A Strong Law of Large Numbers for Super-stable Processes

Michael A. Kouritzin [a]

[a] *Department of Mathematical and Statistical Sciences*
University of Alberta
Edmonton, AB T6G 2G1 Canada

Yan-Xia Ren [b,*],

[b] *LMAM School of Mathematical Sciences & Center for Statistical Science*
Peking University
Beijing 100871, P. R. China

Abstract

Let $X = (X_t, t \geq 0; P_\mu)$ be a supercritical, super-stable process corresponding to the operator $-(-\Delta)^{\alpha/2} u + \beta u - \eta u^2$ on $\mathbb{R}^d$ with constants $\beta, \eta > 0$ and $\alpha \in (0, 2]$, and let ℓ be Lebesgue measure on $\mathbb{R}^d$. Put $\hat{W}_t(\theta) = e^{(\beta-|\theta|^\alpha)t} X_t(e^{i\theta\cdot})$, which is a complex-valued martingale for each $\theta \in \mathbb{R}^d$ with limit $\hat{W}(\theta)$ say. Our main result establishes that for any starting measure μ, which is a finite measure on $\mathbb{R}^d$ such that $\int_{\mathbb{R}^d} x\mu(\mathrm{d}x) < \infty$, $\frac{t^{d/\alpha} X_t}{e^{\beta t}} \to c_\alpha \hat{W}(0)\, \ell$ P_μ-a.s. in a topology, termed the shallow topology, strictly stronger than the vague topology yet weaker than the weak topology. This result can be thought of as an extension to a class of superprocesses of Watanabe's strong law of large numbers for branching Markov processes.

Key words: Super-stable process, Super-Brownian motion, Strong law of large

numbers, Fourier Transform, Vague convergence, α-stable process, Probability measures

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1 Introduction

We use $M_F(\mathbb{R}^d)$ to denote the set of finite measures on $\mathbb{R}^d$. We use $\mu(f)$ to denote $\int f \mathrm{d}\mu$ for a measure μ and integrable function f. It is clear that $\mu(D) = \mu(I_D)$, where I_D is the indicator function of D. Let $C_c(\mathbb{R}^d)$ denote the set of continuous functions on $\mathbb{R}^d$ with compact support.

In 1967, Watanabe [23] first discussed the strong law of large numbers for branching Brownian motion. Let $(X_t, t \geq 0; P_x)$ be a branching Brownian motion on $\mathbb{R}^d$ $(d \geq 1)$ starting from a single point $x \in \mathbb{R}^d$ and corresponding to the operator

$$\frac{1}{2}\triangle u + a(F(u) - u),$$

where a is a positive constant and $F(s) := \sum_{n=0}^{\infty} p_n s^n, s \geq 0$, is the generating function of the offspring distribution $\{p_n, n \geq 0\}$. By explicitly using the Gaussian density, Watanabe [23] proved in the supercritical case, i.e. $\beta := a(F'(1) - 1) > 0$, that under the condition $\sum_{n=0}^{\infty} n^2 p_n < \infty$, it follows that

$$\frac{X_t}{e^{\beta t} t^{-d/2}} \to (2\pi)^{-d/2} \ell \cdot W, \qquad P_x - \text{a.s.} \tag{1}$$

as $t \to \infty$ in the sense of vague convergence, where ℓ is the Lebesgue measure on $\mathbb{R}^d$ and W is the limit of the martingale $W_t := e^{-\beta t} X_t(1)$. Later, based on the ideas in [23], Biggins [2]

* Corresponding author.
Email addresses: `mkouritz@math.ualberta.ca` (Michael A. Kouritzin), `yxren@math.pku.edu.cn` (Yan-Xia Ren).

proved strong law of large numbers for discrete-time branching random walk.

Suppose $(X_t, t \geq 0; P_\mu)$ is a super-Brownian motion on $\mathbb{R}^d$, $d \geq 1$, corresponding to operator $\frac{1}{2}\triangle u + \beta u - \eta u^2$, where $\beta > 0$ and $\eta > 0$ are positive constants, and starting from $\mu \in M_F(\mathbb{R}^d)$. It seems that Englander [10] is the first to discuss the law of large numbers for the supercritical super-Brownian motion $(X_t, t \geq 0; P_\mu)$. It was proved in [10] that for any $f \in C_c(\mathbb{R}^d)$,

$$\frac{X_t(f)}{e^{\beta t} t^{-d/2}} \to (2\pi)^{-d/2} \ell(f) \cdot W, \qquad \text{in } P_\mu\text{-probability}, \tag{2}$$

where W is the limit of the martingale $W_t := e^{-\beta t} X_t(1)$. More recently, Wang [22] improved the convergence in (2) from "in probability" to "P_μ-a.s." in the special case that $\mu = \delta_x$, $x \in \mathbb{R}^d$ by combining the Fourier analysis used [23] and the uniform convergence discussion of martingales used in [2]. Wang's proof depends on the specific density of Brownian motion and the compact support property of super-Brownian motion starting from a compactly supported measure. For more path properties of super-Brownian motion, see Dawson, Iscoe and Perkins [8], Dawson and Perkins [9] and Perkins [19] [20]. But, α-stable processes ($\alpha \in (0, 2)$) do not have specific density expressions. More critically, for any $t > 0$, the support of X_t, the super-stable process with index $\alpha \in (0, 2)$, is the whole space $\mathbb{R}^d$ even when the starting measure μ has compact support (see Dawson and Perkins [9] or Perkins [20]). Therefore, the methods in Wang [22] do not transfer over to general $\mu \in M_F(\mathbb{R}^d)$ nor to super-stable process with index $\alpha \in (0, 2)$.

Note that both for branching Brownian motion and super-Brownian motion, the mean of X_t is described by the linear operator $\frac{1}{2}\triangle + \beta$ on $\mathbb{R}^d$. The denominator $e^{\beta t} t^{-d/2}$ in (1) and (2) is exactly the growth rate of $e^{\beta t} S_t^{\frac{1}{2}\Delta}$, the semigroup corresponding to $\frac{1}{2}\triangle + \beta$ on $\mathbb{R}^d$, as $t \to \infty$. In our more general α-stable case, corresponding to the operator $-(-\triangle)^{\frac{\alpha}{2}} + \beta$, it will again turn out that the correct scaling, $e^{\beta t} t^{-d/\alpha}$, is dictated by the growth rate of $e^{\beta t} S_t^{\triangle^\alpha}$, the semigroup corresponding to $-(-\triangle)^{\frac{\alpha}{2}} + \beta$.

If $\frac{1}{2}\triangle$ is replaced by a diffusion operator L with spatially dependent coefficients or more general operator and β is spatially dependent, the strong (or weak) law of large numbers for branching diffusion (or more general branching Hunt processes) and superdiffusion have been investigated recently by many papers. See [1] and [6] for branching diffusion, [11] for branching Hunt processes, and [5] [10] [13] and [14] and [18] (with general branching mechanism) for superdiffusions. In all of these papers, the mean of the process grows pure exponentially as $e^{\lambda_c t}$ with some positive constant λ_c, usually called the (generalized) principal eigenvalue. The techniques used in these papers can not be applied to handle the case when the mean of the process grows in the non-exponential manner $f(t)e^{\lambda_c t}$, where, for example, $f(t) = t^{-d/\alpha}$ as above.

