Shuffle of min's random variable approximations of bivariate copulas'realization

Yanting Zheng * Jingping Yang † Jianhua Z. Huang ‡

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Abstract

The comonotonicity and countermonotonicity provide intuitive upper and lower dependence relationship between random variables. This paper constructs the shuffle of min's random variable approximations for a given Uniform $[0,1]$ random vector. We find the two optimal orders under which the shuffle of min's random variable approximations obtained are shown to be extensions of comonotonicity and countermonotonicity. We also provide the rate of convergence of these random vectors approximations and apply them to compute Value-at-Risk.
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1 Introduction

A bivariate copula is a joint distribution on the unit square with Uniform [0,1] marginal distributions. Sklar's Theorem (1959) states that for any bivariate distribution H with margins F,G, there exists a copula C such that $H(x,y) = C(F(x),G(y)), \forall x,y \in \mathbb{R}$. Any bivariate copula $C(u,v)$ has a lower and an upper bound, that is,

$$W(u,v) \leq C(u,v) \leq M(u,v), \qquad \forall u,v \in [0,1], \tag{1.1}$$

where the copula $M(u,v) = \min(u,v), u,v \in [0,1]$ is called Fréchet upper copula, and $W(u,v) = \max(u+v-1,0), u,v \in [0,1]$ is called Fréchet lower copula. It is known that the Fréchet upper copula $M(u,v)$ corresponds to the dependence structure called comonotonicity, and the Fréchet lower copula $W(u,v)$ corresponds to the countermonotonicity. Here, a pair of random variables (X,Y) is called comonotonic if there exists a random variable Z and two non-decreasing functions f and g such that $X = f(Z)$ and $Y = g(Z)$, and (X,Y) is called countermonotonic if $(X,-Y)$ is comonotonic; see [7], [8] and [20].

In fact, for practical applications, it is necessary to find simple approximations for copulas that can easily be simulated, in particular for applications in the finance and insurance sectors.

*Department of Finance, Beijing Technology and Business University, Beijing, 100048, China. Email: zhengyanting@th.btbu.edu.cn

†LMEQF, Department of Financial Mathematics, Center for Statistical Science, Peking University, Beijing, 100871, China. Email: yangjp@math.pku.edu.cn

‡Department of Statistics, Texas A&M University. Email: jianhua@stat.tamu.edu

The simple dependent structures of Fréchet upper bound and lower bound copulas can be applied in copula approximations. Such approximations are useful for understanding the dependence structure of complicated copulas and for simplifying computation. For example, the checkmin approximation makes use of the Fréchet upper copula $M(u,v)$ to approximate the original copula; both the checkerboard approximation and Bernstein approximation use the independence copula $\Pi(u,v) = uv, u,v \in [0,1]$ in their copula approximation (see [1] and [16]); [21] constructed a Patched Bivariate Fréchet copula approximation by making use of the linear combination of $M(u,v)$, $\Pi(u,v)$ and $W(u,v)$. See also [12] for the importance of constructing new families of copulas having properties desirable for specific applications.

In this paper, given the target random vector, we will construct a series of random vector approximations by a grid-type copula approximation–shuffle of min approximations. Here 'Shuffle' means a special construction method for new copulas ([9], [11] and [15]) and 'min' represents the Fréchet upper bound copula $M(u,v)$, corresponding to the comonotonic structure among random variables. Please see Chapter 3 of [17] for a detailed discussion of shuffle of min copulas. Under the framework of stochastic measure theory, [15] showed that the approximation error of shuffle of min is $\frac{4}{m}$ under the L_∞ norm; [7] and [8] extended the 'Shuffle' method to obtain shuffle of copula approximations. By extensions of bivariate subcopulas, [2] obtained the general maximum and minimum shuffle of min extensions for the bivariate copula. The purpose of this paper is to characterize the specific random variable expressions of the shuffle of min approximations which is convenient for simulation and financial applications. We use this characterization to re-obtain the shuffle of min copulas as sharp lower and upper bounds for the original copula. The distance of our shuffle of min bounds to the original copula is shown to be $\frac{2}{m}$ under the L_∞ norm.

The rest of the paper is organized as follows. Section 2 gives the preparation works including the definition of bivariate shuffle of min and its probabilistic structures. Section 3 constructs the random vector approximations for a given bivariate copula by shuffle of min under two specific partition orders that provide the narrow bounds of the original copula, and also sharpens their convergent rates. Applications in computation of Value-at-Risk is shown in Section 4.

2 Preparation works for random variable approximations by bivariate shuffle of min

By stochastic measure theory, [10] defined the generalized shuffling copula which includes shuffle of min as a special case; [15] showed that there is a strongly piecewise monotonic function connecting any two of random variables whose joint distribution is a shuffle of min. Based on [10] and [15], this section writes out the closed-form random variable expressions of bivariate shuffle of min to prepare for discussions of its random variable approximations in Section 3.

