1}{n\delta} \left(\frac{1}{\langle u_\theta(n\delta), \psi_0\rangle_m} - \frac{1}{\theta\langle 1, \psi_0\rangle_m} \right) = \nu \tag{3.38}$$

uniformly in $\theta > 0$. For $\theta > 1$, it holds that

$$\frac{1}{n\delta} \frac{1}{\theta\langle 1, \psi_0\rangle_m} \le \frac{1}{n\delta} \frac{1}{\langle 1, \psi_0\rangle_m} \to 0, \quad \text{as } n \to \infty, \tag{3.39}$$

uniformly in $\theta > 1$. It follows from (3.38) and (3.39) that

$$\lim_{n\to\infty} n\delta\langle u_\theta(n\delta), \psi_0\rangle_m = \nu^{-1} \tag{3.40}$$

uniformly in $\theta > 1$. By (3.21) and (3.22), we have, as $n \to \infty$,

$$(n\delta)|u_\theta(n\delta, x) - \langle u_\theta(n\delta), \psi_0\rangle_m \phi_0(x)| \le \|h_\theta(n\delta)\|_\infty\, (n\delta)\, \langle u_\theta(n\delta), \psi_0\rangle_m \|\phi_0\|_\infty \to 0$$

uniformly in $\theta > 1$ and $x \in E$. Thus, for any $\mu \in \mathcal{M}_F(E)$,

$$\lim_{n\to\infty} (n\delta)\langle u_\theta(n\delta), \mu\rangle = \nu^{-1}\langle \phi_0, \mu\rangle \quad \text{uniformly in } \theta > 1. \tag{3.41}$$

By (3.23), we have $\langle u_\theta(n\delta), \mu\rangle \le \langle -\log q_{n\delta}, \mu\rangle \le \| -\log q_{n\delta}\|_\infty \|\mu\| \to 0$, as $n \to \infty$, uniformly in $\theta > 0$. Thus,

$$\lim_{n\to\infty} \frac{1 - \exp\{-\langle u_\theta(n\delta), \mu\rangle\}}{\langle u_\theta(n\delta), \mu\rangle} = 1$$

uniformly in $\theta > 0$. Therefore, it follows from (3.41) that

$$\lim_{n\to\infty} n\delta\, (1 - \exp\{-\langle u_\theta(n\delta), \mu\rangle\}) = \nu^{-1}\langle \phi_0, \mu\rangle \quad \text{uniformly in } \theta > 1.$$

Hence by (3.37), we have

$$\lim_{n\to\infty} (n\delta)\mathbb{P}_\mu\left(\|X_{n\delta}\| \ne 0\right) = \nu^{-1}\langle \phi_0, \mu\rangle. \tag{3.42}$$

Since $\mathbb{P}_\mu\left(\|X_t\| \ne 0\right)$ is decreasing in t, we have for $n\delta \le t < (n+1)\delta$,

$$n\delta\, \mathbb{P}_\mu\left(\|X_{(n+1)\delta}\| \ne 0\right) \le t\, \mathbb{P}_\mu\left(\|X_t\| \ne 0\right) \le (n+1)\delta\, \mathbb{P}_\mu\left(\|X_{n\delta}\| \ne 0\right).$$

Now (1.25) follows easily. □

Now we are ready to prove Theorem 1.6.

Proof of Theorem 1.6: First, we consider the special case when $f(x) = \phi_0(x)$. We only need to show that, for any $\lambda > 0$,

$$\mathbb{P}_\mu\left(\exp\left\{-\lambda t^{-1}\langle \phi_0, X_t\rangle\right\} \mid \|X_t\| \ne 0\right) \to \frac{1}{\lambda\nu + 1}, \qquad \text{as } t \to \infty. \tag{3.43}$$

