# Complementary Design Theory for Uniform Designs

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**Abstract.** Uniform design is to seek its design points to be uniformly scattered on the experimental domain under some discrepancy measure. In this paper all the design points of a full factorial design can be split into two subdesigns. One is called the complementary design of the other. The complementary design theory of characterization one design through the other under the four commonly used discrepancy measures is investigated. Based on this complementary design theory, some general rules for searching for uniform designs through their complementary designs are proposed, and so many new uniform or nearly uniform designs can be obtained via the current small uniform designs. An example is presented to illustrate the usefullness of the proposed theory.

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**Keywords:** Complementary design, discrepancy, uniformity.

## 1 Introduction

Uniform design has been widely used especially for computer experiments since it was proposed by Fang (1980) [1] and Wang and Fang (1981) [14]. Its main idea is to seek its design points to be uniformly scattered on the experimental domain under some discrepancy measure. The commonly used measures of non-uniformity include the centered $L_2$-discrepancy (CD for short), the wrap-around $L_2$-discrepancy (WD for short) and the symmetric $L_2$-discrepancy (SD for short ) introduced by Hickernell (1998a [8], 1998b [9]) and the discrete discrepancy (DD for short) proposed by Hickernell and Liu (2002) [10]. A comprehensive discussion about the relationships among them refer to Fang, Li and Sudjianto (2006) [2]. A design $D$ in a specified design space $\mathcal{D}$ is said to be a unform design under some discrepancy if it minimizes the discrepancy among all designs in $\mathcal{D}$.

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Many methods for the construction of uniform or nearly uniform designs have been proposed. They are broadly classified into two categories: combinatorial algebra methods and algorithm optimizations. Based on the techniques of combinatorial algebras and numeric theories, good lattice method (Fang and Wang, 1994 [7]), Latin square method (Fang, Shiu and Pan, 1999 [5]) and balanced incomplete block design method (Lu and Meng, 2000 [12]) were introduced. Along the line of algorithm optimizations, the threshold accepting heuristic (Winker and Fang, 1997 [15]), simulated annealing (Morris and Mitchell, 1995 [13]), stochastic evolutionary (Jin, Chen and Sudjianto, 2005 [11]) and balance-pursuit heuristic (Fang, Tang and Yin, 2005 [6]) were adopted. For detailed reviews and discussion of all kinds of constructions of uniform or nearly uniform designs refer to Fang, Li and Sudjianto (2006) [2] and the related references therein.

In this paper a full factorial design can be split into two subdesigns according to the design points. One is called the complementary design of the other. The complementary design theory of characterization one design through the other under different discrepancy measures is investigated, and so many new uniform or nearly uniform designs can be obtained via the current small uniform designs.

The remainder of this paper is organized as follows. Section 2 presents the quadratic forms of the four discrepancy measures CD, WD, SD and DD. Section 3 establishes some relationships between the discrepancy measures of one design and its complementary design. Based on this complementary design theory, some general rules for searching for uniform designs through their complementary designs are proposed in Section 4. An example is presented to illustrate the usefulness of the proposed theory. Section 5 concludes this paper with some reviews.

# 2 Quadratic forms of discrepancies

Some notations are introduced here. Let $A \otimes B$ denote the Kronecker product of two matrices $A$ and $B$. For any positive integers $q, q_1, \ldots, q_m$, let $V_q = \{1, 2, \ldots, q\}$, $V^m = V_{q_1} \times \cdots \times V_{q_m}$ and $N = q_1 \cdots q_m$. A mixed-level (or asymmetrical) design of $n$ runs and $m$ factors with levels $q_1, \ldots, q_m$, denoted by $(n, q_1 \cdots q_m)$, is a set of $n$ row vectors (or points) in $V^m$ or an $n \times m$ matrix in which each row represents a run, each column represents a factor and the $j$th column takes values from a set of $q_j$ symbols, say, $V_{q_j}$. In particular, an $(n, q^m)$-design is symmetrical. Two designs are called isomorphic if one can be obtained from the other through permutations of rows, columns and symbols in each column. A design is called balanced or U-type design if all levels of each factor appear equally often and denoted by $D(n, q_1 \cdots q_m)$. In this paper, we only consider the balanced designs which are usually needed in practice. The set of all such balanced designs are denoted by $\mathcal{D}(n, q_1 \cdots q_m)$.