In this paper, we will prove strong law of large numbers for super-stable processes with index $\alpha \in (0, 2]$ corresponding to the operator

$$-\left(-\Delta\right)^{\alpha/2} u + \beta u - \eta u^2,$$

where β and η are positive constants. In the special case $\alpha = 2$, our results extend the main result Theorem 3.2 in [22]. In particular, we extend the starting measure δ_x, $x \in \mathbb{R}^d$, in [22] to any finite μ on $\mathbb{R}^d$ satisfying $\int_{\mathbb{R}^d} x \mathrm{d}\mu < \infty$, and the test function $f \in C_c(\mathbb{R}^d)$ in [22] to more general ones (see Theorem 4 below), and moreover, we improve Wang's result from one specific f to shallow convergence (see Theorem 8 below), which implies vague convergence. Our proof depends mainly on Fourier analysis and stochastic calculations, advancing the methods introduced in [3] in the discussion of Hölder continuity for general measure-valued Markov processes including superprocesses. Our proofs are simpler and more extendable than those in [23], [2] and [22]. Indeed, based upon the fundamental role of the Fourier transform in pde there is reason to be optimistic that our methods can be extended to more general operators and branching mechanism.

The spine method recently developed for measure-valued Markov processes is a powerful probabilistic tool in studying properties of the processes, see [10], [11], [12] [15] and [17] (to list a few but not all). Englander, Harris and Kyprianou [11] used the martingale change of measure and spine decomposition to prove the SLLN for branching diffusions. Their proof depends on how the support of branching diffusion expands (see condition (iii) on page 282 of [11]). But as mentioned above, the support of a super-stable process with index $\alpha \in (0, 2)$ expands to the whole space $\mathbb{R}^d$ immediately, so we can not expect to extend the method in [11] to superprocesses with general underlining processes, like α-stable process. The purpose of this paper is to generalize Watanabe's results in [22] from discrete particle systems to superprocess using techniques from Fourier transform theory and stochastic calculations. We emphasize that we consider all $\alpha \in (0, 2]$ and do not assume our starting measure has compact support. Our only assumption on μ is that $\int_{\mathbb{R}^d} x\mu(\mathrm{d}x) < \infty$.

2 Notation and Model

Recall that we use $\mu(f)$ to denote $\int f \mathrm{d}\mu$ for a measure μ and integrable function f. For simplicity, we let $\mu_r = \int |x|^r \mu(\mathrm{d}r)$ and $\cos_\theta$ denote the function $x \to \cos(\theta x)$ below. We also use the following extended Vinogradov symbol (also used in [16]): Suppose $a(n, m), b(n, m)$ are expressions depending upon two sets of variables n, m. Then,

$$a(n,m) \overset{n}{\ll} b(n,m) \text{ means } \exists\ c_m > 0 \text{ such that } a(n,m) \le c_m b(n,m)\ \forall\ n, m.$$

For clarity, c_m depends only on m.

Throughout this paper, we assume $\mu \in M_F\left(\mathbb{R}^d\right)$ such that $\mu_0, \mu_1 < \infty$. We consider the measure-valued Markov process $X = (X_t, t \ge 0; P_\mu)$ on $\mathbb{R}^d$ such that

$$X_t(f) = \mu(f) + \int_0^t X_s\left(\left(-(-\Delta)^{\alpha/2} + \beta\right) f\right) \mathrm{d}s + M_t(f) \tag{3}$$

for all f bounded and continuous functions with bounded and continuous partial derivatives of order $k \leq 2$, where $M_t(f)$ is a martingale with quadratic variation

$$[M(f)](t) = \int_0^t X_s \left(\eta f^2\right) \mathrm{d}s,$$

and $\eta > 0$ and $\beta > 0$ are positive constants. Note that X starts from μ, the particles move independently according to a symmetric α-stable process on $\mathbb{R}^d$ with generator $-(-\Delta)^{\alpha/2}$ with $\alpha \in (0, 2]$, and the branching mechanism is given by $\eta z^2 - \beta z$. Since $\beta > 0$, X is supercritical.

Substituting $f(x) = e^{-i\theta x}$ in (3) and using notation $\hat{X}(t,\theta) = X_t(\cos_\theta) - iX_t(\sin_\theta)$, we get

$$\hat{X}(t,\theta) = \hat{X}(0,\theta) + \int_0^t \left(-|\theta|^\alpha + \beta\right) \hat{X}(s,\theta)\, \mathrm{d}s + \hat{M}(t,\theta) \tag{4}$$

for all $\theta \in \mathbb{R}^d$, where $\hat{M}(t,\theta)$ is a complex martingale with quadratic variations and covariations:

$$\left[\mathrm{Re}\ \hat{M}(\cdot,\theta)\right](t) = \int_0^t X_s \left(\eta \cos_\theta^2\right) \mathrm{d}s;$$

$$\left[\mathrm{Im}\ \hat{M}(\cdot,\theta)\right](t) = \int_0^t X_s \left(\eta \sin_\theta^2\right) \mathrm{d}s;$$

$$\left[\hat{M}(\cdot,0), \mathrm{Re}\ \hat{M}(\cdot,\theta)\right](t) = \int_0^t X_s \left(\eta \cos_\theta\right) \mathrm{d}s;$$

$$\left[\hat{M}(\cdot,0), \mathrm{Im}\ \hat{M}(\cdot,\theta)\right](t) = \int_0^t X_s \left(\eta \sin_\theta\right) \mathrm{d}s.$$

Using variations of constants, we get

$$\hat{X}(t,\theta) = e^{(\beta - |\theta|^\alpha)t} \hat{X}(0,\theta) + \int_0^t e^{(\beta - |\theta|^\alpha)(t-s)} \hat{M}(\mathrm{d}s,\theta). \tag{5}$$

Define

$$\hat{W}_t(\theta) = \hat{W}(t,\theta) = e^{(|\theta|^\alpha - \beta)t} \hat{X}(t,\theta) = e^{(|\theta|^\alpha - \beta)t} X_t(e^{-i\theta t}). \tag{6}$$

Then, $\hat{W}(t,\theta)$ is a complex martingale for any $\theta \in \mathbb{R}^d$.

3 Results

Our first result describes the limiting object of our scaled super-stable process in frequency domain. It will be used in the subsequent results herein.

Theorem 1 *Suppose $\alpha \in (0,2]$ and $\kappa \in \left(0, \frac{\beta}{2}\right)$. Then, $\hat{W}_t(\theta)$ converges almost surely and in the mean-square sense to limit $\hat{W}(\theta)$ for each $\theta \in I\!R^d$. Moreover, the limit object satisfies*

$$P_\mu\left[\left|\hat{W}\left(\lambda\right)-\hat{W}\left(\theta\right)\right|^2\right] \overset{\lambda,\theta}{\ll} |\theta-\lambda|^{1\wedge\alpha}. \tag{7}$$

for all $|\lambda|^\alpha, |\theta|^alpha \leq \kappa$.

