# Boundary Harnack principle and gradient estimates for fractional Laplacian perturbed by non-local operators

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**Abstract**

Suppose $d \geq 2$ and $0 < \beta < \alpha < 2$. We consider the non-local operator $\mathcal{L}^b = \Delta^{\alpha/2} + \mathcal{S}^b$, where

$$\mathcal{S}^b f(x) := \lim_{\varepsilon \to 0} \mathcal{A}(d, -\beta) \int_{|z|>\varepsilon} (f(x+z) - f(x)) \frac{b(x,z)}{|z|^{d+\beta}} \, dy.$$

Here $b(x,z)$ is a bounded measurable function on $\mathbb{R}^d \times \mathbb{R}^d$ that is symmetric in $z$, and $\mathcal{A}(d,-\beta)$ is a normalizing constant so that when $b(x,z) \equiv 1$, $\mathcal{S}^b$ becomes the fractional Laplacian $\Delta^{\beta/2} := -(-\Delta)^{\beta/2}$. In other words,

$$\mathcal{L}^b f(x) := \lim_{\varepsilon \to 0} \mathcal{A}(d, -\beta) \int_{|z|>\varepsilon} (f(x+z) - f(x)) \, j^b(x,z) \, dz,$$

where $j^b(x,z) := \mathcal{A}(d,-\alpha)|z|^{-(d+\alpha)} + \mathcal{A}(d,-\beta) b(x,z) |z|^{-(d+\beta)}$. It is recently established in Chen and Wang [10] that, when $j^b(x,z) \geq 0$ on $\mathbb{R}^d \times \mathbb{R}^d$, there is a conservative Feller process $X^b$ having $\mathcal{L}^b$ as its infinitesimal generator. In this paper we establish, under certain conditions on $b$, a uniform boundary Harnack principle for harmonic functions of $X^b$ (or equivalently, of $\mathcal{L}^b$) in any $\kappa$-fat open set. We further establish uniform gradient estimates for non-negative harmonic functions of $X^b$ in open sets.

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## 1 Introduction

Let $d \geq 2$, $0 < \beta < \alpha < 2$, and $b(x,z)$ be a bounded measurable function on $\mathbb{R}^d \times \mathbb{R}^d$ with $b(x,z) = b(x,-z)$ for $x, z \in \mathbb{R}^d$. Consider the non-local operator $\mathcal{L}^b = \Delta^{\alpha/2} + \mathcal{S}^b$, where

$$\mathcal{S}^b f(x) := \lim_{\varepsilon \to 0} \mathcal{A}(d, -\beta) \int_{|z|>\varepsilon} (f(x+z) - f(x)) \frac{b(x,z)}{|z|^{d+\beta}} \, dz. \tag{1.1}$$

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Here $\mathcal{A}(d,-\beta)$ is a normalizing constant so that when $b(x,z)\equiv 1$, $\mathcal{S}^b$ becomes the fractional Laplacian $\Delta^{\beta/2}:=-(-\Delta)^{\beta/2}$; in other words, $\mathcal{A}(d,-\beta)=\beta 2^{\beta-1}\pi^{-d/2}\Gamma((d+\beta)/2)/\Gamma(1-\beta/2)$. Thus $\mathcal{L}^b$ can be expressed as

$$\mathcal{L}^b f(x)=\lim_{\varepsilon\to 0}\int_{|z|>\varepsilon}(f(x+z)-f(x))\,j^b(x,z)\,dz \tag{1.2}$$

where

$$j^b(x,z)=\frac{\mathcal{A}(d,-\alpha)}{|z|^{d+\alpha}}+\frac{\mathcal{A}(d,-\beta)b(x,z)}{|z|^{d+\beta}}. \tag{1.3}$$

Note that since $b(x,z)$ is symmetric in $z$, for $f\in C_b^2(\mathbb{R}^d)$,

$$\mathcal{L}^b f(x)=\int_{\mathbb{R}^d}\left(f(x+z)-f(x)-\nabla f(x)\cdot z\mathbf{1}_{\{|z|\le 1\}}\right)j^b(x,z)\,dz. \tag{1.4}$$

Recently, $\mathcal{L}^b$, the fractional Laplacian perturbed by a lower order non-local operator $\mathcal{S}^b$, and its fundamental solution have been studied in Chen and Wang [10]. It is established there that if for every $x\in\mathbb{R}^d$, $j^b(x,z)\ge 0$ (that is, $b(x,z)\ge -\frac{\mathcal{A}(d,-\alpha)}{\mathcal{A}(d,-\beta)}|z|^{\beta-\alpha}$) for a.e. $z\in\mathbb{R}^d$, then $\mathcal{L}^b$ has a unique jointly continuous fundamental solution $p^b(t,x,y)$, which uniquely determines a conservative Feller process $X^b$ on the canonical Skorokhod space $\mathbb{D}([0,+\infty),\mathbb{R}^d)$ such that

$$\mathbb{E}_x\left[f(X_t^b)\right]=\int_{\mathbb{R}^d}f(y)p^b(t,x,y)dy,\qquad x\in\mathbb{R}^d.$$

for every bounded measurable function $f$ on $\mathbb{R}^d$. The Feller process $X^b$ is typically non-symmetric and it has a Lévy system $(J^b(x,y)dy,t)$ (see [10, Proposition 5.4]), where

$$J^b(x,y):=j^b(x,y-x). \tag{1.5}$$

When $b$ takes constant value $\varepsilon>0$, $X^\varepsilon$ has the same distribution as the Lévy process $Y+\varepsilon^{1/\beta}Z$, where $Y$ and $Z$ are rotationally symmetric $\alpha$- and $\beta$-stable processes on $\mathbb{R}^d$ that are independent of each other. Moreover, two-sided heat kernel estimates have been obtained in [10] for $\mathcal{L}^b$, while two-sided Dirichlet kernel estimates in $C^{1,1}$ open sets have recently been obtained in Chen and Yang [11]. In this paper, we investigate boundary Harnack principle and gradient estimates for non-negative harmonic functions of $\mathcal{L}^b$ in open sets.

Boundary Harnack principle (BHP) asserts that non-negative harmonic functions that vanish in an exterior part of a neighborhood at the boundary decay at the same rate. It is an important property in analysis and in probability theory on harmonic functions. We refer the reader to the introduction of [3, 13] for a brief account on the history of BHP that started with Brownian motion and then extended to subordinate Brownian motions and to certain pure jump strong Markov processes. Since $\mathcal{L}^b$ is typically state-dependent and its dual operator may not be Markovian, the BHP results in [3, 13] are not applicable to harmonic functions of $\mathcal{L}^b$. In this paper, we establish uniform boundary Harnack principle on $\kappa$-fat open sets for non-negative harmonic functions of $\mathcal{L}^b$ by estimating Poisson kernels of $\mathcal{L}^b$ in small balls.

Gradient estimates for harmonic functions of elliptic operators and on manifolds have been studied extensively in literature, including the celebrated Li-Yau inequality. See [12] and the references therein and for a coupling argument. Gradient estimates for harmonic functions for non-local operators are quite recently. In [4], a gradient estimate for harmonic functions of symmetric stable processes is obtained. Gradient estimates for harmonic functions of mixed stable processes were derived in [16]. It has recently been extended to a class of isotropic unimodal Lévy process in [15]. For gradient estimate for harmonic functions of the Schrödinger operator $\Delta^{\alpha/2}+q$, see [4] for $\alpha\in(1,2)$ and [14] for $\alpha\in(0,1]$. The second main result of this paper is to establish gradient estimates for positive harmonic functions of $\mathcal{L}^b$. As far as we know, this is the first gradient estimate result for non-Lévy non-local operators.

We now describe our main results in details. In this paper, we use ":=" as a way of definition. For $a,b\in\mathbb{R}$, $a\wedge b:=\min\{a,b\}$ and $a\vee b:=\max\{a,b\}$. Let $|x-y|$ denote the Euclidean distance between $x$ and $y$, and $B(x,r)$ the open ball centered at $x$ with radius $r>0$. For any two positive functions $f$ and $g$, $f\overset{c}{\lesssim} g$ means that there is a positive constant $c$ such that $f\le cg$ on their common domain of definition, and $f\overset{c}{\asymp} g$ means that $c^{-1}g\le f\le cg$. We also write "$\lesssim$" and "$\asymp$" if $c$ is unimportant or understood. If $D\subset\mathbb{R}^d$ is an open set, for every $x,y\in D$, define

$$\delta_D(x):=\operatorname{dist}(x,\partial D)\quad\text{and}\quad r_D(x,y):=\delta_D(x)+\delta_D(y)+|x-y|. \tag{1.6}$$

It is easy to see that

$$r_D(x,y)\asymp\delta_D(x)+|x-y|\asymp\delta_D(y)+|x-y|. \tag{1.7}$$

Denote by $\tau^b_D:=\inf\{t>0: X^b_t\notin D\}$, the exit time from $D$ by $X^b$. When there is no danger of confusion, we will drop the superscript $b$ and simply write $\tau_D$ for $\tau^b_D$.

**Definition 1.1.** A function $f$ defined on $\mathbb{R}^d$ is said to be harmonic in an open set $D$ with respect to $X^b$ if it has the mean-value property: for every bounded open set $U\subset D$ with $\overline{U}\subset D$,

$$f(x)=\mathbb{E}_x\left[f(X^b_{\tau_U})\right]\quad\text{for } x\in U. \tag{1.8}$$

It is said to be regular harmonic in $D$ if (1.8) holds for $U=D$.

Denote by $\partial$ a cemetery point that is added to $D$ as an isolated point. We use the convention that $X^b_\infty:=\partial$ and any function $f$ is extended to the cemetery point $\partial$ by setting $f(\partial)=0$. So $\mathbb{E}_x\left[f(X^b_{\tau_D})\right]$ should be understood as $\mathbb{E}_x\left[f(X^b_{\tau_D}):\tau_D<+\infty\right]$. In Definition 1.1, we always assume implicitly that the expectation in (1.8) is absolutely convergent.

**Assumption 1** Suppose $M_1,M_2\ge 1$. $b(x,z)$ is a bounded function on $\mathbb{R}^d\times\mathbb{R}^d$ satisfying

$$\|b\|_\infty\le M_1\quad\text{and}\quad b(x,z)=b(x,-z)\quad\text{for } x,z\in\mathbb{R}^d, \tag{1.9}$$

and there exists a positive constant $\varepsilon_0\in[0,1]$ such that for every $x,y\in\mathbb{R}^d$,

$$M_2^{-1}J^{\varepsilon_0}(x,y)\le J^b(x,y)\le M_2J^{\varepsilon_0}(x,y). \tag{1.10}$$

Here $J^{\varepsilon_0}(x,y)=\mathcal{A}(d,-\alpha)|x-y|^{-d-\alpha}+\varepsilon_0\mathcal{A}(d,-\beta)|x-y|^{-d-\beta}$. Since $J^{\varepsilon_0}(x,y)$ depends only on $|x-y|$, we also write $J^{\varepsilon_0}(|x-y|)$ for $J^{\varepsilon_0}(x,y)$.

**Definition 1.2.** Let $\kappa \in (0,1)$. An open set $D \subset \mathbb{R}^d$ is said to be $\kappa$-fat if for every $z \in \partial D$ and $r \in (0,1]$, there is some point $x \in D$ so that $B(x, \kappa r) \subset D \cap B(z,r)$.

The following is the first main result of this paper.

**Theorem 1.3** (Uniform boundary Harnack inequality)**.** *Suppose Assumption 1 holds and $D$ is a $\kappa$-fat open set in $\mathbb{R}^d$ with $\kappa \in (0,1)$. There exist constants $r_1 = r_1(d,\alpha,\beta,M_1) \in (0,1]$ and $C_1 = C_1(d,\alpha,\beta,\kappa,M_1,M_2) \geq 1$ such that for every $z_0 \in \partial D$ and $r \in (0, r_1/2]$, and all non-negative functions $u, v$ that are regular harmonic in $D \cap B(z_0, 2r)$ with respect to $X^b$ and vanish in $D^c \cap B(z_0, 2r)$, we have*

$$\frac{u(x)}{v(x)} \leq C_1 \frac{u(y)}{v(y)} \qquad \textit{for } x, y \in D \cap B(z_0, r).$$

We call the above property *uniform* boundary Harnack principle because the constants $r_1$ and $C_1$ in the above theorem are independent of $\varepsilon_0 \in [0,1]$ appeared in condition (1.10). We next study the gradient estimates for non-negative harmonic functions in open sets. We write $\partial_{x_i}$ or $\partial_i$ for $\frac{\partial}{\partial x_i}$ and $\nabla$ for $(\partial_{x_1}, \cdots, \partial_{x_d})$.

**Theorem 1.4.** *Let $D$ be an arbitrary open set in $\mathbb{R}^d$. Under Assumption 1, there is a constant $C_2 = C_2(d,\alpha,\beta,M_1,M_2) > 0$ such that for any non-negative function $f$ in $\mathbb{R}^d$ which is harmonic in $D$ with respect to $X^b$, $\nabla f(x)$ exists for every $x \in D$, and we have*

$$|\nabla f(x)| \leq C_2 \frac{f(x)}{1 \wedge \delta_D(x)} \quad \textit{for } x \in D. \tag{1.11}$$

For $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$ and $1 \leq i \leq d$, we write $\tilde{x}^i$ for $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_d) \in \mathbb{R}^{d-1}$.

