# THE VOTER MODEL IN A RANDOM ENVIRONMENT IN $\mathbb{Z}^d$

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Abstract. We consider the voter model with flip rates determined by $(\mu_e, e \in E_d)$, where $E_d$ is the set of all non-oriented nearest-neighbour edges in the Euclidean lattice $\mathbb{Z}^d$. Suppose that $(\mu_e, e \in E_d)$ are i.i.d. random variables satisfying $\mu_e \geq 1$. We prove that when $d = 2$, almost surely for all random environments the voter model has only two extremal invariant measures: $\delta_0$ and $\delta_1$.

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*Key words:* voter model, random walk, random environment, duality.

## 1. Introduction

The voter model is an interacting particle system, describing the collective behavior of voters who constantly update their political positions. In this paper, voters are represented by vertices of the Euclidean lattice $\mathbb{Z}^d$. The voter at $x$ may hold either of two political positions, denoted by 0 or 1. Let $\eta(x)$ be the political position of voter $x$ and the collection $\eta = \{\eta(x); x \in \mathbb{Z}^d\}$ be an element of $\{0,1\}^{\mathbb{Z}^d}$. The voter at $x$ updates his political position at a random time, following the exponential distribution with parameter $\sum_z \mu_{xz}$, where the summation is over $2d$ nearest neighbors. At the time of update the voter takes the position of his neighbor $y$ with probability $\mu_{xy}/(\sum_z \mu_{xz})$. When $\mu_e \equiv 1$, this is a model well studied in Chapter V of [10].

The voter model can be constructed by the graphical representation, see §3.6 of [10]. This approach not only works for all positive $\mu_{xy}$, but also clearly exhibits the duality relation which will be used in our proof in §3.

We are interested in the case when $(\mu_e, e \in E_d)$ are i.i.d. random variables satisfying $\mu_e \geq 1$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. There have been very few literatures about the voter model in random environments. As far as we know, only the one dimensional voter model in a random environment has been explored in [8], and a voter model in the supercritical cluster of the Bernoulli bond percolation is discussed in [3]. As usual, one would like to identify all invariant measures. When all voters take the same position, there will be no change thereafter. Therefore the configurations that $\eta(x) \equiv 0$ or 1 are traps of the voter model and the measures $\delta_0$ and $\delta_1$ of point mass are invariant. With an extra effort, we are able to identify all invariant measures.

**Theorem 1.1.** *Let $d = 1$ or 2. Suppose that $(\mu_e)$ are i.i.d. and $\mu_e \geq 1$ $\mathbb{P}$-a.s. There exists $\Omega_0 \subseteq \Omega$ with $\mathbb{P}(\Omega_0) = 1$. For any $\omega \in \Omega_0$, the voter model has only two extremal invariant measures: $\delta_0$ and $\delta_1$.*

This extends Corollary 1.13 of Chapter V of [10]. In light of Example 1.5 of Chapter V of [10], the conclusion of the theorem generally does not hold when $d \geq 3$.

The proof of the theorem involves the duality and the dual of the voter model is a coalescing Markov chain taking values on the set of all finite set of vertices of $\mathbb{Z}^d$. When the initial state is a singleton, the coalescing Markov chain always takes value on singletons. If we identify singleton $\{x\}$ with vertex $x$, then the coalescing Markov chain is exactly a continuous-time random walk in a random environment. Intuitively, a walker stays at $x$ for an exponential time with parameter $\mu_x$, jumps to a nearest neighbor, say $y$, with probability $\mu_{xy}/(\sum_z \mu_{xz})$. This is also called the *variable speed random walk* or the *random conductance model.* There is a large amount of literatures on random walks in random environments, e.g., [1], [5] and [6].

Let $\{X_t\}$ and $\{Y_t\}$ be two independent variable speed random walks. The problem we are interested is reduced to the property that $X_t = Y_t$ infinitely often. More specifically, we say $X_t = Y_t$ infinitely often if there exists an infinite sequence of random times $\{t_1, t_2, ...\}$ with $t_1 < t_2 < ...$ and $\lim_{i\to\infty} t_i = \infty$, such that $X(t_i) = Y(t_i)$ for all $i \geq 1$.