Proof. Let $\epsilon = \beta - 2\kappa$. Note that $\hat{W}_t\left(\theta\right)$ and $\hat{W}_t\left(\theta\right) - \hat{W}_t\left(\lambda\right)$ are complex martingales with quadratic variations satisfying

$$\begin{aligned}
\left[\mathrm{Re}\ \hat{W}\left(\theta\right)\right](t) &= \int_0^t e^{2(|\theta|^\alpha-\beta)(s)}X_s\left(\eta\cos_\theta^2\right)\mathrm{d}s;\\
\left[\mathrm{Im}\ \hat{W}\left(\theta\right)\right](t) &= \int_0^t e^{2(|\theta|^\alpha-\beta)(s)}X_s\left(\eta\sin_\theta^2\right)\mathrm{d}s;\\
\left[\mathrm{Re}\ (\hat{W}\left(\theta\right)-\hat{W}\left(\lambda\right))\right](t) &= \int_0^t e^{-2\beta s}X_s\left(\eta(e^{|\theta|^\alpha s}\cos_\theta - e^{|\lambda|^\alpha s}\cos_\lambda)^2\right)\mathrm{d}s;\\
\left[\mathrm{Im}\ (\hat{W}\left(\theta\right)-\hat{W}\left(\lambda\right))\right](t) &= \int_0^t e^{-2\beta s}X_s\left(\eta(e^{|\theta|^\alpha s}\sin_\theta - e^{|\lambda|^\alpha s}\sin_\lambda)^2\right)\mathrm{d}s.
\end{aligned}$$

By the martingale property of $\hat{W}_t(0) = e^{-\beta s}\hat{X}\left(s,0\right) = e^{-\beta s}X_s(1)$, we have for $0 \leq u < t$ that

$$\begin{aligned}
P_\mu\left[\left|\hat{W}_t\left(\theta\right)-\hat{W}_u\left(\theta\right)\right|^2\right] &= \int_u^t e^{2(|\theta|^\alpha-\beta)s}\eta P_\mu[X_s\left(1\right)]\mathrm{d}s \\
&= \eta\mu\left(1\right)\int_u^t e^{(2|\theta|^\alpha-\beta)s}ds \\
&= \begin{cases} \frac{\eta\mu(1)}{2|\theta|^\alpha-\beta}\left(e^{(2|\theta|^\alpha-\beta)t}-e^{(2|\theta|^\alpha-\beta)u}\right), & \text{if } 2\left|\theta\right|^\alpha \neq \beta, \\ \\ \eta\mu\left(1\right)(t-u), & \text{if } 2\left|\theta\right|^\alpha = \beta. \end{cases}
\end{aligned} \tag{8}$$

Therefore, letting $u = 0$, we find $0 < \sup\limits_{t\geq 0} P_\mu\left[\left|\hat{W}_t\left(\theta\right)\right|^2\right] < \infty \quad$ if $2\left|\theta\right|^\alpha < \beta$ (since $P_\mu\left[\left|\hat{X}\left(0,\theta\right)\right|^2\right] \leq |\mu(\sin(\theta))|^2 + |\mu(\cos(\theta))|^2 \leq \mu_0^2 < \infty$). An application of the martingale convergence theorem yields

$$\hat{W}(\theta) \doteq \lim_{t\to\infty} \hat{W}_t(\theta)$$

exists almost surely and in mean square sense for each $\theta \in \mathbb{R}^d$.

Next, we show the Hölder continuity in mean property for $\hat{W}$. $\hat{W}_t(0)$ is a non-negative martingale starting at $\hat{X}(0,0) = \mu_0$ and satisfying

$$\left[\hat{W}(0)\right]_t = \int_0^t \eta e^{-\beta s} \hat{W}_s(0)\,\mathrm{d}s.$$

Hence, we have by the Burkholder-Davis-Gundy inequality that

$$\begin{aligned}
& P_\mu\left[\sup_{u\geq 0}\left|\hat{W}_u(\lambda) - \hat{W}_u(\theta) - \hat{X}(0,\lambda) + \hat{X}(0,\theta)\right|^2\right] \\
\overset{\lambda,\theta}{\ll} \; & P_\mu\left[\left|\int_0^\infty e^{-2\beta s} X_s \left(e^{2|\lambda|^\alpha s} + e^{2|\theta|^\alpha s} - 2e^{|\lambda|^\alpha s + |\theta|^\alpha s}\cos_{\theta-\lambda}\right)\mathrm{d}s\right|\right] \\
\overset{\lambda,\theta}{\ll} \; & \int_0^\infty e^{-\beta s}\left[(e^{2|\lambda|^\alpha s} + e^{2|\theta|^\alpha s} - 2e^{|\lambda|^\alpha s + |\theta|^\alpha s}) P_\mu(\hat{W}_s(0))\right. \\
+ \; & \left. P_\mu\left|e^{(|\lambda|^\alpha + |\theta|^\alpha - \beta)s} X_s(1-\cos_{\theta-\lambda})\right|\right]\mathrm{d}s \\
\overset{\lambda,\theta}{\ll} \; & \int_0^\infty e^{-\beta s}(e^{|\lambda|^\alpha s} - e^{|\theta|^\alpha s})^2 \mathrm{d}s \\
+ \; & \int_0^\infty e^{-\epsilon s} e^{(-|\theta-\lambda|^\alpha s} P_\mu\left|e^{(|\theta-\lambda|^\alpha - \beta)s} X_s(1-\cos_{\theta-\lambda})\right|\mathrm{d}s,
\end{aligned} \tag{9}$$

where in the last inequality we used the facts that $\epsilon = \beta - 2\kappa$ and $|\lambda|^\alpha, |\theta|^\alpha \leq \kappa$. However,

$$P_\mu\left[\left|e^{(|\theta-\lambda|^\alpha - \beta)s} X_s(1-\cos_{\theta-\lambda})\right|\right] = \mu(1-\cos_{\theta-\lambda}), \tag{10}$$

$$P_\mu\left[\left|\hat{X}(0,\lambda) - \hat{X}(0,\theta)\right|^2\right] \overset{\lambda,\theta}{\ll} \mu(1-\cos_{\theta-\lambda}), \tag{11}$$

and it follows by Taylor's theorem that

$$|1 - \cos((\theta-\lambda)x)| \leq |\theta-\lambda|\,|x|, \tag{12}$$

and

$$\left|e^{|\lambda|^\alpha s} - e^{|\theta|^\alpha s}\right|^2 \overset{s,\lambda,\theta}{\ll} s^2 e^{2\kappa s}(|\lambda|^\alpha - |\theta|^\alpha)^2 \overset{s,\lambda,\theta}{\ll} s^2 e^{2\kappa s}|\lambda-\theta|^\alpha \tag{13}$$

since if $|\theta| > |\lambda|$, then $|\theta|^\alpha - |\lambda|^\alpha \leq \left(|\theta|^2 - |\lambda|^2\right)^{\frac{\alpha}{2}} \leq 2\kappa^{\frac{\alpha}{2}} |\theta - \lambda|^{\frac{\alpha}{2}}$.

Substituting bounds (10)-(13) above into (9), we find by the Burkholder-Davis-Gundy inequality that

$$P_\mu \left[\sup_{u \geq 0} \left| \hat{W}(u, \lambda) - \hat{W}(u, \theta) \right|^2 \right] \overset{\lambda,\theta}{\ll} |\theta - \lambda|^\alpha + |\theta - \lambda| . \tag{14}$$

and, letting $u \to \infty$, we get (7). ■

Next, we convert our "frequency domain" result to a SLLN for super-stable processes. Since both the limit and prelimit are measures, we introduce test functions f.