**Assumption 2.** Suppose there is $i \in \{1, \cdots, d\}$ so that for every $x \in \mathbb{R}^d$,

$$b(x,z) = \varphi(\tilde{x}^i)\psi(|z|) \quad \text{a.e. } z \in \mathbb{R}^d, \tag{1.12}$$

where $\varphi : \mathbb{R}^{d-1} \to \mathbb{R}$ is a non-negative measurable function, and $\psi : \mathbb{R}_+ \to \mathbb{R}$ is a measurable function such that

$$\frac{\psi(r)}{r^{d+\beta}} \text{ is non-increasing in } r > 0. \tag{1.13}$$

**Theorem 1.5.** *Suppose Assumption 1 and Assumption 2 hold. Let $D = \left\{x \in \mathbb{R}^d : \ x_i > \Gamma(\tilde{x}^i)\right\}$ be a special Lipschitz domain, where $\Gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ is a Lipschitz function with Lipschitz constant $\lambda_0$ (that is, $|\Gamma(\tilde{x}^i) - \Gamma(\tilde{y}^i)| \leq \lambda_0 |\tilde{x}^i - \tilde{y}^i|$ for every $\tilde{x}^i, \tilde{y}^i \in \mathbb{R}^{d-1}$). Then there are positive constants $R_1 = R_1(d,\alpha,\beta,\lambda_0,M_1,M_2)$ and $C_3 = C_3(d,\alpha,\beta,\lambda_0,M_1,M_2) \geq 1$ such that for every $r \in (0, R_1]$, there is a constant $\eta_1 = \eta_1(d,\alpha,\beta,\lambda_0,M_1,M_2,r) \in (0, r/2)$ so that for every $z_0 \in \partial D$ and every non-negative function $f$ that is harmonic in $D \cap B(z_0, r)$ with respect to $X^b$ and vanishes in $D^c \cap B(z_0, r)$,*

$$C_3^{-1} \frac{f(x)}{\delta_D(x)} \leq |\nabla f(x)| \leq C_3 \frac{f(x)}{\delta_D(x)} \quad \textit{for } x \in D \cap B(z_0, \eta_1). \tag{1.14}$$

Obviously Assumption 2 is implied by

**Assumption 3.** There exists a measurable function $\psi : \mathbb{R}_+ \to \mathbb{R}$ satisfying (1.13) such that for every $x \in \mathbb{R}^d$,

$$b(x,z) = \psi(|z|) \quad \text{a.e. } z \in \mathbb{R}^d. \tag{1.15}$$

**Definition 1.6.** An open set $D \subset \mathbb{R}^d$ is said to be Lipschitz if for every $z_0 \in \partial D$, there is a Lipschitz function $\Gamma_{z_0} : \mathbb{R}^{d-1} \to \mathbb{R}$, an orthonormal coordinate system $CS_{z_0}$ and a constant $R_{z_0} > 0$ such that if $y = (y_1, \cdots, y_{d-1}, y_d)$ in $CS_{z_0}$ coordinates, then

$$D \cap B(z_0, R_{z_0}) = \{y : y_d > \Gamma_{z_0}(y_1, \cdots, y_{d-1})\} \cap B(z_0, R_{z_0}).$$

If there exist positive constants $R_0$ and $\lambda_0$ so that $R_{z_0}$ can be taken to be $R_0$ for all $z_0 \in \partial D$ and the Lipschitz constants of $\Gamma_{z_0}$ are not greater than $\lambda_0$, we call $D$ a Lipschitz open set with characteristics $(\lambda_0, R_0)$.

Clearly, if $D$ is a Lipschitz open set with characteristics $(\lambda_0, R_0)$, then it is $\kappa$-fat for some $\kappa = \kappa(\lambda_0, R_0) \in (0,1)$. The following theorem follows directly from Theorem 1.5.

**Theorem 1.7.** *Let $D$ be a Lipschitz open set in $\mathbb{R}^d$ with characteristics $(\lambda_0, R_0)$. Under Assumptions 1 and 3, there are positive constants $R_2 = R_2(d, \alpha, \beta, \lambda_0, R_0, M_1, M_2)$ and $C_4 = C_4(d, \alpha, \beta, \lambda_0, R_0, M_1, M_2) \ge 1$ such that for every $r \in (0, R_2]$, there is a constant $\eta_2 = \eta_2(d, \alpha, \beta, \lambda_0, R_0, M_1, M_2, r) \in (0, r/2)$ so that for every $z_0 \in \partial D$ and every non-negative function $f$ that is harmonic in $D \cap B(z_0, r)$ with respect to $X^b$ and vanishes in $D^c \cap B(z_0, r)$,*

$$C_4^{-1} \frac{f(x)}{\delta_D(x)} \le |\nabla f(x)| \le C_4 \frac{f(x)}{\delta_D(x)} \qquad \text{for } x \in D \cap B(z_0, \eta_2). \tag{1.16}$$

Results in Theorem 1.4, Theorem 1.5 and Theorem 1.7 can be called *uniform* gradient estimates because the constants $C_k$, $2 \le k \le 4$, and $\eta_i$, $1 \le i \le 2$, are independent of $\varepsilon_0$ of (1.10).

The rest of the paper is organized as follows. Preliminary results on Green functions and Poisson kernels are presented in Section 2. The proof of the uniform boundary Harnack principle is given in Section 3. Section 4 is devoted to the proof of Theorem 1.4, while the proof of Theorem 1.5 is given in Section 5. In this paper, we use capital letters $C_1, C_2, \cdots$ to denote constants in the statements of results. The lower case constants $c_1, c_2, \cdots$, will denote the generic constants used in proofs, whose exact values are not important, and can change from one appearance to another. We use $e_k$ to denote the unit vector along the positive direct of $x_k$-axis.

# 2 Preliminaries

Recall the Lévy system $(J^b(x,y)dy, t)$ from (1.5), which describes the jumps of $X^b$: for any non-negative measurable function $f$ on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ with $f(s,y,y) = 0$ for all $y \in \mathbb{R}^d$, $x \in \mathbb{R}^d$ and stopping time $T$ (with respect to the filtration of $X^b$),

$$\mathbb{E}_x \left[ \sum_{s \le T} f(s, X^b_{s-}, X^b_s) \right] = \mathbb{E}_x \left[ \int_0^T \int_{\mathbb{R}^d} f(s, X^b_s, y) J^b(X^b_s, y) dy\, ds \right]. \tag{2.1}$$

Suppose $D$ is a Greenian open set of $X^b$. Let $G^b_D(x,y)$ denote the Green function of $D$, that is,

$$\int_D f(y)G^b_D(x,y)dy = \mathbb{E}_x\left[\int_0^{\tau_D} f(X^b_s)ds\right]$$

for every bounded measurable function $f$ on $D$ and $x \in D$. It follows from (2.1) that for every bounded open set $D$ in $\mathbb{R}^d$, every $f \geq 0$, and $x \in D$,

$$\mathbb{E}_x\left[f(X^b_{\tau_D}) : X^b_{\tau_D-} \neq X^b_{\tau_D}\right] = \int_{\bar{D}^c} f(z)\left(\int_D G^b_D(x,y)J^b(y,z)dy\right)dz. \tag{2.2}$$

Define

$$K^b_D(x,z) := \int_D G^b_D(x,y)J^b(y,z)dy \quad \text{for } (x,z) \in D \times \bar{D}^c. \tag{2.3}$$

We call $K^b_D(x,z)$ the Poisson kernel of $X^b$ on $D$. Then (2.2) can be written as

$$\mathbb{E}_x\left[f(X^b_{\tau_D}) : X^b_{\tau_D-} \neq X^b_{\tau_D}\right] = \int_{\bar{D}^c} f(z)K^b_D(x,z)dz. \tag{2.4}$$

For any $\lambda > 0$, define

$$b_\lambda(x,z) := \lambda^{\beta-\alpha} b(\lambda^{-1}x, \lambda^{-1}z) \quad \text{for } x,z \in \mathbb{R}^d. \tag{2.5}$$

It is not hard to show that

$$J^{b_\lambda}(x,y) = \lambda^{-(d+\alpha)} J^b(\lambda^{-1}x, \lambda^{-1}y) \quad \text{for } x,y \in \mathbb{R}^d. \tag{2.6}$$

and

$$\{\lambda X^b_{\lambda^{-\alpha}t}; t \geq 0\} \text{ has the same distribution as } \{X^{b_\lambda}_t; t \geq 0\}. \tag{2.7}$$

So for any $\lambda > 0$, we have the following scaling properties:

$$G^b_D(x,y) = \lambda^{d-\alpha} G^{b_\lambda}_{\lambda D}(\lambda x, \lambda y) \quad \text{for } x,y \in D, \tag{2.8}$$

$$K^b_D(x,z) = \lambda^d K^{b_\lambda}_{\lambda D}(\lambda x, \lambda z) \quad \text{for } x \in D,\ z \in \bar{D}^c. \tag{2.9}$$

If $u$ is harmonic in $D$ with respect to $X^b$, then for any $\lambda > 0$, $v(x) := u(x/\lambda)$ is harmonic in $\lambda D$ with respect to $X^{b_\lambda}$.

When $b(x,z) \equiv 0$, $X^0$ is simply an isotropic symmetric $\alpha$-stable process on $\mathbb{R}^d$, which we will denote as $X$. We will also write $J$ for $J^0$. It is known that if $d > \alpha$, the process $X$ is transient and its Green function is given by

$$G(x,y) = \frac{\Gamma(d/2)}{2^\alpha \pi^{d/2} \Gamma(\alpha/2)^2} |x-y|^{\alpha-d} \quad \text{for } x,y \in \mathbb{R}^d. \tag{2.10}$$

It is shown in Blumenthal *et al.* [1] that the Green function of $X$ in a ball $B(0,r)$ is given by

$$G_{B(0,r)}(x,y) = \frac{\Gamma(d/2)}{2^\alpha \pi^{d/2} \Gamma(\alpha/2)^2} \int_0^z (u+1)^{-d/2} u^{\alpha/2-1} du\, |x-y|^{\alpha-d} \quad \text{for } x,y \in B(0,r), \tag{2.11}$$

where $z = (r^2 - |x|^2)(r^2 - |y|^2)|x-y|^{-2}$ and $r > 0$. The above formula yields the following two-sided estimates (see, for example, [5]): Suppose $B$ is an arbitrary ball in $\mathbb{R}^d$ with radius $r > 0$. Then there is a universal constant $c_1 = c_1(d, \alpha) > 1$ so that for every $x, y \in B$,

$$G_B(x,y) \stackrel{c_1}{\asymp} |x-y|^{\alpha-d} \left(1 \wedge \frac{\delta_B(x)}{|x-y|}\right)^{\alpha/2} \left(1 \wedge \frac{\delta_B(y)}{|x-y|}\right)^{\alpha/2}. \tag{2.12}$$

Since for $a, b > 0$, $a \wedge b \asymp \frac{ab}{a+b}$ and $1 \wedge \frac{a}{b} \asymp \frac{a}{a+b}$, in view of (1.7) we can rewrite (2.12) as

$$G_B(x,y) \asymp |x-y|^{\alpha-d} \frac{\delta_B(x)^{\alpha/2} \delta_B(y)^{\alpha/2}}{r_B(x,y)^\alpha}. \tag{2.13}$$

It follows immediately from (2.12) that there is a positive constant $c_2 = c_2(d, \alpha) > 1$ so that for $B = B(x_0, r)$,

$$c_2^{-1} r^\alpha \le \mathbb{E}_x \tau_B \le c_2 r^\alpha \qquad \text{for } x \in B(x_0, r/2).$$

Riesz (see [1]) derived the following explicit formula for the Poisson kernel $K_{B(0,r)}(x,z)$ of $X$ on $B(0,r)$.

$$K_{B(0,r)}(x,z) = \frac{\Gamma(d/2)\sin(\pi\alpha/2)}{\pi^{d/2+1}} \, \frac{(r^2 - |x|^2)^{\alpha/2}}{(|z|^2 - r^2)^{\alpha/2} \, |x-z|^d} \quad \text{for } |x| < r \text{ and } |z| > r, \tag{2.14}$$

We point out that $\mathbb{P}_x(X_{\tau_D} \neq X_{\tau_D -}) = 1$ for every $x \in D$ if $D$ is a domain that satisfies uniform exterior cone condition.

# 3 Boundary Harnack principle

Recall that we write $X$ and $J$ for $X^0$ and $J^0$. First we record the following gradient estimate on the Green function $G_D$ of symmetric $\alpha$-stable process $X$ from [4].

**Lemma 3.1.** ([4, Corollary 3.3]) *Let $D$ be a Greenian domain in $\mathbb{R}^d$ of $X$. Then*

$$|\nabla G_D(x,y)| \le d \frac{G_D(x,y)}{|x-y| \wedge \delta_D(x)} \quad \text{for } x, y \in D, \ x \neq y. \tag{3.1}$$

For $x \neq y$ in $D$, define

$$h_D(x,y) := \begin{cases} |x-y|^{\alpha-\beta-d} \left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right)^{\alpha/2} & \text{if } \alpha > 2\beta, \\ |x-y|^{\beta-d} \left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right)^{\beta} \left(1 \vee \log \frac{|x-y|}{\delta_D(x)}\right) & \text{if } \alpha = 2\beta, \\ |x-y|^{\alpha-\beta-d} \left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right)^{\alpha/2} \left(1 \vee \frac{|x-y|}{\delta_D(x)}\right)^{\beta-\alpha/2} & \text{if } \alpha < 2\beta, \end{cases} \tag{3.2}$$

The following two results are established in [11].

**Lemma 3.2.** ([11, Theorems 4.11 and 4.13]) *Suppose $b$ is a bounded function satisfying* (1.9), *and that for every $x \in \mathbb{R}^d$, $J^b(x,y) \ge 0$ a.e. $y \in \mathbb{R}^d$. There exist positive constants $r_1 = r_1(d, \alpha, \beta, M_1) \in$*

$(0,1]$ *and* $C_5 = C_5(d,\alpha,\beta,M_1)$ *such that for any* $x_0 \in \mathbb{R}^d$ *and any ball* $B = B(x_0,r)$ *with radius* $r \in (0,r_1]$, *we have for* $x,y \in B$,

$$\frac{1}{2}G_B(x,y) \le G_B^b(x,y) \le \frac{3}{2}G_B(x,y) \quad \text{and} \quad |\mathcal{S}_x^b G_B^b(x,y)| \le C_5 h_B(x,y) \tag{3.3}$$

*Moreover,* $\mathbb{P}_x\left(X^b_{\tau_B} \in \partial B\right) = 0$ *for every* $x \in B$. *In this case, for every non-negative measurable function* $f$,

$$\mathbb{E}_x f(X^b_{\tau_B}) = \int_{\bar{B}^c} f(z) K_B^b(x,z)dz \quad \text{for } x \in B.$$

**Lemma 3.3.** ([11, Lemma 3.1]) *Let* $D$ *be a bounded open set in* $\mathbb{R}^d$. *There exists a constant* $C_6 = C_6(d,\alpha,\beta,\mathrm{diam}(D),M_1) > 0$ *such that for any bounded function* $b$ *satisfying* (1.9), *and that for every* $x \in \mathbb{R}^d$, $J^b(x,y) \ge 0$ *for a.e.* $y \in \mathbb{R}^d$, *we have*

$$G_D^b(x,y) \le C_6|x-y|^{\alpha-d} \quad \text{for } x,y \in D.$$

Note that the constant $C_7$ below is independent of $\varepsilon_0 \in [0,1]$ appeared in (1.10).