Note that

$$\begin{aligned}
&\mathbb{P}_\mu\left(\exp\left\{-\lambda t^{-1}\langle \phi_0, X_t\rangle\right\} \mid \|X_t\| \ne 0\right)\\
&= \frac{\mathbb{P}_\mu\left(\exp\left\{-\lambda t^{-1}\langle \phi_0, X_t\rangle\right\}\right) - \mathbb{P}_\mu(\|X_t\| = 0)}{\mathbb{P}_\mu(\|X_t\| \ne 0)}\\
&= 1 - \frac{1 - \mathbb{P}_\mu\left(\exp\left\{-\lambda t^{-1}\langle \phi_0, X_t\rangle\right\}\right)}{\mathbb{P}_\mu(\|X_t\| \ne 0)}.
\end{aligned}$$

By Lemma 1.5, to prove (3.43), it suffices to show that, as $t \to \infty$,

$$t\left(1 - \mathbb{P}_\mu\left(\exp\left\{-\lambda t^{-1}\langle \phi_0, X_t\rangle\right\}\right)\right) = t\left(1 - \exp\left\{-\langle u_{\lambda t^{-1}\phi_0}(t), \mu\rangle\right\}\right) \to \frac{\lambda}{\lambda\nu + 1}\langle \phi_0, \mu\rangle. \tag{3.44}$$

Since $t \to \mathbb{P}_\mu\left(\exp\left\{-\lambda t^{-1}\langle \phi_0, X_t\rangle\right\}\right)$ is a right continuous function, by the Croft-Kingman lemma (see, for example, [2, Section 6.5]), it suffices to show that, for every $\delta > 0$, (3.44) holds for every sequence $n\delta$ as $n \to \infty$. For this, it is enough to prove that for any $\delta > 0$, as $n \to \infty$,

$$n\delta\, \langle u_{\lambda(n\delta)^{-1}\phi_0}(n\delta), \mu\rangle \to \frac{\lambda}{\lambda\nu + 1}\langle \phi_0, \mu\rangle. \tag{3.45}$$

By Lemma 3.5, we have

$$\begin{aligned}
& \lim_{n\to\infty} \frac{1}{(n\delta)\langle u_{\lambda(n\delta)^{-1}\phi_0}(n\delta), \psi_0\rangle_m} \\
= & \lim_{n\to\infty} \frac{1}{n\delta}\left(\frac{1}{\langle u_{\lambda(n\delta)^{-1}\phi_0}(n\delta), \psi_0\rangle_m} - \frac{1}{\langle \lambda(n\delta)^{-1}\phi_0, \psi_0\rangle_m}\right) + \frac{1}{\lambda} \\
= & \ \nu + \lambda^{-1},
\end{aligned}$$

which implies that

$$(n\delta)\langle u_{\lambda(n\delta)^{-1}\phi_0}(n\delta), \psi_0\rangle_m \to \frac{\lambda}{\lambda\nu + 1}, \quad \text{as } n\to\infty. \tag{3.46}$$

Using Lemma 3.4 and (3.46), we get that, as $n\to\infty$,

$$\begin{aligned}
& n\delta \left| \langle u_{\lambda(n\delta)^{-1}\phi_0}(n\delta), \mu\rangle - \langle u_{\lambda(n\delta)^{-1}\phi_0}(n\delta), \psi_0\rangle_m \langle \phi_0, \mu\rangle \right| \\
\leq & \ n\delta \|h_{\lambda(n\delta)^{-1}\phi_0}(n\delta)\|_\infty \langle u_{\lambda(n\delta)^{-1}\phi_0}(n\delta), \psi_0\rangle_m \langle \phi_0, \mu\rangle \to 0.
\end{aligned} \tag{3.47}$$

Now (3.45) follows easily from (3.46) and (3.47).

For a general f, let

$$\tilde{f}(x) = f(x) - \langle f, \psi_0\rangle_m \phi_0(x). \tag{3.48}$$

Then, $\langle \tilde{f}, \psi_0\rangle_m = 0$. It is clear that

$$\mathbb{P}_\mu\left(\left(t^{-1}\langle \tilde{f}, X_t\rangle\right)^2 \mid \|X_t\| \neq 0\right) = \frac{\mathbb{P}_\mu\left(\langle \tilde{f}, X_t\rangle\right)^2}{t^2 \mathbb{P}_\mu(\|X_t\| \neq 0)}. \tag{3.49}$$