Theorem 2 *Suppose $\kappa \in \left(0, \frac{\beta}{2}\right]$ and f satisfies*

$$\hat{c} \doteq \int_{\mathbb{R}^d} e^{\epsilon|\theta|^\alpha} \left| \hat{f}(\theta) \right| \frac{\mathrm{d}\theta}{(2\pi)^d} < \infty \tag{15}$$

for $\epsilon = \beta - 2\kappa$. Then, for any $\delta \in (0, \beta\kappa - 2\kappa^2)$ there is a constant $c > 0$ and a random variable $C_\delta > 0$ such that

$$P_\mu \left[\max_{n\epsilon \leq t \leq (n+1)\epsilon} \left| \frac{X_t(f)}{e^{\beta t} t^{-\frac{d}{\alpha}}} - \int_{|\theta|^\alpha \leq \kappa} e^{-t|\theta|^\alpha} \hat{W}(\theta) \hat{f}(\theta) \frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d} \right| \right] \leq c\sqrt{n} e^{-\left(\beta\kappa - 2\kappa^2\right)n} \tag{16}$$

and

$$\left| \frac{X_t(f)}{e^{\beta t} t^{-\frac{d}{\alpha}}} - \int_{|\theta|^\alpha \leq \kappa} e^{-t|\theta|^\alpha} \hat{W}(\theta) \hat{f}(\theta) \frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d} \right| \leq C_\delta e^{-\delta t} \quad P_\mu\text{-a.s.}, \tag{17}$$

where $\hat{W}$ is defined in the previous theorem.

Remark 1 *This result directly generalizes Wang [22, Theorem 3.1].*

Proof. We first note that

$$\frac{X_t(f)}{e^{\beta t}} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-|\theta|^\alpha t} \hat{W}_t(\theta) \hat{f}(\theta) \, \mathrm{d}\theta.$$

By Doob's L_p-inequality

$$P_\mu \left[\sup_{t > u} \left| \hat{W}_t(\theta) - \hat{W}_u(\theta) \right|^2 \right] \leq 4 \frac{\eta\mu(1)}{\beta - 2|\theta|^\alpha} e^{(2|\theta|^\alpha - \beta)u}.$$

provided $2|\theta|^{\alpha} < \beta$. Letting $t \to \infty$ above, we get

$$P_{\mu}\left[\left|\hat{W}(\theta) - \hat{W}_{u}(\theta)\right|^{2}\right] \leq 4\frac{\eta\mu(1)}{\beta - 2|\theta|^{\alpha}}e^{(2|\theta|^{\alpha}-\beta)u}$$

if $2|\theta|^{\alpha} < \beta$ and combining the last two equations, we get

$$P_{\mu}\left[\sup_{t\geq u}\left|\hat{W}(t,\theta) - \hat{W}(\theta)\right|^{2}\right] \leq 32\frac{\eta\mu(1)}{\beta - 2|\theta|^{\alpha}}e^{(2|\theta|^{\alpha}-\beta)u}$$

provided $2|\theta|^{\alpha} < \beta$. Letting $u = n\epsilon$, we get

$$\begin{aligned}
&\int_{|\theta|^{\alpha}\leq\kappa}\sqrt{P_{\mu}\left(\sup_{t\geq n\epsilon}|\hat{W}_{t}(\theta)\hat{f}(\theta) - \hat{W}(\theta)\hat{f}(\theta)|^{2}e^{-2t|\theta|^{\alpha}}\right)}\mathrm{d}\theta \\
&\leq \int_{|\theta|^{\alpha}\leq\kappa}\sqrt{P_{\mu}\left(\sup_{t\geq n\epsilon}|\hat{W}_{t}(\theta) - \hat{W}(\theta)|^{2}\right)}|\hat{f}(\theta)|e^{-n\epsilon|\theta|^{\alpha}}\mathrm{d}\theta \\
&\leq \int_{|\theta|^{\alpha}\leq\kappa}4\sqrt{\frac{2\eta\mu(1)}{\beta - 2|\theta|^{\alpha}}}e^{\left(|\theta|^{\alpha}-\frac{\beta}{2}\right)n\epsilon}|\hat{f}(\theta)|e^{-n\epsilon|\theta|^{\alpha}}\mathrm{d}\theta \\
&\leq 4\sqrt{\frac{2\eta\mu(1)}{\beta - 2\kappa}}e^{-\frac{\beta}{2}n\epsilon}\int_{|\theta|^{\alpha}\leq\frac{\beta}{2}}|\hat{f}(\theta)|\mathrm{d}\theta \\
&\overset{n}{\ll} e^{\left(-\frac{\beta^{2}}{2}+\beta\kappa\right)n}
\end{aligned}$$

since $\epsilon = \beta - 2\kappa$. Moreover, by our above bound and Doob's L_p-inequality

$$\begin{aligned}
&\int_{|\theta|^{\alpha}>\kappa}\sqrt{P_{\mu}\left(\sup_{n\epsilon\leq t\leq(n+1)\epsilon}|\hat{W}_{t}(\theta)\hat{f}(\theta)|^{2}e^{-2t|\theta|^{\alpha}}\right)}\mathrm{d}\theta \\
&\leq \int_{|\theta|^{\alpha}>\kappa}\sqrt{P_{\mu}\left(\sup_{n\epsilon\leq t\leq(n+1)\epsilon}|\hat{W}_{t}(\theta) - \hat{W}_{0}(\theta)|^{2}\right)}|\hat{f}(\theta)|e^{-n\epsilon|\theta|^{\alpha}}\mathrm{d}\theta \\
&+ \int_{|\theta|^{\alpha}>\kappa}\sqrt{P_{\mu}(|\hat{W}_{0}(\theta)|^{2})}|\hat{f}(\theta)|e^{-n\epsilon|\theta|^{\alpha}}\mathrm{d}\theta \\
&\leq 2\int_{|\theta|^{\alpha}>\kappa}\sqrt{\frac{\eta\mu(1)}{2|\theta|^{\alpha} - \beta}(e^{(2|\theta|^{\alpha}-\beta)(n+1)\epsilon} - 1)}|\hat{f}(\theta)|e^{-n\epsilon|\theta|^{\alpha}}\mathrm{d}\theta \\
&+\mu(1)\int_{|\theta|^{\alpha}>\kappa}|\hat{f}(\theta)|e^{-n\epsilon|\theta|^{\alpha}}\mathrm{d}\theta
\end{aligned}$$

