# TESTS ALTERNATIVE TO HIGHER CRITICISM FOR HIGH DIMENSIONAL MEANS UNDER SPARSITY AND COLUMN-WISE DEPENDENCE 

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#### Abstract

We consider two alternative tests to the Higher Criticism test of Donoho and Jin (2004) for high dimensional means under the sparsity of the non-zero means for sub-Gaussian distributed data with unknown column-wise dependence. The two alternative test statistics are constructed by first thresholding $L_{1}$ and $L_{2}$ statistics based on the sample means, respectively, followed by maximizing over a range of thresholding levels to make the tests adaptive to the unknown signal strength and sparsity. The two alternative tests can attain the same detection boundary of the Higher Criticism test in Donoho and Jin (2004) which was established for uncorrelated Gaussian data. It is demonstrated that the maximal $L_{2}$-thresholding test is at least as powerful as the maximal $L_{1}$-thresholding test, and both the maximal $L_{2}$ and $L_{1}$-thresholding tests are at least as powerful as the Higher Criticism test


1. Introduction. Let $\mathbf{X}_{1}, \cdots, \mathbf{X}_{n}$ be independent and identically distributed (IID) $p$-variate random vectors generated from the following model

$$
\begin{equation*}
\mathbf{X}_{i}=\mathbf{W}_{i}+\boldsymbol{\mu} \quad \text { for } i=1, \cdots, n, \tag{1.1}
\end{equation*}
$$

where $\boldsymbol{\mu}=\left(\mu_{1}, \cdots, \mu_{p}\right)^{T}$ is a $p$-dimensional unknown vector of means, $\mathbf{W}_{i}=$ $\left(W_{i 1}, \cdots, W_{i p}\right)^{T}$ and $\left\{\mathbf{W}_{i}\right\}_{i=1}^{n}$ are IID random vectors with zero mean and common covariance $\boldsymbol{\Sigma}$. For the $i$-th sample, $\left\{W_{i j}\right\}_{j=1}^{p}$ is a sequence of weakly stationary dependent random variables with zero mean and variances $\sigma_{j}^{2}$. Motivated by the high dimensional applications arising in genetics, finance and other fields, the current paper focuses on testing high dimensional hypotheses

$$
\begin{equation*}
H_{0}: \boldsymbol{\mu}=0 \quad \text { vs } \quad H_{1}: \text { non-zero } \mu_{j} \text { are sparse and faint. } \tag{1.2}
\end{equation*}
$$

The specifications for the sparsity and faintness in the above $H_{1}$ are the following. There are $p^{1-\beta}$ non-zero $\mu_{j} \mathrm{~s}$ (signals) for a $\beta \in(1 / 2,1)$, which are sparse

[^0]since the signal bearing dimensions constitute only a small fraction of the total $p$ dimensions. Also under the $H_{1}$, the signal strength is faint in that the non-zero $\mu_{j}=\sqrt{2 r \log (p) / n}$ for $r \in(0,1)$. These specification of the $H_{1}$ have been the most challenging "laboratory" conditions in developing novel testing procedures under high dimensionality.

Donoho and Jin (2004) pioneered the theory of the Higher Criticism (HC) test which was originally conjectured in Tukey (1976), and showed that the HC test can attain the optimal detection boundary established by Ingster (1997) for uncorrelated Gaussian random vectors $\left(\boldsymbol{\Sigma}=\mathbf{I}_{p}\right)$. The optimal detection boundary is a phase-diagram in the space of $(\beta, r)$, the two quantities which define the sparsity and the strength of non-zero $\mu_{j} \mathrm{~s}$ under the $H_{1}$, such that if $(\beta, r)$ lies above the boundary, there exists a test which has asymptotically diminishing probabilities of the type I and type II errors simultaneously; and if $(\beta, r)$ is below the boundary, no such test exists. Hall and Jin (2008, 2010) investigated the impacts of the columnwise dependence on the HC test. In particular, Hall and Jin (2008) found that the HC test is adversely affected if the dependence is of long range dependent. If the dependence is weak, and the covariance matrix is known or can be estimated reliably, the dependence can be utilized to enhance the signal strength of the testing problem so as to improve the performance of the HC test. The improvement is reflected in lowering the needed signal strength $r$ by a constant factor. Delaigle and Hall (2009) evaluated the HC test under a nonparametric setting allowing column-wise dependence, and showed that the detection boundary of Donoho and Jin (2004) for the HC test can be maintained under weak column-wise dependence. Delaigle, Hall and Jin (2011) showed that the standard HC test based on the normality assumption can perform poorly when the underlying data deviate from the normal distribution and studied a version of the HC test based on the $t$-statistics formulation. Cai, Jeng and Jin (2011) considered detecting Gaussian mixtures which differ from the null in both the mean and the variance. Arias-Castro, Bubeck and Lugosi (2012a, 2012b) established the lower and upper bounds for the minimax risk for detecting sparse differences in the covariance.

We show in this paper that there are alternative test procedures for weakly dependent sub-Gaussian data with unknown covariance which attain the same detection boundary as the HC test established in Donoho and Jin (2004) for Gaussian distributed data with $\boldsymbol{\Sigma}=\mathbf{I}_{p}$. The alternative test statistics are obtained by first constructing, for $\gamma=1$ and 2 ,

$$
T_{\gamma n}(s)=\sum_{j=1}^{p}\left|\sqrt{n} \bar{X}_{j}\right|^{\gamma} I\left(\left|\bar{X}_{j}\right| \geq \sqrt{\lambda_{p}(s) / n}\right)
$$

which threshold with respect to $\bar{X}_{j}$ at a level $\sqrt{\lambda_{p}(s) / n}$ for $s \in(0,1)$, where $\lambda_{p}(s)=$ $2 s \log p$ and $\bar{X}_{j}$ is the sample mean of the $j$-th margin of the data vectors and $I(\cdot)$
is the indicator function. We note that $\gamma=1$ and 2 correspond to the $L_{1}$ and $L_{2}$ versions of the thresholding statistics respectively; and $\gamma=0$ corresponds to the HC test statistic. In the literature, the $L_{1}$ statistic is called the hard thresholding in Donoho and Johnstone (1994) and Donoho and Jin (2008), and the $L_{0}$ statistic is called the clipping-thresholding in Donoho and Jin (2008). We then maximize standardized versions of $T_{\gamma n}(s)$ with respect to $s$ over $\mathcal{S}$, a subset of $(0,1)$, which results in the following maximal $L_{\gamma}$-thresholding statistics:

$$
\begin{equation*}
\hat{\mathcal{M}}_{\gamma n}=\max _{s \in \mathcal{S}} \frac{T_{\gamma n}(s)-\hat{\mu}_{T_{\gamma n}, 0}(s)}{\hat{\sigma}_{T_{\gamma n}, 0}(s)} \quad \text { for } \gamma=0,1 \text { and } 2, \tag{1.3}
\end{equation*}
$$

where $\hat{\mu}_{T_{\gamma n}, 0}(s)$ and $\hat{\sigma}_{T_{\gamma n}, 0}(s)$ are, respectively, estimators of the mean $\mu_{T_{\gamma n}, 0}(s)$ and standard deviation $\sigma_{T_{\gamma n}, 0}(s)$ of $T_{\gamma n}(s)$ under $H_{0}$, whose forms will be given later in the paper. By developing the asymptotic distributions of $\hat{\mathcal{M}}_{\gamma n}$, the maximal $L_{\gamma^{-}}$ thresholding tests are formulated for $\gamma=0,1$ and 2 with the maximal $L_{0}$-test being equivalent to the HC test. An analysis on the relative power performance of the three tests reveals that if the signal strength parameter $r \in(0,1)$, the maximal $L_{2^{-}}$ thresholding test is at least as powerful as the maximal $L_{1}$-thresholding test, and both the $L_{1}$ and $L_{2}$-thresholding tests are at least as powerful as the HC test. If we allow slightly stronger signal so that $r>2 \beta-1$, the differential power performance of the three tests is amplified with the maximal $L_{2}$-test being the most advantageous followed by the maximal $L_{1}$-test.

In additional to the connection to the HC test, the maximal $L_{\gamma}$-thresholding test, by its nature of formulation, is related to the high dimensional multivariate testing procedures, for instance the tests proposed by Bai and Saranadasa (1996) and Chen and Qin (2010). While these tests can maintain accurate size approximation under a diverse range of dimensionality and column-wise dependence, their performance is hampered when the non-zero means are sparse and faint. The proposed test formulation is also motivated by a set of earlier work including Donoho and Johnstone (1994) for selecting significant wavelet coefficients, and Fan (1996) who considered testing for the mean of a random vector $\mathbf{X}$ with IID normally distributed components. We note that the second step of maximization with respect to $s \in \mathcal{S} \subset(0,1)$ is designed to make the test adaptive to the underlying signals strength and sparsity, which is the essence of the HC procedure in Donoho and Jin (2004), as well as that of Fan (1996).

The rest of the paper is organized as follows. In Section 2, we provide basic results on the $L_{2}$-thresholding statistic via the large deviation method and the asymptotic distribution of the single-threshold statistic. Section 3 gives the asymptotic distribution of $\hat{\mathcal{M}}_{2 n}$ as well as the associated test procedure. Power comparisons among the HC and the maximal $L_{1}$ and $L_{2}$-thresholding tests are made in Section 4. Section 5 reports simulation results which confirm the theoretical results. Some discussions
are given in Section 6. All technical details are relegated to the Appendix.
2. Single-threshold Test Statistic. Let $\mathbf{X}_{1}, \cdots, \mathbf{X}_{n}$ be independent $p$-dimensional random sample from a common distribution $F$, and $\mathbf{X}_{i}=\mathbf{W}_{i}+\boldsymbol{\mu}$ where $\boldsymbol{\mu}=$ $\left(\mu_{1}, \cdots, \mu_{p}\right)^{T}$ is the vector of means and $\mathbf{W}_{i}=\left(W_{i 1}, \cdots, W_{i p}\right)^{T}$ is a vector consisting of potentially dependent random variables with zero mean and finite variances. The dependence among $\left\{W_{i j}\right\}_{j=1}^{p}$ is called the column-wise dependence in $\mathbf{W}_{i}$. Those non-zero $\mu_{j}$ are called "signals".

Let $\bar{X}_{j}=\frac{1}{n} \sum_{i=1}^{n} X_{i j}, \sigma_{j}^{2}=\operatorname{Var}\left(W_{i j}\right)$ and $s_{j}^{2}=(n-1)^{-1} \sum_{i=1}^{n}\left(X_{i j}-\bar{X}_{j}\right)^{2}$ be the sample variance for the $j$-th margin. The signal strength in the $j$-th margin can be measured by the t-statistics $\sqrt{n} \bar{X}_{j} / s_{j}$ or the z-statistics $\sqrt{n} \bar{X}_{j} / \sigma_{j}$ if $\sigma_{j}$ is known. For easy expedition, the test statistics will be constructed based on the zstatistics by assuming $\sigma_{j}$ is known; and without loss of generality we assume $\sigma_{j}^{2}=1$. Using the $t$-statistics actually leads to less restrictive conditions for the underlying random variables since the large deviation results for the self-normalized $t$-statistics can be established under weaker conditions to allow heavier tails in the underlying distribution as demonstrated in Shao (1997), Jing, Shao and Zhou (2008) and Wang and Hall (2009). See Delaigle, Hall and Jin (2011) for analysis on the sparse signal detection using the $t$-statistics.

We assume the following assumptions in our analysis.
C.1: The dimension $p=p(n) \rightarrow \infty$ as $n \rightarrow \infty$ and $\log (p)=o\left(n^{1 / 3}\right)$.
C.2: There exists a positive constant $H$ such that, for any $j \neq l \in\{1, \cdots, p\}$, $E\left(e^{h^{\prime}\left(W_{1 j}^{d}, W_{1 l}^{d}\right)}\right)<\infty$ for $h \in[-H, H] \times[-H, H]$ and some $1<d \leq 2$.
C.3: For each $i=1, \cdots, n,\left\{W_{i j}\right\}_{j=1}^{p}$ is a weakly stationary sequence such that $E\left(W_{i j}\right)=E\left(W_{i(j+k)}\right)=0$ and $\operatorname{Cov}\left(W_{i j}, W_{i(j+k)}\right)$ do not depend on $j$ for any integer $k$. And $\sum_{k}\left|\rho_{k}\right|<\infty$ where $\rho_{k}=\operatorname{Cov}\left(W_{i 1}, W_{i(k+1)}\right)$.
C.4: Among the $p$ marginal means, there are $m=p^{1-\beta}$ signals for a $\beta \in(1 / 2,1)$ and the signal $\mu_{j}=\sqrt{2 r \log (p) / n}$ for a $r>0$. The signals' locations $\ell_{1}<\ell_{2}<\cdots<$ $\ell_{m}$ are randomly selected from $\{1,2, \cdots, p\}$ without replacement so that

$$
\begin{equation*}
P\left(\ell_{1}=p_{1}, \cdots, \ell_{m}=p_{m}\right)=\binom{p}{m}^{-1} \text { for all } 1 \leq p_{1}<p_{2}<\cdots<p_{m} \leq p \tag{2.1}
\end{equation*}
$$

C. 1 specifies the growth rate of $p$ relative to the sample size $n$ is in the paradigm of "large p , small n ". That $\log p=o\left(n^{1 / 3}\right)$ is the rate we can attain for Gaussian data or cases where we can attain "accurate" enough estimation of $\mu_{T_{\gamma n}, 0}$ which satisfies equation (2.6). When data are not Gaussian and the "accurate" estimators are not attainable, the growth rate of $p$ will be more restrictive at $p=n^{1 / \theta}$ as will be discussed in the next section. C. 2 assumes the joint distributions of ( $W_{i j}, W_{i l}$ ) is subGaussian, which implies each marginal $W_{i j}$ is sub-Gaussian as well. C. 3 prescribes
weakly dependence among $\left\{W_{i j}\right\}_{j=1}^{p}$. The first part of C. 4 reiterates the sparse and faint signal setting. The range of the signal strength includes the case of $r \in(0,1)$, representing the most fainted detectable signal strength, which has been considered in Donoho and Jin (2004) and other research works. The second part of C. 4 provides a random allocation mechanism for the signal bearing dimensions, which is the same as the one assumed in Hall and Jin (2010). Existing research on the detection boundary of the HC test for the sparse mean problem (Donoho and Jin, 2004; Hall and Jin, 2010) are largely conducted for the case of $n=1$ when the data are Gaussian. This is understandable since the sample means are sufficient statistics and there is no loss of generality when we treat the problem as $n=1$ even if we have multiple observations. However, when the underlying distributions are as specified in C.2, we cannot translate the test problem to $n=1$ without incurring a loss of information.