Denote the unit square as $I^2 = I^{(1)} \times I^{(2)}$, where $I^{(i)} = [0,1]$, $i = 1,2$, and the superscript $^{(i)}$ means the i-th dimensional interval of I^2. Let $\mu([a,b])$ be the length of an interval $[a,b]$. First, we introduce the definition of the shuffling structure in [10] for easy explanation. A system $\{\mathscr{J}^i\}_{i=1}^2$ is called shuffling structure, if $\mathscr{J}^1$, $\mathscr{J}^2$ are systems of closed and non-empty intervals of $I^{(1)}$, $I^{(2)}$, and $\mathscr{J}^i = \{J_j^{(i)} = [\underline{J}_j^{(i)}, \overline{J}_j^{(i)}]\}_{j=1}^m$ satisfies

(P1) for every $i \in \{1,2\}$ and $h,k \in \{1,2,\ldots,m\}$, $J_h^{(i)}$ and $J_k^{(i)}$ have at most one endpoint in common;

(P2) $\sum_{j=1}^{m}\mu(J_j^{(1)})=\sum_{j=1}^{m}\mu(J_j^{(2)})=1$;

(P3) for every $k\in\{1,2,\ldots,m\}$, $\mu(J_k^{(1)})=\mu(J_k^{(2)})$.

Based on the shuffling structure above, different shuffling copulas can be obtained by assuming different constructing copulas (see [10]). Here, we focus one specific shuffling copula–shuffle of min, and give its random variable expressions in detail. To facilitate our discussion, we define an interval ordering.

Definition 2.1. *If two intervals $[a,b]$ and $[c,d]$ satisfy $a<b\leq c<d$, then we say the interval $[a,b]$ is ahead of $[c,d]$, denoted as $[a,b]\leqq[c,d]$; if $a<b<c<d$, then we say the interval $[a,b]$ is strictly ahead of $[c,d]$, denoted as $[a,b]\prec[c,d]$.*

Given the shuffling structure system $\{\mathscr{J}^i\}_{i=1}^2$, we use the permutations $\{\sigma_i\}_{i=1}^2$ on $\{1,2,\ldots,m\}$ to represent their respective partition orders, i.e.

$$J^{(i)}_{\sigma_i(1)}\leqq J^{(i)}_{\sigma_i(2)}\leqq\cdots\leqq J^{(i)}_{\sigma_i(m)},\ \forall i=1,2.$$

Note that $J_j^{(1)}\times J_j^{(2)}$, $\forall j\in\{1,2,\ldots,m\}$ is a sub-cube with length $\mu(J_j^{(1)})$ in I^2. However, the endpoints $\overline{J}_j^{(1)}\neq\overline{J}_j^{(2)}$ and $\underline{J}_j^{(1)}\neq\underline{J}_j^{(2)}$ as the respective partition orders are different. For distinction, we write $\underline{J}_j^{(i)}:=\underline{J}_j^{(i)}(\varpi)$ and $\overline{J}_j^{(i)}:=\overline{J}_j^{(i)}(\varpi)$ for $i=1,2$, $j\in\{1,2,\ldots,m\}$. The partition order ϖ plays a key role in the properties of shuffle of min approximations of a given copula and we will further discuss it in Section 3.

A copula function C is a shuffle of min if for the shuffling structure $\{\mathscr{J}^i\}_{i=1}^2$ and the permutation sets $\{\sigma_i\}_{i=1}^2$ above, it spreads the mass $\mu(J_j^{(1)})$ uniformly along one of the diagonals of the sub-square $J_j^{(1)}\times J_j^{(2)}$, $\forall j=1,2,\ldots,m$. For each $j=1,2,\ldots,m$, let ω_j denote the slope of the diagonal of $J_j^{(1)}\times J_j^{(2)}$. If $\omega_j\equiv 1$, C is called a straight shuffle of min. If $\omega_j\equiv -1$, C is called a flipped shuffle of min. Let $\varpi:=\{\sigma_i,i=1,2\}$ and $\omega:=\{\omega_j,j=1,2,\ldots,m\}$. The copula C is said to be a shuffle of min generated by $\big(\{\mathscr{J}^i\}_{i=1}^2,\varpi,\omega\big)$.