Using Taylor's theorem, we continue the above estimate to get

$$\int_{|\theta|^\alpha>\kappa}\sqrt{P_\mu\left(\sup_{n\epsilon\le t\le(n+1)\epsilon}|\hat{W}_t\left(\theta\right)\hat{f}\left(\theta\right)|^2 e^{-2t|\theta|^\alpha}\right)}\mathrm{d}\theta$$

$$\le\left(2\sqrt{\eta\left(n+1\right)\mu_0\epsilon}\right)\cdot\left[e^{-\frac{\beta}{2}(n+1)\epsilon}\int_{|\theta|^\alpha\ge\beta}e^{|\theta|^\alpha\epsilon}\left|\hat{f}\left(\theta\right)\right|\mathrm{d}\theta+e^{-(n+1)\epsilon\kappa}\int_{\kappa<|\theta|^\alpha<\beta}e^{|\theta|^\alpha\epsilon}\left|\hat{f}\left(\theta\right)\right|\mathrm{d}\theta\right]$$

$$+\mu_0 e^{-(n+1)\epsilon\kappa}\hat{c}$$

$$\overset{n}{\ll}\left(\sqrt{n}\right)e^{-n\epsilon\kappa}.$$

Hence, by the previous equations and Cauchy-Schwarz' inequality

$$\begin{aligned}
&P_\mu\left[\sup_{n\epsilon\le t\le(n+1)\epsilon}\left|\frac{X_t(f)}{e^{\beta t}}-\int_{|\theta|^\alpha\le\kappa}e^{-t|\theta|^\alpha}\hat{W}\left(\theta\right)\hat{f}\left(\theta\right)\frac{\mathrm{d}\theta}{\left(2\pi\right)^d}\right|\right]\\
\le&\frac{1}{(2\pi)^d}\int_{|\theta|^\alpha\le\kappa}P_\mu\left(\sup_{t\ge n\epsilon}\left|\hat{W}_t\left(\theta\right)\hat{f}\left(\theta\right)-\hat{W}\left(\theta\right)\hat{f}\left(\theta\right)\right|e^{-t|\theta|^\alpha}\right)\mathrm{d}\theta\\
&+\frac{1}{(2\pi)^d}\int_{|\theta|^\alpha>\kappa}P_\mu\left(\sup_{n\epsilon\le t\le(n+1)\epsilon}\left|\hat{W}_t\left(\theta\right)\hat{f}\left(\theta\right)\right|e^{-t|\theta|^\alpha}\right)\mathrm{d}\theta\\
\overset{n}{\ll}&\sqrt{n}e^{-(\beta\kappa-2\kappa^2)n}
\end{aligned}$$

using $\epsilon=\beta-2\kappa$. Then (16) holds. Multiplying both sides by $t^{\frac{d}{\alpha}}$ and fixing $\delta\in(0,\beta\kappa-2\kappa^2)$, we get that

$$\sum_{n=1}^{\infty}P_\mu\sup_{n\epsilon\le t\le(n+1)\epsilon}\left[\left|\frac{X_t(f)}{e^{\beta t}t^{-\frac{d}{\alpha}}}-\int_{|\theta|^\alpha\le\kappa}e^{-t|\theta|^\alpha}\hat{W}\left(\theta\right)\hat{f}\left(\theta\right)\frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d}\right|e^{\delta t}\right]<\infty.$$

So there is a random $C_\delta>0$ such that

$$\left|\frac{X_t(f)}{e^{\beta t}t^{-\frac{d}{\alpha}}}-\int_{|\theta|^\alpha\le\kappa}e^{-t|\theta|^\alpha}\hat{W}\left(\theta\right)\hat{f}\left(\theta\right)\frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d}\right|\le C_\delta e^{-\delta t}\quad P_\mu\text{-a.s.}$$

■

Finally, we can state our first SLLN (not in frequency domain). The following lemma will be immediately improved by the theorem to follow thereafter.

Lemma 3 *Suppose f has Fourier transform $\hat{f}$ that satisfies*

$$\int_{\mathbb{R}^d} e^{\epsilon|\theta|^\alpha} \left|\hat{f}\left(\theta\right)\right| \mathrm{d}\theta < \infty \tag{18}$$

for all $\epsilon < \beta$. Then,

(1) existence of a $\kappa_0 < \frac{\beta}{2}$ such that $\sup_{|\theta|^\alpha \leq \kappa_0} |\hat{f}(\theta)| < \infty$ implies that

$$\frac{X_t(f)}{e^{\beta t} t^{-\frac{d}{\alpha}}} - \hat{W}\left(0\right) \int e^{-t|\theta|^\alpha} \hat{f}\left(\theta\right) \frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d} \to 0 \quad P_\mu\text{-a.s.}$$

(2) continuity at 0 of $\hat{f}$ implies that

$$\lim_{t\to\infty} \frac{t^{\frac{d}{\alpha}} X_t(f)}{e^{\beta t}} \to c_\alpha \ \hat{W}(0)\hat{f}(0) \quad P_\mu\text{-a.s.},$$

where $c_\alpha = \int_{\mathbb{R}^d} e^{-|y|^\alpha} \frac{\mathrm{d}y}{(2\pi)^d}$.

Remark 2 *1) It is clearly sufficient that*

$$\int_{\mathbb{R}^d} e^{\beta|\theta|^\alpha} \left|\hat{f}\left(\theta\right)\right| \frac{\mathrm{d}\theta}{\left(2\pi\right)^d} < \infty.$$

2) $c_2 = (2\pi)^{-\frac{d}{2}}$.

3) The Fourier transform is defined in a different manner for each $L_p(I\!R^d)$ with $p \in [1,2]$. (Each can be thought of as an extension of the Fourier transform on $\mathcal{S}(I\!R^d)$, the set of rapidly decreasing functions (see [21] for definition).) If $f \in L_1(I\!R^d)$, then $\hat{f}$ is continuous and $\hat{f}(0) = \int_{I\!R^d} f(x)dx$.

Proof. We let $a_i = i^{\frac{3\alpha}{1\wedge\alpha}-1}$ and $s_n = \sum_{i=1}^{n} a_i$. By (7), we have that $P_\mu \left|\hat{W}\left(\theta\right) - \hat{W}\left(0\right)\right| \overset{\theta}{\ll} |\theta|^{\frac{1\wedge\alpha}{2}}$ for $|\theta|^\alpha \leq \kappa_0$ so

$$P_\mu\left[\max_{s_n\le t\le s_{n+1}}\int_{|\theta|^\alpha\le\kappa}e^{-t|\theta|^\alpha}\left|\hat W(\theta)\hat f(\theta)-\hat W(0)\hat f(\theta)\right|\frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d}\right] \tag{19}$$

$$\overset{n,\kappa}{\ll}(s_{n+1})^{\frac{d}{\alpha}}\int_{|\theta|^\alpha\le\kappa}e^{-s_n|\theta|^\alpha}P_\mu\left|\hat W(\theta)-\hat W(0)\right|\frac{\mathrm{d}\theta}{(2\pi)^d}\sup_{|\theta|^\alpha\le\kappa}|\hat f(\theta)|$$

$$\overset{n,\kappa}{\ll}(s_{n+1})^{\frac{d}{\alpha}}\int_{|\theta|^\alpha\le\kappa}e^{-s_n|\theta|^\alpha}|\theta|^{\frac{1\wedge\alpha}{2}}\frac{\mathrm{d}\theta}{(2\pi)^d}$$

$$\overset{n,\kappa}{\ll}\left(\frac{s_{n+1}}{s_n}\right)^{d/\alpha}|s_n|^{-\frac{1\wedge\alpha}{2\alpha}}$$

for all $\kappa\le\kappa_0$ and $n=1,2,...$ Moreover, by (18)