**Theorem 1.2.** *Let $d = 1$ or 2. Suppose that $(\mu_e, e \in E_d)$ are i.i.d. and $\mu_e \geq 1$ $\mathbb{P}$-a.s. There exists $\Omega_0 \subseteq \Omega$ with $\mathbb{P}(\Omega_0) = 1$. Let $\omega \in \Omega_0$ and $P_\omega$ denote the probability conditional on the environment. If $\{X_t\}$ and $\{Y_t\}$ are two independent variable speed random walks starting from $x$ and $y$ respectively, then $P_\omega(X_t = Y_t$ infinitely often$) = 1$.*

The proof for the 2-dimensional case follows a similar idea which is used in [3] and the key to the proof is the heat kernel estimates obtained by Barlow and Deuschel [1]. For more references related to collisions of two random walks, we refer readers to [2] [4] and [9]. Again the conclusion of the theorem holds trivially in dimension one, and fails in general when the dimension is 3 or greater.

This paper is organized as follows. In the next section we quote from [1] the heat kernel bounds as Theorem 2.1, and prove two lemmas that will be used in the proof of the main theorem. Theorems 1.1 and 1.2 will then be proved in §3. Between the two proofs we will review the coalescing Markov chain and its dual relation with the voter model.

For the typographical reason we write $\mu(x)$ instead of $\mu_x$ whenever it is necessary. $[s]$ stands for the largest integer which is smaller than or equal to $s$. $|A|$ means the cardinality of set $A$. $\mathbb{Z}^d = (V_d, E_d)$ is the $d$ dimensional Euclidean lattice. Let $|x-y|$ denote the Euclidean distance between $x$ and $y$. $B_x(s) = \{z : |z-x| \le s\}$ is the ball of vertices, centered at $x$. Note that this also makes sense when the radius $s$ is a positive number. Therefore we prefer $B(s)$ over $B([s])$.

## 2. Lemmas

Suppose that $(\mu_e)$ are i.i.d. and $\mu_e \ge 1$ $\mathbb{P}$-a.s. Let $\{X_t\}$ be the *variable speed random walk* and $q_t^\omega(x,y)$ be the heat kernel of $\{X_t\}$. Namely,

$$q_t^\omega(x,y) = P_\omega^x(X_t = y).$$

**Theorem 2.1.** (*Theorem 1.2 of* [1]) *Let $d \ge 2$ and $\sigma \in (0,1)$. There exist random variables $S_x$, $x \in \mathbb{Z}^d$, such that*

$$\mathbb{P}(S_x(\omega) \ge n) \le c_1 \exp(-c_2 n^\sigma), \tag{2.1}$$

*and constants $c_i$ (depending only on $d$ and the distribution of $\mu_e$) such that the following hold.*
*If $|x-y|^2 \vee t \ge S_x^2$, then*

$$q_t^\omega(x,y) \le c_3 t^{-d/2} \mathrm{e}^{-c_4|x-y|^2/t} \text{ when } t \ge |x-y|, \tag{2.2}$$

$$q_t^\omega(x,y) \le c_3 \exp(-c_4|x-y|(1 \vee \log(|x-y|/t))) \text{ when } t \le |x-y|. \tag{2.3}$$

*If $t \ge S_x^2 \vee |x-y|^{1+\sigma}$, then*

$$q_t^\omega(x,y) \ge c_5 t^{-d/2} \mathrm{e}^{-c_6|x-y|^2/t}. \tag{2.4}$$

**Lemma 2.2.** *Let $A_n(\omega)$ be the random set defined by*

$$A_n(\omega) = \{x : |x| \le n, S_x(\omega) \le 2\log n\}.$$

*Then almost surely there exists a finite random variable $U(\omega)$ such that $|A_n(\omega)| \ge c_7 n^2$ for any $n \ge U(\omega)$.*