Now we give the random variable representation of the shuffle of min. Suppose S_1 and S_2 are the uniform $[0,1]$ random variables whose joint distribution is the shuffle of min C generated by $\big(\{\mathscr{J}^i\}_{i=1}^2,\varpi,\omega\big)$. According to the construction of C, the relationship between S_1 and S_2 on each sub-square $J_i^{(1)}\times J_i^{(2)}$ is linear; specifically,

$$\begin{aligned}S_2=&\big[S_1\sum_{i=1}^{m}\mathbb{I}_{\{S_1\in J_i^{(1)},\omega_i=1\}}-\sum_{i=1}^{m}(\underline{J}_i^{(1)}(\varpi)-\underline{J}_i^{(2)}(\varpi))\mathbb{I}_{\{S_1\in J_i^{(1)},\omega_i=1\}}\big]\\&+\big[\sum_{i=1}^{m}(\overline{J}_i^{(2)}(\varpi)+\underline{J}_i^{(1)}(\varpi))\mathbb{I}_{\{S_1\in J_i^{(1)},\omega_i=-1\}}-S_1\sum_{i=1}^{m}\mathbb{I}_{\{S_1\in J_i^{(1)},\omega_i=-1\}}\big]\\=:&\ f_m(S_1|\varpi,\omega).\end{aligned}\tag{2.1}$$

where $\mathbb{I}$ is the indicator function. We obtain different functions between S_1 and S_2 by setting different m, ϖ and ω. Particularly, we denote $\varpi_{straight}=\{\sigma_1(i)=\sigma_2(i)=i,i=1,2,\ldots,m\}$ and $\varpi_{flipped}=\{\sigma_1(i)=\sigma_2(i)=m-i,i=1,2,\ldots,m\}$ as the straight and flipped orders. Given $m>1$, by equation (2.1), if $\omega_i=1$, $\forall i=1,2,\ldots,m$, then we have $S_1=S_2$ under straight order $\varpi_{straight}$, which means (S_1, S_2) is comonotonicity; If $\omega_i = -1, \forall i = 1, 2, \ldots, m$, then $S_1 = -S_2$ under flipped order $\varpi_{flipped}$, which means (S_1, S_2) is countermonotonicity. Thus, the straight and flipped bivariate shuffle of min under straight order and flipped order can be regarded as a generalization of Fréchet copulas corresponding to comonotonicity and countermonotonicity. We also note that the random variable representation of given in (2.1) is convenient to use for generating random numbers for a shuffle of min. Application of using such random numbers in financial risk management will be illustrated in Section 4.

By the random variable expressions in (2.1), the joint distribution C has the expression

$$P(S_1 \le u_1, S_2 \le u_2) = \sum_{i=1}^{m} \max\big\{ \min\{(u_1 + u_2 - \overline{J}_i^{(2)}(\varpi) - \underline{J}_i^{(1)}(\varpi))\mathbb{I}_{\{\omega_i=-1\}}, \\ u_1 - \underline{J}_i^{(1)}(\varpi), u_2 - \underline{J}_i^{(2)}(\varpi), \overline{J}_i^{(1)}(\varpi) - \underline{J}_i^{(1)}(\varpi)\}, 0\big\}. \quad (2.2)$$

Remark 2.1. *Like discussions of the bivariate case, it is easy to obtain the random variable expressions in multivariate dimensions. That is, the random variable expressions $(S_1, \ldots, S_d)$ of the d dimensional shuffle of min C, generated by $\big(\{\mathscr{J}^i\}_{i=1}^d, \{\sigma_i\}_{i=1}^d, \{\omega_j(h,k), h,k = 1,2,\ldots,d\}_{j=1}^m\big)$, are*

$$S_k = \sum_{i=1}^{m} \big[(S_h - \underline{J}_i^{(h)} + \underline{J}_i^{(k)})\mathbb{I}_{\{\omega_i(h,k)=1\}} + (\overline{J}_i^{(k)} - S_h + \underline{J}_i^{(h)})\mathbb{I}_{\{\omega_i(h,k)=-1\}}\big]\mathbb{I}_{\big\{S_h \in J_i^{(h)}\big\}},$$

where for any $h \neq k$, $h,k = 1,2,\ldots,d$, $\omega_j(h,k) = 1 (or -1)$ means the comonotonicity (or countermonotonicity) between S_h and S_k in sub-square $J_j^{(h)} \times J_j^{(k)}$.

3 Random vector approximation by shuffle of min for a given random vector

In this section, for a given random vector (U, V) with Uniform $[0,1]$ margins, we denote their joint distribution as C and then construct a sequence of random vectors that their joint distributions are shuffle of min approximations of C. Based on the ideas in [16], we apply a two-step partition scheme to construct the random vectors of the shuffle of min approximations for C. Given an integer m, we partition the interval $I^{(i)} = [0,1], i = 1,2$ equally into m subintervals, and obtain m^2 sub-squares

$$\big\{I_i^{(1)} \times I_j^{(2)}, i,j = 0,1,\ldots,m-1\big\},$$

where $I_0^{(i)} = [0, 1/m]$ is a close interval and $I_j^{(i)} = (j/m, (j+1)/m]$, $i = 1,2$, $j = 0,1,\ldots,m-1$. By Definition 2.1, the orders of the subintervals $\{I_i^{(1)}\}$ and $\{I_j^{(2)}\}$ are both with no shuffling, i.e. $I_0^{(i)} \leqq I_1^{(i)} \leqq \cdots \leqq I_{m-1}^{(i)}$, $i = 1,2$. This is our first step partition.