**Theorem 3.4** (Uniform Harnack inequality)**.** *Let* $r_1 \in (0,1]$ *be the constant in Lemma 3.2. Under Assumption 1, there exists a constant* $C_7 = C_7(d,\alpha,\beta,M_1,M_2) \ge 1$ *such that for every* $x_0 \in \mathbb{R}^d$, $r \in (0,r_1]$, *and every non-negative function* $u$ *which is regular harmonic in* $B(x_0,r)$, *we have*

$$\sup_{y \in B(x_0,r/2)} u(y) \le C_7 \inf_{y \in B(x_0,r/2)} u(y).$$

*Proof.* Let $u^*(x) := \mathbb{E}_x\left[u(X^{\varepsilon_0}_{\tau_{B(x_0,r)}})\right]$. Then $u^*$ is regular harmonic in $B(x_0,r)$ with respect to the mixed stable processes $X^{\varepsilon_0}$. In view of Lemma 3.2 and Assumption 1, for every $x_0 \in \mathbb{R}^d$ and $r \in (0,r_1]$, the Poisson kernel $K^b_{B(x_0,r)}(x,z)$ on $B(x_0,r)$ of $X^b$ is comparable to that of $X^{\varepsilon_0}$. Thus for every $x \in B(x_0,r/2)$, $u(x)$ is comparable to $u^*(x)$. Theorem 3.4 then follows from the uniform Harnack inequality for mixed stable processes; see [6, (3.40)]. □

**Lemma 3.5** (Harnack inequality)**.** *Under Assumption 1, there exists a constant* $C_8 = C_8(d,\alpha,\beta,M_1,M_2) > 0$ *such that the following statement is true: If* $x_1,x_2 \in \mathbb{R}^d$, $r \in (0,r_1]$ *and* $k \in \mathbb{N}$ *are such that* $|x_1 - x_2| < 2^k r$, *then for every non-negative function* $u$ *which is harmonic with respect to* $X^b$ *in* $B(x_1,r) \cup B(x_2,r)$, *we have*

$$C_8^{-1} 2^{-k(d+\alpha)} u(x_2) \le u(x_1) \le C_8 2^{k(d+\alpha)} u(x_2). \tag{3.4}$$

*Proof.* Without loss of generality, we may assume $|x_1 - x_2| \ge r/4$. Note that for every $x \in B(x_2,r/8) \subset B(x_1,r/8)^c$, we have $|x - x_1| < 2^{k+1}r$. Thus by Lemma 3.2 and Assumption 1, we have

$$\begin{aligned} K^b_{B(x_1,r/8)}(x_1,x) &\ge \frac{1}{2M_2}\int_{B(x_1,r/8)} G_{B(x_1,r/8)}(x_1,y)J(y,x)dy \\ &= \frac{1}{2M_2} K_{B(x_1,r/8)}(x_1,x) \end{aligned}$$

$$\geq \frac{c_1}{2M_2} 2^{-\alpha} r^{\alpha} (2^{-k-1} r)^{-d-\alpha} = c_2 r^{-d} 2^{-k(d+\alpha)}. \tag{3.5}$$

Recall that by Theorem 3.4, we have $u(x) \geq c_3 u(x_2)$ for every $x \in B(x_2, r/8)$. Thus by (3.5),

$$\begin{aligned} u(x_1) &\geq \int_{B(x_2,r/8)} u(x) K^b_{B(x_1,r/8)}(x_1, x) dx \\ &\geq c_2 c_3 u(x_2) r^{-d} 2^{-k(d+\alpha)} \int_{B(x_2,r/8)} dx \\ &\geq c_4 2^{-k(d+\alpha)} u(x_2), \end{aligned}$$

and (3.4) follows by symmetry. □

**Proof of Theorem 1.3.** Note that there are constants $R_0 = R_0(d, \alpha, \beta, M_1) \in (0, r_1)$ and $c = c(d, \alpha, \beta, M_1) > 1$ so that

$$\frac{c^{-1}}{|x-y|^{d+\alpha}} \leq J^b(x, y) \leq \frac{c}{|x-y|^{d+\alpha}} \quad \text{for all } |x-y| \leq R_0 \tag{3.6}$$

for all $b(x, z)$ satisfying (1.9). Thus using (3.3), we can get uniform estimates on the Poisson kernel

$$K^b_{B(x_0,r)}(x, z) = \int_{B(x_0,r)} G^b_{B(x_0,r)}(x, y) J^b(y, z) dy$$

of any ball $B(x_0, r)$ with respect to $X^b$ with $r \in (0, R_0/3)$, $x \in B(x_0, r)$ and $r < |z - x_0| < 2r$. Specifically, for $r < |z - x_0| < 2r$, $K^b_{B(x_0,r)}(x, z)$ is uniformly comparable to $K_{B(x_0,r)}(x, z)$. Using the explicit formula (2.14) for the Poisson kernel $K_{B(x_0,r)}$, (3.3), Theorem 3.4 and (1.10), we can adapt the arguments in [7, Theorem 2.6] to get our uniform boundary Harnack principle 1.3 (cf. the proof of [6, Theorem 3.9]). Since the proof is almost identical to those in [7, Section 3], we omit the details here. □

**Lemma 3.6.** *Suppose Assumption 1 holds and $D$ is a Lipschitz open set with characteristics $(\lambda_0, R_0)$. Let $r_1 \in (0, 1]$ be the constant in Lemma 3.2. There is a positive constant $C_9 = C_9(d, \alpha, \beta, \lambda_0, R_0, M_1, M_2) \geq 1$ such that for every $z_0 \in \partial D$, $r \in (0, r_1/2)$, and every non-negative harmonic function $u$ that is regular harmonic in $D \cap B(z_0, 2r)$ with respect to $X^b$ and vanishes in $D^c \cap B(z_0, 2r)$,*

$$\mathbb{E}_x \left[ u(X^b_{\tau_{D\cap B_k}}) : X^b_{\tau_{D\cap B_k}} \in B_0^c \right] \leq C_9 2^{-k\alpha} u(x), \quad x \in D \cap B_k, \tag{3.7}$$

*for $B_k := B(z_0, 2^{-k} r)$ and $k \geq 1$.*

*Proof.* Without loss of generality, we may assume $z_0 = 0$. By the uniform inner cone property of a Lipschitz open set, one can find a point $\tilde{z}_0 \in D \cap B(0, r)$ and $\kappa = \kappa(\lambda_0, R_0) \in (0, 1)$ such that $\widetilde{B}_k := B(\tilde{z}_k, \kappa 2^{-k} r) \subset B_k \cap D$ for every $\tilde{z}_k := 2^{-k} \tilde{z}_0$ and $k \geq 0$. Define

$$u_k(x) := \mathbb{E}_x \left[ u(X^b_{\tau_{D\cap B_k}}) : X^b_{\tau_{D\cap B_k}} \in B_0^c \right].$$

Since $u_0 = u$, (3.7) is clearly true for $k = 0$. Henceforth we suppose $k \geq 1$. Note that $u_k \geq 0$ is regular harmonic with respect to $X^b$ in $D \cap B_k$, and $u_k(x) \leq u_{k-1}(x)$ for all $x \in \mathbb{R}^d$. Define

$$I_k(x) := \mathbb{E}_x \left[ u(X^b_{\tau_{B_k}}) : X^b_{\tau_{B_k}} \in B_0^c \right].$$

Clearly by definition $u_k(\tilde{z}_k) \leq I_k(\tilde{z}_k)$. For any $k \geq 1$, by Lemma 3.2 and (2.8), we have

$$\begin{aligned}
K^b_{B_k}(\tilde{z}_k, y) &= \int_{B_k} G^b_{B_k}(\tilde{z}_k, z) J^b(z, y) dz \\
&\leq \frac{3}{2} \int_{B_k} G_{B_k}(\tilde{z}_k, z) J^b(z, y) dz \\
&= \frac{3}{2} \int_{2^{-(k-1)} B_1} G_{2^{-(k-1)} B_1}(2^{-(k-1)} \tilde{z}_1, z) J^b(z, y) dz \\
&= \frac{3}{2} 2^{-(k-1)d} \int_{B_1} G_{2^{-(k-1)} B_1}(2^{-(k-1)} \tilde{z}_1, 2^{-(k-1)} w) J^b(2^{-(k-1)} w, y) dw \\
&= \frac{3}{2} 2^{-(k-1)\alpha} \int_{B_1} G_{B_1}(\tilde{z}_1, w) J^b(2^{-(k-1)} w, y) dw. \qquad (3.8)
\end{aligned}$$

Note that for any $y \in B_0^c$ and $w \in B_1$,

$$\frac{|y - w|}{|y - 2^{-(k-1)} w|} \leq \frac{|y| + |w|}{|y| - 2^{-k+1}|w|} \leq 3.$$

Thus by (1.10) we have

$$J^b(2^{-(k-1)} w, y) \leq M_2 J^{\varepsilon_0}(|y - 2^{-(k-1)} w|) \leq 3^{d+\alpha} M_2 J^{\varepsilon_0}(|y - w|) \leq 3^{d+\alpha} M_2^2 J^b(w, y). \qquad (3.9)$$

It follows from (3.8), (3.9) and Lemma 3.2 that for any $y \in B_0^c$,

$$\begin{aligned}
K^b_{B_k}(\tilde{z}_k, y) &\leq \frac{3^{d+\alpha+1}}{2} M_2^2 2^{-(k-1)\alpha} \int_{B_1} G_{B_1}(\tilde{z}_1, w) J^b(w, y) dw \\
&\leq 3^{d+\alpha+1} M_2^2 2^{-(k-1)\alpha} \int_{B_1} G^b_{B_1}(\tilde{z}_1, w) J^b(w, y) dw \\
&= c_1 2^{-k\alpha} K^b_{B_1}(\tilde{z}_1, y). \qquad (3.10)
\end{aligned}$$

Now we have for $k \geq 1$

$$\begin{aligned}
I_k(\tilde{z}_k) &= \int_{B_0^c} u(y) K^b_{B_k}(\tilde{z}_k, y) dy \\
&\leq c_1 2^{-k\alpha} \int_{B_0^c} u(y) K^b_{B_1}(\tilde{z}_1, y) dy \\
&= c_1 2^{-k\alpha} I_1(\tilde{z}_1). \qquad (3.11)
\end{aligned}$$

Next we compare $I_1(\tilde{z}_1)$ with $u(\tilde{z}_1)$. Using Lemma 3.2, (1.10) and (2.8), we have

$$K^b_{\widetilde{B}_1}(\tilde{z}_1, y) = \int_{|z - \tilde{z}_1| < \kappa r/2} G^b_{\widetilde{B}_1}(\tilde{z}_1, z) J^b(z, y) dz$$

$$
\begin{aligned}
&\geq \frac{1}{2M_2}\int_{|z-\tilde z_1|<\kappa r/2} G_{\widetilde{B}_1}(\tilde z_1,z)J^{\varepsilon_0}(|y-z|)dz\\
&= \frac{1}{2M_2}\int_{|z-\tilde z_1|<\kappa r/2} G_{\kappa B_1}(0,z-\tilde z_1)J^{\varepsilon_0}(|y-z|)dz\\
&= \frac{1}{2M_2}\int_{|w|<\kappa r/2} G_{\kappa B_1}(0,w)J^{\varepsilon_0}(|y-\tilde z_1-w|)dw\\
&= \frac{1}{2M_2}\kappa^d\int_{|z|<r/2} G_{\kappa B_1}(0,\kappa z)J^{\varepsilon_0}(|y-\tilde z_1-\kappa z|)dz\\
&= \frac{1}{2M_2}\kappa^\alpha\int_{B_1} G_{B_1}(0,z)J^{\varepsilon_0}(|y-\tilde z_1-\kappa z|)dz. \qquad (3.12)
\end{aligned}
$$

Again using Lemma 3.2 and (1.10), we have

$$
\begin{aligned}
K^b_{B_1}(\tilde z_1,y) &= \int_{B_1} G^b_{B_1}(\tilde z_1,z)J^b(z,y)dz\\
&\leq \frac{3}{2}M_2\int_{B_1} G_{B_1}(\tilde z_1,z)J^{\varepsilon_0}(|y-z|)dz\\
&= \frac{3}{2}M_2\left(\int_{|z|\leq|\tilde z_1|/2}+\int_{|\tilde z_1|/2<|z|<r/2} G_{B_1}(\tilde z_1,z)J^{\varepsilon_0}(|y-z|)dz\right)\\
&= \frac{3}{2}M_2\left(\int_{|z|\leq|\tilde z_1|/2} G_{B_1}(\tilde z_1,z)J^{\varepsilon_0}(|y-z|)dz\right.\\
&\qquad \left.+2^d\int_{|\tilde z_1|/4<|w+\tilde z_1/2|<r/4} G_{B_1}(\tilde z_1,2w+\tilde z_1)J^{\varepsilon_0}(|y-\tilde z_1-2w|)dw\right). \qquad (3.13)
\end{aligned}
$$

Note that for any $y\in B_0^c$ and $|z|\leq|\tilde z_1|/2$, $|z-\tilde z_1|\geq|\tilde z_1|-|z|\geq|z|$ and $|y-\tilde z_1-\kappa z|/|y-z|\leq(|y|+|\tilde z_1|+\kappa|z|)/(|y|-|z|)\leq 4$. Thus

$$
\begin{aligned}
G_{B_1}(\tilde z_1,z) &\asymp |z-\tilde z_1|^{\alpha-d}\left(1\wedge\frac{\delta_{B_1}(\tilde z_1)^{\alpha/2}\delta_{B_1}(z)^{\alpha/2}}{|z-\tilde z_1|^\alpha}\right)\\
&\leq |z|^{\alpha-d}\left(1\wedge\frac{\delta_{B_1}(0)^{\alpha/2}\delta_{B_1}(z)^{\alpha/2}}{|z|^\alpha}\right)\asymp G_{B_1}(0,z),
\end{aligned}
$$

and

$$
J^{\varepsilon_0}(|y-z|)\leq J^{\varepsilon_0}(\frac{1}{4}|y-\tilde z_1-\kappa z|)\leq 4^{d+\alpha}J^{\varepsilon_0}(|y-\tilde z_1-\kappa z|).
$$

It follows then that for any $y\in B_0^c$,

$$
\int_{|z|\leq|\tilde z_1|/2} G_{B_1}(\tilde z_1,z)J^{\varepsilon_0}(|y-z|)dz\leq c_2\int_{|z|\leq|\tilde z_1|/2} G_{B_1}(0,z)J^{\varepsilon_0}(|y-\tilde z_1-\kappa z|)dz. \qquad (3.14)
$$