$$P_\mu|\hat W(0)|\cdot\max_{s_n\le t\le s_{n+1}}\int_{|\theta|^\alpha>\kappa}e^{-t|\theta|^\alpha}|f(\theta)|\frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d}\overset{n,\kappa}{\ll}(s_{n+1})^{\frac{d}{\alpha}}\int_{|\theta|^\alpha>\kappa}e^{-s_n|\theta|^\alpha}|f(\theta)|\mathrm{d}\theta \tag{20}$$

$$\overset{n,\kappa}{\ll}(s_{n+1})^{\frac{d}{\alpha}}e^{-(s_n+\epsilon)\kappa},$$

here $\epsilon=\beta-2\kappa$ as in Theorem 2, and from (16) of Theorem 2 one finds that

$$P_\mu\left[\max_{s_n\le t\le s_{n+1}}\left|\frac{X_t(f)}{e^{\beta t}t^{-\frac{d}{\alpha}}}-\int_{|\theta|^\alpha\le\kappa}e^{-t|\theta|^\alpha}\hat W(\theta)\hat f(\theta)\frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d}\right|\right] \tag{21}$$

$$\le\sum_{j=\lfloor s_n/\epsilon\rfloor}^{\lfloor s_{n+1}/\epsilon\rfloor}P_\mu\left[\max_{j\epsilon\le t\le(j+1)\epsilon}\left|\frac{X_t(f)}{e^{\beta t}t^{-\frac{d}{\alpha}}}-\int_{|\theta|^\alpha\le\kappa}e^{-t|\theta|^\alpha}\hat W(\theta)\hat f(\theta)\frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d}\right|\right]$$

$$\overset{n,\kappa}{\ll}(s_{n+1}-s_n)\sqrt{s_n}e^{-\left(\frac{\beta^2}{2}-\beta\kappa\right)s_n}$$

Therefore, we have by the previous three equations that

$$\sum_{n=1}^\infty P_\mu\left[\max_{s_n\le t\le s_{n+1}}\left|\frac{X_t(f)}{e^{\beta t}t^{-\frac{d}{\alpha}}}-\hat W(0)\int e^{-t|\theta|^\alpha}\hat f(\theta)\frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d}\right|\right]<\infty \tag{22}$$

for any $\kappa\le\kappa_0<\frac{\beta}{2}$, and so

$$\frac{X_t(f)}{e^{\beta t}t^{-\frac{d}{\alpha}}}-\hat W(0)\int_{\mathbb{R}^d}e^{-t|\theta|^\alpha}\hat f(\theta)\frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d}\to 0\qquad P_\mu\text{-a.s.} \tag{23}$$

Next, given $\gamma > 0$ we have by the continuity of $\hat{f}(\theta)$ at 0 that there is a $\kappa_0 \in (0, \frac{\beta}{2})$ satisfying (18) and $\sup_{|\theta|^\alpha \le \kappa_0} \left| \hat{f}(\theta) - \hat{f}(0) \right| < \gamma$, which implies that

$$\int e^{-t|\theta|^\alpha} |\hat{f}(\theta) - \hat{f}(0)| \frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d} \tag{24}$$

$$= \int_{|\theta|^\alpha \le \kappa_0} e^{-t|\theta|^\alpha} |\hat{f}(\theta) - \hat{f}(0)| \frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d} + \int_{|\theta|^\alpha > \kappa_0} e^{-t|\theta|^\alpha} |\hat{f}(\theta) - \hat{f}(0)| \frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d}$$

$$\overset{\gamma,t}{\ll} \gamma + e^{-(t+\epsilon)\kappa_0} \int_{|\theta|^\alpha > \kappa_0} e^{\epsilon|\theta|^\alpha} |\hat{f}(\theta)| \frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d} + |\hat{f}(0)| \int_{|\theta|^\alpha > \kappa_0} e^{-t|\theta|^\alpha} \frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d}$$

and the result follows from the fact that

$$\int_{|\theta|^\alpha > \kappa_0} e^{-t|\theta|^\alpha} \frac{\mathrm{d}\theta}{\left(2\pi t^{-\frac{1}{\alpha}}\right)^d} = \int_{|y|^\alpha > t\kappa_0} e^{-|y|^\alpha} \frac{\mathrm{d}y}{(2\pi)^d} \to 0$$

as $t \to \infty$. ■

Starting from Watanabe, everybody considered continuous, compactly supported f. It is interesting to see how far we can relax the assumptions on f.

Theorem 4 *Suppose that f is such that its Fourier transform $\hat{f}$ exists, $\hat{f}$ is continuous at 0 and there is an $\epsilon > 0$ such that*

$$\int_{\mathbb{R}^d} e^{\epsilon|\theta|^\alpha} |\hat{f}(\theta)| \mathrm{d}\theta < \infty.$$

Then,

$$\frac{X_t(f)}{e^{\beta t} t^{-\frac{d}{\alpha}}} \to c_\alpha \hat{W}(0) \hat{f}(0), \quad P_\mu\text{-a.s.}$$

Remark 3 *The Fourier transform is only defined as an element of $L^p\left(\mathbb{R}^d\right)$ for some $p \in [1, 2]$ and hence almost everywhere, so continuous at 0 should be interpreted as 'there is a version that is continuous at zero'. Compared to the previous theorems, $\epsilon > 0$ can be arbitrarily small.*

Proof. We define $\hat{\phi}^{\beta}(\theta) = \begin{cases} 1, & |\theta|^{\alpha} \leq \beta, \\ e^{\beta(\beta-|\theta|^{\alpha})}, & |\theta|^{\alpha} > \beta, \end{cases}$ and $f^{\beta} \doteq f * \phi^{\beta}$. Hence, we have $\widehat{f^{\beta}} = \widehat{f}\hat{\phi}^{\beta}$ and

$$\int_{\mathbb{R}^d} e^{\beta|\theta|^{\alpha}} |\widehat{f^{\beta}}(\theta)| \mathrm{d}\theta \leq e^{\beta^2} \int_{|\theta|^{\alpha} \leq \beta} |\widehat{f}(\theta)| \mathrm{d}\theta + e^{\beta^2} \int_{|\theta|^{\alpha} > \beta} |\widehat{f}(\theta)| \mathrm{d}\theta < \infty.$$

Therefore, by Lemma 3, we have that

$$\frac{t^{\frac{d}{\alpha}} X_t(f^{\beta})}{e^{\beta t}} \to c_{\alpha} \hat{W}(0) \hat{f}^{\beta}(0) = c_{\alpha} \hat{W}(0) \hat{f}(0), \quad P_{\mu}\text{-a.s..}$$

Let $a_i = \frac{1}{\sqrt{i}}$ and $s_n = \sum_{i=1}^{n} a_i$ so $s_n \nearrow \infty$. Then, we also have that

$$\begin{aligned}
&(2\pi)^d P_{\mu}\left[\sup_{s_n \leq t \leq s_{n+1}} \left|\frac{t^{\frac{d}{\alpha}} X_t(f-f^{\beta})}{e^{\beta t}}\right|\right] \\
&= P_{\mu}\left[\sup_{s_n \leq t \leq s_{n+1}} \left| t^{\frac{d}{\alpha}} \int_{|\theta|^{\alpha} > \beta} e^{-|\theta|^{\alpha} t} (\hat{W}_t - \hat{W}_0)(\theta) \left(\hat{f} - \hat{f}^{\beta}\right)(\theta) \, \mathrm{d}\theta \right|\right] \\
&\quad + P_{\mu}\left[\sup_{s_n \leq t \leq s_{n+1}} \left| t^{\frac{d}{\alpha}} \int_{|\theta|^{\alpha} > \beta} e^{-|\theta|^{\alpha} t} \hat{W}_0(\theta) \left(\hat{f} - \hat{f}^{\beta}\right)(\theta) \, \mathrm{d}\theta \right|\right].
\end{aligned}$$