We first consider the $L_{2}$ version of the thresholding statistic $T_{2 n}$ in this section. The study of the $T_{1 n}$ version is outlined in Section 4 when we compare the power performance to the HC test. Let $Y_{j, n}=n \bar{X}_{j}^{2}$. Then, the $L_{2}$-thresholding statistic can be written as

$$
\begin{equation*}
T_{2 n}(s)=\sum_{j=1}^{p} Y_{j, n} I\left\{Y_{j, n} \geq \lambda_{p}(s)\right\} \tag{2.2}
\end{equation*}
$$

where $s$ is the thresholding parameter that takes values over a range within $(0,1)$. There is no need to consider $s \geq 1$ in the thresholding since large deviation results given in Petrov (1995) imply that under $H_{0}, P\left(\max _{1 \leq j \leq p} Y_{j, n} \leq \lambda_{p}(s)\right) \rightarrow 1$.

Define a set of slowing varying functions: $L_{p}^{(1)}=2 r \log p+1, L_{p}^{(2)}=2 \sqrt{s \log p / \pi}$, $L_{p}^{(3)}=s(\sqrt{s}-\sqrt{r})^{-1} \sqrt{\log p / \pi}, L_{p}^{(4)}=8 r \log p, L_{p}^{(5)}=4 s^{3 / 2} \pi^{-\frac{1}{2}}(\log p)^{3 / 2}$ and $L_{p}^{(6)}=$ $2 s^{2}(\log p)^{3 / 2} / \sqrt{\pi}(\sqrt{s}-\sqrt{r})$.

Let $\mu_{T_{2 n}, 0}(s)$ and $\sigma_{T_{2 n}, 0}^{2}(s)$ be the mean and variance of $T_{2 n}(s)$ under $H_{0}$, respectively; and $\mu_{T_{2 n}, 1}(s)$ and $\sigma_{T_{2 n}, 1}^{2}(s)$ be those respectively under the $H_{1}$ as specified in C.4. The following proposition depicts the mean and variance of $T_{2 n}(s)$ by applying Fubini's theorem and the large deviation results (Petrov, 1995 and Lemma A. 1 in Zhong et al., 2013).

Proposition 1. Under C.1-C.4, $E\left\{T_{2 n}(s)\right\}$ and $\operatorname{Var}\left\{T_{2 n}(s)\right\}$ are respectively,

$$
\begin{align*}
\mu_{T_{2 n}, 0}(s) & =p\left\{2 \lambda_{p}^{1 / 2}(s) \phi\left(\lambda_{p}^{1 / 2}(s)\right)+2 \bar{\Phi}\left(\lambda_{p}^{1 / 2}(s)\right)\right\}\left\{1+O\left\{n^{-1 / 2} \lambda_{p}^{3 / 2}(s)\right\}\right\}  \tag{2.3}\\
\sigma_{T_{2 n}, 0}^{2}(s) & =p\left\{2\left[\lambda_{p}^{3 / 2}(s)+3 \lambda_{p}^{1 / 2}(s)\right] \phi\left(\lambda_{p}^{1 / 2}(s)\right)+6 \bar{\Phi}\left(\lambda_{p}^{1 / 2}(s)\right)\right\}\{1+o(1)\} \tag{2.4}
\end{align*}
$$

under the $H_{0}$; and

$$
\begin{aligned}
& \mu_{T_{2 n}, 1}(s)=\left\{L_{p}^{(1)} p^{1-\beta} I(s<r)+L_{p}^{(3)} p^{1-\beta-(\sqrt{s}-\sqrt{r})^{2}} I(s>r)\right\}\{1+o(1)\}+\mu_{T_{2 n}, 0}(s), \\
& \sigma_{T_{2 n}, 1}^{2}(s)=\left\{L_{p}^{(4)} p^{1-\beta} I(s<r)+L_{p}^{(5)} p^{1-s}+L_{p}^{(6)} p^{1-\beta-(\sqrt{s}-\sqrt{r})^{2}} I(s>r)\right\}\{1+o(1)\}
\end{aligned}
$$

under the $H_{1}$ specified in C.4.
Expressions (2.3) and (2.4) provide the first and the second order terms of $\mu_{T_{2 n}, 0}(s)$ and $\sigma_{T_{2 n}, 0}^{2}(s)$, which are needed when we consider their empirical estimation under $H_{0}$ when formulating the $L_{2}$ thresholding test statistic. Note that $\mu_{T_{2 n}, 0}(s)=$ $L_{p}^{(2)} p^{1-s}\{1+o(1)\}$ and $\sigma_{T_{2 n}, 0}^{2}(s)=L_{p}^{(5)} p^{1-s}\{1+o(1)\}$. Only the first order terms for the variance are needed under $H_{1}$ but the approximation to $\mu_{T_{2 n}, 1}(s)$ has to be more accurate so as to know the order of the difference between $\mu_{T_{2 n}, 1}(s)$ and $\mu_{T_{2 n}, 0}(s)$. Proposition 1 indicates that the column-wise dependence as specified in C. 3 does not have much leading order impact on the variance of $T_{2 n}(s)$. The leading order variance is almost the same when $\mathbf{W}_{i}$ are column-wise independent. The difference only appears in the coefficients of the slow-varying functions $L_{p}^{(4)}, L_{p}^{(5)}$ and $L_{p}^{(6)}$, while their orders of magnitude remain unchanged. The reason behind this phenomena is the thresholding. It can be understood by an analogue for multivariate Gaussian distributions with non-zero correlation. Despite the dependence in the Gaussian distribution, exceedances beyond high thresholds are asymptotically independent (Sibuya, 1960 and Joe, 1997).

We now study the asymptotic distribution of $T_{2 n}(s)$ to prepare for the proposal of the maximal $L_{2}$-thresholding statistic. Write

$$
T_{2 n}(s)=\sum_{j=1}^{p} Z_{j, n}(s)
$$

where $Z_{j, n}(s):=Y_{j, n} I\left\{Y_{j, n}>\lambda_{p}(s)\right\}$ and $\lambda_{p}(s)=2 s \log (p)$. For integers $a, b \in$ $[-\infty, \infty]$ such that $a<b$, define $\mathscr{F}_{a}^{b}=\sigma\left\{Z_{l, n}(s): l \in(a, b)\right\}$ be the $\sigma$-algebra generated by $\left\{Z_{l, n}(s)\right\}_{l=a}^{b}$ and define the $\rho$-mixing coefficients

$$
\begin{equation*}
\rho_{Z(s)}(k)=\sup _{l, \xi \in L^{2}\left(\mathscr{F}_{-\infty}^{l}\right), \zeta \in L^{2}\left(\mathscr{F}_{l+k}^{\infty}\right)}|\operatorname{Corr}(\xi, \zeta)| . \tag{2.5}
\end{equation*}
$$

See Doukhan (1994) for comprehensive discussions on the mixing concept. The following is a condition regarding the dependence among $\left\{Z_{j, n}(s)\right\}_{j=1}^{p}$.
C.5: For any $s \in(0,1)$, the sequence of random variables $\left\{Z_{j, n}(s)\right\}_{j=1}^{p}$ is $\rho$-mixing such that $\rho_{Z(s)}(k) \leq C \alpha^{k}$ for some $\alpha \in(0,1)$ and a positive constant $C$.

The requirement of $\left\{Z_{j, n}(s)\right\}_{j=1}^{p}$ being $\rho$-mixing for each $s$ is weaker than requiring the original data columns $\left\{X_{i j}\right\}_{j=1}^{p}$ being $\rho$-mixing, whose mixing coefficient $\rho_{X_{i}}(k)$ can be similarly defined as (2.5). This is because, according to Theorem 5.2 in Bradley (2005),

$$
\rho_{Z(s)}(k) \leq \sup _{i \leq n} \rho_{X_{i}}(k)=\rho_{X_{1}}(k), \text { for each } k=1, \cdots, p \text { and } s \in(0,1)
$$

The following theorem reports the asymptotic normality of $T_{2 n}(s)$ under both $H_{0}$ and $H_{1}$.

Theorem 1. Assume C.1-C.5. Then, for any $s \in(0,1)$,

$$
\begin{aligned}
& (i): \sigma_{T_{2 n}, 0}^{-1}(s)\left\{T_{2 n}(s)-\mu_{T_{2 n}, 0}(s)\right\} \xrightarrow{d} N(0,1) \quad \text { under } H_{0} ; \\
& (i i): \sigma_{T_{2 n}, 1}^{-1}(s)\left\{T_{2 n}(s)-\mu_{T_{2 n}, 1}(s)\right\} \xrightarrow{d} N(0,1) \quad \text { under } H_{1} .
\end{aligned}
$$

From (2.3) and (2.4), define the leading order terms of $\mu_{T_{2 n}, 0}(s)$ and $\sigma_{T_{2 n}, 0}^{2}(s)$, respectively,

$$
\begin{aligned}
& \tilde{\mu}_{T_{2 n}, 0}(s)=p\left\{2 \lambda_{p}^{1 / 2}(s) \phi\left(\lambda_{p}^{1 / 2}(s)\right)+2 \bar{\Phi}\left(\lambda_{p}^{1 / 2}(s)\right)\right\} \text { and } \\
& \tilde{\sigma}_{T_{2 n}, 0}^{2}(s)=p\left\{2\left[\lambda_{p}^{3 / 2}(s)+3 \lambda_{p}^{1 / 2}(s)\right] \phi\left(\lambda_{p}^{1 / 2}(s)\right)+6 \bar{\Phi}\left(\lambda_{p}^{1 / 2}(s)\right)\right\} .
\end{aligned}
$$

It is clear that the asymptotic normality in Theorem 1(i) remains if we replace $\sigma_{T_{2 n}, 0}(s)$ by $\tilde{\sigma}_{T_{2 n}, 0}(s)$.

To formulate a test procedure based on the thresholding statistic $T_{2 n}(s)$, we need to estimate $\mu_{T_{2 n}, 0}(s)$ by a $\hat{\mu}_{T_{2 n}, 0}(s)$, say. Ideally, if

$$
\begin{equation*}
\mu_{T_{2 n}, 0}(s)-\hat{\mu}_{T_{2 n}, 0}(s)=o\left\{\tilde{\sigma}_{T_{2 n}, 0}(s)\right\}, \tag{2.6}
\end{equation*}
$$

the first part of Theorem 1 remains valid if we replace $\mu_{T_{2 n}, 0}(s)$ with $\hat{\mu}_{T_{2 n}, 0}(s)$. An obvious choice of $\hat{\mu}_{T_{2 n}, 0}(s)$ is $\tilde{\mu}_{T_{2 n}, 0}(s)$ which is known upon given $p$ and $s$. Indeed, if $W_{i j} \mathrm{~s}$ are the standard normally distributed, we have

$$
\mu_{T_{2 n}, 0}(s)=\tilde{\mu}_{T_{2 n}, 0}(s) \quad \text { for } \quad s \in(0,1),
$$

implying the leading order is exactly $\mu_{T_{2 n}, 0}(s)$ for the Gaussian data. Hence, if we take $\hat{\mu}_{T_{2 n}, 0}(s)=\tilde{\mu}_{T_{2 n}, 0}(s),(2.6)$ is satisfied for the Gaussian data.

For non-Gaussian observations, the difference between $\mu_{T_{2 n}, 0}(s)$ and $\tilde{\mu}_{T_{2 n}, 0}(s)$ may not be a smaller order of $\sigma_{T_{2 n}, 0}(s)$. Specifically, from (2.3) and (2.4), we have

$$
\frac{\mu_{T_{2 n}, 0}(s)-\tilde{\mu}_{T_{2 n}, 0}(s)}{\sigma_{T_{2 n}, 0}(s)}=O\left\{\lambda_{p}^{5 / 4}(s) p^{(1-s) / 2} n^{-1 / 2}\right\} .
$$

To make the above ratio diminishing to zero, the strategy of Delaigle, Hall and Jin (2011) can be adopted by restricting $p=n^{1 / \theta}$ and $s \in\left((1-\theta)_{+}, 1\right)$ for a positive $\theta$, where $(a)_{+}=a$ if $a>0$ and $(a)_{+}=0$ if $a \leq 0$. Under this circumstance

$$
\begin{equation*}
\frac{\mu_{T_{2 n}, 0}(s)-\tilde{\mu}_{T_{2 n}, 0}(s)}{\sigma_{T_{2 n}, 0}(s)}=O\left\{(2 s / \theta \log n)^{5 / 4} n^{\frac{1-s-\theta}{2 \theta}}\right\} \rightarrow 0 . \tag{2.7}
\end{equation*}
$$

Clearly, for not so high dimension with $\theta \geq 1$, (2.7) holds for all $s \in(0,1)$, and $\tilde{\mu}_{T_{2 n}, 0}(s)$ satisfies (2.6). For higher dimensions with $\theta<1$, the thresholding level $s$ has to be restricted to ensure (2.7). The restriction can alter the detection boundary of the test we will propose in the next section. This echoes a similar phenomena for the HC test given in Delaigle, Hall and Jin (2011). To expedite our discussion, we assume in the rest of the paper that (2.6) is satisfied by the $\hat{\mu}_{T_{2 n}, 0}(s)$. We note such an arrangement is not entirely unrealistic as separate effort may be made to produce more accurate estimators. Assuming so allows us to stay focused on the main agenda of the testing problem.