For a design $D = (d_{ij}) \in \mathcal{D}(n, q_1 \cdots q_m)$, the detailed computational formula of $\mathrm{CD}^2(D)$, $\mathrm{WD}^2(D)$, $\mathrm{SD}^2(D)$ and $\mathrm{DD}^2(D)$ were derived by Hickernell (1998a [8], b [9]) and Fang, Lin, and Liu (2003) [3], respectively. Note that all the four formula can be expressed as the following unified form

$$\text{constant} + n^{-2}\sum_{i=1}^{N}\sum_{j=1}^{N}\prod_{k=1}^{m} f(d_{ik}, d_{jk}, q_k) - 2n^{-1}\sum_{i=1}^{N}\prod_{k=1}^{m} g(d_{ik}, q_k),$$

where $f(\cdot,\cdot,\cdot)$ and $g(\cdot,\cdot)$ are different types of functions according to different discrepancies.

For a $D(n, q_1 \cdots q_m)$ design $D$, let $n(i_1, \ldots, i_m)$ denote the number of times that the point $(i_1, \ldots, i_m)$ occurs in $D$. Then the design $D$ can be uniquely determined by the column vector of length $N$ given by

$$\mathbf{y}_D = (n(i_1, \ldots, i_m))_{(i_1,\ldots,i_m)\in V^m}\,, \tag{1}$$

where all points $(i_1, \ldots, i_m)$'s in $V^m$ are arranged in the lexicographical order. In particular, a full factorial $D(N, q_1 \cdots q_m)$ design corresponds to $\mathbf{1}_N$, the $N$-vector of ones. By noticing

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\prod_{k=1}^{m} f(d_{ik}, d_{jk}, q_k) = \sum_{(i_1,\ldots,i_m)\in V^m}\ \sum_{(j_1,\ldots,j_m)\in V^m} n(i_1, \ldots, i_m)\, n(j_1, \ldots, j_m)\prod_{k=1}^{m} f(i_k, j_k, q_k)$$

and

$$\sum_{i=1}^{N}\prod_{k=1}^{m} g(d_{ik}, q_k) = \sum_{(i_1,\ldots,i_m)\in V^m} n(i_1, \ldots, i_m)\prod_{k=1}^{m} g(i_k, q_k),$$

the following quadratic forms of $\mathrm{CD}^2(D)$, $\mathrm{WD}^2(D)$, $\mathrm{SD}^2(D)$ and $\mathrm{DD}^2(D)$ in terms of $\mathbf{y}_D$ can be obtained.

**Lemma 1.** *For a design $D \in \mathcal{D}(n, q_1 \cdots q_m)$, we have*

$$\mathrm{WD}^2(D) = -\left(\frac{4}{3}\right)^m + \frac{1}{n^2}\mathbf{y}_D^T\mathbf{W}\mathbf{y}_D, \tag{2}$$

$$\mathrm{CD}^2(D) = \left(\frac{13}{12}\right)^m - \frac{2}{n}\mathbf{c}^T\mathbf{y}_D + \frac{1}{n^2}\mathbf{y}_D^T\mathbf{C}\mathbf{y}_D, \tag{3}$$

$$\mathrm{SD}^2(D) = \left(\frac{4}{3}\right)^m - \frac{2}{n}\mathbf{s}^T\mathbf{y}_D + \frac{1}{n^2}\mathbf{y}_D^T\mathbf{S}\mathbf{y}_D \tag{4}$$