By the branching property and (2.20), we have,

$$\sup_{t>2} \mathbb{V}\mathrm{ar}_\mu \langle \tilde{f}, X_t\rangle = \sup_{t>2} \langle \mathbb{V}\mathrm{ar}_{\delta_\cdot} \langle \tilde{f}, X_t\rangle, \mu\rangle < \infty.$$

It follows from (2.14) that

$$\sup_{t>1} \left| \mathbb{P}_\mu \langle \tilde{f}, X_t\rangle \right| = \sup_{t>1} \left| \langle T_t \tilde{f}, \mu\rangle \right| < \infty.$$

Combining the last two displays, we get that $\sup_{t>2} \mathbb{P}_\mu\left(\langle \tilde{f}, X_t\rangle\right)^2 < \infty$. Thus by (1.25) and (3.49), we get that as $t\to\infty$,

$$\mathbb{P}_\mu\left(\left(t^{-1}\langle \tilde{f}, X_t\rangle\right)^2 \mid \|X_t\| \neq 0\right) \to 0, \quad \text{as } t\to\infty,$$

which implies that, as $t\to\infty$,

$$t^{-1}\langle \tilde{f}, X_t\rangle|_{\mathbb{P}_{t,\mu}} \to 0, \text{ in probability.} \tag{3.50}$$

Thus, by (3.48), we have

$$t^{-1}\langle f, X_t\rangle|_{\mathbb{P}_{t,\mu}} \xrightarrow{d} \langle f, \psi_0\rangle_m W.$$

□

Corollary 3.6 *For any $f \in \mathcal{C}_2$, it holds that, as $t \to \infty$,*

$$\frac{\langle f, X_t \rangle}{\langle \phi_0, X_t \rangle}\Big|_{\mathbb{P}_{t,\mu}} \to \langle f, \psi_0 \rangle_m \quad \textit{in probability.} \tag{3.51}$$

Proof: Recall that $\tilde{f}$ was defined in (3.48). Thus

$$\frac{\langle f, X_t \rangle}{\langle \phi_0, X_t \rangle} - \langle f, \psi_0 \rangle_m = \frac{\langle \tilde{f}, X_t \rangle}{\langle \phi_0, X_t \rangle}.$$

For any $\epsilon > 0$ and $\delta > 0$, by (3.50) and (1.26), we have,

$$\begin{aligned}
&\mathbb{P}_\mu \left(\frac{|\langle \tilde{f}, X_t \rangle|}{\langle \phi_0, X_t \rangle} > \epsilon \mid \|X_t\| \neq 0 \right) \\
&\leq \mathbb{P}_\mu \left(t^{-1} |\langle \tilde{f}, X_t \rangle| > \delta \mid \|X_t\| \neq 0 \right) + \mathbb{P}_\mu \left(t^{-1} \langle \phi_0, X_t \rangle < \delta/\epsilon \mid \|X_t\| \neq 0 \right) \\
&\to 0 + P\left(W < \delta/\epsilon \right), \quad \text{as } t \to \infty.
\end{aligned}$$

Letting $\delta \to 0$, we get that

$$\lim_{t \to \infty} \mathbb{P}_{t,\mu} \left(\frac{|\langle \tilde{f}, X_t \rangle|}{\langle \phi_0, X_t \rangle} > \epsilon \right) = 0.$$

Now, (3.51) follows immediately. □

3.2 Proof of Theorem 1.8

In this subsection, we give the proof of Theorem 1.8. We prove a simple lemma first.

Lemma 3.7 *Suppose that $\{F_t : t > 0\}$ is a family of bounded random variables, that is, there is a constant M such that $|F_t| \leq M$ for all $t > 0$, then any $s > 0$,*

$$\lim_{t \to \infty} |\mathbb{P}_{t+s,\mu}(F_{t+s}) - \mathbb{P}_{t,\mu}(F_{t+s})| = 0. \tag{3.52}$$

Proof: By Lemma 1.5, we have

$$\lim_{t \to \infty} \frac{\mathbb{P}_\mu(\|X_t\| \neq 0)}{\mathbb{P}_\mu(\|X_{t+s}\| \neq 0)} = 1. \tag{3.53}$$