The asymptotic normality established in Theorem 1 allows an asymptotic $\alpha$-level test that rejects $H_{0}$ if

$$
\begin{equation*}
T_{2 n}(s)-\hat{\mu}_{T_{2 n}, 0}(s)>z_{\alpha} \tilde{\sigma}_{T_{2 n}, 0}(s) \tag{2.8}
\end{equation*}
$$

where $z_{\alpha}$ is the upper $\alpha$ quantile of the standard normal distribution.
3. Maximal Thresholding. While the asymptotic normality of $T_{2 n}(s)$ in Theorem 1 ensures the single thresholding level test in (2.8) a correct size asymptotically, the power of the test depends on $s$, the underlying signal strength $r$ and the sparsity $\beta$. A test procedure is said to be able to separate a pair of null and alternative hypotheses asymptotically if the sum of the probabilities of the type I and type II errors converges to zero as $n \rightarrow \infty$. Let $\alpha_{n}$ be a sequence of the probabilities of type I error, which can be made converging to zero as $n \rightarrow \infty$. The sum of the probabilities of the type I and type II errors for the test given in (2.8) with nominal size $\alpha_{n}$ is approximately

$$
\begin{equation*}
\operatorname{Err}_{\alpha_{n}}:=\alpha_{n}+P\left(\left.\frac{T_{2 n}(s)-\mu_{T_{2 n}, 0}(s)}{\sigma_{T_{2 n}, 0}(s)} \leq z_{\alpha_{n}} \right\rvert\, H_{1}\right) \tag{3.1}
\end{equation*}
$$

which is attained based on the facts that (i) the size $\alpha_{n}$ is attained asymptotically and (ii) $\hat{\mu}_{T_{2 n}, 0}(s)$ and $\tilde{\sigma}_{T_{2 n}, 0}(s)$ are sufficiently accurate estimators in the test procedure (2.8).

Our strategy is to first make $\alpha_{n} \rightarrow 0$ such that $z_{\alpha_{n}}=C(\log p)^{\epsilon}$ for an arbitrarily small $\epsilon>0$ and a constant $C>0$. The second term on the right hand side of (3.1) is

$$
\begin{equation*}
\operatorname{Err}_{I I}:=P\left(\frac{T_{2 n}(s)-\mu_{T_{2 n}, 1}(s)}{\sigma_{T_{2 n}, 1}(s)} \leq z_{\alpha_{n}} \frac{\sigma_{T_{2 n}, 0}(s)}{\sigma_{T_{2 n}, 1}(s)}-\frac{\mu_{T_{2 n}, 1}(s)-\mu_{T_{2 n}, 0}(s)}{\sigma_{T_{2 n}, 1}(s)}\right) \tag{3.2}
\end{equation*}
$$

Because $z_{\alpha_{n}}$ is slowly varying, $0<\sigma_{T_{2 n}, 0}(s) / \sigma_{T_{2 n}, 1}(s) \leq 1$ and $\left(T_{2 n}(s)-\mu_{T_{2 n}, 1}(s)\right) / \sigma_{T_{2 n}, 1}(s)$ is stochastically bounded, a necessary and sufficient condition that ensures $E r r_{\alpha_{n}} \rightarrow$ 0 is

$$
\begin{equation*}
\Delta_{2}(s ; r, \beta):=\frac{\mu_{T_{2 n}, 1}(s)-\mu_{T_{2 n}, 0}(s)}{\sigma_{T_{2 n}, 1}(s)} \rightarrow \infty . \tag{3.3}
\end{equation*}
$$

From Proposition 1, it follows that, up to a factor $1+o(1)$,

$$
\Delta_{2}(s ; r, \beta)= \begin{cases}C_{1} p^{(1+s-2 \beta) / 2} & \text { if } s \leq r \text { and } s \leq \beta ; \\ C_{2} p^{(1-\beta) / 2} & \text { if } s \leq r \text { and } s>\beta ; \\ C_{3} p^{1 / 2-\beta+r-(\sqrt{s}-2 \sqrt{r})^{2} / 2} & \text { if } s>r \text { and } s \leq(\sqrt{s}-\sqrt{r})^{2}+\beta ; \\ C_{4} p^{\left(1-\beta-(\sqrt{s}-2 \sqrt{r})^{2}\right) / 2} & \text { if } s>r \text { and } s>(\sqrt{s}-\sqrt{r})^{2}+\beta\end{cases}
$$

where $C_{1}=\sqrt{2}(\pi s)^{\frac{1}{4}}\left(\frac{r}{s}\right)(\log p)^{\frac{1}{4}}, C_{2}=\frac{1}{2}(r \log p)^{1 / 2}, C_{3}=s^{1 / 4}(\log p)^{-\frac{1}{4}} /\left\{\sqrt{2} \pi^{1 / 4}(\sqrt{s}-\right.$ $\sqrt{r})\}$ and $C_{4}=(2 \sqrt{\pi}(\sqrt{s}-\sqrt{r}))^{-\frac{1}{2}}(\log p)^{-\frac{1}{4}}$.

Let

$$
\varrho^{*}(\beta)= \begin{cases}\beta-1 / 2, & 1 / 2<\beta \leq 3 / 4 \\ (1-\sqrt{1-\beta})^{2}, & 3 / 4<\beta<1\end{cases}
$$

As demonstrated in Donoho and Jin (2004) and Ingster (1997), the phase diagram $r=\varrho^{*}(\beta)$ is the optimal detection boundary for testing the hypotheses we are considering in this paper when the data are Gaussian and $\boldsymbol{\Sigma}=\mathbf{I}_{p}$. Here the optimality means that for any $r>\varrho^{*}(\beta)$, there exists at least one test such that the sum of the probabilities of the type I and type II errors diminishes to zero as $n \rightarrow \infty$; but for $r<\varrho^{*}(\beta)$, no such test exists. For correlated Gaussian data such that $\boldsymbol{\Sigma} \neq \mathbf{I}_{p}$, Hall and Jin (2010) found that the detection boundary $r=\varrho^{*}(\beta)$ may be lowered by transforming the data via the inverse of Cholesky factorization $\mathbf{L}$ such that $\mathbf{L} \Sigma \mathbf{L}^{T}=\mathbf{I}_{p}$. More discussion on the optimality is given in Section 6.

From the expression of $\Delta_{2}(s ; r, \beta)$ given above, it can be shown (see the proof of Theorem 3 in the Appendix) that if $r>\varrho^{*}(\beta)$ there exists at least one $s \in(0,1)$ for each pair of $(r, \beta)$ such that (3.3) is satisfied and hence the thresholding test would be powerful. This is the key for the maximal $L_{2}$-thresholding test that we will propose later to attain the detection boundary.

It is clear that we have to make the thresholding level $s$ adaptive to the unknown $r$ and $\beta$. One strategy is to use a range of thresholding levels, say $s \in \mathcal{S} \subset(0,1)$ so that the underlying $(r, \beta)$ can be "covered". This is the very idea of the HC test.

Let $\hat{\mathcal{T}}_{2, n}(s)=\tilde{\sigma}_{T_{2 n}, 0}^{-1}(s)\left\{T_{2 n}(s)-\hat{\mu}_{T_{2 n}, 0}(s)\right\}$ be the standardized version of $T_{2 n}(s)$. Define the maximal thresholding statistic

$$
\hat{\mathcal{M}}_{2 n}=\sup _{s \in \mathcal{S}} \hat{\mathcal{T}}_{2, n}(s)
$$

where $\mathcal{S}=(0,1-\eta]$ for an arbitrarily small positive $\eta$. Let

$$
\begin{equation*}
\mathcal{S}_{n}=\left\{s_{i}: s_{i}=Y_{i, n} /(2 \log p) \text { and } 0<Y_{i, n}<2(1-\eta) \log p\right\} \cup\{1-\eta\} . \tag{3.4}
\end{equation*}
$$

Since both $\hat{\mu}_{T_{2 n}, 0}(s)$ and $\tilde{\sigma}_{T_{2 n}, 0}(s)$ are monotone decreasing functions of $s$, it can be shown that $\hat{\mathcal{M}}_{2 n}$ can be attained on $\mathcal{S}_{n}$, namely

$$
\begin{equation*}
\hat{\mathcal{M}}_{2 n}=\max _{s \in \mathcal{S}_{n}} \hat{\mathcal{T}}_{2, n}(s) \tag{3.5}
\end{equation*}
$$

This largely reduces the computational burden of $\hat{\mathcal{M}}_{2 n}$. The asymptotic distribution of $\hat{\mathcal{M}}_{2 n}$ is established in the following theorem.

Theorem 2. Assume C.1-C.3, C.5 and (2.6) hold. Then, under $H_{0}$,

$$
P\left(a(\log p) \hat{\mathcal{M}}_{2 n}-b(\log p, \eta) \leq x\right) \rightarrow \exp \left(-e^{-x}\right)
$$

where $a(y)=(2 \log (y))^{1 / 2}$ and $b(y, \eta)=2 \log (y)+2^{-1} \log \log (y)-2^{-1} \log \left(\frac{4 \pi}{(1-\eta)^{2}}\right)$.
The theorem leads to an asymptotic $\alpha$-level test that rejects $H_{0}$ if

$$
\begin{equation*}
\hat{\mathcal{M}}_{2 n}>\mathcal{B}_{\alpha}=\left(\mathcal{E}_{\alpha}+b(\log p, \eta)\right) / a(\log p) \tag{3.6}
\end{equation*}
$$

where $\mathcal{E}_{\alpha}$ is the upper $\alpha$ quantile of the Gumbel distribution $\exp \left(-e^{-x}\right)$. We name the test the maximal $L_{2}$-thresholding test. The following theorem shows that its detection boundary is $r=\varrho^{*}(\beta)$.

Theorem 3. Under conditions C.1-C. 5 and assume (2.6) holds, then (i): if $r>\varrho^{*}(\beta)$, the sum of the type I and II errors of the maximal $L_{2}$-thresholding tests converges to 0 when the nominal sizes $\alpha_{n}=\bar{\Phi}\left((\log p)^{\epsilon}\right) \rightarrow 0$ for an arbitrarily small $\epsilon>0$ as $n \rightarrow \infty$.
(ii): if $r<\varrho^{*}(\beta)$, the sum of the type I and II errors of the maximal $L_{2}$-thresholding test converges to 1 when the nominal sizes $\alpha_{n} \rightarrow 0$ as $n \rightarrow \infty$.

It is noted that when $r>\varrho^{*}(\beta)$ in Part (i) of Theorem 3, we need to restrict the rate of the nominal type I error $\alpha_{n}$ 's convergence to 0 , since the conclusion of Part (i) may not be true for all $\alpha_{n} \rightarrow 0$. However, in Part (ii) where $r<\varrho^{*}(\beta)$, no restriction for $\alpha_{n}$ is required, which has to be the case as otherwise there is no guarantee that $r=\varrho^{*}(\beta)$ is the detection boundary of the test.

If the estimator $\hat{\mu}_{T_{2 n, 0}}(s)$ cannot attain (2.6) and $\tilde{\mu}_{T_{2 n, 0}}(s)$ is used as the estimator, we have to restrict $p=n^{1 / \theta}$ for a $\theta \in(0,1)$ and limit $s \in(1-\theta, 1)$. In this case, the above theorem is valid if we replace $\varrho^{*}(\beta)$ by $\varrho_{\theta}^{*}(\beta)$ where

$$
\varrho_{\theta}^{*}(\beta)= \begin{cases}(\sqrt{1-\theta}-\sqrt{1-\beta-\theta / 2})^{2} & \text { if } 1 / 2<\beta \leq(3-\theta) / 4 \\ \beta-1 / 2 & \text { if }(3-\theta) / 4<\beta \leq 3 / 4 \\ (1-\sqrt{1-\beta})^{2} & \text { if } 3 / 4<\beta<1\end{cases}
$$

which is clearly inferior to $\varrho^{*}(\beta)$. The boundary $\varrho_{\theta}^{*}(\beta)$ is the same as the one in Delaigle, Hall and Jin (2011) based on the marginal t-statistics, whereas our result is based on the $z$-statistics. The $t$-statistic formulation reduces the demand on the tails of the distributions as shown in Delaigle, Hall and Jin (2011). We note that if $\theta \geq 1$, Theorem 3 remains so that the Gaussian detection boundary is still valid.
4. Power Comparison. We compare the power of the maximal $L_{2}$-thresholding test with those of the HC test and the maximal $L_{1}$-thresholding test in this section. Let us first introduced these two tests.

The HC test is based on

$$
\begin{equation*}
\hat{\mathcal{T}}_{0, n}(s)=\frac{T_{0 n}(s)-2 p \bar{\Phi}\left(\lambda_{p}^{1 / 2}(s)\right)}{\sqrt{2 p \bar{\Phi}\left(\lambda_{p}^{1 / 2}(s)\right)\left(1-2 \bar{\Phi}\left(\lambda_{p}^{1 / 2}(s)\right)\right)}} \tag{4.1}
\end{equation*}
$$

where $T_{0 n}(s)=\sum_{j=1}^{p} I\left(Y_{j, n} \geq \lambda_{p}(s)\right)$. Like Delaigle and Hall (2009), we consider here a two-sided HC test instead of a one-sided test treated in Donoho and Jin (2004). With the same reasoning as Donoho and Jin (2004, p968), we define the HC test statistic

$$
\hat{\mathcal{M}}_{0 n}=\max _{s \in \mathcal{S}} \hat{\mathcal{T}}_{0, n}(s),
$$

where $\mathcal{S}=(0,1-\eta]$ for an arbitrary small $\eta$ and is the same as the maximal $L_{2}$ thresholding statistic. Using the same argument for the maximal $L_{2}$-thresholding statistic, it can be shown that $\hat{\mathcal{M}}_{0 n}$ attains its maximum value on $\mathcal{S}_{n}$ given in (3.4) as well.