*Proof.* Set $W_n(\omega) := |A_n(\omega)| = \sum_{|x|\le n} \mathbf{1}_{\{S_x(\omega)\le 2\log n\}}$. By (2.1),

$$\mathbb{E}W_{2^n} = \sum_{|x|\le 2^n} (1 - \mathbb{P}(S_x > n\log 4)) \ge \pi 2^{2n}(1 - c_1\exp\{-c_2(n\log 4)^\sigma\}) .$$

Since $0 \le W_{2^n} \le \pi 2^{2n}$, it follows that

$$\mathbb{P}(W_{2^n} < \pi 2^{2n}(1 - c_1 \exp\{-\frac{c_2}{2}(n\log 4)^\sigma\}) \le \exp\{-\frac{c_2}{2}(n\log 4)^\sigma\} .$$

By the Borel-Cantelli Lemma, $\{W_{2^n} < \pi 2^{2n}(1 - c_1 \exp\{-c_2(n\log 4)^\sigma/2\})\}$ happens finitely many times almost surely. Then there exists $\Omega_0 \subseteq \Omega$ with $\mathbb{P}(\Omega_0) = 1$ and a random variable $U(\omega)$ such that $U(\omega) < \infty$ for all $\omega \in \Omega_0$, and for all $n \geq U(\omega)$,

$$W_{2^n}(\omega) \geq \pi 2^{2n}(1 - c_1 \exp\{-c_2(n\log 4)^\sigma/2\}) \geq c_8 2^{2n} \ .$$

For all $2^n < k < 2^{n+1}$, $W_k \geq W_{2^n} \geq c_8 2^{2n} \geq c_8 k^2/4$. □

**Lemma 2.3.** *Let $\omega \in \Omega_0$, and $\{X_t\}$, $\{Y_t\}$ be independent variable speed random walks in $\mathbb{Z}^2$ starting from $x$ and $y$ respectively. Then*

$$P_\omega(X_t = Y_t \textit{ for some } t \geq 1) \geq \delta > 0,$$

*where $\delta$ is a constant independent of $\omega$, $x$ and $y$.*

*Proof.* Fix $\omega \in \Omega_0$. Recall that $B_x(n) = \{z : |z - x| \leq n\}$, and set

$$M_\omega(n) = B_x(n) \cap B_y(n) \cap A_n(\omega) \ .$$

For $n \geq U(\omega) + (|x| \vee |y|)(1 + 12\pi c_7^{-1})$ ,

$$\begin{aligned} |M_\omega(n)| &= |B_x(n)| + |B_y(n)| + |A_n(\omega)| - |B_x(n) \cup B_y(n)| - |B_x(n) \cup A_n(\omega)| \\ &\quad - |B_y(n) \cup A_n(\omega)| + |B_x(n) \cup B_y(n) \cup A_n(\omega)| \\ &\geq |B_x(n)| + |B_y(n)| + |A_n(\omega)| - 2|B_0(|x| \vee |y| + n)| \\ &\geq \pi n^2 + \pi n^2 + c_7 n^2 - 2\pi(|x| \vee |y| + n)^2 \\ &= c_7 n^2 - 4\pi n(|x| \vee |y|) - 2\pi(|x| \vee |y|)^2 \\ &\geq c_7 n^2 - 6\pi n(|x) \vee |y|) \geq \frac{c_7}{2} n^2 \ . \end{aligned} \tag{2.5}$$

Set $T = \exp(\frac{2}{1+\sigma}\log t_0)$, where $\sigma$ is given in Theorem 2.1, and

$$t_0 = [S_x(\omega) \vee S_y(\omega)]^2 + [U(\omega) + (|x| \vee |y|)(1 + 12\pi c_7^{-1})]^2.$$

Define the random variable

$$H := \int_{t_0}^{T} \mathbf{1}_{\{X_s = Y_s \in M(s^{1/2})\}} \, \mathrm{d}s \ .$$

We shall establish a lower bound of $\mathbb{E}H$ and an upper bound of $\mathbb{E}H^2$ below as lemmas.