*and*

$$\mathrm{DD}^2(D) = -\prod_{j=1}^{m}\left[\frac{a + (q_j - 1)b}{q_j}\right] + \frac{1}{n^2}\mathbf{y}_D^T\mathbf{E}\mathbf{y}_D, \tag{5}$$

*where for $i, j = 1, \ldots, q_k$ and $k = 1, \ldots, m$, $\mathbf{W} = \mathbf{W}_1 \otimes \cdots \otimes \mathbf{W}_m$, $\mathbf{W}_k = (w_{ij}^k)$,*

$$w_{ij}^k = 1.5 - |i-j|(q_k - |i-j|)/q_k^2,$$

$\mathbf{C} = \mathbf{C}_1 \otimes \cdots \otimes \mathbf{C}_m,\ \mathbf{C}_k = (c_{ij}^k),$

$$c_{ij}^k = 1 + |2i-1-q_k|/(4q_k) + |2j-1-q_k|/(4q_k) - |i-j|/(2q_k),$$

$\mathbf{c} = \mathbf{c}_1 \otimes \cdots \otimes \mathbf{c}_m,\ \mathbf{c}_k = (c_1^k, \ldots, c_{q_k}^k)^T,$

$$c_i^k = 1 + |2i-1-q_k|/(4q_k) - |2i-1-q_k|^2/(8q_k^2),$$

$\mathbf{S} = \mathbf{S}_1 \otimes \cdots \otimes \mathbf{S}_m,\ \mathbf{S}_k = (s_{ij}^k),$

$$s_{ij}^k = 2 - 2|i-j|/q_k,$$

$\mathbf{s} = \mathbf{s}_1 \otimes \cdots \otimes \mathbf{s}_m,\ \mathbf{s}_k = (s_1^k, \ldots, s_{q_k}^k)^T,$

$$s_i^k = 1 + (2i-1)/q_k - (2i-1)^2/(2q_k^2),$$

*and* $\mathbf{E} = \mathbf{E}_1 \otimes \cdots \otimes \mathbf{E}_m,\ \mathbf{E}_k = (e_{ij}^k),$

$$e_{ij}^k = \begin{cases} a, & \text{if} \quad i = j, \\ b, & \text{otherwise,} \end{cases}$$

$a > b > 0$ *are two constants used in the definition of the kernel function of the discrepancy measure DD.*

It should be mentioned that the expressions of $\mathrm{CD}^2(D)$ and $\mathrm{WD}^2(D)$ were first presented in Fang and Qin (2003) [4].

# 3 Complementary design theory

Let $H = D(N, q_1 \cdots q_m)$, i.e., the full factorial $(N, q_1 \cdots q_m)$ design. Here we consider only the designs in $\mathcal{D}(n, q_1 \cdots q_m)$ with no repeated points. For a $D(n, q_1 \cdots q_m)$ design $D$, let $H = (D^T, \overline{D}^T)^T$ be a row partition of $H$ after several row permutations such that $D$ consists of the first $n$ runs of $H$ and $\overline{D}$ consists of the remaining $N-n$ runs. The design $\overline{D}$ is called the complementary design of $D$. Note that $\mathbf{y}_D = \mathbf{1}_N - \mathbf{y}_{\overline{D}}$. Based on Lemma 1, the relationships between the discrepancies of design $D$ and its complementary design $\overline{D}$ can be obtained and summarized in the following Theorem 1, whose detailed proof is postponed to Appendix.

**Theorem 1.** *For any design* $D \in \mathcal{D}(n, q_1 \cdots q_m)$, *the discrepancies of design* $D$ *and its* *complementary design $\overline{D}$ have the following relationships*

$$\begin{aligned}\mathrm{WD}^2(D) &= -\left(\frac{4}{3}\right)^m + \frac{2n-N}{n^2}\prod_{k=1}^{m}\left(\frac{4q_k}{3}+\frac{1}{6q_k}\right) \\ &\quad + \frac{(N-n)^2}{n^2}\left[\mathrm{WD}^2(\overline{D}) + \left(\frac{4}{3}\right)^m\right], \end{aligned} \tag{6}$$