By the definition of $\mathbb{P}_{t,\mu}$, we have

$$\begin{aligned}
\mathbb{P}_{t+s,\mu}(F_{t+s}) &= \mathbb{P}_{t,\mu}(F_{t+s}, \ \|X_{t+s}\| \neq 0) \frac{\mathbb{P}_\mu(\|X_t\| \neq 0)}{\mathbb{P}_\mu(\|X_{t+s}\| \neq 0)} \\
&= \mathbb{P}_{t,\mu}(F_{t+s}) \frac{\mathbb{P}_\mu(\|X_t\| \neq 0)}{\mathbb{P}_\mu(\|X_{t+s}\| \neq 0)} - \mathbb{P}_{t,\mu}(F_{t+s}, \|X_{t+s}\| = 0) \frac{\mathbb{P}_\mu(\|X_t\| \neq 0)}{\mathbb{P}_\mu(\|X_{t+s}\| \neq 0)}.
\end{aligned}$$

Thus, as $t \to \infty$,

$$|\mathbb{P}_{t+s,\mu}(F_{t+s}) - \mathbb{P}_{t,\mu}(F_{t+s})| \leq M \left| \frac{\mathbb{P}_\mu(\|X_t\| \neq 0)}{\mathbb{P}_\mu(\|X_{t+s}\| \neq 0)} - 1 \right| + M \mathbb{P}_{t,\mu}(\|X_{t+s}\| = 0) \frac{\mathbb{P}_\mu(\|X_t\| \neq 0)}{\mathbb{P}_\mu(\|X_{t+s}\| \neq 0)}$$

$$= \quad 2M\left|\frac{\mathbb{P}_\mu(\|X_t\|\neq 0)}{\mathbb{P}_\mu(\|X_{t+s}\|\neq 0)}-1\right|\to 0.$$

□

We now recall some facts about weak convergence which will be used later. For $f:\mathbb{R}^d\to\mathbb{R}$, let $\|f\|_L := \sup_{x\neq y}|f(x)-f(y)|/\|x-y\|$ and $\|f\|_{BL} := \|f\|_\infty + \|f\|_L$. For any distributions ν_1 and ν_2 on $\mathbb{R}^d$, define

$$\beta(\nu_1,\nu_2) := \sup\left\{\left|\int f\,d\nu_1 - \int f\,d\nu_2\right| \;:\; \|f\|_{BL}\leq 1\right\}.$$

Then β is a metric. It follows from [7, Theorem 11.3.3] that the topology generated by β is equivalent to the weak convergence topology. From the definition, we can easily see that, if ν_1 and ν_2 are the distributions of two $\mathbb{R}^d$-valued random variables X and Y respectively, then

$$\beta(\nu_1,\nu_2)\leq E\|X-Y\|\leq\sqrt{E\|X-Y\|^2}. \tag{3.54}$$

The following simple fact will be used several times later in this section:

$$\left|e^{ix}-\sum_{m=0}^{n}\frac{(ix)^m}{m!}\right|\leq\min\left(\frac{|x|^{n+1}}{(n+1)!},\frac{2|x|^n}{n!}\right). \tag{3.55}$$

Now we are ready to prove Theorem 1.8.

Proof of Theorem 1.8: Define an $\mathbb{R}^2$-valued random variable:

$$U_1(t) := \left(t^{-1}\langle\phi_0,X_t\rangle, t^{-1/2}\langle f,X_t\rangle\right).$$

For $s,t>2$ we have

$$U_1(s+t) = \left((t+s)^{-1}\langle\phi_0,X_{t+s}\rangle,(t+s)^{-1/2}\langle f,X_{s+t}\rangle\right).$$

First, we consider another $\mathbb{R}^2-$valued random variable $U_2(s,t)$ defined by

$$U_2(s,t) = \left(t^{-1}\langle\phi_0,X_t\rangle, t^{-1/2}\left(\langle f,X_{s+t}\rangle-\langle T_sf,X_t\rangle\right)\right).$$