According to Donoho and Jin (2004), under $H_{0}$

$$
P\left(a(\log p) \hat{\mathcal{M}}_{0 n}-b(\log p, \eta) \leq x\right) \rightarrow \exp \left(-e^{-x}\right)
$$

with the same normalizing sequences as those in Theorem 2. Let $\mathcal{B}_{\alpha}$ be the same as that of the maximal $L_{2}$-thresholding test given in (3.6). An $\alpha$ level HC test rejects $H_{0}$ if

$$
\begin{equation*}
\hat{\mathcal{M}}_{0 n}>\mathcal{B}_{\alpha} \tag{4.2}
\end{equation*}
$$

Let us introduce the maximal $L_{1}$-thresholding test statistic. Recall that

$$
T_{1 n}(s)=\sum_{j=1}^{p}\left|\sqrt{n} \bar{X}_{j}\right| I\left(\left|\bar{X}_{j}\right|>\sqrt{\lambda_{p}(s) / n}\right) .
$$

It can be shown that the mean and variance of $T_{1 n}(s)$ under $H_{0}$ are, respectively,

$$
\mu_{T_{1 n}, 0}(s)=\sqrt{2 / \pi} p^{1-s}\{1+o(1)\} \quad \text { and } \quad \sigma_{T_{1 n}, 0}^{2}(s)=\left\{2 p^{1-s} \sqrt{(s / \pi) \log p}\right\}\{1+o(1)\} .
$$

Define

$$
\hat{\mathcal{T}}_{1, n}(s)=\frac{T_{1 n}(s)-\hat{\mu}_{T_{1 n}, 0}(s)}{\tilde{\sigma}_{T_{1 n}, 0}(s)}
$$

where $\hat{\mu}_{T_{1 n}, 0}(s)$ is a sufficiently accurate estimator of $\mu_{T_{1 n}, 0}(s)$ in a similar sense to (2.6) and $\tilde{\sigma}_{T_{1 n}, 0}^{2}(s)=2 p^{1-s} \sqrt{(s / \pi) \log p}$. The maximal $L_{1}$-thresholding statistic is

$$
\hat{\mathcal{M}}_{1 n}=\max _{s \in \mathcal{S}} \hat{\mathcal{T}}_{1, n}(s)
$$

where, again $\mathcal{S}=(0,1-\eta]$. It can be shown that $\hat{\mathcal{M}}_{1 n}=\max _{s \in \mathcal{S}_{n}} \hat{\mathcal{T}}_{1, n}(s)$ for the same $S_{n}$ in (3.4).

Using a similar approach to that in Theorem 2, we can show that

$$
P\left(a(\log p) \hat{\mathcal{M}}_{1 n}-b(\log p, \eta) \leq x\right) \rightarrow \exp \left(-e^{-x}\right) .
$$

Hence, an $\alpha$-level maximal $L_{1}$-thresholding test rejects the $H_{0}$ if

$$
\begin{equation*}
\hat{\mathcal{M}}_{1 n}>\mathcal{B}_{\alpha} . \tag{4.3}
\end{equation*}
$$

From (3.6), (4.2) and (4.3), the three tests have the same critical values $\mathcal{B}_{\alpha}$ at nominal level $\alpha$. This brings convenience for the power comparison. Let us define the power of the three tests

$$
\Omega_{\gamma}(r, \beta):=P\left(\hat{\mathcal{M}}_{\gamma n}>\mathcal{B}_{\alpha}\right)
$$

for $\gamma=0,1$ and 2 respectively. Notice that

$$
\begin{equation*}
\hat{\mathcal{M}}_{\gamma n}=\max _{s \in \mathcal{S}_{n}}\left\{\mathcal{I}_{\gamma n}(s) \tilde{e}_{\gamma}(s)+\tilde{\sigma}_{T_{\gamma n}, 0}^{-1}(s)\left(\mu_{T_{\gamma n}, 0}(s)-\hat{\mu}_{T_{\gamma n}, 0}(s)\right)\right\}, \tag{4.4}
\end{equation*}
$$

where $\tilde{e}_{\gamma}(s)=\sigma_{T_{\gamma n}, 0}(s) / \tilde{\sigma}_{T_{\gamma n}, 0}(s)$ and

$$
\mathcal{I}_{\gamma n}(s)=\sigma_{T_{\gamma n}, 0}^{-1}(s)\left(T_{\gamma n}(s)-\mu_{T_{\gamma n}, 0}(s)\right)=\mathcal{I}_{\gamma n, 1}(s) R_{\gamma}(s)+\Delta_{\gamma, 0}(s ; r, \beta),
$$

in which $R_{\gamma}(s)=\sigma_{T_{\gamma n}, 1}(s) / \sigma_{T_{\gamma n}, 0}(s), \mathcal{T}_{\gamma n, 1}(s)=\sigma_{T_{\gamma n}, 1}^{-1}(s)\left(T_{\gamma n}(s)-\mu_{T_{\gamma n}, 1}(s)\right)$ and $\Delta_{\gamma, 0}(s ; r, \beta)=\sigma_{T_{\gamma n}, 0}^{-1}(s)\left(\mu_{T_{\gamma n}, 1}(s)-\mu_{T_{\gamma n}, 0}(s)\right)$. As shown in (A.8), (A.22) and (A.24) in the Appendix

$$
\begin{aligned}
& \Delta_{0,0}(s ; r, \beta)=(s \pi \log p)^{\frac{1}{4}} p^{1 / 2-\beta+s / 2} I(r>s)+L_{p}^{(6)} p^{1 / 2-\beta-(\sqrt{s}-\sqrt{r})^{2}+s / 2} I(r<s) \\
& \Delta_{1,0}(s ; r, \beta)=(s \pi \log p)^{\frac{1}{4}}(r / s)^{\frac{1}{4}} p^{1 / 2-\beta+s / 2} I(r>s)+L_{p}^{(6)} p^{1 / 2-\beta-(\sqrt{s}-\sqrt{r})^{2}+s / 2} I(r<s) .
\end{aligned}
$$

and
$\Delta_{2,0}(s ; r, \beta)=(s \pi \log p)^{\frac{1}{4}}(r / s) p^{1 / 2-\beta+s / 2} I(r>s)+L_{p}^{(6)} p^{1 / 2-\beta-(\sqrt{s}-\sqrt{r})^{2}+s / 2} I(r<s)$,
where $L_{p}^{(6)}=\{2(\sqrt{s}-\sqrt{r})\}^{-1} s^{1 / 4}(\pi \log p)^{-1 / 4}$.
Derivations given in the proof of Theorem 4 in the Appendix show that for $\gamma=0,1$ and 2

$$
\begin{equation*}
\hat{\mathcal{M}}_{\gamma n} \sim \max _{s \in \mathcal{S}_{n}} \Delta_{\gamma, 0}(r, s, \beta) \tag{4.5}
\end{equation*}
$$

where ' $a \sim b$ ' means that the $a / b=1+o_{p}(1)$. This implies that we only need to compare $\max _{s \in \mathcal{S}_{n}} \Delta_{\gamma, 0}(r, s, \beta)$ in the power comparison.

From the established expressions of $\Delta_{\gamma, 0}(s ; r, \beta)$, we note two facts. One is that if $r>2 \beta-1$, for any $s \in(2 \beta-1, r)$,
$\Delta_{2,0}(s ; r, \beta) / \Delta_{1,0}(s ; r, \beta)=(r / s)^{3 / 4}>1$ and $\Delta_{1,0}(s ; r, \beta) / \Delta_{0,0}(s ; r, \beta)=(r / s)^{1 / 4}>1$.
The other is if $r \in\left(\varrho^{*}(\beta), 2 \beta-1\right]$, asymptotically,

$$
\begin{equation*}
\Delta_{0,0}(s ; r, \beta)=\Delta_{1,0}(s ; r, \beta)=\Delta_{2,0}(s ; r, \beta) \quad \text { for all } s \in \mathcal{S} . \tag{4.7}
\end{equation*}
$$

Hence, when $(r, \beta)$ lies just above the detection boundary, the three $\Delta_{\gamma, 0}$ functions are the same. If $(r, \beta)$ moves further away from the detection boundary so that $r>2 \beta-1$, there will be a clear ordering among the $\Delta_{\gamma, 0}$ functions. The following theorem summarizes the relative power performance.

Theorem 4. Assume C.1-C.5 and (2.6) hold. For any given significant level $\alpha \in(0,1)$, the powers of the $H C$, the maximal $L_{1}$ and $L_{2}$-thresholding tests under $H_{1}$ as specified in C.4 satisfy, as $n \rightarrow \infty$,

$$
\begin{equation*}
\Omega_{0}(r, \beta) \leq \Omega_{1}(r, \beta) \leq \Omega_{2}(r, \beta) \quad \text { for } r>2 \beta-1 \tag{4.8}
\end{equation*}
$$

and $\Omega_{\gamma}(r, \beta)$ s are asymptotic equivalent for $r \in\left(\varrho^{*}(\beta), 2 \beta-1\right]$.
The theorem indicates that when $(r, \beta)$ is well above the detection boundary such that $r>2 \beta-1$ there is a clear ordering in the power among the three tests with the $L_{2}$ being the most powerful followed by the $L_{1}$ test. However, when $(r, \beta)$ is just above the detection boundary such that $r \in\left(\varrho^{*}(\beta), 2 \beta-1\right]$, the three tests have asymptotically equivalent powers. In the latter case, comparing the second order terms of $\hat{\mathcal{M}}_{\gamma n}$ may lead to differentiations among the powers of the three tests. However, it is a rather technical undertaking to assess the impacts of the second order terms. The analysis conducted in Theorem 4 is applicable to the setting of Gaussian data with $n=1$ and $\boldsymbol{\Sigma}$ satisfying C.3, which is the setting commonly assumed in the investigation of the detection boundary for the HC test (Donoho and Jin, 2004; Hall and Jin, 2010; and Arias-Castro et al., 2012a). Specifically, the power ordering among the three maximal thresholding tests in Theorem 4 remains
but under lesser conditions C.3-C.5. Condition C. 1 is not needed since the Gaussian assumption allows us to translate the problem to $n=1$ since the sample mean is sufficient. Condition C. 2 is automatically satisfied for the Gaussian distribution. The condition (2.6) is met for the Gaussian data as we have discussed in Section 2.
5. Simulation Results. We report results from simulation experiments which were designed to evaluate the performance of the maximal $L_{1}$ and $L_{2}$-thresholding tests and the HC test. The purpose of the simulation study is to confirm the theoretical findings that there is an ordering in the power among the three tests discovered in Theorem 4.

Independent and identically distributed $p$-dim random vectors $\mathbf{X}_{i}$ were generated according to

$$
\mathbf{X}_{i}=\mathbf{W}_{i}+\boldsymbol{\mu}, \quad i=1, \cdots, n
$$

where $\mathbf{W}_{i}=\left(W_{i 1}, \cdots, W_{i p}\right)^{T}$ is a stationary random vector and $\left\{W_{i j}\right\}_{j=1}^{p}$ have the same marginal distribution $F$. In the simulation, $\mathbf{W}_{i}$ was generated from a $p$-dimensional multivariate Gaussian distribution with zero mean and covariance $\boldsymbol{\Sigma}=\left(\sigma_{i j}\right)_{p \times p}$ where $\sigma_{i j}=\rho^{|i-j|}$ for $\rho=0.3$ and 0.5 , respectively.

The simulation design on $\boldsymbol{\mu}$ had the sparsity parameter $\beta=0.6,0.7$ and 0.8 , respectively, and the signal strength $r=0.1,0.3,0.5,0.6,0.8,0.9,1.1$ and 1.2 , respectively. We chose two scenarios on the dimension and sample size combinations: (a) a large $p$, small $n$ setting and (b) both $p$ and $n$ are moderately large. For Scenario (a), we chose $p=\exp \left(c_{0} n^{0.3}+c_{1}\right)$ where $c_{0}=1.90$ and $c_{1}=2.30$ so that the dimensions $p$ were 2000 and 20,000 , and the sample sizes $n$ are 30 and 100 , respectively. We note that under the setting $\beta=0.8$, there were only 4 and 7 non-zero means, respectively, among the 2000 and 20,000 dimensions. And those for $\beta=0.7$ were 9 and 19 respectively, and those for $\beta=0.6$ were 20 and 52 , respectively. These were quite sparse. For Scenario (b), we chose $p=n^{1.25}+184$ such that $(p, n)=(500,100)$ and $(p, n)=(936,200)$.

The maximal $L_{2}$-test statistic $\hat{\mathcal{M}}_{2 n}$ was constructed using $\tilde{\mu}_{T_{2 n}, 0}(s)$ and $\tilde{\sigma}_{T_{2 n}, 0}(s)$ given in (2.3) and (2.4), respectively, as the mean and standard deviation estimators. The maximal $L_{1}$ test statistic and the HC test statistic, $\hat{\mathcal{M}}_{1 n}$ and $\hat{\mathcal{M}}_{0 n}$, were constructed similarly using the leading order mean and standard deviation under $H_{0}$. The set of thresholding level $\mathcal{S}$ was chosen to be $(0,1-\eta]$ with $\eta=0.05$.

Figures 1-4 display the average empirical sizes and powers of the HC, the maximal $L_{1}$ and $L_{2}$-thresholding tests based on 20,000 simulations, with Figures 1-2 for Scenario (a) and Figures 3-4 for Scenario (b). To make the power comparison fair and conclusive, we adjusted the nominal level of the tests so that the simulated sizes of the tests were all around $\alpha=0.05$ with the HC had slightly larger sizes than those of the maximal $L_{1}$ test, and the sizes of the maximal $L_{1}$ test were slightly
larger than those of the maximal $L_{2}$ test. These were designed to rule out potential "favoritism" in the power comparison due to advantages in the sizes of the maximal $L_{2}$ and/or $L_{1}$ tests.