$$\begin{aligned}\mathrm{CD}^2(D) &= \left(\frac{13}{12}\right)^m + \frac{1}{n^2}\left(\mathbf{1}_N^T\mathbf{C}\mathbf{1}_N - 2n\mathbf{c}^T\mathbf{1}_N\right) \\ &\quad + \frac{1}{n^2}\left[-2\mathbf{y}_D^T(\mathbf{C}\mathbf{1}_N - N\mathbf{c}) + (N-n)^2\left(\mathrm{CD}^2(\overline{D}) - \left(\frac{13}{12}\right)^m\right)\right], \end{aligned} \tag{7}$$

$$\begin{aligned}\mathrm{SD}^2(D) &= \left(\frac{4}{3}\right)^m + \frac{1}{n^2}\left(\mathbf{1}_N^T\mathbf{S}\mathbf{1}_N - 2n\mathbf{s}^T\mathbf{1}_N\right) \\ &\quad + \frac{1}{n^2}\left[-2\mathbf{y}_D^T(\mathbf{S}\mathbf{1}_N - N\mathbf{s}) + (N-n)^2\left(\mathrm{SD}^2(\overline{D}) - \left(\frac{4}{3}\right)^m\right)\right], \end{aligned} \tag{8}$$

*and*

$$\begin{aligned}\mathrm{DD}^2(D) &= -\prod_{k=1}^{m}\left[\frac{a+(q_k-1)b}{q_k}\right] + \frac{2n-N}{n^2}\prod_{k=1}^{m}[a+(q_k-1)b] \\ &\quad + \frac{(N-n)^2}{n^2}\left(\mathrm{DD}^2(\overline{D}) + \prod_{k=1}^{m}\left[\frac{a+(q_k-1)b}{q_k}\right]\right), \end{aligned} \tag{9}$$

*where* $\mathbf{C}$, $\mathbf{c}$, $\mathbf{S}$ *and* $\mathbf{s}$ *are defined in Lemma 1.*

# 4 Applications to the search for uniform designs

In this section some effective rules for searching for a uniform design under the foregoing four uniformity measures through its complementary design are established.

According to the relationships (6) and (9) in Theorem 1, the rules for determining a uniform design $D$ under the discrepancy measures WD and DD through its complementary design can be easily obtained.

**Theorem 2.** *Under the discrepancy measure WD or DD, the design $D$ is a uniform design in $\mathcal{D}(n, q_1 \cdots q_m)$ if and only if its complementary design $\overline{D}$ is a uniform design in $\mathcal{D}(N-n, q_1 \cdots q_m)$.*

Based on the relationship (7) in Theorem 1, the similar result to Theorem 2 can be obtained under a specified condition that the value $\mathbf{y}_D^T(\mathbf{C}\mathbf{1}_N - N\mathbf{c})$ is a constant for all designs in $\mathcal{D}(n, q_1 \cdots q_m)$.

**Theorem 3.** *Under the discrepancy measure CD, if the value* $\mathbf{y}_D^T(\mathbf{C}\mathbf{1}_N - N\mathbf{c})$ *is a constant for all designs in* $\mathcal{D}(n, q_1 \cdots q_m)$*, then a design* $D$ *is a uniform design in* $\mathcal{D}(n, q_1 \cdots q_m)$ *if and only if its complementary design* $\overline{D}$ *is a uniform design in* $\mathcal{D}(N-n, q_1 \cdots q_m)$*.*

According to the sufficient condition given in Theorem 3, we can obtain the following three corollaries, whose detailed proofs are postponed to Appendix.