**Lemma 2.4.** $\mathbb{E}_\omega H \geq c_9 \log T$ .

**Lemma 2.5.** $\mathbb{E}_\omega H^2 \leq (4\pi c_3^2 + 2\pi^2 c_3^4/c_4)(\log T)^2$.

Then by the Hölder's inequality,

$$P_\omega(H > 0) \geq \frac{(\mathbb{E}_\omega H)^2}{\mathbb{E}_\omega H^2} \geq \frac{(c_9 \log T)^2}{(4\pi c_3^2 + 2\pi^2 c_3^4/c_4)(\log T)^2} = \frac{c_9^2 c_4}{4\pi c_3^2 c_4 + 2\pi^2 c_3^4}.$$

It completes the proof by choosing $\delta$ to be $c_9^2c_4/(4\pi c_3^2c_4+2\pi^2c_3^4)$ which is positive. □

*Proof of Lemma 2.4.*

$$\begin{aligned}
\mathbb{E}_\omega H &= \int_{t_0}^T \mathbb{E}_\omega \mathbf{1}_{\{X_s=Y_s\in M(s^{1/2})\}}\,\mathrm{d}s \\
&= \int_{t_0}^T \sum_{z\in M(s^{1/2})} P_\omega(X_s=z, Y_s=z)\,\mathrm{d}s \\
&= \int_{t_0}^T \sum_{z\in M(s^{1/2})} q_s^\omega(x,z)q_s^\omega(y,z)\,\mathrm{d}s\ .
\end{aligned}$$

Since $z\in M(s^{1/2})$, we have $|x-z|^2\le s\le T=\exp(\frac{2}{1+\sigma}\log t_0)$. Thus $s\ge t_0\ge S_x^2(\omega)\vee|x-z|^{1+\sigma}$. Similarly $s\ge S_y^2(\omega)\vee|y-z|^{1+\sigma}$. Hence the condition of (2.4) is satisfied, and $\mathbb{E}_\omega H$ can be bounded below.

$$\begin{aligned}
\mathbb{E}_\omega H &\ge \int_{t_0}^T \sum_{z\in M(s^{1/2})} c_5^2 s^{-2}\exp\left(-c_6\frac{|x-z|^2}{s}-c_6\frac{|y-z|^2}{s}\right)\mathrm{d}s \\
&\ge c_5^2\mathrm{e}^{-2c_6}\int_{t_0}^T \sum_{z\in M(s^{1/2})} s^{-2}\,\mathrm{d}s \ge \frac{c_5^2c_7\mathrm{e}^{-2c_6}}{2}\int_{t_0}^T s^{-1}\,\mathrm{d}s\ge c_9\log T\ .
\end{aligned}$$

Here we get the first inequality by (2.4), the second inequality by the fact that $|x-z|^2\le s$ for $z\in M(s^{1/2})$, and the third inequality by (2.5) that $|M(s^{1/2})|\ge c_7s/2$ since $s^{1/2}\ge t_0^{1/2}\ge U(\omega)+(|x|\vee|y|)(1+12\pi c_7^{-1})$. □

*Proof of Lemma 2.5.*

$$\begin{aligned}
\mathbb{E}_\omega H^2 &= 2\mathbb{E}_\omega\left(\int_{t_0}^T \mathrm{d}t\int_t^T \mathbf{1}_{\{X_t=Y_t\in M(t^{1/2})\}}\mathbf{1}_{\{X_s=Y_s\in M(s^{1/2})\}}\,\mathrm{d}s\right) \\
&=2\int_{t_0}^T \mathrm{d}t\int_t^T \mathbb{E}_\omega \sum_{z\in M(t^{1/2})}\sum_{w\in M(s^{1/2})} \mathbf{1}_{\{X_t=Y_t=z\}}\mathbf{1}_{\{X_s=Y_s=w\}}\mathrm{d}s \\
&=2\int_{t_0}^T \mathrm{d}t\int_t^T \sum_{z\in M(t^{1/2})} P_\omega^{(x,y)}(X_t=Y_t=z)\sum_{w\in M(s^{1/2})} P_\omega^{(z,z)}(X_{s-t}=Y_{s-t}=w)\mathrm{d}s \\
&=2\int_{t_0}^T \mathrm{d}t\int_t^T \sum_{z\in M(t^{1/2})} q_t^\omega(x,z)q_t^\omega(y,z)\sum_{w\in M(s^{1/2})} q_{s-t}^\omega(z,w)q_{s-t}^\omega(z,w)\mathrm{d}s \\
&\le 2\int_{t_0}^T \mathrm{d}t\Big[\sum_{z\in M(t^{1/2})} q_t^\omega(x,z)q_t^\omega(y,z)\int_0^T \sum_{w\in M((s+t)^{1/2})} (q_s^\omega(z,w))^2\mathrm{d}s\Big]\ .
\end{aligned}$$