Figures $1-4$ show that the power of the tests were the most influenced by the signal strength parameter $r$, followed by the sparsity $\beta$. The powers were insensitive to the level of dependence $\rho$, which confirmed our finding that the thresholding largely removes the dependence. The observed ordering in the empirical power shown in Figures 1-4 were consistent to the conclusions in Theorem 4. We observed that in all the simulation settings, despite some size advantages by the HC test and/or the maximal $L_{1}$ test, the maximal $L_{2}$ test had better power than the maximal $L_{1}$ and the HC test, and the maximal $L_{1}$ test test had better power than the HC test. We find that for each fixed level of sparsity $\beta$, when the signal strength $r$ was increased so that $(r, \beta)$ move away from the detection boundary $r=\varrho^{*}(\beta)$, the difference among the powers of the three test were enlarged. This was especially the case for the most sparse case of $\beta=0.8$ and was indeed confirmatory to Theorem 4. The simulated powers of the three tests were very much the same at $r=0.1$ and were barely changed even both $n$ and $p$ were increased. This was consistent with the fact that $r=0.1$ is below the detection boundary for $\beta=0.7$ and 0.8 considered in the simulation.
6. Discussion. Our analysis shows that there are alternative $L_{1}$ and $L_{2}$ formulations to the HC test which attain the detection boundary $r=\varrho^{*}(\beta)$ of the HC test. The tests based on the $L_{1}$ and $L_{2}$ formulations are more powerful than the HC test when $(r, \beta)$ pair is away from the detection boundary such that $r>2 \beta-1$. The three tests have asymptotically equivalent power when $(r, \beta)$ is just above the detection boundary.

The detection boundary $r=\varrho^{*}(\beta)$ coincides with that of the HC test discovered in Donoho and Jin (2004) for the Gaussian data with independent components. That the three tests considered in this paper attain the detection boundary $r=\varrho^{*}(\beta)$ under the considered sub-Gaussian setting with column-wise dependence can be understood in two aspects. One is that the three test statistics are all directly formulated via the marginal sample means $\bar{X}_{j}$ which are asymptotically normally distributed; the other is that the thresholding statistics are asymptotically uncorrelated as implied from Proposition 1.

According to Ingster (1997) and Donoho and Jin (2004), $r=\varrho^{*}(\beta)$ is the optimal detection boundary for Gaussian distributed data with independent components. However, it may not be optimal for the dependent nonparametric setting considered in this paper. Indeed, for weakly dependent Gaussian data, Hall and Jin (2010) showed that the detection boundary $r=\varrho^{*}(\beta)$ can be lowered by utilizing the dependence. The latter was carried out by pre-transforming the data with $\mathbf{L}$, the

Fig 1. Empirical sizes and powers of the $H C$ (dotted lines with squares), the maximal $L_{1}$ - (dashed lines with dots) and $L_{2}$-(solid lines with circles) thresholding tests when $p=2000$ and $n=30$ with the marginal distribution is the standard normal.






Fig 2. Empirical sizes and powers of the $H C$ (dotted lines with squares), the maximal $L_{1}$ - (dashed lines with dots) and $L_{2}$-(solid lines with circles) thresholding tests when $p=20,000$ and $n=100$ with the marginal distribution is the standard normal.







Fig 3. Empirical sizes and powers of the $H C$ (dotted lines with squares), the maximal $L_{1}$ - (dashed lines with dots) and $L_{2}$-(solid lines with circles) thresholding tests when $p=500$ and $n=100$ with the marginal distribution is the standard normal.




Fig 4. Empirical sizes and powers of the HC (dotted lines with squares), the maximal $L_{1}$ - (dashed lines with dots) and $L_{2}$-(solid lines with circles) thresholding tests when $p=936$ and $n=200$ with the marginal distribution is the standard normal.





inverse of Cholesky decomposition of $\boldsymbol{\Sigma}$, or an empirically estimate of $\mathbf{L}$ and then conducting the HC test based on the transformed data. It is expected that the main results of this paper on the relative performance of the three tests would remain valid for the transformed data. Hall and Jin (2008) and Delaigle and Hall (2009) studied the detection boundary for dependent data and Cai and Wu (2012) studied the boundary for detecting mixtures with a general known distribution. However, the optimal detection boundary under the dependent sub-Gaussian distribution setting is still an open problem.

## APPENDIX A: TECHNICAL DETAILS

In this Appendix, we provide proofs to Theorems 2, 3 and 4 reported in Sections 3 and 4. Throughout this Appendix we use $L_{p}=C \log ^{b}(p)$ to denote slow varying functions for some constant $b$ and positive constant $C$, and $\phi(\cdot)$ and $\bar{\Phi}(\cdot)$ for the density and survival functions of the standard normal distribution, respectively. Let $\rho_{k}$ be the correlation coefficient between $W_{i 1}$ and $W_{i(k+1)}$, and write $\rho_{1}=\rho$ for simplicity and $\mu_{j}=E\left(X_{i j}\right)$ for $i \in\{1, \cdots, n\}$ and $j \in\{1, \cdots, p\}$. Put $\lambda_{p}(s)=$ $2 s \log p$.

Proof of Theorem 2. Let $u=\bar{\Phi}\left(\lambda_{p}^{1 / 2}(s)\right)$. Write $\mathcal{J}_{2}(u):=\hat{\mathcal{T}}_{2, n}(s)$ and

$$
\mathcal{M}_{2 n}=\max _{s \in(0,1-\eta]} \hat{\mathcal{T}}_{2, n}(s)=\max _{u \in\left[u_{0}, 1 / 2\right)} \mathcal{J}_{2}(u)
$$

where $u_{0}=\bar{\Phi}\left(\lambda_{p}^{1 / 2}(1-\eta)\right)$. Using the same technique for the proof of Theorem 1 in Zhong et al. (2013), it may be shown that the joint asymptotic normality of $\mathcal{T}_{2, n}(s)$ at any finite points $\underline{s}=\left(s_{1}, \cdots, s_{d}\right)^{T}$. This is equivalent to the joint asymptotic normality of $\mathcal{J}_{2}(u)$ at $u_{i}=\bar{\Phi}\left(\sqrt{2 s_{i} \log p}\right)$ for $i=1, \cdots, d$.

We want to show the tightness of the process $\mathcal{J}_{2}(u)$. Let $f_{n, u}(x)=\sigma_{0}^{-1}(u) x^{2} I\{|x|>$ $g(u)\}$ where $\sigma_{0}^{2}(u)=\sigma_{0}^{2}(p ; s)$ and $\sigma_{0}^{2}(p ; s)=\sigma_{T_{2 n}, 0}^{2}(s) / p$. Write

$$
\mathcal{J}_{2}(u)=p^{-1 / 2} \sum_{j=1}^{p}\left\{f_{n, u}\left(\left|\sqrt{n} \bar{X}_{j}\right|\right)-E\left(f_{n, u}\left(\left|\sqrt{n} \bar{X}_{j}\right|\right)\right)\right\},
$$

where $g(u)=\bar{\Phi}^{-1}(u)$. Based on the finite dimensional convergence of $\mathcal{J}_{2}(u)$ and Theorem 1.5.6 in Van der Vaart and Wellner (1996), we only need to show the asymptotically equicontinuous of $\mathcal{J}_{2}(u)$, that is, for any $\varepsilon>0$ and $\eta>0$ there exists a finite partition $\Lambda=\cup_{i=1}^{k} \Lambda_{i}$ such that

$$
\begin{equation*}
\lim \sup _{n \rightarrow \infty} P^{*}\left\{\max _{1 \leq i \leq k} \sup _{u, v \in \Lambda_{i}}\left|\mathcal{J}_{2}(u)-\mathcal{J}_{2}(v)\right|>\varepsilon\right\}<\eta, \tag{A.1}
\end{equation*}
$$

where $P^{*}$ is the outer probability measure.

Define $\mathscr{F}_{n}=\left\{f_{n, u}\left(\left|\sqrt{n} \bar{X}_{j}\right|\right)=\sigma_{0}^{-1}(u)\left|\sqrt{n} \bar{X}_{j}\right|^{2} I\left\{\left|\sqrt{n} \bar{X}_{j}\right|>g(u)\right\}: u \in \Lambda:=\right.$ $\left.\left[u_{0}, 1 / 2\right)\right\}$ and $\rho\left(f_{n, u}-f_{n, v}\right)=\left[E\left\{f_{n, u}\left(\left|\sqrt{n} \bar{X}_{j}\right|\right)-f_{n, v}\left(\left|\sqrt{n} \bar{X}_{j}\right|\right)\right\}^{2}\right]^{1 / 2}$. It can be shown that if $u>v$,

$$
\rho\left(f_{n, u}-f_{n, v}\right)^{2}=\left\{2-2 \sigma_{0}^{-1}(u) \sigma_{0}(v)\right\}\{1+o(1)\} .
$$

Thus, for every $\delta_{n} \rightarrow 0, \sup _{|u-v|<\delta_{n}} \rho\left(f_{n, u}-f_{n, v}\right) \rightarrow 0$, which implies that for each $\delta>0, \Lambda$ can be partitioned into finitely many sets $\Lambda_{1}, \cdots, \Lambda_{k}$ satisfy

$$
\max _{1 \leq i \leq k} \sup _{u, v \in \Lambda_{i}} \rho\left(f_{n, u}-f_{n, v}\right)<\delta .
$$

Let $N\left(\varepsilon, \mathscr{F}_{n}, \rho\right)$ be the bracketing number $N_{0}$, the smallest number of functions $f_{1}, \cdots, f_{N_{0}}$ in $\mathscr{F}_{n}$ such that for each $f$ in $\mathscr{F}_{n}$ there exists an $f_{i}\left(i \in\left\{1, \cdots, N_{0}\right\}\right)$ satisfying $\rho\left(f-f_{i}\right) \leq \varepsilon$. Applying Theorem 2.2 in Andrews and Pollard (1994), if the following two conditions hold for an even integer $Q \geq 2$ and a real number $\gamma>0$ such that

$$
\begin{array}{r}
\sum_{d=1}^{\infty} d^{Q-2} \alpha(d)^{\frac{\gamma}{Q+\gamma}}<\infty \quad \text { and } \\
\int_{0}^{1} \varepsilon^{-\frac{\gamma}{2+\gamma}} N\left(\varepsilon, \mathscr{F}_{n}, \rho\right)^{1 / Q} d \varepsilon<\infty, \tag{A.3}
\end{array}
$$

we have for $n$ large enough $\left\|\sup _{\substack{\rho\left(f_{n, u}-f_{n, v}\right)<\delta \\ u, v \in \mathcal{L}_{i}}}\left|\mathcal{J}_{2}(u)-\mathcal{J}_{2}(v)\right|\right\|_{Q}<k^{-1 / Q} \eta \varepsilon$.
Invoking the maximal inequality of Pisier (1983), it follows that

$$
\left\|\max _{1 \leq i \leq k} \sup _{\substack{\left.\rho\left(f_{n, u}, f_{n, v}, t \in \Lambda_{i}\right)<\delta \\ s, t\right)_{i}}}\left|\mathcal{J}_{2}(u)-\mathcal{J}_{2}(v)\right|\right\|_{Q}<\eta \varepsilon
$$

Now using the Markov inequality, we get for $n$ large enough

$$
\begin{aligned}
& P^{*}\left\{\max _{1 \leq i \leq k} \sup _{u, v \in \Lambda_{i}}\left|\mathcal{J}_{2}(u)-\mathcal{J}_{2}(v)\right|>\varepsilon\right\} \\
& \quad \leq\left\|\max _{1 \leq i \leq k} \sup _{\substack{\rho\left(f_{n}, u-f_{n, v}\right)<\delta \\
u, v \in \Lambda_{i}}}\left|\mathcal{J}_{2}(u)-\mathcal{J}_{2}(v)\right|\right\|_{Q} / \varepsilon<\eta .
\end{aligned}
$$

Hence, the condition (A.1) holds and $\mathcal{J}_{2}(u)$ is asymptotically tight.
It remains to show (A.2) and (A.3) hold. For (A.3), we note that $\mathscr{F}_{n}$ is a V-C class for each $n$. This is because

$$
\mathscr{G}_{n}=\left\{f_{n, u}(x)=\sigma_{0}^{-1}(u) I(x>g(u)): u \in\left(u_{0}, 1 / 2\right)\right\}
$$

is a V-C class with VC index 2 . Let $\varphi(x)=x^{2}$. Then $\mathscr{F}_{n}=\varphi \cdot \mathscr{G}_{n}$ is a V-C class by Lemma 2.6.18 in Van der Vaart and Wellner (1996). Let $G_{n}\left(x, u_{0}\right)=\sup _{u \in \Lambda}\left|f_{n, u}(x)\right|$
be the envelop function for class $\mathscr{F}_{n}$. Clearly, we can take $G_{n}\left(x, u_{0}\right)=\sigma_{0}^{-1}\left(u_{0}\right) x^{2}$. It is easy to see that $\rho\left\{G_{n}\left(\left|\sqrt{n} \bar{X}_{i}\right|, u_{0}\right)\right\}<\infty$ for a constant $u_{0}>0$. Applying a result on covering number of V-C classes (Theorem 2.6.7, Van der Vaart and Wellner, 1996), we get $N\left(\varepsilon, \mathscr{F}_{n}, \rho\right) \leq K \varepsilon^{-2}$ for a universal constant $K$. It can be verified that if $Q>2+\gamma$, then (A.3) holds. The condition (A.2) follows from the assumption that $\rho_{Z}(d) \leq C \alpha^{d}$.