**Corollary 1.** *Under the discrepancy measure CD, if the levels* $q_1, \ldots, q_m$ *are all odd, then a design* $D$ *is a uniform design in* $\mathcal{D}(n, q_1 \cdots q_m)$ *if and only if its complementary design* $\overline{D}$ *is a uniform design in* $\mathcal{D}(N-n, q_1 \cdots q_m)$*.*

**Corollary 2.** *Under the discrepancy measure CD, if* $m = 2$*, then a design* $D$ *is a uniform design in* $\mathcal{D}(n, q_1 q_2)$ *if and only if its complementary design* $\overline{D}$ *is a uniform design in* $\mathcal{D}(N-n, q_1 q_2)$*.*

**Corollary 3.** *Under the discrepancy measure CD, if* $q_1 = \cdots = q_{m-1} = 2$*, then a design* $D$ *is a uniform design in* $\mathcal{D}(n, 2^{m-1} q_m)$ *if and only if its complementary design* $\overline{D}$ *is a uniform design in* $\mathcal{D}(N-n, 2^{m-1} q_m)$*.*

When the discrepancy measure SD is used, the similar result to Theorem 3 can be given in the following Theorem 4.

**Theorem 4.** *Under the discrepancy measure SD, if the value* $\mathbf{y}_D^T(\mathbf{S}\mathbf{1}_N - N\mathbf{s})$ *is a constant for all designs in* $\mathcal{D}(n, q_1 \cdots q_m)$*, then a design* $D$ *is a uniform design in* $\mathcal{D}(n, q_1 \cdots q_m)$ *if and only if its complementary design* $\overline{D}$ *is a uniform design in* $\mathcal{D}(N-n, q_1 \cdots q_m)$*.*

Two corollaries follow directly from Theorem 4, whose detailed proofs are also postponed to Appendix.

**Corollary 4.** *Under the discrepancy measure SD, if the levels* $q_1, \ldots, q_m$ *are all odd, then a design* $D$ *is a uniform design in* $\mathcal{D}(n, q_1 \cdots q_m)$ *if and only if its complementary design* $\overline{D}$ *is a uniform design in* $\mathcal{D}(N-n, q_1 \cdots q_m)$*.*

**Corollary 5.** *Under the discrepancy measure SD, if* $q_1 = \cdots = q_{m-1} = 2$*, then a design* $D$ *is a uniform design in* $\mathcal{D}(n, 2^{m-1} q_m)$ *if and only if its complementary design* $\overline{D}$ *is a uniform design in* $\mathcal{D}(N-n, 2^{m-1} q_m)$*.*

**Example 1.** *Consider the construction of uniform design* $D$ *in* $\mathcal{D}(18, 3^3)$ *under the discrepancy measure CD. From the web site http://www.stat.psu.edu/˜rli/UniformDesign/, it is known that the following design* $\overline{D}$ *is a uniform design in* $\mathcal{D}(9, 3^3)$ *under the CD.*

$$\overline{D} = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \\ 1 & 3 & 2 & 3 & 2 & 1 & 2 & 1 & 3 \end{pmatrix}^T.$$

*By the complementary design theory, it is concluded that the design $D$ given by*

$$D = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \\ 1 & 1 & 2 & 2 & 3 & 3 & 1 & 1 & 2 & 2 & 3 & 3 & 1 & 1 & 2 & 2 & 3 & 3 \\ 2 & 3 & 1 & 2 & 1 & 3 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 \end{pmatrix}^T$$

*is a uniform design in $\mathcal{D}(18, 3^3)$ under the discrepancy measure CD. On the other hand, another uniform design $D_1$ in $\mathcal{D}(18, 3^3)$ can also be found on the web site http://www.stat.psu.edu/ ˜rli/UniformDesign/. After rearranging the rows in the lexicographical order, the design $D_1$ is given by*

$$D_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \\ 1 & 1 & 2 & 2 & 3 & 3 & 1 & 1 & 2 & 2 & 3 & 3 & 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 2 & 1 & 3 & 2 & 3 & 2 & 3 & 1 & 3 & 1 & 2 & 1 & 3 & 2 & 2 & 1 & 3 \end{pmatrix}^T.$$