Since $z \in M(t^{1/2})$, $|x-z| \le t^{1/2} \le t$. Similarly $|y-z| \le t$. Moreover $t \ge t_0 \ge [S_x \vee S_y]^2$. Thus the condition of (2.2) is satisfied. Hence

$$q_t^\omega(x,z)q_t^\omega(y,z) \le c_3^2 t^{-2} \ .$$

On the other hand, since $z \in M(t^{1/2})$, $z \in A_{t^{1/2}}(\omega)$. Thus by the definition of $A_n(\omega)$ we conclude that $S_z \le \log t$, and

$$\begin{aligned}&\int_0^T \sum_{w\in M((s+t)^{1/2})} (q_s^\omega(z,w))^2 \mathrm{d}s \\ &\le \log t + \int_{\log t}^T \sum_{w\in B_z(s)} (q_s^\omega(z,w))^2 \mathrm{d}s + \int_{\log t}^T \sum_{w\notin B_z(s)} (q_s^\omega(z,w))^2 \mathrm{d}s \ . \end{aligned} \tag{2.6}$$

For $w \notin B_z(s)$, we have $S_z(\omega) \le \log t \le s \le |z-w|$, thus the condition of (2.3) is satisfied. Hence

$$\begin{aligned}\int_{\log t}^T \sum_{v\notin B_z(s)} (q_s^\omega(z,w))^2 ds &\le \int_{\log t}^T \sum_{v\notin B_z(s)} c_3^2 \exp(-2c_4|z-w|)\,\mathrm{d}s \\ &\le \int_{\log t}^T \sum_{n=[s]}^{\infty} 2\pi n c_3^2 \exp(-2c_4 n)\,\mathrm{d}s \le c_{10} \ .\end{aligned}$$

For $w \in B_z(s)$, $s \ge |z-w|$ and $s \ge S_z(\omega)$, the condition of (2.2) is satisfied. Hence

$$\begin{aligned}&\int_{\log t}^T \sum_{w\in B_z(s)} (q_s^\omega(z,w))^2\,\mathrm{d}s \\ &\le \int_{\log t}^T \sum_{w\in B_z(s)} c_3^2 s^{-2} \exp\left(-2c_4|z-w|^2/s\right)\,\mathrm{d}s \\ &\le \int_{\log t}^T [c_3^2 s^{-2} + \sum_{n=1}^{[s]} c_3^2 2\pi n s^{-2} \exp\left(-2c_4 n^2/s\right)]\,\mathrm{d}s \\ &\le c_3^2\left(\frac{1}{\log t} - \frac{1}{T}\right) + 2\pi c_3^2 \sum_{n=1}^{[T]} n \int_n^T s^{-2} \exp\left(-2c_4 n^2/s\right)\,\mathrm{d}s \\ &\le \frac{c_3^2}{\log t} + 2\pi c_3^2 \sum_{n=1}^{[T]} n \int_{T^{-1}}^{n^{-1}} \exp(-2c_4 n^2 u)\,\mathrm{d}u \\ &\le \frac{c_3^2}{\log t} + \pi \frac{c_3^2}{c_4} \sum_{n=1}^{[T]} n^{-1} \le \frac{c_3^2}{\log t} + \frac{\pi c_3^2}{c_4} \log T \ .\end{aligned}$$