We claim that,

$$U_2(s,t)|_{\mathbb{P}_{t,\mu}}\xrightarrow{d}\left(W,\sqrt{W}G_1(s)\right),\qquad\text{as } t\to\infty, \tag{3.56}$$

where $G_1(s)\sim\mathcal{N}(0,\sigma_f^2(s))$ with $\sigma_f^2(s)=\langle\mathbb{V}\mathrm{ar}_{\delta_\cdot}\langle f,X_s\rangle,\psi_0\rangle_m$ and W is the random variable defined in Theorem 1.6. Denote the characteristic function of $U_2(s,t)$ under $\mathbb{P}_{t,\mu}$ by $\kappa_1(\theta_1,\theta_2,s,t)$:

$$\begin{aligned}
&\kappa_1(\theta_1,\theta_2,s,t)\\
=\;&\mathbb{P}_{t,\mu}(\exp\{i\theta_1t^{-1}\langle\phi_0,X_t\rangle+i\theta_2t^{-1/2}\left(\langle f,X_{s+t}\rangle-\langle T_sf,X_t\rangle\right)\}\\
=\;&\mathbb{P}_{t,\mu}\Big(\exp\Big\{i\theta_1t^{-1}\langle\phi_0,X_t\rangle\\
&+\int_E\int_{\mathbb{D}}\left(e^{i\theta_2t^{-1/2}\langle f,\omega_s\rangle}-1-i\theta_2t^{-1/2}\langle f,\omega_s\rangle\right)\mathbb{N}_x(d\omega)X_t(dx)\Big\}\Big),
\end{aligned} \tag{3.57}$$

where in the last equality we used the Markov property of X, (2.15) and (2.10). Define

$$J_s(\theta,x) := \int_{\mathbb{D}} \left(\exp\{\langle i\theta f, \omega_s\rangle\} - 1 - i\theta\langle f,\omega_s\rangle\right)\mathbb{N}_x(d\omega)$$

and

$$I_s(\theta,x) := \int_{\mathbb{D}} \left(\exp\{\langle i\theta f, \omega_s\rangle\} - 1 - i\theta\langle f,\omega_s\rangle + \frac{1}{2}\theta^2\langle f,\omega_s\rangle^2\right)\mathbb{N}_x(d\omega).$$

Let $V_s(x) = \mathbb{V}\mathrm{ar}_{\delta_x}\langle f, X_s\rangle \in \mathcal{C}_2^+$. Then, by (2.11), we have

$$\begin{aligned} J_s(\theta,x) &= -\frac{1}{2}\theta^2 V_s(x) + I_s(\theta,x) \\ &= -\frac{1}{2}\theta^2\langle V_s,\psi_0\rangle_m\phi_0(x) - \frac{1}{2}\theta^2\widetilde{V}_s(x) + I_s(\theta,x), \end{aligned}$$

where $\widetilde{V}_s = V_s - \langle V_s,\psi_0\rangle_m\phi_0(x) \in \mathcal{C}_2$. Thus, we have

$$\begin{aligned} & i\theta_1 t^{-1}\langle\phi_0, X_t\rangle + \langle J_s(t^{-1/2}\theta_2,\cdot), X_t\rangle \\ = & \left(i\theta_1 - \frac{1}{2}\theta_2^2\langle V_s,\psi_0\rangle_m\right)t^{-1}\langle\phi_0,X_t\rangle - \frac{1}{2}\theta_2^2 t^{-1}\langle\widetilde{V}_s, X_t\rangle + \langle I_s(t^{-1/2}\theta_2,\cdot),X_t\rangle. \end{aligned} \tag{3.58}$$

By (3.50), we know that, as $t\to\infty$,

$$t^{-1}\langle\widetilde{V}_s, X_t\rangle|_{\mathbb{P}_{t,\mu}} \to 0 \quad \text{in probability.} \tag{3.59}$$

By (3.55), we have

$$\left|I_s(t^{-1/2}\theta_2,x)\right| \le \theta_2^2 t^{-1}\mathbb{N}_x\left(\langle f,\omega_s\rangle^2\left(\frac{t^{-1/2}\theta_2\langle f,\omega_s\rangle}{6}\wedge 1\right)\right). \tag{3.60}$$

Let

$$h(x,s,t) = \mathbb{N}_x\left(\langle f,\omega_s\rangle^2\left(\frac{t^{-1/2}\theta_2\langle f,\omega_s\rangle}{6}\wedge 1\right)\right).$$