*Obviously, the design point $(3, 2, 2)$ appears twice in $D_1$. Simple computation gives that both $CD^2(D)$ and $CD^2(D_1)$ are equal to the same value $0.0325$, but $WD^2(D) = 0.1003 < WD^2(D_1) = 0.1004$. Therefore, design $D$ is superior to the design $D_1$ provided on the web site.*

# 5 Concluding remarks

In this paper a full factorial design is split into two subdesigns with no repeated design points. Some identities relating the discrepancy measures of one subdesign to those of the other subdesign are derived under the four commonly used discrepancy measures CD, WD, SD and DD. Based on this complementary design theory, some general rules for searching for uniform or nearly uniform designs through their complementary designs are established. They are very powerful to search for a larger uniform design when the corresponding complementary design is smaller. An example shows that the obtained uniform design under the discrepancy measure CD is superior to the design provided on the web site of uniform designs, since the new design not only contains no repeated design points, but has a smaller WD value.

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# Appendix

**Proof of Theorem 1**

It is easy to check that $\mathbf{W}\mathbf{1}_N = \prod_{k=1}^m \left(\frac{4q_k}{3} + \frac{1}{6q_k}\right)\mathbf{1}_N$. Based on the equation (2) and the fact that $\mathbf{y}_{\overline{D}}^T \mathbf{1}_N = N - n$, we have

$$\begin{aligned} \mathrm{WD}^2(D) + \left(\frac{4}{3}\right)^m &= \frac{1}{n^2}(\mathbf{1}_N - \mathbf{y}_{\overline{D}})^T \mathbf{W}(\mathbf{1}_N - \mathbf{y}_{\overline{D}}) \\ = \quad \frac{2n-N}{n^2} \prod_{k=1}^m \left(\frac{4q_k}{3} + \frac{1}{6q_k}\right) &+ \frac{(N-n)^2}{n^2} \left(\mathrm{WD}^2(\overline{D}) + \left(\frac{4}{3}\right)^m\right). \end{aligned}$$

Similarly, from (3) we have

$$\begin{aligned} \mathrm{CD}^2(D) - \left(\frac{13}{12}\right)^m &= \frac{1}{n^2}(\mathbf{1}_N - \mathbf{y}_{\overline{D}})^T \mathbf{C}(\mathbf{1}_N - \mathbf{y}_{\overline{D}}) - \frac{2}{n}\mathbf{c}^T(\mathbf{1}_N - \mathbf{y}_{\overline{D}}) \\ = \quad \frac{1}{n^2}\Bigg[(\mathbf{1}_N^T \mathbf{C}\mathbf{1}_N - 2n\mathbf{c}^T\mathbf{1}_N) &- 2\mathbf{y}_{\overline{D}}^T(\mathbf{C}\mathbf{1}_N - N\mathbf{c}) + (N-n)^2\left(\mathrm{CD}^2(\overline{D}) - \left(\frac{13}{12}\right)^m\right)\Bigg]. \end{aligned}$$

According to (4), we can obtain

$$\begin{aligned} \mathrm{SD}^2(D) - \left(\frac{4}{3}\right)^m &= \frac{1}{n^2}(\mathbf{1}_N - \mathbf{y}_{\overline{D}})^T \mathbf{S}(\mathbf{1}_N - \mathbf{y}_{\overline{D}}) - \frac{2}{n}\mathbf{s}^T(\mathbf{1}_N - \mathbf{y}_{\overline{D}}) \\ = \quad \frac{1}{n^2}\Bigg[(\mathbf{1}_N^T \mathbf{S}\mathbf{1}_N - 2n\mathbf{s}^T\mathbf{1}_N) &- 2\mathbf{y}_{\overline{D}}^T(\mathbf{S}\mathbf{1}_N - N\mathbf{s}) + (N-n)^2\left(\mathrm{SD}^2(\overline{D}) - \left(\frac{4}{3}\right)^m\right)\Bigg]. \end{aligned}$$