Putting two estimates together into (2.6),

$$\int_0^T \sum_w (q_s^\omega(z,w))^2 \mathrm{d}s \le \log t + c_{10} + \frac{c_3^2}{\log t} + \frac{\pi c_3^2}{c_4}\log T \le (2 + \frac{\pi c_3^2}{c_4})\log T.$$

Therefore,

$$\begin{aligned}\mathbb{E}_\omega H^2 &\le 2\int_{t_0}^T \left(\sum_{z\in M(t^{1/2})} c_3^2 t^{-2}(2+\frac{\pi c_3^2}{c_4})\log T\right) \mathrm{d}t \\ &\le 2\int_{t_0}^T \frac{c_3^2\pi(2+\pi c_3^2/c_4)\log T}{t}\mathrm{d}t \le (4\pi c_3^2 + \frac{2\pi^2 c_3^4}{c_4})(\log T)^2 \ .\end{aligned}$$

□

## 3. Proof of Theorems

*Proof of Theorem 1.2.* When $d = 1$, $(X_t, Y_t)$ can be viewed as a random walk in the 2-dimensional lattice with random conductance, and is therefore recurrent. In particular $(X_t, Y_t)$ will certainly hit the diagonal line. Namely, $P_\omega(X_t = Y_t, \text{ for some } t > 0) = 1$.

For d= 2, let $\delta > 0$ be defined in Lemma 2.3. Fix $\omega \in \Omega_0$. By Lemma 2.3, there exists a function $f : V_2 \times V_2 \mapsto [1,\infty)$, such that for all $x$, $y \in V_2$,

$$P_\omega^{(x,y)}(X_t = Y_t \text{ for some } 1 < t \le f(x,y)) \ge \frac{\delta}{2}\ . \tag{3.1}$$

Set $x_0 = x$, $y_0 = y$ and $t_0 = 0$. Define $x_i$, $y_i$ and $t_i$ inductively for $i \ge 1$ as follows. Suppose that $x_i$, $y_i$ and $t_i$ are already defined. Let $\{\tilde{X}_t\}$ and $\{\tilde{Y}_t\}$ be two independent continuous-time random walks starting from $x_i$ and $y_i$. Define

$$x_{i+1} := \tilde{X}(f(x_i,y_i)), \quad y_{i+1} := \tilde{Y}(f(x_i,y_i)), \text{ and } t_{i+1} := t_i + f(x_i,y_i).$$

Define $\mathcal{E}_i$ to be the event that $X_t = Y_t$ for some $t \in (t_i + 1, t_{i+1}]$ for $i \ge 0$. By (3.1) and the strong Markov property,

$$P_\omega(\mathcal{E}_i | X_t, Y_t, t \le t_i) = P_\omega^{(x_i,y_i)}(\tilde{X}_t = \tilde{Y}_t \text{ for some } 1 < t \le f(x_i,y_i)) \ge \frac{\delta}{2}\ .$$

By the second Borel-Cantelli lemma (extended version, see Page 237 of [7]), $P_\omega(\mathcal{E}_i$ infinitely often)=1. Furthermore,

$$P_\omega(X_t = Y_t \text{ infinitely often}) \ge P_\omega(\mathcal{E}_i \text{ infinitely often}) = 1\ .$$

Thus we complete the proof of Theorem 1.2. □

As a dual of the voter model, the coalescing Markov chain $\{A_t\}$ takes value on the set of all finite sets of vertices of $\mathbb{Z}^d$. Suppose the initial state

$A$ is a finite set of vertices. Construct $A_t$ as follows: Image that there is a particle at each $x \in A$. Each particle performs a variable speed random walk starting from a point in $A$, independent of each other until they meet. Once two particles collide, they coalesce into one particle. Then $A_t$ is the set of locations of all particles at time $t$.