For the first term, we find by Doob's L_p-inequality, (8) and Taylor's theorem (in the second last inequality) that

$$\begin{aligned}
&P_{\mu}\left[\sup_{s_n \leq t \leq s_{n+1}} \left| t^{\frac{d}{\alpha}} \int_{|\theta|^{\alpha} > \beta} e^{-|\theta|^{\alpha} t} (\hat{W}_t - \hat{W}_0)(\theta) \left(\hat{f} - \hat{f}^{\beta}\right)(\theta) \, \mathrm{d}\theta \right|\right] \\
&\leq (s_{n+1})^{\frac{d}{\alpha}} \int_{|\theta|^{\alpha} > \beta} e^{-|\theta|^{\alpha} s_n} P_{\mu}^{\frac{1}{2}} \left[\sup_{s_n \leq t \leq s_{n+1}} \left|\hat{W}_t - \hat{W}_0\right|^2 (\theta)\right] \left|\hat{f} - \hat{f}^{\beta}\right|(\theta) \, \mathrm{d}\theta \\
&\leq 2\sqrt{\eta\mu_0} (s_{n+1})^{\frac{d}{\alpha}} \int_{|\theta|^{\alpha} > \beta} e^{-|\theta|^{\alpha} s_n} \sqrt{\frac{e^{(2|\theta|^{\alpha} - \beta) s_{n+1}} - 1}{2|\theta|^{\alpha} - \beta}} \left|\hat{f} - \hat{f}^{\beta}\right|(\theta) \, \mathrm{d}\theta \\
&\leq 2\sqrt{\eta\mu_0} (s_{n+1})^{\frac{d}{\alpha} + \frac{1}{2}} e^{-\frac{\beta}{2} s_{n+1}} \int_{|\theta|^{\alpha} > \beta} e^{a_n |\theta|^{\alpha}} \left|\hat{f} - \hat{f}^{\beta}\right|(\theta) \, \mathrm{d}\theta \\
&\overset{n}{\ll} (s_{n+1})^{\frac{d}{\alpha} + \frac{1}{2}} \sqrt{e^{-\beta s_{n+1}}}.
\end{aligned}$$

(Here, we used the fact that $a_n \leq \epsilon$ for large n in the last bound.) For the second term, we find

$$
\begin{aligned}
& P_\mu\left[\sup_{t\geq s_n}\left|t^{\frac{d}{\alpha}}\int_{|\theta|^\alpha>\beta} e^{-|\theta|^\alpha t}\hat{W}_0\left(\theta\right)\left(\hat{f}-\hat{f}^\beta\right)\left(\theta\right)\mathrm{d}\theta\right|\right] \\
& \leq \sup_{t\geq s_n}\left|t^{\frac{d}{\alpha}}e^{-\beta t}\int_{|\theta|^\alpha>\beta}\mu\left(e^{i\theta\cdot(\cdot)}\right)\left(\hat{f}-\hat{f}^\beta\right)\left(\theta\right)\mathrm{d}\theta\right| \\
& \leq s_n^{\frac{d}{\alpha}}e^{-\beta s_n}\mu_0^2\int\left(1-\hat{\phi}^\beta\right)\left|\hat{f}\left(\theta\right)\right|\mathrm{d}\theta
\end{aligned}
$$

for large enough n. Combining the previous three equations, we find

$$
\sum_{n=1}^{\infty}P_\mu\left[\sup_{s_n\leq t\leq s_{n+1}}\left|\frac{t^{\frac{d}{\alpha}}X_t(f-f^\beta)}{e^{\beta t}}\right|\right]<\infty,
$$

which implies that $\frac{t^{\frac{d}{\alpha}}X_t(f-f^\beta)}{e^{\beta t}}\to 0$ P_μ-a.s. (since $s_n\to\infty$) and therefore

$$
\frac{t^{\frac{d}{\alpha}}X_t(f)}{e^{\beta t}}\to c_\alpha\hat{W}\left(0\right)\hat{f}\left(0\right),\quad P_\mu\text{-a.s.}
$$

■

Now that we removed the β-dependence on the decay on $\hat{f}$, we can easily generalize Wang's and Watanabe's works from a single continuous, compactly supported function to vague convergence and beyond. We start by considering the case where $f\in L_1$. (Until now, we only assumed existence of the Fourier transform.)

Let

$$
\mathcal{G}^\alpha=\left\{g:\ g\in L^1(\mathbb{R}^d)\text{ such that }\int e^{\epsilon|\theta|^\alpha}\hat{g}\left(\theta\right)\mathrm{d}\theta<\infty\text{ for some }\epsilon>0\right\}
$$

For any $g\in\mathcal{G}^\alpha$, it follows that the Fourier transform $\hat{g}$, is continuous by the L_1-property.

Corollary 5 *Suppose ℓ is Lebesgue measure, that $f\in L_1$ and, for each $\epsilon>0$, there exists $f_1,f_2\in\mathcal{G}^\alpha$ such that $f_1\leq f\leq f_2$ and $\ell\left(f_2-f_1\right)<\epsilon$. Then,*

$$
\frac{t^{\frac{d}{\alpha}}X_t(f)}{e^{\beta t}}\to c_\alpha\hat{W}\left(0\right)\int_{\mathbb{R}^d}f\left(x\right)\mathrm{d}x\quad P_\mu\text{-a.s.}
$$

Proof. By Theorem 4, we have that

$$\frac{t^{\frac{d}{\alpha}} X_t(f_i)}{e^{\beta t}} \to c_\alpha \hat{W}(0)\, \lambda(f_i) \quad P_\mu\text{-a.s.}$$

for $i = 1, 2$. However, this then implies

$$c_\alpha \hat{W}(0)\, \ell(f_1) \leq \liminf_{t\to\infty} \frac{t^{\frac{d}{\alpha}} X_t(f)}{e^{\beta t}} \leq \limsup_{t\to\infty} \frac{t^{\frac{d}{\alpha}} X_t(f)}{e^{\beta t}} \leq c_\alpha \hat{W}(0)\, \ell(f_2)$$

and the Corollary follows. ■

A further useful corollary follows:

Corollary 6 *For any $f \in C_c\left(\mathbb{R}^d\right)$, it follows that*

$$\frac{t^{\frac{d}{\alpha}} X_t(f)}{e^{\beta t}} \to c_\alpha \hat{W}(0) \int_{\mathbb{R}^d} f(x)\,\mathrm{d}x \quad P_\mu\text{-}a.s.$$