We note that $h(x,s,t)\downarrow 0$ as $t\uparrow\infty$. By (2.20), we have

$$h(x,s,t) \le \mathbb{N}_x(\langle f, X_s\rangle^2) = \mathbb{V}\mathrm{ar}_{\delta_x}\langle f,X_s\rangle \lesssim \phi_0(x)\in\mathcal{C}_2.$$

Thus, by (1.25) and (2.13), we have, for any $u<t$,

$$t^{-1}\mathbb{P}_{t,\mu}\langle h(\cdot,s,t),X_t\rangle \le t^{-1}\mathbb{P}_{t,\mu}\langle h(\cdot,s,u),X_t\rangle = \frac{\mathbb{P}_\mu\langle h(\cdot,s,u),X_t\rangle}{t\mathbb{P}_\mu(\|X_t\|\neq 0)} \to \nu\langle h(\cdot,s,u),\psi_0\rangle_m,$$

as $t\to\infty$. Letting $u\to\infty$, we get $\langle h(\cdot,s,u),\psi_0\rangle_m\to 0$. Thus, by (3.60), we get that

$$\lim_{t\to\infty}\mathbb{P}_{t,\mu}|\langle I_s(t^{-1/2}\theta_2,\cdot),X_t\rangle| = 0,$$

which implies that, as $t\to\infty$,

$$\langle I_s(t^{-1/2}\theta_2,\cdot),X_t\rangle|_{\mathbb{P}_{t,\mu}} \to 0 \quad \text{in probability.} \tag{3.61}$$

Thus, by (3.59), (3.61) and (3.58), we get

$$i\theta_1 t^{-1}\langle \phi_0, X_t\rangle + \langle J_s(t^{-1/2}\theta_2,\cdot), X_t\rangle|_{\mathbb{P}_{t,\mu}} \stackrel{d}{\to} \left(i\theta_1 - \frac{1}{2}\theta_2^2\langle V_s,\psi_0\rangle_m\right)W.$$

Since the real part of $J_s(t^{-1/2}\theta_2, x)$ is non-positive, we have

$$|\exp\{i\theta_1 t^{-1}\langle \phi_0, X_t\rangle + \langle J_s(t^{-1/2}\theta_2,\cdot), X_t\rangle\}| \le 1.$$

Therefore, by (3.57) and the dominated convergence theorem, we get

$$\lim_{t\to\infty}\kappa_1(\theta_1,\theta_2,s,t) = P\left(\exp\left\{\left(i\theta_1 - \frac{1}{2}\theta_2^2\langle V_s,\psi_0\rangle_m\right)W\right\}\right),$$

which implies our claim (3.56).

By (3.56), we have

$$U_3(s,t) := \Big((t+s)^{-1}\langle \phi_0, X_t\rangle, (t+s)^{-1/2}\left(\langle f, X_{s+t}\rangle - \langle T_s f, X_t\rangle\right)\Big)|_{\mathbb{P}_{t,\mu}} \stackrel{d}{\to} \left(W, \sqrt{W}G_1(s)\right), \quad (3.62)$$

as $t\to\infty$. It follows from (2.13) and (1.25) that, as $t\to\infty$,

$$(t+s)^{-2}\mathbb{P}_{t,\mu}\left(\langle \phi_0, X_{t+s}\rangle - \langle \phi_0, X_t\rangle\right)^2 = \frac{\mathbb{P}_\mu\left(\langle \phi_0, X_{t+s}\rangle - \langle \phi_0, X_t\rangle\right)^2}{(t+s)^2\mathbb{P}_\mu(\|X_t\|\neq 0)}$$
$$= \frac{\mathbb{P}_\mu\left(\langle \mathbb{V}\mathrm{ar}_{\delta_\cdot}\langle \phi_0, X_s\rangle, X_t\rangle\right)}{(t+s)^2\mathbb{P}_\mu(\|X_t\|\neq 0)} \to 0.$$