Writing $A_{i,j} = \{(U,V) \in I_i^{(1)} \times I_j^{(2)}\}$, $i,j = 0,1,\ldots,m-1$, then $A_{i,j}, 0 \le i,j \le m-1$ is a partition of the probability space. Considering the cell probability $P(A_{i,j}) = P(U \in I_i^{(1)}, V \in I_j^{(2)})$ for fixed

$i,j\in\{0,1,\ldots,m-1\}$, it is easy to see that

$$\sum_{k=0}^{m-1}P(A_{i,k})=P(U\in I_i^{(1)})=\frac{1}{m}\quad\text{and}\quad\sum_{l=0}^{m-1}P(A_{l,j})=P(V\in I_j^{(2)})=\frac{1}{m}.\tag{3.1}$$

In the second step, for each $I_i^{(1)}\times I_j^{(2)}, i,j\in\{0,1,\ldots,m-1\}$, we use $P(A_{i,k})$, $k=0,1,\ldots,m-1$ to partition $I_i^{(1)}$ and $P(A_{l,j})$, $l=0,1,\ldots,m-1$ to partition $I_j^{(2)}$. Specifically, for the subinterval $I_i^{(1)}$, we find one partition $I_{i,k}^{(1)}=:(\underline{I}_{i,k}^{(1)},\overline{I}_{i,k}^{(1)}]$, $k=0,1,\ldots,m-1$ such that $\bigcup_{0\leq k\leq m-1}I_{i,k}^{(1)}=I_i^{(1)}$ and $\mu(I_{i,k}^{(1)})=P(A_{i,k})$. To obtain the end points $\underline{I}_{i,k}^{(1)}$, $k=0,1,\ldots,m-1$, we use a permutation $\sigma_i^{(1)}$ on $\{0,1,\ldots,m-1\}$ to order $I_{i,k}^{(1)}$, $k=0,1,\ldots,m-1$ as

$$I^{(1)}_{i,\sigma_i^{(1)}(0)}\leqq I^{(1)}_{i,\sigma_i^{(1)}(1)}\leqq\cdots\leqq I^{(1)}_{i,\sigma_i^{(1)}(m-1)}.\tag{3.2}$$

Similarly, for the subinterval $I_j^{(2)}$, we find its partition $I_{l,j}^{(2)}=:(\underline{I}_{l,j}^{(2)},\overline{I}_{l,j}^{(2)}]$, $l=0,1,\ldots,m-1$ such that $\bigcup_{0\leq l\leq m-1}I_{l,j}^{(2)}=I_j^{(2)}$ and $\mu(I_{l,j}^{(2)})=P(A_{l,j})$. Here, we use another permutation $\sigma_j^{(2)}$ on $\{0,1,\ldots,m-1\}$ to order $I_{l,j}^{(2)}$, $l=0,1,\ldots,m-1$ as

$$I^{(2)}_{\sigma_j^{(2)}(0),j}\leqq I^{(2)}_{\sigma_j^{(2)}(1),j}\leqq\cdots\leqq I^{(2)}_{\sigma_j^{(2)}(m-1),j}.\tag{3.3}$$

Note that for each sub-square $I_i^{(1)}\times I_j^{(2)}$, $I_{i,j}^{(1)}\times I_{i,j}^{(2)}\subset I_i^{(1)}\times I_j^{(2)}$ is also a sub-square with length $\mu(I_{i,j}^{(1)})=\mu(I_{i,j}^{(2)})=P(A_{i,j})$ which can be determined by the original random vector (U,V). Then it is easy to verify that $\{I_{i,j}^{(k)}, i,j=0,1,\ldots,m-1,k=1,2\}$ is a shuffling structure. In the following, we re-denote $\sigma=\{\sigma_j^{(i)}, i=1,2; j=0,1,\ldots,m-1\}$. From equation (3.2) and (3.3), σ plays an important role in the lower end points of $I_{i,j}^{(1)}$ and $I_{i,j}^{(2)}$. Particularly,

$$\underline{I}_{i,j}^{(1)}(\sigma)=\frac{i}{m}+\sum_{q=0}^{[\sigma_i^{(1)}]^{-1}(j)-1}P(A_{i,\sigma_i^{(1)}(q)});\ \ \underline{I}_{i,j}^{(2)}(\sigma)=\frac{j}{m}+\sum_{q=0}^{[\sigma_j^{(2)}]^{-1}(i)-1}P(A_{\sigma_j^{(2)}(q),j}),\tag{3.4}$$

where $[\sigma_i^{(k)}]^{-1}$ means the inverse function of $\sigma_i^{(k)}$ for $k=1,2$, $i=0,1,\ldots,m-1$.