As a result, $\mathcal{J}_{2}(u)$ converge to a zero mean Gaussian process $\mathcal{N}_{2}(u)$ with

$$
\operatorname{Cov}\left(\mathcal{N}_{2}(u), \mathcal{N}_{2}(v)\right)=\frac{\sigma_{0}(u)}{\sigma_{0}(v)}=\exp \left(-\frac{1}{2}\left[\log \left\{\sigma_{0}^{2}(v)\right\}-\log \left\{\sigma_{0}^{2}(u)\right\}\right]\right)
$$

for $u<v$. It can be shown that there exists an Ornstein-Uhlenbeck $(\mathrm{O}-\mathrm{U})$ process $\mathcal{U}_{2}(\cdot)$ with mean zero 0 and $E\left(\mathcal{U}_{2}(u) \mathcal{U}_{2}(v)\right)=\exp \{-|u-v|\}$ such that $\mathcal{N}_{2}(u)=$ $\mathcal{U}_{2}\left(\frac{1}{2} \log \left\{\sigma_{0}^{2}(u)\right\}\right)$. Therefore, by a result for O-U process in Leadbetter et al. (1983, p217),

$$
\begin{aligned}
P\left(\max _{s \in \mathcal{S}} \hat{\mathcal{T}}_{2, n}(s)<B_{\tau_{n}}(x)\right) & =P\left(\max _{u \in \Lambda} \mathcal{N}_{2}(u)<B_{\tau_{n}}(x)\right) \\
& =P\left(\max _{u \in\left(0, \tau_{n}\right)} \mathcal{U}_{2}(u)<B_{\tau_{n}}(x)\right) \rightarrow \exp \{-\exp (-x)\}
\end{aligned}
$$

where $\tau_{n}=\frac{1}{2} \log \left\{\sigma_{0}^{2}\left(\frac{1}{2}\right) / \sigma_{0}^{2}\left(u_{0}\right)\right\}, B_{\tau_{n}}(x)=\left(x+b^{*}\left(\tau_{n}\right)\right) / a\left(\tau_{n}\right), a(t)=(2 \log (t))^{1 / 2}$ and $b^{*}(t)=2 \log (t)+2^{-1} \log \log (t)-\frac{1}{2} \log (\pi)$. From (2.4), we have $\tau_{n}=\frac{1-\eta}{2} \log p\{1+$ $o(1)\}$. Since

$$
\begin{aligned}
a\left(\tau_{n}\right) \max _{u \in\left(0, \tau_{n}\right)} \mathcal{U}_{2}(u)-b^{*}\left(\tau_{n}\right)= & \frac{a\left(\tau_{n}\right)}{a(\log p)}\left[a(\log p) \max _{u \in\left(0, \tau_{n}\right)} \mathcal{U}_{2}(u)-b^{*}(\log p)\right] \\
& +\frac{a\left(\tau_{n}\right)}{a(\log p)} b^{*}(\log p)-b^{*}\left(\tau_{n}\right),
\end{aligned}
$$

$a\left(\tau_{n}\right) / a(\log p) \rightarrow 1$ and

$$
\begin{aligned}
\frac{a\left(\tau_{n}\right)}{a(\log p)} b^{*}(\log p)-b^{*}\left(\tau_{n}\right)= & \frac{a\left(\tau_{n}\right)}{a(\log p)}\left[b^{*}(\log p)-b^{*}\left(\tau_{n}\right)\right] \\
& +b^{*}\left(\tau_{n}\right)\left[\frac{a\left(\tau_{n}\right)}{a(\log p)}-1\right] \rightarrow-\log \frac{(1-\eta)}{2},
\end{aligned}
$$

we have

$$
a\left(\tau_{n}\right) \max _{u \in\left(0, \tau_{n}\right)} \mathcal{U}_{2}(u)-b^{*}\left(\tau_{n}\right)=a(\log p) \max _{u \in\left(0, \tau_{n}\right)} \mathcal{U}_{2}(u)-\left(b^{*}(\log p)+\log \frac{(1-\eta)}{2}\right) .
$$

Finally, note that $b^{*}(\log p)+\log \frac{(1-\eta)}{2}=b(\log p, \eta)$. This finishes the proof of Theorem 2.

Proof of Theorem 3. (i). The proof is made under four cases. For each case, we find the corresponding detectable region and the union of the four regions is the overall detectable region of the thresholding test. Basically, we show for any ( $\beta, r$ ) above $\varrho^{*}(\beta)$ within one of the four cases, there exists at least one threshold level $s$ such that $H_{1}$ is detectable. For notation simplification, we only keep the leading order terms for $\mu_{T_{2 n}, 1}(s)-\mu_{T_{2 n}, 0}(s), \sigma_{T_{2 n, 1}}(s), \sigma_{T_{2 n, 0}}(s)$ and $\Delta_{2}(s ; r, \beta)$.

Case 1: $s \leq r$ and $s \leq \beta$. In this case, $\mu_{T_{2 n}, 1}(s)-\mu_{T_{2 n}, 0}(s)=L_{p} p^{1-\beta}$ and $\sigma_{T_{2 n}, 1}(s)=\sigma_{T_{2 n}, 0}(s)=L_{p} p^{(1-s) / 2}$. Hence

$$
\Delta_{2}(s ; r, \beta)=\frac{\mu_{T_{2 n}, 1}(s)-\mu_{T_{2 n}, 0}(s)}{\sigma_{T_{2 n}, 1}(s)}=L_{p} p^{(1+s-2 \beta) / 2} .
$$

So to make $\left(\mu_{T_{2 n}, 1}(s)-\mu_{T_{2 n}, 0}(s)\right) / \sigma_{T_{2 n}, 1}(s) \rightarrow \infty$, we need $s>2 \beta-1$. It follows that the detectable region for this case is $r \geq 2 \beta-1$. Specifically, if we select $s=\min \{r, \beta\}$, we arrive at the best divergence rate for $\Delta_{2}(s ; r, \beta)$ of order $L_{p} p^{(1+\min \{r, \beta\}-2 \beta) / 2}$.

Case 2: $s \leq r$ and $s>\beta$. In this case, $\mu_{T_{2 n}, 1}(s)-\mu_{T_{2 n}, 0}(s)=L_{p} p^{1-\beta}, \sigma_{T_{2 n}, 1}(s)=$ $L_{p} p^{(1-\beta) / 2}$, and $\sigma_{T_{2 n}, 0}(s)=L_{p} p^{(1-s) / 2}$. Then,

$$
\Delta_{2}(s ; r, \beta)=\frac{\mu_{T_{2 n}, 1}(s)-\mu_{T_{2 n}, 0}(s)}{\sigma_{T_{2 n}, 1}(s)}=L_{p} p^{(1-\beta) / 2}
$$

So the detectable region in ( $\beta, r$ ) plane is $r>\beta$. In this region, the best divergence rate of $\Delta_{2}$ is of order $L_{p} p^{(1-\beta) / 2}$ for any $\beta<s \leq r$.

Case 3: $s>r$ and $s \leq(\sqrt{s}-\sqrt{r})^{2}+\beta$. The case is equivalent to $\sqrt{r}<\sqrt{s} \leq$ $(r+\beta) /(2 \sqrt{r})$ and $\mu_{T_{2 n}, 1}(s)-\mu_{T_{2 n}, 0}(s)=L_{p} p^{1-(\sqrt{s}-\sqrt{r})^{2}-\beta}, \sigma_{T_{2 n}, 1}(s)=\sigma_{T_{2 n}, 0}=$ $L_{p} p^{(1-s) / 2}$. Then

$$
\begin{equation*}
\Delta_{2}(s ; r, \beta)=\frac{\mu_{T_{2 n}, 1}(s)-\mu_{T_{2 n}, 0}(s)}{\sigma_{T_{2 n}, 1}(s)}=L_{p} p^{\frac{1}{2}-\beta+r-(\sqrt{s}-2 \sqrt{r})^{2} / 2} \tag{A.4}
\end{equation*}
$$

To ensure (A.4) diverging to infinity, we need

$$
2 \sqrt{r}-\sqrt{1-2 \beta+2 r}<\sqrt{s}<2 \sqrt{r}+\sqrt{1-2 \beta+2 r} .
$$

Thus, the detectable region must satisfy

$$
\sqrt{r}<(r+\beta) /(2 \sqrt{r}), 1-2 \beta+2 r>0 \text { and } 2 \sqrt{r}-\sqrt{1-2 \beta+2 r} \leq(r+\beta) /(2 \sqrt{r})
$$

This translates to

$$
\left.\beta-\frac{1}{2}<r<\beta \text { and either } r \leq \beta / 3 \text { or } r>\beta / 3 \text { and } r \geq(1-\sqrt{1-\beta})^{2}\right) \text {. }
$$

Case 4: $s>r$ and $s>(\sqrt{s}-\sqrt{r})^{2}+\beta$. This is equivalent to $\sqrt{s}>\max \{(r+$ $\beta) /(2 \sqrt{r}), \sqrt{r}\}$. In this case, $\mu_{T_{2 n}, 1}(s)-\mu_{T_{2 n}, 0}(s)=L_{p} p^{1-(\sqrt{s}-\sqrt{r})^{2}-\beta}, \sigma_{T_{2 n}, 1}(s)=$ $L_{p} p^{\left(1-(\sqrt{s}-\sqrt{r})^{2}-\beta\right) / 2}$. Then

$$
\Delta_{2}(s ; r, \beta)=\frac{\mu_{T_{2 n}, 1}(s)-\mu_{T_{2 n}, 0}(s)}{\sigma_{T_{2 n}, 1}(s)}=L_{p} p^{\left(1-(\sqrt{s}-\sqrt{r})^{2}-\beta\right) / 2} .
$$

Hence, it requires that

$$
\sqrt{r}-\sqrt{1-\beta}<\sqrt{s}<\sqrt{r}+\sqrt{1-\beta} .
$$

In order to find a $s$, we need $\sqrt{r}+\sqrt{1-\beta}>\max \{(r+\beta) /(2 \sqrt{r}), \sqrt{r}\}$. If $\sqrt{r}>$ $(r+\beta) /(2 \sqrt{r})$, namely $r>\beta$, the above inequality is obviously true. If $r \leq \beta$, then $\sqrt{r}+\sqrt{1-\beta}>(r+\beta) /(2 \sqrt{r})$ is equivalent to $r>(1-\sqrt{1-\beta})^{2}$. So the detectable region is $r>(1-\sqrt{1-\beta})^{2}$ in this case.

In summary of Cases 1-4, the union of the detectable regions in above four cases is $r>\varrho^{*}(\beta)$, as illustrated in Figure 5 .

Now we are ready to prove the theorem. We only need to show that the sum of type I and II errors of the maximal test goes to 0 when $r>\varrho^{*}(\beta)$. Because the maximal test is of asymptotic $\alpha_{n}$ level, it suffices to show that the power goes to 1 in the detectable region as $n \rightarrow \infty$ and $\alpha_{n} \rightarrow 0$. Recall that the $\alpha_{n}$ level rejection region is $R_{\alpha_{n}}=\left\{\hat{\mathcal{M}}_{2 n}>\mathcal{B}_{\alpha_{n}}\right\}$. From Theorem 2, we notice that $\mathcal{B}_{\alpha_{n}}=O\left\{(\log \log p)^{1 / 2}\right\}:=$ $L_{p}^{*}$. Then, it is sufficient if

$$
\begin{equation*}
P\left(\mathcal{M}_{2 n} / L_{p}^{*} \rightarrow \infty\right) \rightarrow 1, \quad \text { as } n \rightarrow \infty \tag{A.5}
\end{equation*}
$$

at every $(\beta, r)$ in the detectable region. Since $\mathcal{M}_{2 n} \geq \mathcal{T}_{2}(s)$ for any $s \in \mathcal{S}$. Therefore, (A.5) is true if for any point in the detectable region, there exists a $\lambda_{p}(s)=2 s \log p$ such that

$$
\begin{equation*}
\mathcal{T}_{2}(s) / L_{p}^{*} \xrightarrow{p} \infty . \tag{A.6}
\end{equation*}
$$

Therefore, we want to show

$$
\begin{equation*}
\frac{T_{2 n}(s)-\mu_{T_{2 n}, 0}(s)}{L_{p}^{*} \sigma_{T n, 0}(s)}=\left(\frac{T_{2 n}(s)-\mu_{T_{2 n}, 1}(s)}{L_{p}^{*} \sigma_{T_{2 n}, 1}(s)}+\frac{\mu_{T_{2 n}, 1}(s)-\mu_{T_{2 n}, 0}(s)}{L_{p}^{*} \sigma_{T_{2 n}, 1}(s)}\right) \frac{\sigma_{T_{2 n}, 1}(s)}{\sigma_{T_{2 n}, 0}(s)} \xrightarrow{p} \infty . \tag{A.7}
\end{equation*}
$$

Because $\left(T_{2 n}(s)-\mu_{T_{2 n}, 1}(s)\right) / L_{p}^{*} \sigma_{T_{2 n}, 1}(s)=o_{p}(1)$ and $\sigma_{T_{2 n}, 0}(s) \leq \sigma_{T_{2 n}, 1}(s)$, (A.7) is true if $\left(\mu_{T_{2 n}, 1}(s)-\mu_{T_{2 n}, 0}(s)\right) / L_{p}^{*} \sigma_{T_{2 n}, 1}(s) \rightarrow \infty$. As we have shown in the early proof, for every $(r, \beta)$ in the detectable region, there exist a $s$ such that $\frac{\mu_{T_{2 n}, 1}(s)-\mu_{T_{2 n}, 0}(s)}{L_{p} \sigma_{T_{2 n}, 1}(s)} \rightarrow$ $\infty$ for any slow varying function $L_{p}$. This concludes (A.6) and hence (A.5), which completes the proof of Part (i).