Hence, as $t\to\infty$,

$$U_4(s,t) := \Big((t+s)^{-1}\langle \phi_0, X_{t+s}\rangle, (t+s)^{-1/2}\left(\langle f, X_{s+t}\rangle - \langle T_s f, X_t\rangle\right)\Big)|_{\mathbb{P}_{t,\mu}} \stackrel{d}{\to} \left(W, \sqrt{W}G_1(s)\right). \quad (3.63)$$

By (3.52), we have

$$U_4(s,t)|_{\mathbb{P}_{t+s,\mu}} \stackrel{d}{\to} \left(W, \sqrt{W}G_1(s)\right), \quad \text{as } t\to\infty. \quad (3.64)$$

Now, we deal with $J_2(t,s) := \frac{\langle T_s f, X_t\rangle}{(t+s)^{1/2}}$. We claim that

$$\lim_{s\to\infty}\lim_{t\to\infty}\mathbb{P}_{t+s,\delta_x}\left(|J_2(t,s)|^2\right) = 0. \quad (3.65)$$

By (2.13), we have that $\mathbb{P}_\mu\langle T_s f, X_t\rangle = \langle T_{t+s}f, \mu\rangle \to 0$ as $t\to\infty$. Thus, by (3.52), (1.25) and (2.19), we have

$$\lim_{t\to\infty}\mathbb{P}_{t+s,\delta_x}\left(|J_2(t,s)|^2\right) = \lim_{t\to\infty}\mathbb{P}_{t,\delta_x}\left(|J_2(t,s)|^2\right) = \lim_{t\to\infty}\frac{\mathbb{P}_\mu\langle T_s f, X_t\rangle^2}{(t+s)\mathbb{P}_\mu(\|X_t\|\neq 0)} = \nu\sigma^2_{(T_s f)}. \quad (3.66)$$

It follows (1.28) that, as $s\to\infty$,

$$\sigma^2_{(T_s f)} = \int_s^\infty \langle A(T_u f)^2, \psi_0\rangle_m\,du \to 0.$$

Now (3.65) follows immediately.

By (2.19), we have $\lim_{s\to\infty} V_s(x) = \sigma_f^2\phi_1(x)$, thus $\lim_{s\to\infty}\sigma_f^2(s) = \sigma_f^2$. Hence,

$$\lim_{s\to\infty}\beta(G_1(s), G_1(f)) = 0. \tag{3.67}$$

Let $\mathcal{D}(s+t)$ and $\widetilde{\mathcal{D}}(s,t)$ be the distributions of $U_1(s+t)$ and $U_4(s,t)$ under $\mathbb{P}_{t+s,\mu}$ respectively, and let $\widehat{\mathcal{D}}(s)$ and $\mathcal{D}$ be the distributions of $(W, \sqrt{W}G_1(s))$ and $(W, \sqrt{W}G_1(f))$ respectively. Then, using (3.54), we have

$$\begin{aligned}\limsup_{t\to\infty}\beta(\mathcal{D}(s+t), \mathcal{D}) &\le \limsup_{t\to\infty}[\beta(\mathcal{D}(s+t), \widetilde{\mathcal{D}}(s,t)) + \beta(\widetilde{\mathcal{D}}(s,t), \widehat{\mathcal{D}}(s)) + \beta(\widehat{\mathcal{D}}(s), \mathcal{D})] \\ &\le \limsup_{t\to\infty}(\sqrt{\mathbb{P}_{t+s,\mu}((t+s)^{-1}\langle T_s f, X_t\rangle^2)} + 0 + \beta(\widehat{\mathcal{D}}(s), \mathcal{D}). \end{aligned} \tag{3.68}$$

Using this and the definition of $\limsup_{t\to\infty}$, we easily get that

$$\limsup_{t\to\infty}\beta(\mathcal{D}(t), \mathcal{D}) = \limsup_{t\to\infty}\beta(\mathcal{D}(s+t), \mathcal{D}) \le \lim_{t\to\infty}(\sqrt{\mathbb{P}_{t+s,\mu}(J_2(s,t)^2)} + \beta(\widehat{\mathcal{D}}(s), \mathcal{D}).$$

Letting $s\to\infty$, by (3.65) and (3.67), we get

$$\limsup_{t\to\infty}\beta(\mathcal{D}(t), \mathcal{D}) = 0.$$

The proof of Theorem 1.8 is now complete. □

References

[1] Adamczak, R. and Miłoś, P.: CLT for Ornstein-Uhlenbeck branching particle system. Preprint, 2011. arXiv:1111.4559.