Remark 3.1. *As in Section 2, the permutations are introduced to illustrate the partition order of subintervals. However, in the two-step construction of shuffle of min approximation, we make no shuffling in the first step partition, but use different permutations to order the subintervals of $I_i^{(1)}$ and $I_j^{(2)}$ in the second step partition.*

Re-denote $\omega=\{\omega_{i,j}, i,j=0,1,\ldots,m-1\}$. Based on the partitions of $I^{(1)}\times I^{(2)}$ under the two-step scheme, we can define the uniform $[0,1]$ random variable $f_m(U|\sigma,\omega)$ from the original random variable U as follows.

$$
\begin{aligned}
& f_m(U|\varpi,\omega) \\
&= \Bigg[U\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\mathbb{I}_{\{U\in I_{i,j}^{(1)},\omega_{i,j}=1\}}-\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}[\underline{I}_{i,j}^{(1)}(\varpi)-\underline{I}_{i,j}^{(2)}(\varpi)]\mathbb{I}_{\{U\in I_{i,j}^{(1)},\omega_{i,j}=1\}}\Bigg] \\
&\quad+\Bigg[\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}[\bar{I}_{i,j}^{(2)}(\varpi)+\underline{I}_{i,j}^{(1)}(\varpi)]\mathbb{I}_{\{U\in I_{i,j}^{(1)},\omega_{i,j}=-1\}}-U\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\mathbb{I}_{\{U\in I_{i,j}^{(1)},\omega_{i,j}=-1\}}\Bigg]
\end{aligned}
\tag{3.5}
$$

where $\mathbb{I}_{\{U\in I_{i,j}^{(1)},\omega_{i,j}=1\}}$ ($\mathbb{I}_{\{U\in I_{i,j}^{(1)},\omega_{i,j}=-1\}}$) is the indicator of a comonotonic (countermonotonic) relationship between U and $f_m(U|\varpi,\omega)$ on $I_{i,j}^{(1)}\times I_{i,j}^{(2)}$. By equation (3.5), it is easy to obtain the joint distribution of $(U,f_m(U|\varpi,\omega))$ as

$$
\begin{aligned}
& C_{SM}^{(m)}(u,v|\varpi,\omega) \\
&= \sum_{i=1}^{m-1}\sum_{j=0}^{m-1}\max\{\min\{[u+v-\bar{I}_{i,j}^{(2)}(\varpi)-\underline{I}_{i,j}^{(1)}(\varpi)]\mathbb{I}_{\{\omega_{i,j}=-1\}},u-\underline{I}_{i,j}^{(1)}(\varpi),v-\underline{I}_{i,j}^{(2)}(\varpi),P(A_{i,j})\},0\}.
\end{aligned}
\tag{3.6}
$$

Constructed from the original random vector (U,V) under the two-step scheme, $C_{SM}^{(m)}(u,v|\varpi,\omega)$ spreads the mass $P(A_{i,j})$ uniformly along one of the diagonals of the sub-square $I_{i,j}^{(1)}\times I_{i,j}^{(2)}$. Then $C_{SM}^{(m)}(u,v|\varpi,\omega)$ is a shuffle of min approximation generated by $\big\{\{I_{i,j}^{(k)}\}_{i,j=0}^{m-1},\varpi,\omega\big\}$ from discussions in Section 2. Based on the discussions in [15],[16], [10] and so on, $(U,f_m(U|\varpi,\omega))$ converges to (U,V) in distribution as m goes to infinity.

Denote the Spearman's rho of $(U,f_m(U|\varpi,\omega))$ as $\rho_{SM}(\varpi,\omega)$. By equation (3.5), we have

$$
\begin{aligned}
\rho_{SM}(\varpi,\omega)=\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\Bigg\{&\big[4-12\big(\underline{I}_{i,j}^{(1)}(\varpi)-\underline{I}_{i,j}^{(2)}(\varpi)\big)E[U\mathbb{I}_{\{U\in I_{i,j}^{(1)}\}}]\big]\mathbb{I}_{\{\omega_{i,j}=1\}} \\
&+\big[12\big(\bar{I}_{i,j}^{(2)}(\varpi)+\underline{I}_{i,j}^{(1)}(\varpi)\big)E[U\mathbb{I}_{\{U\in I_{i,j}^{(1)}\}}]-4\big]\mathbb{I}_{\{\omega_{i,j}=-1\}}\Bigg\}-3.
\end{aligned}
\tag{3.7}
$$

Let $\omega\equiv\mathbb{1}(-\mathbb{1})$ if $\omega_{i,j}=1(-1)$ for all $i,j\in\{0,1,\ldots,m-1\}$. Thus, $(U,f_m(U|\varpi,\mathbb{1}))$ is comonotonic and $(U,f_m(U|\varpi,-\mathbb{1}))$ is countermonotonic. Next, we find the optimal partition orders ϖ^* maximizing $\rho_{SM}(\varpi,\mathbb{1})$ and $\varpi^\#$ minimizing $\rho_{SM}(\varpi,-\mathbb{1})$. Suppose that $\varpi^*:=\{\sigma_i^{(1)*}(j),\sigma_i^{(2)*}(j),i,j=0,1,\ldots,m-1\}$ and $\varpi^\#:=\{\sigma_i^{(1)\#}(j),\sigma_i^{(2)\#}(j),i,j=0,1,\ldots,m-1\}$.