Proof. Let $M = \sup_x |f(x)|$, $K = \sup\{|x| : f(x) \neq 0\}$ and $\epsilon \in (0,1)$. Then, by uniform continuity there is a $\delta > 0$ (with $\delta < K$) such that $|f(x) - f(y)| < \frac{\epsilon}{8}$ for all $|x - y| < \delta$ and an $r \in (0,1)$ such that $2M \int_{B(0,\delta)^c} \phi_{r\delta}(y)\,\mathrm{d}y < \dfrac{\epsilon}{8}$, where $\phi_p(y) = \frac{1}{\left(\sqrt{2\pi}p\right)^d} e^{-\frac{|y|^2}{2p^2}}$. Finally, there is an $n > 1$ such that $\int_{B(0,nK)} \phi_{r\delta}(x-y)\,\mathrm{d}y > \dfrac{1}{2}$ for all $|x| \leq K$. Now, we define

$$\begin{aligned}
f_2(x) &= \int_{B(0,nK)} \left(\frac{\epsilon}{2} + f(y)\right) \phi_{r\delta}(x-y)\,\mathrm{d}y \\
f_1(x) &= \int_{B(0,nK)} \left(f(y) - \frac{\epsilon}{2}\right) \phi_{r\delta}(x-y)\,\mathrm{d}y
\end{aligned}$$

Then, noting

$$\begin{aligned}
f(x) &= \int_{\mathbb{R}^d} f(x)\phi_{r\delta}(x-y)\mathrm{d}y \\
&= \int_{B(0,nK)} f(x)\phi_{r\delta}(x-y)\mathrm{d}y + \int_{B(0,nK)^c} f(x)\phi_{r\delta}(x-y)\mathrm{d}y,
\end{aligned}$$

we have

$$
\begin{aligned}
& f_2\left(x\right)-f\left(x\right) \\
= & \frac{\epsilon}{2}\int_{B(0,nk)}\phi_{r\delta}(x-y)\mathrm{d}y+\int_{B(0,nk)\cap B(x,\delta)}(f(y)-f(x))\phi_{r\delta}(x-y)\mathrm{d}y \\
& +\int_{B(0,nk)\cap B(x,\delta)^c}(f(y)-f(x))\phi_{r\delta}(x-y)\mathrm{d}y-\int_{B(0,nk)^c}f(x)\phi_{r\delta}(x-y)\mathrm{d}y \\
\geq & \frac{\epsilon}{4}-\int_{B(x,\delta)}\underbrace{\left|f\left(y\right)-f\left(x\right)\right|}_{<\frac{\epsilon}{8}}\phi_{r\delta}\left(x-y\right)\mathrm{d}y-2M\int_{B(x,\delta)^c}\phi_{r\delta}\left(x-y\right)\mathrm{d}y \\
> & 0.
\end{aligned}
$$

Similarly we have $f_1 \leq f$. By construction, we have that $f_1, f_2 \in \mathcal{G}^{\alpha}$, $f_1 \leq f \leq f_2$ and $\ell\left(f_2 - f_1\right) < \epsilon$. Hence, this corollary follows from the previous one. ■

We will use the following lemma to go from single f convergence to vague convergence and beyond by setting $\mathcal{M}$ to be a countable subset of $C_c\left(\mathbb{R}^d\right)$ that generates the Borel topology on $\mathbb{R}^d$. In what follows, $(E,\mathcal{T})$ will denote a topological space, and $B(E)$ and $\overline{C}(E)$ will denote the bounded Borel measurable and the bounded continuous $\mathbb{R}$-valued functions on E, respectively.

Lemma 7 *Suppose that $(E,\mathcal{T})$ is a topological space with a countable base, and $\{\mu_t\}\cup\{\mu\}$ are (possibly non-finite) Borel measures; $f \in B\left(E\right)$ satisfies $0 < \mu\left(f\right) < \infty$; $\mathcal{M} \subset B(E)$ strongly separates points, is countable and is closed under multiplication; and*

$$\mu_t\left(gf\right) \to \mu\left(gf\right)$$

for all $g \in \mathcal{M}\cup\{1\}$. Then,

$$\mu_t\left(gf\right) \to \mu\left(gf\right)$$

for all $g \in \overline{C}(E)$.

Proof. We define the probability measures by

$$\nu_t\left(g\right) = \tfrac{\mu_t(gf)}{\mu_t(f)} \text{ and } \nu\left(g\right) = \tfrac{\mu(gf)}{\mu(f)}$$

for all $g \in B(E)$ and find by hypothesis that $\nu_t(g) \to \nu(g)$ for all $g \in \mathcal{M}$. Now, it follows from Blount and Kouritzin [4, Theorem 6] that

$$\nu_t \to \nu \text{ weakly as } t \to \infty$$

or, equivalently $\mu_t(gf) \to \mu(gf)$ as $t \to \infty$, for all $g \in \overline{C}(E)$. ■

Definition 1 We call $\mathcal{H} = \{h \in C(\mathbb{R}^d) : \exists \epsilon > 0 \text{ so that } \sup_{x \in \mathbb{R}^d} e^{\epsilon|x|^2}|h(x)| < \infty\}$ the *swiftly decreasing* functions on $\mathbb{R}^d$ and say Borel measures $\{\mu_t\}$ *converge shallowly* to Borel measure μ if $\mu_t(h) \to \mu(h)$ as $t \to \infty$ for all $h \in \mathcal{H}$.

Theorem 8 $\frac{t^{\frac{d}{\alpha}} X_t}{e^{\beta t}} \to c_\alpha \hat{W}(0)\,\ell$, *$P_\mu$-a.s. in the shallow topology, where ℓ is Lebesgue measure.*

Proof. Let $f_n(x) = \left(\frac{1}{\sqrt{\pi n}}\right)^d e^{-\frac{|x|^2}{n}}$ so $\hat{f}_n(\theta) = e^{-n|\theta|^2}$ and

$$\frac{t^{\frac{d}{\alpha}} X_t(f_n)}{e^{\beta t}} \to c_\alpha \hat{W}(0)\,\ell(f_n) \quad P_\mu\text{-a.s.}$$

by Theorem 4. Moreover, $C_c\left(\mathbb{R}^d\right)$ is an algebra that strongly separates points. Therefore, it follows by Blount and Kouritzin [4, Lemma 2] that there is a countable subcollection $\mathcal{M}$ that strongly separates points and is closed under multiplication. From Corollary 6, we have that

$$\frac{t^{\frac{d}{\alpha}} X_t(gf_n)}{e^{\beta t}} \to c_\alpha \hat{W}(0)\, l(gf_n) \quad P_\mu\text{-a.s.}$$

for all $g \in \mathcal{M}$. Fix an ω such that convergence takes place for all f_n and $g \in \mathcal{M}$. Now, it follows by Lemma 7 that

$$\frac{t^{\frac{d}{\alpha}} X_t(gf_n)}{e^{\beta t}} \to c_\alpha \hat{W}(0)\, l(gf_n) \text{ for all } g \in \overline{C}(E) \text{ and } n = 1, 2, ..., \quad P_\mu\text{-a.s.}$$

The theorem follows. ■

An immediate corollary of this Theorem is the following analog of Watanabe's result:

Corollary 9 $\dfrac{t^{\frac{d}{\alpha}} X_t}{e^{\beta t}} \to c_\alpha \hat{W}(0)\, \ell$ P_μ*-a.s. in the vague topology, where* ℓ *is Lebesgue measure.*

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