As for the discrepancy measure DD, by noting that $\mathbf{E}\mathbf{1}_N = \prod_{k=1}^m [a + (q_k - 1)b]\mathbf{1}_N$, we have

$$\begin{aligned} \mathrm{DD}^2(D) + \prod_{k=1}^m \left[\frac{a + (q_k-1)b}{q_k}\right] &= \frac{1}{n^2}(\mathbf{1}_N - \mathbf{y}_{\overline{D}})^T \mathbf{E}(\mathbf{1}_N - \mathbf{y}_{\overline{D}}) \\ = \quad \frac{2n-N}{n^2}\prod_{k=1}^m [a + (q_k-1)b] &+ \frac{(N-n)^2}{n^2}\left(\mathrm{DD}^2(\overline{D}) + \prod_{k=1}^m \left[\frac{a + (q_k-1)b}{q_k}\right]\right) \end{aligned}$$

The proof of Theorem 1 is complete.

**Proof of Corollary 1**

Note that when $q_1, \ldots, q_m$ are all odd, $\mathbf{C}_k \mathbf{1}_{q_k} = q_k \mathbf{c}_k$, for $k = 1, \ldots, m$. Then we have $\mathbf{C}\mathbf{1}_N = N\mathbf{c}$. So the proof of Corollary 1 is complete.

**Proof of Corollary 2**

It can be easily verified that

$$\mathbf{C}_k \mathbf{1}_{q_k} - q_k \mathbf{c}_k = \frac{1}{8q_k}\mathbf{1}_{q_k}\, I_{\{q_k \text{ is even}\}} \ \text{ for } \ k = 1, \ldots, m,$$

where $I_{\{\cdot\}}$ is the indicator function. By using these equations, we have

$$\begin{aligned}
&\mathbf{C}\mathbf{1}_N - N\mathbf{c} \\
= \; & \left(q_1\mathbf{c}_1 + \frac{1}{8q_1}\mathbf{1}_{q_1}\, I_{\{q_1 \text{ is even}\}}\right) \otimes \left(q_2\mathbf{c}_2 + \frac{1}{8q_2}\mathbf{1}_{q_2}\, I_{\{q_2 \text{ is even}\}}\right) - N\mathbf{c} \\
= \; & \frac{q_2}{8q_1}\mathbf{1}_{q_1} \otimes \mathbf{c}_2\, I_{\{q_1 \text{ is even}\}} + \frac{q_1}{8q_2}\mathbf{c}_1 \otimes \mathbf{1}_{q_2}\, I_{\{q_2 \text{ is even}\}} + \frac{1}{64q_1q_2}\mathbf{1}_N\, I_{\{\text{both } q_1 \text{ and } q_2 \text{ are even}\}},
\end{aligned}$$

Since the three values $\mathbf{y}_D^T\mathbf{1}_N = n$, $\mathbf{y}_D^T(\mathbf{1}_{q_1} \otimes \mathbf{c}_2)$ and $\mathbf{y}_D^T(\mathbf{c}_1 \otimes \mathbf{1}_{q_2})$ are all constants for all designs in $\mathcal{D}(n, q_1q_2)$, Corollary 2 follows directly from Theorem 3.

**Proof of Corollary 3**

By noting that $\mathbf{c}_k = \frac{35}{32}\mathbf{1}_2$ for $k = 1, \ldots, m-1$, we have $\mathbf{c} = \left(\frac{35}{32}\right)^{m-1}\mathbf{1}_{2^{m-1}} \otimes \mathbf{c}_m$. Note that $\mathbf{C}_k\mathbf{1}_2 = \frac{9}{4}\mathbf{1}_2$ for $k = 1, \ldots, m-1$ and $\mathbf{C}_m\mathbf{1}_{q_m} - q_m\mathbf{c}_m = \frac{1}{8q_m}\mathbf{1}_{q_m}\, I_{\{q_m \text{ is even}\}}$. Combining the above equations, we obtain

$$\begin{aligned}
\mathbf{C}\mathbf{1}_N - N\mathbf{c} &= \left(\left(\frac{9}{4}\right)^{m-1}\mathbf{1}_{2^{m-1}}\right) \otimes (\mathbf{C}_m\mathbf{1}_{q_m}) - N\mathbf{c} \\
= \; & \left(q_m\left(\frac{9}{4}\right)^{m-1} - N\left(\frac{35}{32}\right)^{m-1}\right) 1_{2^{m-1}} \otimes \mathbf{c}_m + \left(\frac{9}{4}\right)^{m-1}\frac{1}{8q_m}1_N\, I_{\{q_m \text{ is even}\}},
\end{aligned}$$