Theorem 3.1. *For the shuffle of min random variables $(U,f_m(U|\varpi,\omega))$ obtained in the two-step scheme, if $\varpi^*=\arg\max_\varpi\rho_{SM}(\varpi,\mathbb{1})$ and $\varpi^\#=\arg\min_\varpi\rho_{SM}(\varpi,-\mathbb{1})$, then for all $i,j\in\{0,1,\ldots,m-1\}$, we have $\sigma_i^{(1)*}(j)=\sigma_i^{(2)*}(j)=j$ and $\sigma_i^{(1)\#}(j)=\sigma_i^{(2)\#}(j)=m-j-1$.*

The proof of Theorem 3.1 is shown in the Appendix. For illustration, we call ϖ^* as the straight order and $\varpi^\#$ as the flipped order. By Theorem 3.1, under the straight order, equations (3.2) and

(3.3) become

$$I_{i,0}^{(1)} \leqq I_{i,1}^{(1)} \leqq \cdots \leqq I_{i,m-2}^{(1)} \leqq I_{i,m-1}^{(1)} \text{ and } I_{0,j}^{(2)} \leqq I_{1,j}^{(2)} \leqq \cdots \leqq I_{m-2,j}^{(2)} \leqq I_{m-1,j}^{(2)}.$$

Under the flipped order, equations (3.2) and (3.3) become

$$I_{i,m-1}^{(1)} \leqq I_{i,m-2}^{(1)} \leqq \cdots \leqq I_{i,1}^{(1)} \leqq I_{i,0}^{(1)} \text{ and } I_{m-1,j}^{(2)} \leqq I_{m-2,j}^{(2)} \leqq \cdots \leqq I_{1,j}^{(2)} \leqq I_{0,j}^{(2)}.$$

Based on Theorem 3.1, it is easy to obtain the properties of the partition intervals' end points in the following corollary.

Corollary 3.1. *For any $i,j = 1,2,\ldots,m$, under the straight order σ^*, we have $\overline{I}_{i,j-1}^{(1)}(\sigma^*) = \underline{I}_{i,j}^{(1)}(\sigma^*)$ and $\overline{I}_{i-1,j}^{(2)}(\sigma^*) = \underline{I}_{i,j}^{(2)}(\sigma^*)$; under the flipped order $\sigma^\#$, we have $\overline{I}_{i,j}^{(1)}(\sigma^\#) = \underline{I}_{i,j-1}^{(1)}(\sigma^\#)$ and $\overline{I}_{i,j}^{(2)}(\sigma^\#) = \underline{I}_{i-1,j}^{(2)}(\sigma^\#)$.*

Remark 3.2. *The order (left to right, bottom to top) mentioned in [2] which extends the bivariate sub-copula by blocks is essentially the same as the straight order. In our paper, we further obtain the flipped order and then show the probability properties of the straight and flipped orders.*

Denote $C_{SSM+}^{(m)}$ and $C_{FSM-}^{(m)}$ as the joint distributions of $(U, f_m(U|\sigma^*, \mathbb{1}))$ and $(U, f_m(U|\sigma^\#, -\mathbb{1}))$. Then by equation (3.6), we have

$$C_{SSM+}^{(m)}(u,v) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} \max\left\{\min\{u - \underline{I}_{i,j}^{(1)}(\sigma^*), v - \underline{I}_{i,j}^{(2)}(\sigma^*), P(A_{i,j})\}, 0\right\}$$

and

$$\begin{aligned} C_{FSM-}^{(m)}(u,v) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} \max\Big\{ \min\{&u + v - \overline{I}_{i,j}^{(2)}(\sigma^\#) - \underline{I}_{i,j}^{(1)}(\sigma^\#), \\ &u - \underline{I}_{i,j}^{(1)}(\sigma^\#), v - \underline{I}_{i,j}^{(2)}(\sigma^\#), P(A_{i,j})\}, 0\Big\}. \end{aligned}$$

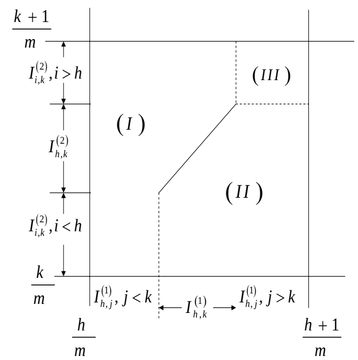
By [2], we know that $C_{SSM+}^{(m)}(u,v)$ and $C_{FSM-}^{(m)}(u,v)$ are lower and upper bounds of the original distribution C. In Theorem 3.1, we further gives the convergent rate of the two bounds.