Fig 5. The detectable sub-regions of the $L_{2}$ threshold test. Case 1: the union of $\{I, I I, I I I, I V\} ;$ Case 2, the region is I; Case 3: the union of $\{I I, I I I, I V, V, V I, V I I\} ;$ Case 4: the union of $\{I, I I$, III, VI, VII\}.
(ii) Note that

$$
\hat{\mathcal{M}}_{2 n}=\max _{s \in \mathcal{S}_{n}}\left\{\left(\mathcal{T}_{2 n, 1}(s) R_{2}(s)+\Delta_{2,0}(s ; r, \beta)\right) \tilde{e}_{2}(s)+\frac{\mu_{T_{\gamma n}, 0}(s)-\hat{\mu}_{T_{\gamma n}, 0}(s)}{\tilde{\sigma}_{T_{\gamma n}, 0}(s)}\right\}
$$

where $R_{2}(s), \tilde{e}_{2}(s)$ and $\mathcal{T}_{2 n, 1}(s)$ are defined in (4.4) and
(A.8)

$$
\begin{aligned}
\Delta_{2,0}(s ; r, \beta)= & \frac{\mu_{T_{2 n}, 1}(s)-\mu_{T_{2 n}, 0}(s)}{\sigma_{T_{2 n}, 0}(s)}=(s \pi \log p)^{1 / 4}(r / s) p^{1 / 2-\beta+s / 2} I(r>s) \\
& +\frac{s^{1 / 4}(\pi \log p)^{-1 / 4}}{2(\sqrt{s}-\sqrt{r})} p^{1 / 2-(\sqrt{s}-\sqrt{r})^{2}-\beta+s / 2} I(r<s) .
\end{aligned}
$$

If $r<\varrho^{*}(\beta)$, then $r<\beta$ and $r<(r+\beta)^{2} /(4 r)$. Hence

$$
R_{2}(s)= \begin{cases}1+o(1) & \text { if } s \leq r \\ 1+o(1) & \text { if } r<s \leq \frac{(r+\beta)^{2}}{4 r} \\ s^{\frac{1}{4}}(\sqrt{s}-\sqrt{r})^{-\frac{1}{2}} p^{\frac{1}{2}}(2 \sqrt{s r}-r-\beta) & \{1+o(1)\} \\ \text { if } s>\frac{(r+\beta)^{2}}{4 r}\end{cases}
$$

It is also noticed that $r<\varrho^{*}(\beta)$ implies that $(r+\beta)^{2} /(4 r)>1$. Therefore, for all $s \in \mathcal{S}_{n}, R_{2}(s)=1+o(1)$.

If $r<\varrho^{*}(\beta)$, then $r<2 \beta-1$. Hence, $\max _{s \leq r} \Delta_{2,0}(s ; r, \beta) \leq L_{p} p^{1 / 2-\beta+r / 2} \rightarrow 0$ as $p(n) \rightarrow \infty$.

If $r<\varrho^{*}(\beta)$ and $r<1 / 4$, then $r<\beta-1 / 2$. It follows that, for all $s>r$,

$$
1 / 2-(\sqrt{s}-\sqrt{r})^{2}-\beta+s / 2=1 / 2+r-\beta-\frac{1}{2}(\sqrt{s}-2 \sqrt{r})^{2} \leq 1 / 2+r-\beta<0
$$

If $r<\varrho^{*}(\beta)$ and $r>1 / 4$, then for all $s>r$,

$$
1 / 2-(\sqrt{s}-\sqrt{r})^{2}-\beta+s / 2 \leq 1 / 2+r-\beta-\frac{1}{2}(1-2 \sqrt{r})^{2}<0
$$

Hence, $\max _{s>r} \Delta_{2,0}(s ; r, \beta) \leq L_{p} p^{1 / 2+r-\beta} I\{r<1 / 4\}+L_{p} p^{1-\beta-(1-\sqrt{r})^{2}} I\{r>1 / 4\} \rightarrow$ 0 as $p(n) \rightarrow \infty$. In summary, we have $R_{2}(s)=1+o(1)$ and $\max _{s \in \mathcal{S}_{n}} \Delta_{2,0}(s ; r, \beta) \rightarrow 0$ if $r<\varrho^{*}(\beta)$. Therefore, together with the assumption (2.6), $\hat{\mathcal{M}}_{2 n}=\max _{s \in \mathcal{S}_{n}} \mathcal{T}_{2 n, 1}(s)\{1+$ $\left.o_{p}(1)\right\}$.

We note that, by employing the same argument of Theorem 2, it can be shown that

$$
P\left(a(\log p) \max _{s \in \mathcal{S}} \mathcal{T}_{2 n, 1}(s)-b(\log p, \delta) \leq x\right) \rightarrow \exp \left(-e^{-x}\right)
$$

where $\delta$ is defined just above (A.11). Then the power of the test

$$
\begin{aligned}
& P\left(\hat{\mathcal{M}}_{2 n}>\left(\mathcal{E}_{\alpha_{n}}+b(\log p, \eta)\right) / a(\log p)\right) \\
& =P\left(\hat{\mathcal{M}}_{2 n}>\left(\mathcal{E}_{\alpha_{n}}+b(\log p, \delta)\right) / a(\log p)\right)\{1+o(1)\}=\alpha_{n}\{1+o(1)\} \rightarrow 0 .
\end{aligned}
$$

Thus the sum of type I and II errors goes to 1 . This completes the proof of Part (ii).

Proof of Theorem 4. We first prove that $\hat{\mathcal{M}}_{\gamma n} \sim \max _{s \in \mathcal{S}_{n}} \Delta_{\gamma, 0}(r, s, \beta)$, which will be proved in two parts.

$$
\begin{array}{r}
\hat{\mathcal{M}}_{\gamma n} \sim \mathcal{M}_{\gamma n} \quad \text { and } \\
\mathcal{M}_{\gamma n} \sim \max _{s \in \mathcal{S}_{n}} \Delta_{\gamma, 0}(r, s, \beta) \tag{A.10}
\end{array}
$$

where $\mathcal{M}_{\gamma n}=\max _{s \in \mathcal{S}_{n}} \mathcal{T}_{\gamma n}(s)=\max _{s \in \mathcal{S}_{n}}\left\{\mathcal{T}_{\gamma n, 1}(s) R_{\gamma}(s)+\Delta_{\gamma, 0}(s ; r, \beta)\right\}$.
To show (A.9), note the decomposition for $\mathcal{M}_{\gamma n}$ in (4.4). Let $\mathcal{M}_{\gamma n}=\max _{s \in \mathcal{S}_{n}}\left\{\mathcal{T}_{\gamma n}(s) \tilde{e}_{\gamma}(s)\right\}$. We can firstly show that $\hat{\mathcal{M}}_{\gamma n} \sim \tilde{\mathcal{M}}_{\gamma n}$ because of the following inequality

$$
\tilde{\mathcal{M}}_{\gamma n}-\left|\max _{s \in \mathcal{S}_{n}} \frac{\mu_{T_{\gamma n}, 0}(s)-\hat{\mu}_{T_{\gamma n}, 0}(s)}{\tilde{\sigma}_{T_{\gamma n}, 0}(s)}\right| \leq \hat{\mathcal{M}}_{\gamma n} \leq \tilde{\mathcal{M}}_{\gamma n}+\left|\max _{s \in \mathcal{S}_{n}} \frac{\mu_{T_{\gamma n}, 0}(s)-\hat{\mu}_{T_{\gamma n}, 0}(s)}{\tilde{\sigma}_{T_{\gamma n}, 0}(s)}\right| .
$$

Under condition (2.6), i.e., $\max _{s \in \mathcal{S}} \tilde{\sigma}_{T_{\gamma n}, 0}^{-1}(s)\left(\mu_{T_{\gamma n}, 0}(s)-\hat{\mu}_{T_{\gamma n}, 0}(s)\right)=o(1)$, hence, $\hat{\mathcal{M}}_{\gamma n} \sim \tilde{\mathcal{M}}_{\gamma n}$. Second, we can show $\mathcal{M}_{\gamma n} \sim \tilde{\mathcal{M}}_{\gamma n}$. Note the following inequality

$$
\begin{aligned}
\min \left\{\mathcal{M}_{\gamma n} \min _{s \in \mathcal{S}_{n}} \tilde{e}_{\gamma}(s), \mathcal{M}_{\gamma n} \max _{s \in \mathcal{S}_{n}} \tilde{e}_{\gamma}(s)\right\} & \leq \tilde{\mathcal{M}}_{\gamma n} \\
& \leq \max \left\{\mathcal{M}_{\gamma n} \min _{s \in \mathcal{S}_{n}} \tilde{e}_{\gamma}(s), \mathcal{M}_{\gamma n} \max _{s \in \mathcal{S}_{n}} \tilde{e}_{\gamma}(s)\right\} .
\end{aligned}
$$

Under conditions C.1-C.4, $\min _{s \in \mathcal{S}_{n}} \tilde{e}_{\gamma}(s)=\max _{s \in \mathcal{S}_{n}} \tilde{e}_{\gamma}(s)=1+o(1)$. So we have

$$
\tilde{\mathcal{M}}_{\gamma n} \sim \mathcal{M}_{\gamma n} \min _{s \in \mathcal{S}_{n}} \tilde{e}_{\gamma}(s) \sim \mathcal{M}_{\gamma n} \min _{s \in \mathcal{S}_{n}} \tilde{e}_{\gamma}(s) \sim \mathcal{M}_{\gamma n} .
$$

In summary, we have $\hat{\mathcal{M}}_{\gamma n} \sim \tilde{\mathcal{M}}_{\gamma n} \sim \mathcal{M}_{\gamma n}$. Therefore, $\hat{\mathcal{M}}_{\gamma n} \sim \mathcal{M}_{\gamma n}$.
The path leading to (A.10) is the following. First of all, it can be shown using an argument similar to the one used in the proof of Theorem 2 that

$$
P\left(a(\log p) \max _{s \in \mathcal{S}} \mathcal{I}_{\gamma n, 1}(s)-b(\log p, \delta) \leq x\right) \rightarrow \exp \left(-e^{-x}\right)
$$

where $\delta=\max \{\eta-r+2 r \sqrt{1-\eta}-\beta, \eta\} I(r<1-\eta)+\max \{1-\beta, \eta\} I(r>1-\eta)$. Thus, for $\gamma=0,1$ and 2 ,

$$
\begin{equation*}
\max _{s \in \mathcal{S}} \mathcal{I}_{\gamma n, 1}(s)=O_{p}\left\{\log ^{1 / 2}(\log p)\right\} . \tag{A.11}
\end{equation*}
$$

Equations (A.13) to (A.20) in the following reveal that for all $s \in \mathcal{S}$ and $r>\varrho^{*}(\beta)$, we can classify $s \in \mathcal{S}$ into two sets $\mathcal{S}_{1}$ and $\mathcal{S}_{2}$ such that

$$
\begin{aligned}
& \text { (i) } \Delta_{\gamma, 0}(s ; r, \beta) \gg R_{\gamma}(s) \text { for } s \in \mathcal{S}_{1} \\
& \text { (ii) } \Delta_{\gamma, 0}(s ; r, \beta) \rightarrow 0 \text { and } R_{\gamma}(s)=1+o(1) \text { for } s \in \mathcal{S}_{2}
\end{aligned}
$$

where ' $c \gg d$ ' means that $c / d=L_{p} p^{\xi}$ for some $\xi>0$. Because $r$ is above the detection boundary $\varrho^{*}(\beta)$, there exists at least one $s \in \mathcal{S}_{1}$ such that $\Delta_{\gamma, 0}(s ; r, \beta) \rightarrow$ $\infty$. Hence,

$$
\begin{equation*}
\max _{s \in \mathcal{S}} \Delta_{\gamma, 0}(s ; r, \beta)=\max _{s \in \mathcal{S}_{1}} \Delta_{\gamma, 0}(s ; r, \beta) \gg \max _{s \in \mathcal{S}} R_{\gamma}(s) . \tag{A.12}
\end{equation*}
$$

Namely, the maximum of $\Delta_{\gamma, 0}(s ; r, \beta)$ is reached on Set $\mathcal{S}_{1}$ where $\Delta_{\gamma, 0}(s ; r, \beta)$ diverges at a much faster rate than that of $\tilde{R}_{\gamma}(s)$, if the latter ever diverges.

Let $A(s)=\mathcal{T}_{2 n, 1}(s) R_{\gamma}(s)$. Combining (A.11) and (A.12), we have

$$
\left|\max _{s \in \mathcal{S}_{n}} \mathcal{I}_{\gamma n, 1}(s)\right|\left|\max _{s \in \mathcal{S}_{n}} R_{\gamma}(s)\right|=o_{p}\left\{\max _{s \in \mathcal{S}_{n}} \Delta_{\gamma, 0}(s ; r, \beta)\right\}
$$

This implies that $\left|\max _{s \in \mathcal{S}_{n}} A(s)\right|=o_{p}\left\{\max _{s \in \mathcal{S}_{n}} \Delta_{\gamma, 0}(s ; r, \beta)\right\}$. Together with the following inequality,

$$
\begin{aligned}
\max _{s \in \mathcal{S}_{n}} \Delta_{\gamma, 0}(s ; r, \beta)-\left|\max _{s \in \mathcal{S}_{n}} A(s)\right| & \leq \max _{s \in \mathcal{S}_{n}}\left\{A(s)+\Delta_{\gamma, 0}(s ; r, \beta)\right\} \\
& \leq \max _{s \in \mathcal{S}_{n}} \Delta_{\gamma, 0}(s ; r, \beta)+\max _{s \in \mathcal{S}_{n}} A(s)
\end{aligned}
$$

we conclude that (A.10) holds.
It remains to show the existence of $\mathcal{S}_{1}$ and $\mathcal{S}_{2}$ in arriving (A.12). We only prove it for the $L_{2}$ test. To complete that, we compare the relative order between $\Delta_{2,0}(s ; r, \beta)$ and $R_{2}(s)$ for three regions above the detection boundary $\varrho^{*}(\beta)$ : (i) $r>\beta$ (ii) $r \in$ $(2 \beta-1, \beta]$ and (iii) $r \in\left(\varrho^{*}(\beta), 2 \beta-1\right]$. In regions (i) and (ii) with $r>(1-\sqrt{1-\beta})^{2}$, we can show that

$$
\begin{align*}
& \Delta_{2,0}(s ; r, \beta) \gg R_{2}(s) \text { for } s>2 \beta-1  \tag{A.13}\\
& \Delta_{2,0}(s ; r, \beta) \rightarrow 0 \text { and } R_{2}(s)=1+o(1) \text { for } s \leq 2 \beta-1 . \tag{A.14}
\end{align*}
$$