Note that for $y\in B_0^c$ and $|\tilde z_1|/4<|w+\tilde z_1/2|<r/4$, $\delta_{B_1}(2w+\tilde z_1)=r/2-|2w+\tilde z_1|\leq 2(r/2-|w|)=2\delta_{B_1}(w)$, and $|y-\tilde z_1-\kappa w|/|y-\tilde z_1-2w|\leq(|y|+\kappa|w+\tilde z_1/2|+(1-\kappa/2)|\tilde z_1|)/(|y|-|\tilde z_1+2w|)\leq 2$. Thus

$$
G_{B_1}(\tilde z_1,2w+\tilde z_1) \asymp |2w|^{\alpha-d}\left(1\wedge\frac{\delta_{B_1}(\tilde z_1)^{\alpha/2}\delta_{B_1}(2w+\tilde z_1)^{\alpha/2}}{|2w|^\alpha}\right)
$$

$$\lesssim \quad |w|^{\alpha-d}\left(1\wedge\frac{\delta_{B_1}(0)^{\alpha/2}\delta_{B_1}(w)^{\alpha/2}}{|w|^\alpha}\right)\asymp G_{B_1}(0,w),$$

and

$$J^{\varepsilon_0}(|y-\tilde{z}_1-2w|)\le J^{\varepsilon_0}(|y-\tilde{z}_1-\kappa w|/2)\le 2^{d+\alpha}J^{\varepsilon_0}(|y-\tilde{z}_1-\kappa w|).$$

Thus for any $y\in B_0^c$,

$$\begin{aligned}
&\int_{|\tilde{z}_1|/4<|w+\tilde{z}_1/2|<r/4} G_{B_1}(\tilde{z}_1,2w+\tilde{z}_1)J^{\varepsilon_0}(|y-\tilde{z}_1-2w|)dw\\
\le\ & c_3\int_{|\tilde{z}_1|/4<|w+\tilde{z}_1/2|<r/4} G_{B_1}(0,w)J^{\varepsilon_0}(|y-\tilde{z}_1-\kappa w|)dw\\
\le\ & c_3\int_{B_1} G_{B_1}(0,w)J^{\varepsilon_0}(|y-\tilde{z}_1-\kappa w|)dw. \qquad (3.15)
\end{aligned}$$

Using (3.14) and (3.15), we can continue the estimates in (3.13) to get that for any $y\in B_0^c$

$$K^b_{B_1}(\tilde{z}_1,y)\le c_4\int_{B_1} G_{B_1}(0,z)J^{\varepsilon_0}(|y-\tilde{z}_1-\kappa z|)dz \qquad (3.16)$$

Combining (3.12) and (3.16), we get

$$K^b_{B_1}(\tilde{z}_1,y)\le c_5\kappa^{-\alpha}K^b_{\widetilde{B}_1}(\tilde{z}_1,y),\quad \text{for } y\in B_0^c.$$

It follows that

$$\begin{aligned}
I_1(\tilde{z}_1) &= \int_{B_0^c} u(y)K^b_{B_1}(\tilde{z}_1,y)dy\le c_5\kappa^{-\alpha}\int_{B_0^c} u(y)K^b_{\widetilde{B}_1}(\tilde{z}_1,y)dy\\
&\le c_5\kappa^{-\alpha}\int_{\widetilde{B}_1^c} u(y)K^b_{\widetilde{B}_1}(\tilde{z}_1,y)dy = c_5\kappa^{-\alpha}u(\tilde{z}_1). \qquad (3.17)
\end{aligned}$$

Consequently by (3.11) and (3.17) we have for all $k\ge 1$,

$$u_k(\tilde{z}_k)\le I_k(\tilde{z}_k)\le c_1c_5\kappa^{-\alpha}2^{-k\alpha}u(\tilde{z}_1). \qquad (3.18)$$

By the monotonicity of $u_k$ in $k$, Theorem 1.3, (3.18) and Lemma 3.5, we conclude that for any $x\in D\cap B_k$ and $k\ge 1$

$$\frac{u_k(x)}{u(x)}\le\frac{u_{k-1}(x)}{u(x)}\le c_6\frac{u_{k-1}(\tilde{z}_{k-1})}{u(\tilde{z}_{k-1})}\le c_6c_1c_5\kappa^{-\alpha}2^{-(k-1)\alpha}\frac{u(\tilde{z}_1)}{u(\tilde{z}_{k-1})}\le c_7 2^{-k\alpha}.$$

The proof is now complete. □

The following lemma follows from Theorem 1.3 and Lemma 3.6 (instead of [2, Lemma 13 and Lemma 14]) in the same way as for the case of symmetric $\alpha$-stable process in [2, Lemma 16]. We omit the details here.

For a Lipschitz open set $D$ with characteristics $(\lambda_0, R_0)$, let $\kappa=\kappa(\lambda_0,R_0)\in(0,1)$ so that $D$ is $\kappa$-fat. For $z_0\in\partial D$ and $r\in(0,1]$, we use $A_r(z_0)$ to denote a point in $D$ such that $B(A_r(z_0),\kappa r)\subset D\cap B(z_0,r)$.

**Lemma 3.7.** *Suppose Assumption 1 holds and $D$ is a Lipschitz open set with characteristics $(\lambda_0, R_0)$. Let $r_1 \in (0,1]$ be the constant in Lemma 3.2. There exist positive constants $\gamma_1 = \gamma_1(d,\alpha,\beta,\lambda_0,R_0,M_1,M_2)$ and $C_{10} = C_{10}(d,\alpha,\beta,\lambda_0,R_0,M_1,M_2)$ such that for every $z_0 \in \partial D$, $r \in (0, r_1/2)$ and all non-negative functions $u$, $v$ that are regular harmonic in $D \cap B(z_0, 2r)$ and vanish in $D^c \cap B(z_0, 2r)$ with $u(A_r(z_0)) = v(A_r(z_0)) > 0$, we have*

(i) *$h(z_0) := \lim_{D \ni x \to z_0} u(x)/v(x)$ exists;*

(ii) *$|\frac{u(x)}{v(x)} - h(z_0)| \le C_{10} \left(\frac{|x-z_0|}{r}\right)^{\gamma_1}$ for $x \in D \cap B(z_0, r)$.*

# 4 Gradient upper bound estimates

We now study gradient estimates for non-negative harmonic functions of $X^b$ in open sets.

**Lemma 4.1.** *Suppose $b$ is a bounded function satisfying (1.9), and that for every $x \in \mathbb{R}^d$, $J^b(x,y) \ge 0$ a.e. $y \in \mathbb{R}^d$. Let $r_1 \in (0,1]$ be the constant in Lemma 3.2, and $B = B(x_0, r)$ with $r \in (0, r_1]$. Then for every $x \in B$, $z \in \bar{B}^c$ and $1 \le i \le d$,*

$$\partial_{x_i} \int_B G_B(x,y) J^b(y,z)dy = \int_B \partial_{x_i} G_B(x,y) J^b(y,z) dy, \tag{4.1}$$

$$\partial_{x_i} \int_B \left( \int_B G_B(x,y) \mathcal{S}^b_y G^b_B(y,w)dy \right) J^b(w,z)dw = \int_B \left( \int_B \partial_{x_i} G_B(x,y) \mathcal{S}^b_y G^b_B(y,w)dy \right) J^b(w,z)dw, \tag{4.2}$$

*and*

$$\partial_{x_i} K^b_B(x,z) = \int_B \partial_{x_i} G_B(x,y) J^b(y,z)dy + \int_B \left( \int_B \partial_{x_i} G_B(x,y) \mathcal{S}^b_y G^b_B(y,w)dy \right) J^b(w,z)dw. \tag{4.3}$$

*Proof.* Without loss of generality we assume $i = d$. Fix $x \in B$ and $z \in \bar{B}^c$. We have

$$\sup_{y \in B} |J^b(y,z)| \le \frac{\mathcal{A}(d,-\alpha)}{\delta_B(z)^{d+\alpha}} + \frac{\|b\|_\infty \mathcal{A}(d,-\beta)}{\delta_B(z)^{d+\beta}} < +\infty.$$

Thus (4.1) follows directly from [4, Lemma 5.2].

Let $g_z(y) := \int_B \mathcal{S}^b_y G^b_B(y,w) J^b(w,z)dw$ for $y \in B$. We have

$$\partial_{x_d} \int_B G_B(x,y) g_z(y)dy = \lim_{\lambda \to 0} \int_B \left[ \frac{G_B(x + \lambda e_d, y) - G_B(x,y)}{\lambda} \right] g_z(y)dy. \tag{4.4}$$

To prove (4.2), we only need to show that the integrand in the right hand side of (4.4) is uniformly integrable on $B$ in $\lambda \in (0, \delta_B(x)/2)$. Note that we have

$$G_B(x,y) = G(x,y) - \mathbb{E}_y \left[ G(x, X_{\tau_B}) \right] =: G(x,y) - H(x,y).$$

Thus

$$\frac{|G_B(x+\lambda e_d, y) - G_B(x,y)|}{\lambda} \quad \le \quad \frac{|G(x+\lambda e_d, y) - G(x,y)|}{\lambda} + \frac{|H(x+\lambda e_d, y) - H(x,y)|}{\lambda}$$

$$=: \quad I + II.$$

Obviously by (2.10) we have

$$I \le c_1 \left( |x+\lambda e_d - y|^{\alpha-1-d} + |x-y|^{\alpha-d-1} \right) \quad \text{for } y \in B, \tag{4.5}$$

for some positive constant $c_1 = c_1(d,\alpha)$. Since $H(x,y) = \mathbb{E}_y[G(x, X_{\tau_B})]$, by the mean-value theorem, there is a point $x_\lambda$ in the line segment connecting $x$ with $x+\lambda e_d$ so that

$$II = \partial_{x_d} H(x_\lambda, y) = \mathbb{E}_y[\partial_{x_d} G(x_\lambda, X_{\tau_B})] \le c_2 \delta_B(x)^{\alpha-1-d}.$$

Thus for some positive constant $c_3 = c_3(d,\alpha,x)$, we have

$$\frac{|G_B(x+\lambda e_d, y) - G_B(x,y)|}{\lambda} \le c_1 \left( |x+\lambda e_d - y| \wedge |x-y| \right)^{\alpha-d-1} + c_3. \tag{4.6}$$

Let $h(y) := \int_B h_B(y,w)dw$ for $y \in B$. Note that by Lemma 3.2 and the boundedness of $w \mapsto J^b(w,z)$ on $B$,

$$|g_z(y)| \le c_4 \int_B h_B(y,w) J^b(w,z) dw \le c_5 h(y). \tag{4.7}$$

Thus by (4.6) and (4.7) the integrand in the right hand side of (4.4) is uniformly integrable on $B$ in $h \in (0, \delta_B(x)/2)$ if the following three conditions are true:

(i) $\int_B h(y)dy < +\infty$;

(ii) $\sup_{w \in B(x,\delta_B(x)/2)} \int_B h(y)|y-w|^{\alpha-1-d}\, dy < +\infty$;

(iii) $\lim_{\varepsilon \downarrow 0} \sup_{w \in B(x,\delta_B(x)/2)} \int_{\{y \in B : |y-w|<\varepsilon\}} h(y)|y-w|^{\alpha-1-d} = 0$.

If $\alpha > 2\beta$, then for any $y \in B$,

$$h(y) \le \int_{w \in B} |y-w|^{\alpha-\beta-d} dw \le \int_{|u|<2r} |u|^{\alpha-\beta-d} du < +\infty;$$

that is, $h(y)$ is bounded from above on $B$. Obviously (i)-(iii) hold for $h$. If $\alpha = 2\beta$, then

$$\begin{aligned} h(y) &= \int_{w \in B} |w-y|^{\beta-d} \left( 1 \wedge \frac{\delta_B(w)^\beta}{|y-w|^\beta} \right) \left( 1 \vee \log \frac{|w-y|}{\delta_B(y)} \right) dw \\ &= \int_{w \in B, |w-y| > e\delta_B(y)} |w-y|^{\beta/2-d} \frac{\delta_B(w)^\beta}{\delta_B(y)^{\beta/2}} \left( \frac{\delta_B(y)^{\beta/2}}{|w-y|^{\beta/2}} \log \frac{|w-y|}{\delta_B(y)} \right) dw \\ &\quad + \int_{w \in B, |w-y| \le e\delta_B(y)} |w-y|^{\beta-d} dw \\ &\lesssim \delta_B(y)^{-\beta/2} + 1. \end{aligned} \tag{4.8}$$

Using this upper bound, it is easy to check $h$ satisfies (i) and (ii). As for (iii), note that $\delta_B(w) \ge \delta_B(x)/2$ for every $w \in B(x, \delta_B(x)/2)$. Consider an arbitrary $\varepsilon \in (0, \delta_B(x)/4)$. Then $B(w,\varepsilon) \subset B$ and $\delta_B(y) \ge \delta_B(x)/4$ for every $y \in B(w,\varepsilon)$. We have by (4.8),

$$\int_{|y-w|<\varepsilon} h(y)|y-w|^{\alpha-1-d} dy \quad \lesssim \quad \int_{|y-w|<\varepsilon} (\delta_B(y)^{-\beta/2} + 1)|w-y|^{\alpha-1-d} dy$$

$$\lesssim \int_{|y-w|<\varepsilon} (\delta_B(x)^{-\beta/2}+1)|w-y|^{\alpha-1-d}dy.$$

Thus condition (iii) is implied by the fact that

$$\lim_{\varepsilon\downarrow 0} \sup_{w\in B(x,\delta_B(x)/2)} \int_{|y-w|<\varepsilon} (\delta_B(x)^{-\beta/2}+1)|w-y|^{\alpha-1-d}dy = 0.$$

When $\alpha<2\beta$, similar to (4.8) we have

$$h(y) \asymp \int_{w\in B} |w-y|^{\beta-d}\left(1\wedge \frac{\delta_B(w)^\beta}{|y-w|^\beta}\right)\left(1\vee \frac{|y-w|^{\beta-\alpha/2}}{\delta_B(y)^{\beta-\alpha/2}}\right)dw \lesssim \delta_B(y)^{\beta-\alpha/2}+1.$$

By a similar calculations as in the case $\alpha=2\beta$, we can show that (i)-(iii) hold for $h$. This completes the proof. $\square$