The process $\{A_t\}$ can also be constructed by the same graphical representation as used in the construction of the voter model. For $\eta \in \{0,1\}^{\mathbb{Z}^2}$ and a finite set $A \subseteq \mathbb{Z}^2$, define

$$H(\eta, A) = \mathbf{1}_{\{\eta(z)=1 \text{ for all } z\in A\}} \ .$$

Then the duality relation holds. That is,

$$\mathbb{E}^{\eta}_{\omega} H(\eta_t, A) = \mathbb{E}^{A}_{\omega} H(\eta, A_t).$$

For the details of the derivation of this equality, see page 230 of [10]. Equivalently,

$$\mathbb{P}^{\eta}_{\omega}(\eta_t(x) = 1 \text{ for all } x \in A) = \mathbb{P}^{A}_{\omega}(\eta(x) = 1 \text{ for all } x \in A_t) \ . \tag{3.2}$$

Now we are ready to prove that for $\omega \in \Omega_0$, the corresponding voter model has only two extremal invariant measures: $\delta_0$ and $\delta_1$.

*Proof of Theorem 1.1.* The following argument is free of dimension. So we will only deal with the 2 dimensional case. Suppose that $\pi$ is an probability measure on $\{0,1\}^{\mathbb{Z}^2}$ and $\pi$ is invariant for the voter model. Let $\alpha(x) = \pi(\eta(x) = 1)$. Then $\alpha$ is a bounded harmonic function. Namely $\alpha$ satisfies

$$\mu_x \alpha(x) = \sum_z \mu_{xz} \alpha(z).$$

The fact that the *variable speed random walk* in $\mathbb{Z}^2$ is recurrent implies that all bounded harmonic functions are constants. Let $\alpha = \pi(\eta(0) = 1)$. We want to prove by (3.2) that for any finite subset $A$,

$$\pi(\eta(x) = 1 \text{ for all } x \in A) = \alpha. \tag{3.3}$$

Let $\tau$ be the first time that the coalescing random walk starting from $A$ coalesces into a singleton. By Theorem 1.2, $P_\omega(\tau < \infty) = 1$. If $t > \tau$, then $A_t = \{X_t\}$, where $X_t$ is the variable speed random walk starting from some point $x \in A$. By (3.2), for any $t$

$$\begin{aligned}
&\pi(\eta(y) = 1 \text{ for all } y \in A) = \mathbb{P}_\omega^\pi(\eta_t(y) = 1 \text{ for all } y \in A) \\
=& \int \mathbb{P}_\omega^\eta(\eta_t(y) = 1 \text{ for all } y \in A)\pi(d\eta) \\
=& \int \mathbb{P}_\omega^A(\eta(y) = 1 \text{ for all } y \in A_t)\pi(d\eta) \\
=& \int [\mathbb{P}_\omega^A(\eta(y) = 1 \text{ for all } y \in A_t, \tau \le t) \\
&\quad + \mathbb{P}_\omega^A(\eta(y) = 1 \text{ for all } y \in A_t, \tau > t)]\pi(d\eta) \\
=& \int [\mathbb{P}_\omega^A(\eta(X_t) = 1, \tau \le t) + \mathbb{P}_\omega^A(\eta(y) = 1 \text{ for all } y \in A_t, \tau > t)]\pi(d\eta) \\
=& \int [\alpha\mathbb{P}_\omega^A(\tau \le t) + \mathbb{P}_\omega^A(\eta(y) = 1 \text{ for all } y \in A_t, \tau > t)]\pi(d\eta) \\
=& \alpha + \int [-\alpha\mathbb{P}_\omega^A(\tau > t) + \mathbb{P}_\omega^A(\eta(y) = 1 \text{ for all } y \in A_t, \tau > t)]\pi(d\eta).
\end{aligned}$$

Thus

$$|\pi(\eta(y) = 1 \text{ for all } y \in A) - \alpha| \le 2\mathbb{P}_\omega^A(\tau > t).$$

Letting $t \to \infty$, we obtain the desired conclusion (3.3), from which we conclude that $\pi = \alpha\delta_1 + (1-\alpha)\delta_0$. □

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