Proposition 3.1. *For any random vector (U,V) with Uniform $[0,1]$ margins, given integer m, the shuffle of min approximations $C_{SSM+}^{(m)}$ under the straight order and $C_{FSM-}^{(m)}$ under the flipped order in the two-step partition of I^2 satisfy*

$$\sup_{(u,v)\in[0,1]^2} \left\{C_{SSM+}^{(m)}(u,v) - C(u,v)\right\} \leq \frac{2}{m} \quad \textit{and} \quad \sup_{(u,v)\in[0,1]^2} \left\{C(u,v) - C_{FSM-}^{(m)}(u,v)\right\} \leq \frac{2}{m}.$$

Proof. We only prove the results of $C_{SSM+}^{(m)}(u,v)$. The proof for $C_{FSM-}^{(m)}(u,v)$ is similar. For fixed $(u,v) \in [0,1]^2$, there exists h,k such that $(u,v) \in I_h^{(1)} \times I_k^{(2)} = (\frac{h}{m}, \frac{h+1}{m}] \times (\frac{k}{m}, \frac{k+1}{m}]$. Inspired by [2], we divide the probability support of $C_{SSM+}^{(m)}$ in $I_h^{(1)} \times I_k^{(2)}$ into three sub-regions which are marked

Figure 1: The subregions of $I_h^{(1)} \times I_k^{(2)}$ under the two-step partitions, where $I_{h,k}^{(1)} \times I_{h,k}^{(2)}$ is the sub-square. The straight shuffle of min approximation $C_{SSM+}^{(m)}(u,v)$ distributes the probability $P(A_{h,k})$ uniformly on the main diagonal of $I_{h,k}^{(1)} \times I_{h,k}^{(2)}$.



in Figure 1. We will compare $C_{SSM+}^{(m)}$ with C in each region. First, consider the case that (u,v) is in the sub-region (I). We have

$$C_{SSM+}^{(m)}(u,v) = u - \frac{h}{m} + C_{SSM+}^{(m)}(\frac{h}{m}, v) = \begin{cases} u - \frac{h}{m} + v - \frac{k}{m} + C(\frac{h}{m}, \frac{k}{m}), & \frac{k}{m} \le v \le \frac{k}{m} + \lambda_1 \\ u - \frac{h}{m} + C(\frac{h}{m}, \frac{k+1}{m}) & \frac{k}{m} + \lambda_1 \le v \le \frac{k+1}{m}, \end{cases}$$

where $\lambda_1 = C(\frac{h}{m}, \frac{k+1}{m}) - C(\frac{h}{m}, \frac{k}{m})$ (see [2]). Then

$$\begin{aligned} C_{SSM+}^{(m)}(u,v) - C(u,v) &= \begin{cases} P(\frac{h}{m} \le U \le u, V \ge v) + P(U \ge u, \frac{k}{m} \le V \le v), & \frac{k}{m} \le v \le \frac{k}{m} + \lambda_1 \\ P(U \le \frac{h+1}{m}, v \le V \le \frac{k+1}{m}) + P(\frac{h}{m} \le U \le u, V \ge v), & \frac{k}{m} + \lambda_1 \le v \le \frac{k+1}{m} \end{cases} \\ &\le \begin{cases} u - \frac{h}{m} + v - \frac{k}{m}, & \frac{k}{m} \le v \le \frac{k}{m} + \lambda_1 \\ \frac{k+1}{m} - v + u - \frac{h}{m}, & \frac{k}{m} + \lambda_1 \le v \le \frac{k+1}{m} \end{cases} \\ &\le \frac{2}{m} \end{aligned}$$

Next, consider the case that (u,v) is in the sub-region (II). We have

$$C_{SSM+}^{(m)}(u,v) = v - \frac{k}{m} + C_{SSM+}^{(m)}(u, \frac{k}{m}) = \begin{cases} u - \frac{h}{m} + v - \frac{k}{m} + C(\frac{h}{m}, \frac{k}{m}), & \frac{h}{m} \le v \le \frac{h}{m} + \lambda_2 \\ v - \frac{k}{m} + C(\frac{h+1}{m}, \frac{k}{m}) & \frac{h}{m} + \lambda_2 \le v \le \frac{h+1}{m}, \end{cases}$$

where $\lambda_2 = C(\frac{h+1}{m}, \frac{k}{m}) - C(\frac{h}{m}, \frac{k}{m})$. Similarly, we have

$$\begin{aligned} C_{SSM+}^{(m)}(u,v) - C(u,v) &= \begin{cases} P(\frac{h}{m} \le U \le u, V \ge v) + P(U \ge u, \frac{k}{m} \le V \le v), & \frac{h}{m} \le u \le \frac{h}{m} + \lambda_2 \\ P(u \le U \le \frac{h+1}{m}, V \le \frac{k+1}{m}) + P(U \ge u, \frac{k}{m} \le V \le v), & \frac{h}{m} + \lambda_2 \le u \le \frac{h+1}{m} \end{cases} \\ &\le \frac{2}{m} \end{aligned}$$