**Lemma 4.2.** *Under Assumption 1, there exists a constant $C_{11}=C_{11}(d,\alpha,\beta,M_1,M_2)>0$ such that for every $B=B(x_0,1)$ and $1\le i\le d$,*

$$\int_B |\partial_{x_i} G_B(x,y)|J^b(y,z)dy \le C_{11}\int_B G_B(x_0,y)J^b(y,z)dy \quad \textit{for } x\in B(x_0,1/4) \textit{ and } z\in \bar{B}^c. \tag{4.9}$$

*Proof.* Without loss of generality, we assume $x_0=0$ and $i=d$. For every $|x|<1/4$ and $|y|<1$, we have $|x-y|\wedge\delta_B(x)\asymp|x-y|$. Thus by (3.1),

$$\begin{aligned}
\int_B |\partial_{x_d} G_B(x,y)|J^b(y,z)dy &\lesssim \int_{|y|<1} \frac{G_B(x,y)}{|x-y|\wedge\delta_B(x)} J^b(y,z)dy \\
&\asymp \int_{|y|<1} \frac{G_B(x,y)}{|x-y|} J^b(y,z)dy \\
&= \left(\int_{1/2\le|y|<1} + \int_{|y|<1/2}\right) \frac{G_B(x,y)}{|x-y|} J^b(y,z)dy \\
&=: I(x,z)+II(x,z).
\end{aligned} \tag{4.10}$$

For $1/2\le|y|<1$ and $|x|<1/4$, we have $|x-y|\asymp|y|\asymp 1$ and $\delta_B(x)\asymp 1$. Thus

$$G_B(x,y)\asymp |y|^{\alpha-d}\left(1\wedge\frac{\delta_B(y)^{\alpha/2}}{|y|^\alpha}\right)\asymp G_B(0,y),$$

and consequently

$$I(x,z)\asymp \int_{1/2\le|y|<1} G_B(0,y)J^b(y,z)dy. \tag{4.11}$$

For every $|y|<1/2$, $|x|<1/4$ and $|z|>1$, we have $|z-y|>1/2$, $|z-y|\asymp|z-y+x|$ and $\delta_B(y)=1-|y|\asymp 1-|y-x|\asymp 1$. Thus by (1.10)

$$II(x,z) \overset{c_1(M_2)}{\asymp} \int_{|y|<1/2} |x-y|^{-d+\alpha-1}\left(1\wedge\frac{\delta_B(x)^{\alpha/2}\delta_B(y)^{\alpha/2}}{|x-y|^\alpha}\right)J^{\varepsilon_0}(|z-y|)dy$$

$$
\begin{aligned}
&\asymp \int_{|y|<1/2} |x-y|^{-d+\alpha-1}\left(1\wedge \frac{\delta_B(x)^{\alpha/2}(1-|y-x|)^{\alpha/2}}{|x-y|^{\alpha}}\right) J^{\varepsilon_0}(|z-y+x|)dy \\
&\asymp \int_{|w+x|<1/2} |w|^{-d+\alpha-1}\left(1\wedge \frac{(1-|w|)^{\alpha/2}}{|w|^{\alpha}}\right) J^{\varepsilon_0}(|z-w|)dw \\
&\le \int_{|w|<3/4} |w|^{-d+\alpha-1}\left(1\wedge \frac{(1-|w|)^{\alpha/2}}{|w|^{\alpha}}\right) J^{\varepsilon_0}(|z-w|)dw \\
&\stackrel{c_2(M_2)}{\asymp} \int_{|w|<3/4} |w|^{-d+\alpha-1}\left(1\wedge \frac{(1-|w|)^{\alpha/2}}{|w|^{\alpha}}\right) J^{b}(w,z)dw \\
&=: g_1(z). \qquad (4.12)
\end{aligned}
$$

For every $z>1$, let

$$
g_2(z) := \int_{|w|<3/4} |w|^{\alpha-d}\left(1\wedge \frac{(1-|w|)^{\alpha/2}}{|w|^{\alpha}}\right) J^{b}(w,z)dw.
$$

Obviously

$$
g_2(z) \asymp \int_{|w|<3/4} G_B(0,w)J^b(w,z)dw \le \int_B G_B(0,w)J^b(w,z)dw. \qquad (4.13)
$$

Note that $J^{\varepsilon_0}(|x-y|)$ is non-increasing in $|x-y|$. Thus by (1.10)

$$
\sup_{|z|>1}\frac{g_1(z)}{g_2(z)} \le \sup_{|z|>1}\frac{M_2 J^{\varepsilon_0}(|z|-3/4)\int_{|w|<3/4}|w|^{\alpha-d-1}\left(1\wedge\frac{(1-|w|)^{\alpha/2}}{|w|^{\alpha}}\right)dw}{M_2^{-1}J^{\varepsilon_0}(|z|+3/4)\int_{|w|<3/4}|w|^{\alpha-d}\left(1\wedge\frac{(1-|w|)^{\alpha/2}}{|w|^{\alpha}}\right)dw} \le M<+\infty, \qquad (4.14)
$$

where $M=M(d,\alpha,\beta,M_2)>0$. Thus by (4.13) and (4.14) we prove that

$$
II(x,z) \stackrel{c_3(M_2)}{\lesssim} \int_B G_B(0,w)J^b(z,w)dw \quad \text{for } |x|<1/4,\ |z|>1. \qquad (4.15)
$$

Therefore (4.9) follows from (4.11) and (4.15). □

Recall the definition of $r_D(x,y)$ and $h_D(x,y)$ from (1.6) and (3.2), respectively.

**Lemma 4.3.** *For $B=B(0,1)$, there exists a constant $C_{12}=C_{12}(d,\alpha,\beta)>0$ such that for every $1\le i\le d$, $|x|<1/4$ and $|w|<1$,*

$$
\int_B |\partial_{x_i}G_B(x,y)|h_B(y,w)dy \le C_{12}|x-w|^{-d+(\alpha-1)\wedge(\alpha-\beta)}\delta_B(w)^{\alpha/2}. \qquad (4.16)
$$

*Proof.* Without loss of generality, we assume $i=d$. For any $|x|<1/4$ and $|y|<1$, we have $\delta_B(x)\asymp 1$ and $|x-y|\wedge\delta_B(x)\asymp|x-y|$. Thus by (3.1),

$$
\begin{aligned}
\int_B |\partial_{x_d}G_B(x,y)|h_B(y,w)dy &\lesssim \int_{|y|<1}\frac{G_B(x,y)}{|x-y|\wedge\delta_B(x)}h_B(y,w)dy \\
&\asymp \int_{|y|<1}\frac{G_B(x,y)}{|x-y|}h_B(y,w)dy. \qquad (4.17)
\end{aligned}
$$

We note that for $|x| < 1/4$ and $|y| < 1$,

$$G_B(x,y) \asymp |x-y|^{\alpha-d}\left(1 \wedge \frac{\delta_B(x)^{\alpha/2}}{|x-y|^{\alpha/2}}\right)\left(1 \wedge \frac{\delta_B(y)^{\alpha/2}}{|x-y|^{\alpha/2}}\right) \asymp |x-y|^{\alpha-d}\frac{\delta_B(y)^{\alpha/2}}{r_B(x,y)^{\alpha}}, \tag{4.18}$$

and $r_B(x,y) \geq \delta_B(x) \geq 3/4$. Now we calculate the integral in (4.17) using (4.18) and the explicit formula of $h_B(y,w)$. If $\alpha > 2\beta$, we have

$$\begin{aligned}
(4.17) \quad &\asymp \int_{|y|<1} \frac{\delta_B(w)^{\alpha/2}}{|x-y|^{d-\alpha+1}|w-y|^{d-\alpha+\beta}}\,\frac{\delta_B(y)^{\alpha/2}}{r_B(x,y)^{\alpha} r_B(y,w)^{\alpha/2}}\,dy \\
&\lesssim \delta_B(w)^{\alpha/2}\int_{|y|<1} |x-y|^{-d+\alpha-1}|y-w|^{-d+\alpha-\beta}dy \\
&\lesssim |x-w|^{-d+(\alpha-1)\wedge(\alpha-\beta)}\delta_B(w)^{\alpha/2} \\
&\leq |x-w|^{-d+(\alpha-1)\wedge(\alpha-\beta)}\delta_B(w)^{\beta}.
\end{aligned} \tag{4.19}$$

If $\alpha = 2\beta$, we have by (2.13) and (3.2)

$$\begin{aligned}
&(4.17) \\
\asymp\ & \int_{|y|<1} |x-y|^{2\beta-1-d}\frac{\delta_B(y)^{\beta}}{r_B(x,y)^{2\beta}}\,|y-w|^{\beta-d}\frac{\delta_B(w)^{\beta}}{r_B(y,w)^{\beta}}\left(1 \vee \log\frac{|y-w|}{\delta_B(y)}\right)dy \\
=\ & \int_{|y|<1,|y-w|\leq e\delta_B(y)} \frac{\delta_B(w)^{\beta}}{|x-y|^{d+1-2\beta}|y-w|^{d-\beta}}\,\frac{\delta_B(y)^{\beta}}{r_B(x,y)^{2\beta} r_B(y,w)^{\beta}}\,dy \\
&+\int_{|y|<1,|y-w|>e\delta_B(y)} \frac{\delta_B(w)^{\beta}}{|x-y|^{d+1-2\beta}|w-y|^{d-\beta}}\,\frac{|y-w|^{\beta/2}\delta_B(y)^{\beta/2}}{r_B(x,y)^{2\beta} r_B(y,w)^{\beta}}\left(\frac{\delta_B(y)^{\beta/2}}{|y-w|^{\beta/2}}\log\frac{|y-w|}{\delta_B(y)}\right)dy \\
\lesssim\ & \delta_B(w)^{\beta}\int_{|y|<1,|y-w|\leq e\delta_B(y)} |x-y|^{2\beta-1-d}|y-w|^{\beta-d}dy \\
&+\delta_B(w)^{\beta}\int_{|y|<1,|y-w|>e\delta_B(y)} |x-y|^{2\beta-1-d}|y-w|^{2\beta-d}dy \\
\lesssim\ & \delta_B(w)^{\beta}|x-w|^{-d+2\beta-1} = |x-w|^{-d+(\alpha-1)\wedge(\alpha-\beta)}\delta_B(w)^{\beta}.
\end{aligned} \tag{4.20}$$

If $\alpha < 2\beta$, we have

$$\begin{aligned}
(4.17) \quad &\asymp \int_{|y|<1} |x-y|^{-d+\alpha-1}\frac{\delta_B(y)^{\alpha/2}}{r_B(x,y)^{\alpha}}\,|y-w|^{-d+\alpha-\beta}\frac{\delta_B(w)^{\alpha/2}}{r_B(y,w)^{\alpha/2}}\left(1 \vee \frac{|y-w|^{\beta-\alpha/2}}{\delta_B(y)^{\beta-\alpha/2}}\right)dy \\
&\leq \int_{|y|<1,|y-w|\geq\delta_B(y)} \frac{\delta_B(w)^{\alpha/2}}{|x-y|^{d-\alpha+1}|w-y|^{d-\alpha+\beta}}\,\frac{|w-y|^{\beta-\alpha/2}\delta_B(y)^{\alpha-\beta}}{r_B(y,w)^{\alpha/2}}\,dy \\
&\quad+\int_{|y|<1,|y-w|<\delta_B(y)} \frac{\delta_B(w)^{\alpha/2}}{|x-y|^{d-\alpha+1}|w-y|^{d-\alpha+\beta}}\,\frac{\delta_B(y)^{\alpha/2}}{r_B(x,y)^{\alpha} r_B(y,w)^{\alpha/2}}\,dy \\
&\lesssim \delta_B(w)^{\alpha/2}\int_{|y|<1,|y-w|\geq\delta_B(y)} |x-y|^{-d+\alpha-1}|y-w|^{-d+\alpha-\beta}dy \\
&\quad+\delta_B(w)^{\alpha/2}\int_{|y|<1,|y-w|<\delta_B(y)} |x-y|^{-d+\alpha-1}|y-w|^{-d+\alpha-\beta}dy \\
&= \delta_B(w)^{\alpha/2}\int_{|y|<1} |x-y|^{-d+\alpha-1}|y-w|^{-d+\alpha-\beta}dy
\end{aligned}$$

$$\begin{aligned}
&\lesssim \delta_B(w)^{\alpha/2}|x-w|^{-d+(\alpha-1)\wedge(\alpha-\beta)}\int_{|y|<1}(|x-y|^{\alpha-1-(\alpha-1)\wedge(\alpha-\beta)}+|y-w|^{\alpha-\beta-(\alpha-1)\wedge(\alpha-\beta)})dy \\
&\lesssim |x-w|^{-d+(\alpha-1)\wedge(\alpha-\beta)}\delta_B(w)^{\alpha/2}. \qquad (4.21)
\end{aligned}$$

Lemma 4.3 follows from (4.20), (4.21) and (4.19). □

**Lemma 4.4.** *Under Assumption 1, there exists a constant $C_{13} = C_{13}(d, \alpha, \beta, M_1, M_2) > 0$ such that for $B = B(x_0, 1)$, $1 \le i \le d$, $x \in B(x_0, 1/4)$ and $z \in \bar{B}^c$,*

$$\int_B \left[\int_B |\partial_{x_i} G_B(x,y)\mathcal{S}_y^b G_B^b(y,w)|dy\right] J^b(w,z)dw \le C_{13}\int_B G_B(x_0,w)J^b(w,z)dw. \qquad (4.22)$$

*Proof.* Without loss of generality, we assume that $x_0 = 0$ and $i = d$. Let $r_1 \in (0, 1]$ be the constant in Lemma 3.2. By Lemma 3.2 and the scaling property, we have for $y, w \in B$,

$$|S_y^b G_B^b(y,w)| = r_1^d |S_y^{b_{r_1}} G_{r_1 B}^{b_{r_1}}(r_1 y, r_1 w)| \le c_1 r_1^d |h_{r_1 B}(r_1 y, r_1 w)| = c_1 r_1^{\alpha-\beta} h_B(y,w) \le c_1 h_B(y,w).$$