[2] Anderson, W. J.: *Continuous Time Markov Chains*. Springer-Verlag, 1991.

[3] Asmussen, S. and Hering, H.: *Branching Processes*. Birkhäuser, Boston, 1983.

[4] Athreya, K. and Ney, P.: *Branching Processes*. Springer-Verlag, 1972.

[5] Athreya, K. and Ney, P.: Functionals of critical multitype branching processes. *Ann. Probab.*, **2** (1974), 339-343.

[6] Dawson, D. A. : *Measure-Valued Markov Processes*. Springer-Verlag, 1993.

[7] Dudley, R. M.: *Real Analysis and Probability*. Cambridge University Press, 2002.

[8] Dynkin, E. B.: Superprocesses and partial differential equations. *Ann. Probab.* **21** (1993), 1185–1262.

[9] Dynkin, E. B. and Kuznetsov, S. E.: ℕ-measure for branching exit Markov system and their applications to differential equations. *Probab. Theory Rel. Fields* **130** (2004), 135–150.

[10] El Karoui, N. and Roelly, S.: Propriétés de martingales, explosion et représentation de Lévy-Khintchine d'une classe de processus de branchment à valeurs mesures. *Stoch. Proc. Appl.* **38** (1991), 239–266.

[11] Joffe, A. and Spitzer, F: On multitype branching processes with $\rho \le 1$. *JMAA* **19** (1967), 409–430.

[12] Kestern, H., Ney, P. and Spitzer, F.: The Galton-Watson process with mean one and finite variance. *Theory Probab. Appl.* **11** (1966), 513–540.

[13] Kim, P. and Song, R. : Intrinsic ultracontractivity of non-symmetric diffusion semigroups in bounded domains. *Tohoku Math. J.* **60** (2008), 527-547.

[14] Li, Z.: Skew convolution semigroups and related immigration processes. *Theory Probab. Appl.* **46** (2003), 274–296.

[15] Li, Z.: *Measure-valued Branching Markov Processes*. Springer, Heidelberg, 2011.

[16] Liu, R.-L., Ren, Y.-X. and Song, R.: $L\log L$ criterion for supercritical branching Hunt processes, *J. Theor. Probab. 24* (2011), 170-193.

[17] Lyons, R., Pemantle, R. and Peres, Y.: Conceptual proofs of $L\log L$ criteria for mean behavior of branching processes, *Ann. Probab. 23* (1995), 1125-1138.

[18] Miłoś, P.: Spatial CLT for the supercritical Ornstein-Uhlenbeck superprocess. Preprint, 2012. arXiv:1203:6661

[19] Ney, P.: Critical muti-type degenerate branching processes. *Technical report, Univ. of Wisc.*, 1967.

[20] Ren, Y.-X., Song, R. and Zhang, R.: Central limit theorems for super Ornstein-Uhlenbeck processes, *Acta Appl. Math.*, to appear, 2014. DOI 10.1007/s10440-013-9837-0

[21] Ren, Y.-X., Song, R. and Zhang, R.: Central limit theorems for supercritical branching Markov processes. *J. Funct. Anal.*, **266** (2014), 1716–1756.

[22] Ren, Y.-X., Song, R. and Zhang, R.: Central limit theorems for supecritical superprocesses. Preprint, 2013.

[23] Schaefer, H. H.: *Banach lattices and Positive Operators*. Springer, New York, 1974

Yan-Xia Ren: LMAM School of Mathematical Sciences & Center for Statistical Science, Peking University, Beijing, 100871, P.R. China. Email: yxren@math.pku.edu.cn

Renming Song: Department of Mathematics, University of Illinois, Urbana, IL 61801, U.S.A. Email: rsong@math.uiuc.edu

Rui Zhang: LMAM School of Mathematical Sciences, Peking University, Beijing, 100871, P.R. China. Email: ruizhang8197@gmail.com