Since the two values $\mathbf{y}_D^T\mathbf{1}_N = n$ and $\mathbf{y}_D^T(\mathbf{1}_{2^{m-1}} \otimes \mathbf{c}_m)$ are all constants for all designs in $\mathcal{D}(n, 2^{m-1}q_m)$, Corollary 3 follows directly from Theorem 3.

**Proof of Corollary 4**

It is easy to check that

$$\mathbf{S}_k\mathbf{1}_{q_k} - q_k\mathbf{s}_k = \frac{1}{2q_k}\mathbf{1}_{q_k} \;\text{ for }\; k = 1, \ldots, m.$$

By using these equations, we have

$$\begin{aligned}
\mathbf{S}\mathbf{1}_N - N\mathbf{s} &= \left(q_1\mathbf{s}_1 + \frac{1}{2q_1}\mathbf{1}_{q_1}\right) \otimes \left(q_2\mathbf{s}_2 + \frac{1}{2q_2}\mathbf{1}_{q_2}\right) - N\mathbf{s} \\
= \; & \frac{q_2}{2q_1}\mathbf{1}_{q_1} \otimes \mathbf{s}_2 + \frac{q_1}{2q_2}\mathbf{s}_1 \otimes \mathbf{1}_{q_2} + \frac{1}{4q_1q_2}\mathbf{1}_N.
\end{aligned}$$

Since the three values $\mathbf{y}_D^T\mathbf{1}_N = n$, $\mathbf{y}_D^T(\mathbf{1}_{q_1} \otimes \mathbf{s}_2)$ and $\mathbf{y}_D^T(\mathbf{s}_1 \otimes \mathbf{1}_{q_2})$ are all constants for all designs in $\mathcal{D}(n, q_1q_2)$, Corollary 4 follows directly from Theorem 4.

**Proof of Corollary 5**

By noting that $\mathbf{s}_k = \frac{11}{8}\mathbf{1}_2$ for $k = 1, \ldots, m-1$, we have $\mathbf{s} = \left(\frac{11}{8}\right)^{m-1}\mathbf{1}_{2^{m-1}} \otimes \mathbf{s}_m$. Note that $\mathbf{S}_k\mathbf{1}_2 = 3\mathbf{1}_2$ for $k = 1, \ldots, m-1$ and $\mathbf{S}_m\mathbf{1}_{q_m} - q_m\mathbf{s}_m = \frac{1}{2q_m}\mathbf{1}_{q_m}$. Combining the above equations, we obtain

$$
\begin{aligned}
\mathbf{S}\mathbf{1}_N - N\mathbf{s} &= (3^{m-1}\mathbf{1}_{2^{m-1}}) \otimes \left( q_m \mathbf{s}_m + \frac{1}{2q_m}\mathbf{1}_{q_m} \right) - N\mathbf{s} \\
&= \left( q_m 3^{m-1} - N\left(\frac{11}{8}\right)^{m-1} \right) \mathbf{1}_{2^{m-1}} \otimes \mathbf{s}_m + \frac{3^{m-1}}{2q_m}\mathbf{1}_N.
\end{aligned}
$$

Since the two values $\mathbf{y}_D^T\mathbf{1}_N = n$ and $\mathbf{y}_D^T(\mathbf{1}_{2^{m-1}} \otimes \mathbf{s}_m)$ are all constants for all designs in $\mathcal{D}(n, 2^{m-1}q_m)$, Corollary 5 follows directly from Theorem 4.

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