Here $c_1 = c_1(d, \alpha, \beta, M_1) > 0$. Hence to prove (4.22), it suffices to prove that for $x \in B(0, 1/4)$ and $z \in \bar{B}^c$,

$$\int_B \left[\int_B |\partial_{x_d} G_B(x,y)| h_B(y,w)dy\right] J^b(w,z)dw \le c_2 \int_B G_B(x_0,w)J^b(w,z)dw \qquad (4.23)$$

for some $c_2 = c_2(d, \alpha, \beta, M_1, M_2) > 0$. By Lemma 4.3, we have

$$\begin{aligned}
&\int_B \left[\int_B |\partial_{x_d} G_B(x,y)| h_B(y,w)dy\right] J^b(w,z)dw \\
\lesssim\ & \int_{|w|<1} \delta_B(w)^{\alpha/2}|w-x|^{-d+(\alpha-1)\wedge(\alpha-\beta)} J^b(w,z)dw \\
=\ & \int_{|w|\le 1/2} + \int_{1/2<|w|<1} \delta_B(w)^{\alpha/2}|w-x|^{-d+(\alpha-1)\wedge(\alpha-\beta)} J^b(w,z)dw \\
=:\ & I(x,z) + II(x,z).
\end{aligned}$$

Fix $|x| < 1/4$ and $|z| > 1$. For $1/2 < |w| < 1$, we have $|w-x| \asymp |w| \asymp 1$, and consequently $G_B(0,w) \asymp \delta_B(w)^{\alpha/2}$. Thus

$$II(x,z) \asymp \int_{1/2<|w|<1} \delta_B(w)^{\alpha/2} J^b(w,z)dw \asymp \int_{1/2<|w|<1} G_B(0,w)J^b(w,z)dw. \qquad (4.24)$$

For any $|w| \le 1/2$, we have $1/2 \le \delta_B(w) \le 1$, $|z-w| \ge 1/2$ and $|z-w+x| \asymp |z-w|$. Hence by (1.10)

$$\begin{aligned}
I(x,z) &\overset{M_2}{\asymp} \int_{|w|\le 1/2} |w-x|^{-d+(\alpha-1)\wedge(\alpha-\beta)} J^{\varepsilon_0}(|z-w|)dw \\
&\asymp \int_{|w|\le 1/2} |w-x|^{-d+(\alpha-1)\wedge(\alpha-\beta)} J^{\varepsilon_0}(|z-w+x|)dw \\
&= \int_{|v+x|\le 1/2} |v|^{-d+(\alpha-1)\wedge(\alpha-\beta)} J^{\varepsilon_0}(|z-v|)dv \\
&\le \int_{|v|\le 3/4} |v|^{-d+(\alpha-1)\wedge(\alpha-\beta)} J^{\varepsilon_0}(|z-v|)dv =: g_1(z). \qquad (4.25)
\end{aligned}$$

We first consider the case $|z| > 2$. Let $g_2(z) := \int_{|v|\le 3/4} |v|^{\alpha-d} J^{\varepsilon_0}(|z-v|)dv$. Note that for any $|v| \le 3/4$, we have $G_B(0,v) \asymp |v|^{\alpha-d}$. Thus

$$g_2(z) \asymp \int_{|v|\le 3/4} G_B(0,v) J^{\varepsilon_0}(|z-v|)dv \le M_2 \int_{|v|\le 3/4} G_B(0,v) J^b(v,z)dv. \tag{4.26}$$

In addition since $J^{\varepsilon_0}(|y|)$ is non-increasing in $|y|$, we have

$$\sup_{|z|>2} \frac{g_1(z)}{g_2(z)} \le \sup_{|z|>2} \frac{J^{\varepsilon_0}(|z|-3/4) \int_{|v|\le 3/4} |v|^{-d+(\alpha-1)\wedge(\alpha-\beta)} dv}{J^{\varepsilon_0}(|z|+3/4) \int_{|v|\le 3/4} |v|^{\alpha-d} dv} \le M < +\infty, \tag{4.27}$$

where $M = M(d,\alpha,\beta) > 0$. Therefore by (4.25), (4.26) and (4.27) we have

$$I(x,z) \overset{c_3(M_2)}{\lesssim} \int_B G_B(0,w) J^b(w,z)dw \quad \text{for } |x| < 1/4 \text{ and } |z| > 2. \tag{4.28}$$

On the other hand if $1 < |z| \le 2$, we have $0 < \delta_B(z) \le 1$, and by (2.14)

$$\int_B G_B(0,w) J^b(w,z)dw \ge M_2^{-1} \int_B G_B(0,w) J(w,z)dw = M_2^{-1} K_B(0,z) \asymp M_2^{-1} \delta_B(z)^{-\alpha/2} \ge M_2^{-1}. \tag{4.29}$$

Note that $|z-w| \ge 1/4$ for any $|w| \le 3/4$. Thus

$$g_1(z) \le J^{\varepsilon_0}(1/4) \int_{|w|\le 3/4} |w|^{-d+(\alpha-1)\wedge(\alpha-\beta)} dw \lesssim 1. \tag{4.30}$$

Thus by (4.25), (4.29) and (4.30) we have

$$I(x,z) \overset{c_4(M_2)}{\lesssim} \int_B G_B(0,w) J^b(w,z)dw \quad \text{for } |x| < 1/4 \text{ and } 1 < |z| \le 2. \tag{4.31}$$

Now (4.23) follows from (4.31) and (4.24). □

**Theorem 4.5.** *Let $r_1 \in (0,1]$ be the constant in Lemma 3.2. Under Assumption 1, there exists a constant $C_{14} = C_{14}(d,\alpha,\beta,M_1,M_2) > 0$ such that for every ball $B_r = B(x_0,r)$ with radius $r \in (0,r_1]$ and $1 \le i \le d$,*

$$|\partial_{x_i} K^b_{B_r}(x,z)| \le \frac{C_{14}}{r} K^b_{B_r}(x_0,z), \quad \textit{for } x \in B(x_0,r/4) \textit{ and } z \in \bar{B}_r^{\,c}. \tag{4.32}$$

*Proof.* Let $\lambda := 1/r \ge 1/r_1 \ge 1$ and define $b_\lambda(x,z) = \lambda^{\beta-\alpha} b(\lambda^{-1}x, \lambda^{-1}z)$. Observe that $\|b_\lambda\|_\infty = r^{\alpha-\beta}\|b\|_\infty \le r_1^{\alpha-\beta} M_1 \le M_1$. By the scaling properties (2.6) and (2.9), $b_\lambda(x,z)$ satisfies Assumption 1 and it suffices to show that for the ball $B = B(x_0,1)$,

$$|\partial_{x_i} K^{b_\lambda}_B(x,z)| \le C_{14} K^{b_\lambda}_B(x_0,z) \qquad \text{for } x \in B(x_0,1/4) \text{ and } z \in \bar{B}^c. \tag{4.33}$$

We know from [11, Lemma 4.9] that

$$G^{b_\lambda}_B(x,y) = G_B(x,y) + \int_B G_B(x,z) \mathcal{S}^{b_\lambda}_z G^{b_\lambda}_B(z,y)dz \quad \text{for } x,y \in B.$$

Thus by (2.3), for $i = 1, \cdots, d$, every $x \in B$, and $z \in \bar{B}^c$,

$$\partial_{x_i} K_B^{b_\lambda}(x,z) = \partial_{x_i} \int_B G_B(x,y) J^{b_\lambda}(y,z)dy + \partial_{x_i} \int_B \left( \int_B G_B(x,y) \mathcal{S}_z^{b_\lambda} G_B^{b_\lambda}(y,w)dy \right) J^{b_\lambda}(w,z)dw. \tag{4.34}$$

Thus by Lemma 4.1

$$\begin{aligned} |\partial_{x_i} K_B^{b_\lambda}(x,z)| &\leq \int_B |\partial_{x_i} G_B(x,y)| J^{b_\lambda}(y,z)dy \\ &\quad + \int_B \left( \int_B |\partial_{x_i} G_B(x,y) \mathcal{S}_z^{b_\lambda} G_B^{b_\lambda}(y,w)|dy \right) J^{b_\lambda}(w,z)dw. \end{aligned} \tag{4.35}$$

On the other hand, by (3.3) and (2.8), we have

$$\frac{1}{2} G_B(x,y) \leq G_B^{b_\lambda}(x,y) \leq \frac{3}{2} G_B(x,y) \quad \text{for } x,y \in B.$$

Thus

$$K_B^{b_\lambda}(x,z) = \int_B G_B^{b_\lambda}(x,y) J^{b_\lambda}(y,z)dy \geq \frac{1}{2} \int_B G_B(x,y) J^{b_\lambda}(y,z)dy \quad \text{for } z \in \bar{B}^c. \tag{4.36}$$

Now (4.33) is implied by (3.3) and Lemmas 4.2-4.4. This completes the proof of the theorem. □

**Lemma 4.6.** *Suppose Assumption 1 holds and $f$ is regular harmonic with respect to $X^b$ in $B(x,r)$ for some $x \in \mathbb{R}^d$ and $r \in (0, r_1]$. Then $\partial_{x_i} f(x)$ exists for every $1 \leq i \leq d$ and*

$$\partial_{x_i} f(x) = \int_{\overline{B(x,r)}^c} f(z) \partial_{x_i} K_{B(x,r)}^b(x,z)dz. \tag{4.37}$$

*Proof.* Recall that $e_i$ is the unit vector along the positive $x_i$-axis. Choose $\varepsilon > 0$ sufficiently small so that $x + \varepsilon e_i \in B(x, r/4)$. By the regular harmonicity of $f$, we have

$$\frac{f(x+\varepsilon e_i) - f(x)}{\varepsilon} = \int_{\overline{B(x,r)}^c} f(z) \left[ \frac{K_{B(x,r)}^b(x+\varepsilon e_i, z) - K_{B(x,r)}^b(x,z)}{\varepsilon} \right] dz.$$

Therefore (4.37) follows from (4.32) and the dominated convergence theorem. □

**Proof of Theorem 1.4:** Let $x \in D$ and $0 < r < (\delta_D(x) \wedge r_1)/2$. Note that under our assumption, $f$ is regular harmonic in $B(x,r)$ with respect to $X^b$. By (4.37) and (4.32), we have

$$\begin{aligned} |\partial_{x_i} f(x)| &\leq \int_{\overline{B(x,r)}^c} f(z) |\partial_{x_i} K_{B(x,r)}^b(x,z)|dz \\ &\leq \frac{C_{14}}{r} \int_{\overline{B(x,r)}^c} f(z) K_{B(x,r)}^b(x,z)dz \\ &= \frac{C_{14}}{r} f(x) \to \frac{2C_{14}}{r_1 \wedge \delta_D(x)} f(x) \quad \text{as } r \uparrow (r_1 \wedge \delta_D(x))/2. \end{aligned}$$

□

# 5 Gradient lower bound estimate

For $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, we write $x = (\tilde{x}, x_d)$, where $\tilde{x} = (x_1, \cdots, x_{d-1})$. In this section, we fix a Lipschitz function $\Gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ with Lipschitz constant $\lambda_0$ so that $|\Gamma(\tilde{x}) - \Gamma(\tilde{y})| \le \lambda_0 |\tilde{x} - \tilde{y}|$ for all $\tilde{x}, \tilde{y} \in \mathbb{R}^{d-1}$. Put $\rho(x) := x_d - \Gamma(\tilde{x})$. Unless stated otherwise, $D$ denotes the special Lipschitz open set defined by $D = \{x \in \mathbb{R}^d : \rho(x) > 0\}$. When $x \in D$, $\rho(x)$ serves as the vertical distance from $x \in D$ to $\partial D$, and it satisfies

$$\rho(x)/\sqrt{1+\lambda_0^2} \le \delta_D(x) \le \rho(x) \quad \text{for } x \in D. \tag{5.1}$$

We define the "box" $D^+(x, h, r) := \{y \in \mathbb{R}^d : 0 < \rho(y) < h,\ |\tilde{x} - \tilde{y}| < r\}$, and the "inverted box" $D^-(x, h, r) := \{y \in \mathbb{R}^d : -h < \rho(y) \le 0,\ |\tilde{x} - \tilde{y}| < r\}$, where $x \in \mathbb{R}^d$ and $h,\ r > 0$.

**Lemma 5.1.** *Let $r_1 \in (0, 1]$ be the constant in Lemma 3.2. Suppose Assumption 1 holds, $z_0 \in \partial D$ and $r \in (0, r_1/2]$. Let $A_{z_0} \in D$ be such that $\rho(A_{z_0}) = |A_{z_0} - z_0| = r/2$. Then there exist positive constants $C_{15} = C_{15}(d, \alpha, \beta, \lambda_0, M_1, M_2)$ and $\gamma_2 = \gamma_2(d, \alpha, \beta, \lambda_0, M_1, M_2)$ such that for every non-negative function $u$ which is harmonic in $D \cap B(z_0, 2r)$ and vanishes in $D^c \cap B(z_0, 2r)$, we have*

$$\frac{u(x)}{u(A_{z_0})} \ge C_{15} \left( \frac{\rho(x)}{\rho(A_{z_0})} \right)^{\alpha - \gamma_2} \qquad \textit{for } x \in D \cap B(z_0, r).$$

*Proof.* Note that by Lemma 3.2 and Assumption 1, we have for every $x \in \mathbb{R}^d$ and $y \in \overline{B(x, r)}^c$,

$$K^b_{B(x,r)}(x, y) \overset{M_2}{\asymp} \int_{B(x,r)} G_{B(x,r)}(x, z) J^{\varepsilon_0}(z, y) dz \asymp K^{\varepsilon_0}_{B(x,r)}(x, y). \tag{5.2}$$

Lemma 5.1 follows from (5.2), the uniform Harnack inequality (Theorem 3.4) and a standard argument of induction in the same way as for the case of symmetric $\alpha$-stable process in [2, Lemma 5] (see also [4, Lemma 4.2]). We omit the details here. □

Hereafter we assume Assumption 1 and Assumption 2 with $i = d$ hold. In this case, the jumping kernel $J^b(x, y)$ of the process $X^b$ satisfies that for every $x \in \mathbb{R}^d$,

$$J^b(x, y) = \frac{\mathcal{A}(d, -\alpha)}{|y - x|^{d+\alpha}} + \mathcal{A}(d, -\beta) \frac{\varphi(\tilde{x}) \psi(|y - x|)}{|y - x|^{d+\beta}} =: j^b(\tilde{x}, |y - x|) \quad \text{a.e. } y \in \mathbb{R}^d. \tag{5.3}$$

We note that by condition (1.13) of Assumption 2, $j^b(\tilde{x}, |z|)$ is non-increasing in $|z|$ for every $\tilde{x} \in \mathbb{R}^{d-1}$. Fix $z_0 \in \partial D$, $r \in (0, r_1]$. We define $D^+ := D^+(z_0, 4r\sqrt{1+\lambda_0^2}, 2r) \supset D \cap B(z_0, 2r)$, and

$$g_{b,r}(x) := \mathbb{P}_x \left( X^b_{\tau_{D^+}} \notin D^-(z_0, \infty, 2r) \right) \quad \text{for } x \in \mathbb{R}^d. \tag{5.4}$$

Clearly, $g_{b,r}$ is regular harmonic in $D^+$ with respect to $X^b$, $g_{b,r}(x) = 0$ in $D^-(z_0, \infty, 2r)$, and $g_{b,r}(x) = 1$ in $(D^+ \cup D^-(z_0, \infty, 2r))^c$.