In region (ii) with $r<(1-\sqrt{1-\beta})^{2}$, we have

$$
\begin{align*}
& \Delta_{2,0}(s ; r, \beta) \gg R_{2}(s) \text { for } 2 \beta-1<s \leq(2 \sqrt{r}+\sqrt{1+2 r-2 \beta})^{2} ;  \tag{A.15}\\
& \Delta_{2,0}(s ; r, \beta) \rightarrow 0 \text { and } R_{2}(s)=1+o(1) \text { for } s \leq 2 \beta-1 \tag{A.16}
\end{align*}
$$

$$
\text { and }(2 \sqrt{r}+\sqrt{1+2 r-2 \beta})^{2}<s<1 .
$$

For $r \in\left(\varrho^{*}(\beta), 2 \beta-1\right]$ in region (iii). If $r>(1-\sqrt{1-\beta})^{2}$, define $D_{1}=(0,(2 \sqrt{r}-$ $\left.\sqrt{1+2 r-2 \beta})^{2}\right)$ and $D_{2}=\left((2 \sqrt{r}-\sqrt{1+2 r-2 \beta})^{2}, 1\right)$. Then it may be shown that

$$
\begin{align*}
& \Delta_{2,0}(s ; r, \beta) \rightarrow 0 \text { and } R_{2}(s)=1+o(1) \text { for } s \in D_{1} ;  \tag{A.17}\\
& \Delta_{2,0}(s ; r, \beta) \gg R_{2}(s) \text { for } s \in D_{2} .
\end{align*}
$$

If $r<(1-\sqrt{1-\beta})^{2}$, define $D_{3}=\left(0,(2 \sqrt{r}-\sqrt{1+2 r-2 \beta})^{2}\right) \cup\left((2 \sqrt{r}+\sqrt{1+2 r-2 \beta})^{2}, 1\right)$ and $D_{4}=\left((2 \sqrt{r}-\sqrt{1+2 r-2 \beta})^{2},(2 \sqrt{r}+\sqrt{1+2 r-2 \beta})^{2}\right)$. Then, it can be shown that

$$
\begin{align*}
& \Delta_{2,0}(s ; r, \beta) \rightarrow 0 \text { and } R_{2}(s)=1+o(1) \text { for } s \in D_{3} ;  \tag{A.19}\\
& \Delta_{2,0}(s ; r, \beta) \gg R_{2}(s) \text { for } s \in D_{4} . \tag{A.20}
\end{align*}
$$

The results in (A.13)-(A.20) indicate that in each region listed above, $\max \Delta_{2,0}(s ; r, \beta)$ will be attained in situations covered by (A.13), (A.15), (A.18) and (A.20), which together imply (A.12).

Next, we compute $\Delta_{\gamma, 0}(s ; r, \beta)$ for the $\mathrm{HC}(\gamma=0)$ and the $L_{1}(\gamma=1)$ test. For the HC test, let $G_{p, 1}(s)=P\left(Y_{i, n}>2 s \log p\right)$. Under assumptions C.1-C.2, applying the large deviation results (Petrov, 1995), it may be shown that

$$
\begin{aligned}
& G_{p, 1}(s)=\left\{(2 \sqrt{\pi \log p}(\sqrt{s}-\sqrt{r}))^{-1} p^{-(\sqrt{s}-\sqrt{r})^{2}}\right\}\{1+o(1)\} \quad \text { if } r<s \text { and } \\
& G_{p, 1}(s)=\left\{1-(2 \sqrt{\pi \log p}(\sqrt{r}-\sqrt{s}))^{-1} p^{-(\sqrt{r}-\sqrt{s})^{2}}\right\}\{1+o(1)\} \quad \text { if } r>s .
\end{aligned}
$$

The mean and variance of $T_{0 n}(s)$ under $H_{0}$ are $\mu_{T_{0 n}, 0}(s)=(\sqrt{s \pi \log p})^{-1} p^{1-s}\{1+$ $o(1)\}$ and $\sigma_{T_{0 n}, 0}^{2}(s)=(\sqrt{s \pi \log p})^{-1} p^{1-s}\{1+o(1)\}$ respectively. The mean and variance of $T_{0 n}(s)$ under the $H_{1}$ as specified in C. 4 are, respectively,

$$
\begin{aligned}
\mu_{T_{0 n}, 1}(s) & =p^{1-\beta} G_{p, 1}(s)+\left(p-p^{1-\beta}\right) 2 \bar{\Phi}\left(\lambda_{p}^{1 / 2}(s)\right)\{1+o(1)\} \quad \text { and } \\
\sigma_{T_{0 n}, 1}^{2}(s) & =p^{1-\beta} G_{p, 1}(s)\left(1-G_{p, 1}(s)\right)+p\left(1-p^{-\beta}\right) 2 \bar{\Phi}\left(\lambda_{p}^{1 / 2}(s)\right)\left(1-2 \bar{\Phi}\left(\lambda_{p}^{1 / 2}(s)\right)\right)
\end{aligned}
$$

These imply that, up to a factor $\{1+o(1)\}$,
$\mu_{T_{0 n}, 1}(s)-\mu_{T_{0 n}, 0}(s)=\left\{(2 \sqrt{\pi \log p}(\sqrt{s}-\sqrt{r}))^{-1} p^{1-\beta-(\sqrt{s}-\sqrt{r})^{2}} I(r<s)+p^{1-\beta} I(r>s)\right\}$.
and

$$
R_{0}(s)= \begin{cases}1 & \text { if } s \leq(\sqrt{s}-\sqrt{r})^{2}+\beta \\ s^{1 / 4}|2(\sqrt{s}-\sqrt{r})|^{-\frac{1}{2}} p^{-\frac{1}{2}\left((\sqrt{s}-\sqrt{r})^{2}+\beta-s\right)} & \text { if } s>(\sqrt{s}-\sqrt{r})^{2}+\beta\end{cases}
$$

Hence,

$$
\begin{align*}
\Delta_{0,0}(s ; r, \beta)= & \frac{s^{1 / 4}}{2(\sqrt{s}-\sqrt{r})(\pi \log p)^{1 / 4}} p^{1 / 2-\beta-(\sqrt{s}-\sqrt{r})^{2}+s / 2} I(r<s) \\
& +(s \pi \log p)^{1 / 4} p^{1 / 2-\beta+s / 2} I(r>s) . \tag{A.22}
\end{align*}
$$

For $L_{1}$ test, the mean and variances of $T_{1 n}(s)$ under $H_{1}$ specified in C. 4 are respectively, up to a factor $1+o(1)$,

$$
\begin{aligned}
\mu_{T_{1 n}, 1}(s)= & \frac{\sqrt{s}}{\sqrt{2 \pi}(\sqrt{s}-\sqrt{r})} p^{1-\beta-(\sqrt{s}-\sqrt{r})^{2}} I(r<s) \\
& +(\sqrt{2 r \log p}) p^{1-\beta} I(r>s)+\sqrt{2 / \pi} p^{1-s} \text { and } \\
\sigma_{T_{1 n}, 1}^{2}(s)= & \frac{s \sqrt{\log p}}{\sqrt{\pi}(\sqrt{s}-\sqrt{r})} p^{1-\beta-(\sqrt{s}-\sqrt{r})^{2}} I(r<s)+p^{1-\beta} I(r>s)+2 \sqrt{(s / \pi) \log p} p^{1-s} .
\end{aligned}
$$

It follows that, up to a factor $1+o(1)$,
(A.23)
$\mu_{T_{1 n}, 1}(s)-\mu_{T_{1 n}, 0}(s)=\frac{\sqrt{s}}{\sqrt{2 \pi}(\sqrt{s}-\sqrt{r})} p^{1-\beta-(\sqrt{s}-\sqrt{r})^{2}} I(r<s)+(\sqrt{2 r \log p}) p^{1-\beta} I(r>s)$
and

$$
R_{1}(s)= \begin{cases}1 & \text { if } s \leq r \text { and } s \leq \beta \\ (\sqrt{2})^{-1}\left(\frac{s}{\pi}\right)^{-\frac{1}{4}}(\log p)^{-\frac{1}{4}} p^{(s-\beta) / 2} & \text { if } s \leq r \text { and } s \geq \beta \\ 1 & \text { if } s>r \text { and } s \leq(\sqrt{s}-\sqrt{r})^{2}+\beta \\ s^{\frac{1}{4}}(2 \sqrt{s}-2 \sqrt{r})^{-\frac{1}{2}} p^{-\frac{1}{2}\left((\sqrt{s}-\sqrt{r})^{2}+\beta-s\right)} & \text { if } s>r \text { and } s>(\sqrt{s}-\sqrt{r})^{2}+\beta\end{cases}
$$

Therefore,

$$
\begin{align*}
\Delta_{1,0}(s ; r, \beta)= & \frac{s^{\frac{1}{4}}}{2(\pi \log p)^{\frac{1}{4}}(\sqrt{s}-\sqrt{r})} p^{1 / 2-\beta-(\sqrt{s}-\sqrt{r})^{2}+s / 2} I(r<s) \\
& +(s \pi \log p)^{\frac{1}{4}}(r / s)^{\frac{1}{4}} p^{1 / 2-\beta+s / 2} I(r>s) . \tag{A.24}
\end{align*}
$$

Replicating the above proof for the $L_{2}$ test, it can be shown that, for $\gamma=0$ and 1 ,

$$
\hat{\mathcal{M}}_{\gamma n} \sim \max _{s \in \mathcal{S}_{n}} \Delta_{\gamma, 0}(s ; r, \beta)
$$

At last, we will compare $\max _{s \in \mathcal{S}_{n}} \Delta_{\gamma, 0}(s ; r, \beta)$ for $\gamma=0,1$ and 2 when $r>2 \beta-1$. Let $s_{n}^{*}=\arg \max \left\{s: s \in \mathcal{S}_{n} \cap(2 \beta-1, r)\right\}$ be a threshold in $(2 \beta-1, r)$ that is closest to $r$. Then the maximal value of $\Delta_{\gamma, 0}(s, r, \beta)$ over $\mathcal{S}_{n}$ is attained at $s_{n}^{*}$. Note that such $s_{n}^{*}$ exists with probability 1. To show this point, it is enough to show that $\mathcal{S}_{n} \cap(2 \beta-1, r) \neq \emptyset$, which is equivalent to show that $P\left(\cup_{i=1}^{p}\left\{Y_{i, n} \in\right.\right.$ $((4 \beta-2) \log p, 2 r \log p)\}) \rightarrow 1$. Let $\left\{k_{1}, \cdots, k_{q}\right\} \in(1, \cdots, p)$ be a sub-sequence such that $q \rightarrow \infty$ and $k_{\text {min }}=\min _{j}\left|k_{j}-k_{j-1}\right| \rightarrow \infty$. Let $D_{n}=\prod_{k=k_{1}}^{k_{q}} P\left(\left\{Y_{i, n} \in\right.\right.$ $\left.\left.((4 \beta-2) \log p, 2 r \log p)^{c}\right\}\right)-P\left(\cap_{i=k_{1}}^{k_{q}}\left\{Y_{i, n} \in((4 \beta-2) \log p, 2 r \log p)^{c}\right\}\right)$. By mixing assumption (C.5) and triangle inequality, it can be seen that $\left|D_{n}\right| \leq q \alpha_{Z}\left(k_{\text {min }}\right) \rightarrow 0$ as $n \rightarrow \infty$. Then it follows that

$$
\begin{aligned}
& P\left(\cup_{i=1}^{p}\left\{Y_{i, n} \in((4 \beta-2) \log p, 2 r \log p)\right\}\right) \\
& \geq P\left(\cup_{i=k_{1}}^{k_{q}}\left\{Y_{i, n} \in((4 \beta-2) \log p, 2 r \log p)\right\}\right) \\
& =1-P\left(\cap_{i=k_{1}}^{k_{q}}\left\{Y_{i, n} \in((4 \beta-2) \log p, 2 r \log p)^{c}\right\}\right) \\
& =1-\prod_{k=k_{1}}^{k_{q}} P\left(\left\{Y_{i, n} \in((4 \beta-2) \log p, 2 r \log p)^{c}\right\}\right)+D_{n} \rightarrow 1
\end{aligned}
$$

where we used $P\left(\left\{Y_{i, n} \in((4 \beta-2) \log p, 2 r \log p)^{c}\right\}\right)<1$ for all $i=1, \cdots, p$. Comparing (A.8), (A.22) and (A.24), we see that $\Delta_{0,0}\left(s_{n}^{*} ; r, \beta\right)<\Delta_{1,0}\left(s_{n}^{*} ; r, \beta\right)<$ $\Delta_{2,0}\left(s_{n}^{*} ; r, \beta\right)$.

It follows that, for $r>2 \beta-1$,

$$
\max _{s \in \mathcal{S}_{n}} \Delta_{0,0}(s ; r, \beta)<\max _{s \in \mathcal{S}_{n}} \Delta_{1,0}(s ; r, \beta)<\max _{s \in \mathcal{S}_{n}} \Delta_{2,0}(s ; r, \beta) .
$$

Therefore, asymptotically with probability $1, \hat{\mathcal{M}}_{0 n}<\hat{\mathcal{M}}_{1 n}<\hat{\mathcal{M}}_{2 n}$, which results in $\Omega_{0}(r, \beta) \leq \Omega_{1}(r, \beta) \leq \Omega_{2}(r, \beta)$. This completes the proof.

## SUPPLEMENTARY MATERIAL

A supplemental to "Tests alternative to higher criticism for high dimensional means under sparsity and column-wise dependence"
(http://www.e-publications.org/ims/support/dowload/imsart-ims.zip). The supplementary material contains proofs for Proposition 1 and Theorem 1 in Section 2.

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