**Lemma 5.2.** *The function $g_{b,r}(x)$ is non-decreasing in $x_d$.*

*Proof.* Note that $g_{b,r}(x) = 1 - \mathbb{P}_x\left(X^b_{\tau_{D^+}} \in D^-(z_0, \infty, 2r)\right)$ for every $x \in \mathbb{R}^d$. Take $x, y \in D^+$ such that $\tilde{x} = \tilde{y}$ and $y_d < x_d$. Consider the process $(X^b_t, \mathbb{P}_x)$ starting from $x$ (i.e. $\mathbb{P}_x(X^b_0 = x) = 1$). For every $t \geq 0$, define

$$Y^b_t := X^b_t - (x_d - y_d)e_d.$$

Then $(Y^b_t, \mathbb{P}_x)$ is a Markov process starting from $y$. Let $\mathcal{S}(\mathbb{R}^d)$ denote the totality of tempered functions on $\mathbb{R}^d$. For every $f \in \mathcal{S}(\mathbb{R}^d)$, if we define $f_d(z) := f(z - (x_d - y_d)e_d)$ for $z \in \mathbb{R}^d$, then

$$\mathcal{L}^b f(x - (x_d - y_d)e_d) = \mathcal{L}^b f_d(x) = \lim_{\varepsilon \to 0} \int_{|y-x|>\varepsilon} (f_d(y) - f_d(x)) j^b(\tilde{x}, |y-x|) dy \text{ for } x \in \mathbb{R}^d.$$

Thus

$$f(Y^b_t) - f(Y^b_0) - \int_0^t \mathcal{L}^b f(Y^b_s) ds = f_d(X^b_t) - f_d(X^b_0) - \int_0^t \mathcal{L}^b f_d(X^b_s) ds$$

is a $\mathbb{P}_x$-martingale. We know from [10, Theorem 5.6] that the solution of the martingale problem $(\mathcal{L}^b, \mathcal{S}(\mathbb{R}^d))$ with initial value $y$ is unique. Hence $(Y^b_t, \mathbb{P}_x)$ has the same distribution as $(X^b_t, \mathbb{P}_y)$. Consider the trajectory $\omega$ of $X^b_t$ starting from $x$. If $\omega$ exits $D^+$ by going into $D^-(0, +\infty, 2r)$, then so does $\omega - (x_d - y_d)e_d$ which is the trajectory of $Y^b_t$ starting from $y$. Hence

$$\mathbb{P}_x\left(X^b_{\tau_{D^+}} \in D^-(z_0, \infty, 2r)\right) \leq \mathbb{P}_x\left(Y^b_{\tau_{D^+}} \in D^-(z_0, \infty, 2r)\right) = \mathbb{P}_y\left(X^b_{\tau_{D^+}} \in D^-(z_0, \infty, 2r)\right).$$

This completes the proof. $\square$

**Lemma 5.3.** *Let $r_1 \in (0, 1]$ be the constant in Lemma 3.2. There are constants $C_{16} = C_{16}(d, \alpha, \beta, \lambda_0, M_1, M_2) > 0$ and $r_2 = r_2(d, \alpha, \beta, \lambda_0, M_1, M_2) \in (0, r_1]$ such that for every $z_0 \in \partial D$ and $r \in (0, r_2]$,*

$$\partial_{x_d} g_{b,r}(x) \geq C_{16} \frac{g_{b,r}(x)}{\delta_D(x)} \quad \textit{for } x \in D \cap B(z_0, r/2). \tag{5.5}$$

*Proof.* Without loss of generality we assume $z_0 = 0$. Let $r_2 \in (0, r_1]$ to be specified later. For $r \in (0, r_2]$, fix $x \in D \cap B(0, r/2)$. By (5.1), $r_0 := \rho(x)/2\sqrt{1 + \lambda_0^2} \leq \delta_D(x)/2 \leq r/4 \leq r_2/4$. Set

$$\widehat{x} = x + 2\rho(x)e_d \quad \text{and} \quad \check{x} = x - 2\rho(x)e_d.$$

Observe that $B(x, r_0), B(\widehat{x}, r_0) \subset D^+$ and $B(\check{x}, r_0) \subset D^-(0, +\infty, 2r)$. By (4.37) and (4.3), we have

$$\begin{aligned}
\partial_{x_d} g_{b,r}(x) &= \int_{\overline{B(x,r_0)}^c} g_{b,r}(z) \partial_{x_d} K^b_{B(x,r_0)}(x, z) dz \\
&\geq \int_{\overline{B(x,r_0)}^c} g_{b,r}(z) \left[\int_{B(x,r_0)} \partial_{x_d} G_{B(x,r_0)}(x, y) J^b(y, z) dy\right] dz \\
&\quad - \int_{\overline{B(x,r_0)}^c} g_{b,r}(z) \left(\int_{B(x,r_0) \times B(x,r_0)} \left|\partial_{x_d} G_{B(x,r_0)}(x, y) S^b_y G^b_{B(x,r_0)}(y, w)\right| J^b(w, z) dy\, dw\right) dz.
\end{aligned} \tag{5.6}$$

Let $\lambda := 1/r_0$ and $B_1 := \lambda B(x, r_0)$. By scaling property, Lemma 4.4 and Lemma 3.2, we have

$$\int_{B(x,r_0)} \int_{B(x,r_0)} \left|\partial_{x_d} G_{B(x,r_0)}(x, y) S^b_y G^b_{B(x,r_0)}(y, w)\right| J^b(w, z) dy\, dw$$

$$
\begin{aligned}
&= \lambda^{d+1}\int_{B_1}\int_{B_1}\left|\partial_{x_d}G_{B_1}(\lambda x,y)S_y^{b_\lambda}G_{B_1}^{b_\lambda}(y,w)\right| J^{b_\lambda}(w,\lambda z)dy\,dw \\
&\le c_1\lambda^{d+1-\alpha+\beta}\int_{B_1}G_{B_1}(\lambda x,w)J^{b_\lambda}(w,\lambda z)dw \\
&= c_1\lambda^{1-\alpha+\beta}\int_{B(x,r_0)}G_{B(x,r_0)}(x,v)J^b(v,z)dv \\
&\le 2c_1\lambda^{1-\alpha+\beta}\int_{B(x,r_0)}G^b_{B(x,r_0)}(x,v)J^b(v,z)dv \\
&= 2c_1 r_0^{-1+\alpha-\beta}K^b_{B(x,r_0)}(x,z). \qquad (5.7)
\end{aligned}
$$

Here $c_1 = c_1(d,\alpha,\beta,M_1,M_2) > 0$. Thus we can continue the estimate in (5.6) to get

$$
\begin{aligned}
&\partial_{x_d}g_{b,r}(x) \\
\ge\ &\int_{\overline{B(x,r_0)}^c} g_{b,r}(z)\left(\int_{B(x,r_0)}\partial_{x_d}G_{B(x,r_0)}(x,y)J^b(y,z)dy\right)dz \\
&-\frac{2c_1}{r_0}r_0^{\alpha-\beta}\int_{\overline{B(x,r_0)}^c} g_{b,r}(z)K^b_{B(x,r_0)}(x,z)dz \\
=\ &\int_{\overline{B(x,r_0)}^c} g_{b,r}(z)\left(\int_{B(x,r_0)}\partial_{x_d}G_{B(x,r_0)}(x,y)J^b(y,z)dy\right)dz - \frac{2c_1}{r_0}r_0^{\alpha-\beta}g_{b,r}(x). \qquad (5.8)
\end{aligned}
$$

Note that by (2.11),

$$
\begin{aligned}
&\partial_{x_d}G_{B(x,r_0)}(x,y) \\
=\ &2^{1-\alpha}\pi^{-d/2}\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)^{-2}\frac{y_d-x_d}{|y-x|^2}r_0^\alpha(r_0^2-|y-x|^2)^{\alpha/2}\left(r_0^2(r_0^2-|y-x|^2)+|y-x|^2\right)^{-d/2} \\
&+(d-\alpha)\frac{y_d-x_d}{|y-x|^2}G_{B(x,r_0)}(x,y). \qquad (5.9)
\end{aligned}
$$

Obviously $\partial_{x_d}G_{B(x,r_0)}(x,y)$ is anti-symmetric in $y$ with respect to the hyperplane $\mathcal{H} := \{y\in\mathbb{R}^d : y_d = x_d\}$. For every $z\in\overline{B(x,r_0)}^c$, define $h_x(z) := \int_{B(x,r_0)}\partial_{x_d}G_{B(x,r_0)}(x,y)J^b(y,z)dy$. By (5.3) we have for a.e. $z\in\overline{B(x,r_0)}^c$,

$$
\begin{aligned}
h_x(z) &= \int_{\{y\in B(x,r_0),y_d>x_d\}}\partial_{x_d}G_{B(x,r_0)}(x,y)\left(J^b(y,z)-J^b(\bar{y},z)\right)dy \\
&= \int_{\{y\in B(x,r_0),y_d>x_d\}}\partial_{x_d}G_{B(x,r_0)}(x,y)\left(j^b(\tilde{y},|z-y|)-j^b(\tilde{y},|z-\bar{y}|)\right)dy, \qquad (5.10)
\end{aligned}
$$

where $\bar{y} = (\tilde{y}, 2x_d - y_d)$. We observe that the right hand side of (5.10) is antisymmetric in $z$ with respect to the hyperplane $\mathcal{H}$. Recall that $j^b(\tilde{y},r)$ is non-increasing in $r$. Following from this, the monotonicity of $g_{b,r}(x)$ in $x_d$ and the Harnack inequality, we have

$$
\begin{aligned}
\int_{\overline{B(x,r_0)}^c} g_{b,r}(z)h_x(z)dz &\ge \int_{B(\widehat{x},r_0)\cup B(\check{x},r_0)} g_{b,r}(z)h_x(z)dz = \int_{B(\widehat{x},r_0)} g_{b,r}(z)h_x(z)dz \\
&\ge c_2 g_{b,r}(x)\int_{B(\widehat{x},r_0)} h_x(z)dz \qquad (5.11)
\end{aligned}
$$

for some $c_2 = c_2(d, \alpha, \beta, \lambda_0, M_1, M_2) > 0$. Note that by (5.9) and (5.10), for a.e. $z \in B(\widehat{x}, r_0)$,

$$\begin{aligned} h_x(z) &\geq (d-\alpha) \int_{\{y \in B(x,r_0): y_d > x_d\}} \frac{y_d - x_d}{|y-x|^2} G_{B(x,r_0)}(x,y) \left( j^b(\tilde{y}, |z-y|) - j^b(\tilde{y}, |z-\hat{y}|) \right) dy \\ &= (d-\alpha) \int_{B(x,r_0)} \frac{y_d - x_d}{|y-x|^2} G_{B(x,r_0)}(x,y) j^b(\tilde{y}, |z-y|) dy > 0. \end{aligned}$$

Therefore by the scaling property of the Green function $G_{B(x,r_0)}$,

$$\begin{aligned} & \int_{B(\widehat{x},r_0)} h_x(z) dz \\ \geq\ & (d-\alpha) \int_{B(\widehat{x},r_0)} \int_{B(x,r_0)} \frac{y_d - x_d}{|y-x|^2} G_{B(x,r_0)}(x,y) J^b(y,z) dy\, dz \\ \geq\ & \frac{d-\alpha}{M_2} \int_{|v - \frac{\widehat{x}}{r_0}|<1} \int_{|w - \frac{x}{r_0}|<1} \frac{1}{r_0} \frac{w_d - x_d/r_0}{|w - x/r_0|^2} G_{B(0,r_0)}(0, r_0(w - x/r_0)) J(r_0|v-w|) r_0^{2d} dw\, dv \\ =\ & \frac{d-\alpha}{r_0 M_2} \int_{|v - \frac{\widehat{x}}{r_0}|<1} \int_{|w - \frac{x}{r_0}|<1} \frac{w_d - x_d/r_0}{|w - x/r_0|^2} G_{B(0,1)}(0, w - x/r_0) J(|v-w|) dw\, dv \\ =\ & \frac{d-\alpha}{r_0 M_2} \int_{|v|<1} \int_{|w|<1} \frac{w_d}{|w|^2} G_{B(0,1)}(0,w) J(|v - w + \frac{\widehat{x} - x}{r_0}|) dw\, dv \\ =\ & \frac{d-\alpha}{r_0 M_2} \int_{|v|<1} \int_{|w|<1} \frac{w_d}{|w|^2} G_{B(0,1)}(0,w) J(|v - w + 4\sqrt{1+\lambda_0^2} e_d|) dw\, dv =: \frac{c_3}{r_0}, \end{aligned} \tag{5.12}$$

with $c_3 = c_3(d, \alpha, M_2) > 0$. It follows from (5.8), (5.9), (5.11) and (5.12) that

$$\partial_{x_d} g_{b,r}(x) \geq \frac{1}{r_0} \left( c_2 c_3 - 2c_1 r_0^{\alpha-\beta} \right) g_{b,r}(x) \geq \frac{2}{\delta_D(x)} \left( c_2 c_3 - 2c_1 \left( r_2/4 \right)^{\alpha-\beta} \right) g_{b,r}(x). \tag{5.13}$$

The lemma now follows from (5.13) by setting $r_2$ so small that $2c_1 \left( r_2/4 \right)^{\alpha-\beta} \leq c_2 c_3/2$. □

**Lemma 5.4.** *Suppose Assumption 1 holds and let $r_2 \in (0, r_1] \subset (0, 1]$ be the constant in Lemma 5.3. There is a positive constant $C_{17} = C_{17}(d, \alpha, \beta, \lambda_0, M_1, M_2)$ such that for every $r \in (0, r_2]$, there is a constant $r_3 = r_3(d, \alpha, \beta, \lambda_0, M_1, M_2, r) \in (0, r/2)$ so that for every $z_0 \in \partial D$ and every non-negative function $f$ that is regular harmonic in $D \cap B(z_0, 2r)$ with respect to $X^b$ and vanishes in $D^c \cap B(z_0, 2r)$,*

$$|\partial_{x_d} f(x)| \geq C_{17} \frac{f(x)}{\delta_D(x)} \quad \text{for } x \in D \cap B(z_0, r_3).$$

*Proof.* Without loss of generality we assume $z_0 = 0$. For $r \in (0, r_2]$, fix an arbitrary $x \in D \cap B(0, r/(2\sqrt{1+\lambda_0^2}))$. Let $z_x \in \partial D$ be such that $|x - z_x| = \rho(x)$. Define $c := \lim_{D \ni y \to z_x} f(y)/g_{b,r}(y)$ and $u(y) := c\, g_{b,r}(y)$. Obviously $B(z_x, 3r/2) \subset B(0, 2r)$, and thus $f$, $u$ are harmonic in $D \cap B(z_x, 3r/2)$ and vanish in $D^c \cap B(z_x, 3r/2)$. Since $\lim_{D \ni y \to z_x} u(y)/f(y) = \lim_{D \ni y \to z_x} f(y)/u(y) = 1$, by Lemma 3.7, for any $y \in D \cap B(z_x, 3r/4)$,

$$\left| \frac{u(y)}{f(y)} - 1 \right| \vee \left| \frac{f(y)}{u(y)} - 1 \right| \leq c_1 \left( \frac{|y - z_x|}{r} \right)^{\gamma_1} \leq c_2 \tag{5.14}$$

for some positive constants $c_i = c_i(d, \alpha, \beta, \lambda_0, M_1, M_2)$, $i = 1, 2$. Consequently,

$$(1+c_2)^{-1} f(y) \le u(y) \le (1+c_2) f(y) \quad \text{for } y \in D \cap B(z_x, 3r/4). \tag{5.15}$$

Note that $x \in D \cap B(z_x, 3r/4)$ since $\rho(x) \le \delta_D(x)\sqrt{1+\lambda_0^2} \le r/2$. By Lemma 5.3 and (5.15), we have

$$\begin{aligned} \partial_{x_d} f(x) &\ge \partial_{x_d} u(x) - |\partial_{x_d}(f-u)(x)| \ge c_3 \frac{u(x)}{\delta_D(x)} - |\partial_{x_d}(f-u)(x)| \\ &\ge c_4 \frac{f(x)}{\delta_D(x)} - |\partial_{x_d}(f-u)(x)|. \end{aligned} \tag{5.16}$$

We assume $\rho(x) < 3r/128$. Set $v(y) := f(y) - u(y)$ and $\xi := 2\rho(x)$. Let $\eta \in (16\rho(x), 3r/8)$ to be specified later. In the rest of this proof, we set $D_1 := D^+(z_x, \xi, \xi)$ and $D_2 := D^+(z_x, \eta, \eta)$. Then $B(x, \delta_D(x)) \subset D_1 \subset D_2 \subset D \cap B(z_x, 3r/4)$ and $\delta_D(x) = \delta_{D_1}(x)$. Define $V(y) := \mathbb{E}_y\left[|v|(X^b_{\tau_{D_1}})\right]$. Clearly $V$ is regular harmonic in $D_1$ with respect to $X^b$ and $|v(y)| \le V(y)$ for all $y \in \mathbb{R}^d$. By Theorem 1.4, we have

$$|\partial_{x_d} v(x)| \le |\partial_{x_d} V(x)| + |\partial_{x_d}(V - v)(x)| \le c_5 \frac{V(x)}{\delta_{D_1}(x)} = c_5 \frac{V(x)}{\delta_D(x)}. \tag{5.17}$$

We aim to estimate $V(x)$. Note that

$$\begin{aligned} V(x) &\le \mathbb{E}_x\left[|v|(X^b_{\tau_{D_1}}) : X^b_{\tau_{D_1}} \in D_2\right] + \mathbb{E}_x\left[f(X^b_{\tau_{D_1}}) : X^b_{\tau_{D_1}} \in D_2^c\right] + \mathbb{E}_x\left[u(X^b_{\tau_{D_1}}) : X^b_{\tau_{D_1}} \in D_2^c\right] \\ &=: I(x) + II(x) + III(x). \end{aligned}$$

By (5.14), for any $y \in D_2 \subset D \cap B(z_x, 2\eta) \subset D \cap B(z_x, 3r/4)$, we have

$$|v(y)| = f(y)\left|\frac{u(y)}{f(y)} - 1\right| \le c_1 f(y) \left(\frac{|y - z_x|}{r}\right)^{\gamma_1} \le c_6 \left(\frac{\eta}{r}\right)^{\gamma_1} f(y).$$

Thus

$$I(x) \le c_6 \left(\frac{\eta}{r}\right)^{\gamma_1} \mathbb{E}_x\left[f(X^b_{\tau_{D_1}})\right] = c_6 \left(\frac{\eta}{r}\right)^{\gamma_1} f(x). \tag{5.18}$$

Let $A_x \in D$ be such that $\rho(A_x) = |A_x - z_x| = \eta/16$. Define $D_3 := B(A_x, \eta/16\sqrt{1+\lambda_0^2})$. We observe that $D_3 \subset D \cap B(z_x, \eta/2) \subset D_2$ and $D_1 \subset D \cap B(z_x, \eta/4)$. For any $y \in D_2^c \cap \text{supp} f$ and $z \in D_1$, we have $|y - A_x| \ge 7\eta/16$, $|A_x - z| \le 5\eta/16$, and

$$|y - z| \ge |y - A_x| - |A_x - z| \ge \frac{2}{7}|y - A_x|. \tag{5.19}$$

If we let $\lambda := 1/\text{diam}(D_1)$, then $\|b_\lambda\|_\infty = \text{diam}(D_1)^{\alpha-\beta}\|b\|_\infty \le (8\rho(x))^{\alpha-\beta} M_1 \le M_1$. Thus by (2.8) and Lemma 3.3, we have

$$G^b_{D_1}(x, z) = \lambda^{d-\alpha} G^{b_\lambda}_{\lambda D}(\lambda x, \lambda z) \le c_7 |x - z|^{\alpha - d}, \quad z \in D_1 \tag{5.20}$$

for some constant $c_7 = c_7(d, \alpha, \beta, M_1) > 0$. So by (5.19) and (5.20), for any $y \in D_2^c \cap \text{supp} f$,

$$K^b_{D_1}(x, y) = \int_{D_1} G^b_{D_1}(x, z) J^b(z, y) dz \le c_7 M_2 \int_{D_1} |x - z|^{\alpha - d} J^{\varepsilon_0}(|y - z|) dz$$

$$\lesssim \ c_7 M_2 J^{\varepsilon_0}(|y-A_x|) \int_{B(z_x,2\xi)} |x-z|^{\alpha-d} dz \lesssim c_7 M_2 \xi^\alpha J^{\varepsilon_0}(|y-A_x|). \qquad (5.21)$$

On the other hand for any $y \in D_2^c \cap \mathrm{supp} f$ and $z \in D_3$, we have $|y-z| \le |y-A_x| + |A_x - z| \le 12|y-A_x|/7$. Thus by Lemma 3.2 and (1.10)

$$\begin{aligned}
K^b_{D_3}(A_x, y) &= \int_{D_3} G^b_{D_3}(A_x, z) J^b(z, y) dz \\
&\gtrsim M_2^{-1} \left( \int_{D_3} G_{D_3}(A_x, z) dz \right) J^{\varepsilon_0}(|y-A_x|) \\
&\asymp M_2^{-1} \eta^\alpha J^{\varepsilon_0}(|y-A_x|). \qquad (5.22)
\end{aligned}$$

Combining (5.21) and (5.22), we have

$$K^b_{D_1}(x, y) \lesssim \frac{\xi^\alpha}{\eta^\alpha} K^b_{D_3}(A_x, y) \quad \text{for } y \in D_2^c \cap \mathrm{supp} f. \qquad (5.23)$$

Consequently, by (5.23), Lemma 5.1 and (5.1), we have

$$\begin{aligned}
II(x) &= \int_{D_2^c \cap \mathrm{supp} f} f(y) K^b_{D_1}(x, y) dy \lesssim \frac{\xi^\alpha}{\eta^\alpha} \int_{D_2^c \cap \mathrm{supp} f} f(y) K^b_{D_3}(A_x, y) dy \\
&\le \frac{\xi^\alpha}{\eta^\alpha} \int_{D_3^c} f(y) K^b_{D_3}(A_x, y) dy \ = \ \frac{\xi^\alpha}{\eta^\alpha} f(A_x) \\
&\lesssim \frac{\xi^\alpha}{\eta^\alpha} \frac{\rho(A_x)^{\alpha-\gamma_2}}{\rho(x)^{\alpha-\gamma_2}} f(x) \asymp \frac{\rho(x)^{\gamma_2}}{\eta^{\gamma_2}} f(x). \qquad (5.24)
\end{aligned}$$

Similarly we can prove that

$$III(x) \lesssim \frac{\rho(x)^{\gamma_2}}{\eta^{\gamma_2}} u(x) \lesssim \frac{\rho(x)^{\gamma_2}}{\eta^{\gamma_2}} f(x). \qquad (5.25)$$

Combining (5.18), (5.24) and (5.25), we have

$$V(x) \le \left( c_6 \left(\frac{\eta}{r}\right)^{\gamma_1} + c_8 \left(\frac{\rho(x)}{\eta}\right)^{\gamma_2} \right) f(x). \qquad (5.26)$$

Thus by (5.16), (5.17) and (5.26) we have

$$\partial_{x_d} f(x) \ge \left( c_4 - c_5 \left( c_6 \frac{\eta^{\gamma_1}}{r^{\gamma_1}} + c_8 \frac{\rho(x)^{\gamma_2}}{\eta^{\gamma_2}} \right) \right) \frac{f(x)}{\delta_D(x)}. \qquad (5.27)$$

Let $\eta = 16\rho(x)^{\gamma_2/(\gamma_1+\gamma_2)}$. The lemma now follows from (5.27) and (5.1) provided we choose $r_3$ small enough such that $c_5 \left(c_6 16^{\gamma_1} r^{-\gamma_1} + c_8 16^{-\gamma_2}\right) \left(r_3(1+\lambda_0)\right)^{\gamma_1\gamma_2/(\gamma_1+\gamma_2)} \le c_4/2$. □

**Proof of Theorem 1.5:** The upper bound in (1.14) was established more generally in Theorem 1.4. The lower bound follows from Lemma 5.4 and the inequality $|\nabla f| \ge |\partial_{x_d} f|$. □

**Remark 5.5.** Taking $b(x, z) = \varepsilon \in (0, M_2]$ in Assumption 1, we get from Theorem 1.4 and Theorem 1.7 the uniform gradient estimate for mixed stable processes, which in particular recovers a main result of [16] on gradient estimates.

## 6 Examples

In this section, we give some concrete examples where Assumptions 1 and 3 hold.

**Example 6.1.** If $b(x, z) = 1_{\{|z| \le c_1\}}$ for some $c_1 > 0$, the jumping kernel of the corresponding Feller process $X^b$ is

$$J^b(x, y) = \frac{\mathcal{A}(d, -\alpha)}{|x - y|^{d+\alpha}} + \frac{\mathcal{A}(d, -\beta)}{|x - y|^{d+\beta}} 1_{\{|x-y| \le c_1\}}.$$

In this case $X^b$ is the independent sum of a symmetric $\alpha$-stable process and a truncated symmetric $\beta$-stable process, and Assumptions 1 and 3 hold with $\varepsilon_0 = 0$ and $\psi(r) = 1_{\{r \le c_1\}}$, respectively.

More generally, suppose $b(x, z) = b_1(x, z) 1_{\{|z| \le c_1\}}$ for some $c_1 > 0$ and a bounded function $b_1(x, z)$ on $\mathbb{R}^d \times \mathbb{R}^d$ that is symmetric in $z$ and is bounded between two positive constants. Then Assumption 1 holds with $\varepsilon_0 = 0$. □

**Example 6.2.** If $b(x, z) = 1 + \frac{\mathcal{A}(d, -\gamma)}{\mathcal{A}(d, -\beta)} |z|^{\beta-\gamma} 1_{\{|z| \le c_2\}}$ for some $c_2 > 0$ and $0 < \gamma < \beta$, the jumping kernel of the corresponding Feller process $X^b$ is

$$J^b(x, y) = \frac{\mathcal{A}(d, -\alpha)}{|x - y|^{d+\alpha}} + \frac{\mathcal{A}(d, -\beta)}{|x - y|^{d+\beta}} + \frac{\mathcal{A}(d, -\gamma)}{|x - y|^{d+\gamma}} 1_{\{|x-y| \le c_2\}}.$$

In this case $X^b$ is the independent sum of a mixed-stable process and a truncated symmetric $\gamma$-stable process, and Assumptions 1 and 3 hold with $\varepsilon_0 = 1$ and $\psi(r) = 1 + \frac{\mathcal{A}(d, -\gamma)}{\mathcal{A}(d, -\beta)} r^{\beta-\gamma} 1_{\{r \le c_2\}}$, respectively.

More generally, suppose $b(x, z)$ is a bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ that is symmetric in $z$ and is bounded between two positive constants. Then Assumption 1 holds with $\varepsilon_0 = 1$. □

**Example 6.3.** We consider the following stochastic differential equation on $\mathbb{R}^d$:

$$dX_t = dY_t + C(X_{t-}) dZ_t, \tag{6.1}$$

where $Y$ is a symmetric $\alpha$-stable process, $Z$ is an independent $\beta$-stable process with $0 < \beta < \alpha$, and $C$ is a bounded Lipschitz function on $\mathbb{R}^d$. Using Picard's iteration method, one can show that for every $x \in \mathbb{R}^d$, SDE (6.1) has a unique strong solution with $X_0 = x$. The collection of the solutions $(X_t, \mathbb{P}_x, x \in \mathbb{R}^d)$ forms a strong Markov process $X$ on $\mathbb{R}^d$. Using Ito's formula, one concludes that the infinitesimal generator of $X$ is $\mathcal{L}^b$ with $b(x, z) = |C(x)|^\beta$. If there exists $c_3 > 0$ such that $|C(x)| \ge c_3$ for $x \in \mathbb{R}^d$, then our Assumption 1 holds with $\varepsilon_0 = 1$. □

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