Finally, consider the case that (u,v) is in the sub-region (III). We have $C^{(m)}_{SSM+}(u,v) = C(\frac{h+1}{m}, \frac{k+1}{m})$. By the Lipschitz property of a copula, we have $C^{(m)}_{SSM+}(u,v) - C(u,v) = C(\frac{h+1}{m}, \frac{k+1}{m}) - C(u,v) \leq \frac{2}{m}$. Combining the above results, we conclude that $0 \leq C^{(m)}_{SSM+}(u,v) - C(u,v) \leq \frac{2}{m}$ for any $(u,v) \in I^2$. ◻

Remark 3.3. *[15] proved*

$$\sup_{(u,v)\in[0,1]^2} |C^{(m)}_{SM}(u,v) - C(u,v)| \leq \frac{4}{m}$$

using the fact that $C^{(m)}_{SM}(\frac{k}{m}, \frac{l}{m}) = C(\frac{k}{m}, \frac{l}{m}), k,l \in \{0,1,\ldots,m\}$. *Proposition 3.1 improves this result by providing a lower and an upper bound of* $C(u,v)$ *and sharpening the order of approximation.*

The following result gives the upper and lower bound of $E[g(U,V)]$ for a smooth function g.

Corollary 3.2. *Suppose that the measurable function* g *defined on* I^2 *has the second order partial derivatives* $\frac{\partial^2 g(u,v)}{\partial u \partial v}$. *Then we have*

$$\big|E[g(U,V)] - E[g(U, f_m(U|\sigma,\omega))]\big| \leq \frac{2}{m}\int_0^1\int_0^1 \big|\frac{\partial^2 g(s,t)}{\partial s \partial t}\big| dsdt, \tag{3.8}$$

where the random variable $f_m(U|\sigma,\omega)$ *represents either* $f_m(U|\sigma^*, \mathbb{1})$ *or* $f_m(U|\sigma^{\#}, -\mathbb{1})$. *Furthermore, if* g *is a convex function, then*

$$E[g(U, f_m(U|\sigma^{\#}, -\mathbb{1}))] \leq E[g(U,V)] \leq E[g(U, f_m(U|\sigma^*, \mathbb{1}))]. \tag{3.9}$$

Proof. Note that the measurable function g can be rewritten as

$$g(u,v) = \int_0^u\int_0^v \frac{\partial^2 g(s,t)}{\partial s \partial t} dsdt + g(0,v) + g(u,0) - g(0,0).$$

Thus,

$$\begin{aligned}
&E[g(U,V)] = \int_0^1\int_0^1 g(u,v)C(du,dv) \\
&= \int_0^1\int_0^1\int_0^u\int_0^v \frac{\partial^2 g(s,t)}{\partial s \partial t} dsdt\, C(du,dv) + \int_0^1 g(0,v)dv + \int_0^1 g(u,0)du - g(0,0) \\
&= \int_0^1\int_0^1\int_0^1\int_0^1 \frac{\partial^2 g(s,t)}{\partial s \partial t} I_{\{u\geq s, v\geq t\}} C(du,dv)\, dsdt + \int_0^1 g(0,v)dv + \int_0^1 g(u,0)du - g(0,0) \\
&= \int_0^1\int_0^1 \frac{\partial^2 g(s,t)}{\partial s \partial t} \overline{C}(s,t)dsdt + \int_0^1 g(0,v)dv + \int_0^1 g(u,0)du - g(0,0),
\end{aligned}$$

where $\overline{C}(s,t) = P(U > s, V > t) = 1 - s - t + C(s,t)$. Similarly, we have

$$E[g(U, f_m(U|\sigma^*, \mathbb{1}))] = \int_0^1\int_0^1 \frac{\partial^2 g(s,t)}{\partial s \partial t} \overline{C}^{(m)}_{SSM+}(s,t)\, dsdt + D$$

and

$$E[g(U,f_m(U|\varpi^{\#},-\mathbb{1})]=\int_0^1\int_0^1\frac{\partial^2 g(s,t)}{\partial s\partial t}\overline{C}^{(m)}_{FSM-}(s,t)\,dsdt+D,$$

where $D=:\int_0^1 g(0,v)dv+\int_0^1 g(u,0)du-g(0,0)$. Combining this with Proposition 3.1, we have

$$\begin{aligned}\big|E[g(U,V)]-E[g(U,f_m(U|\varpi^*,\mathbb{1}))]\big| &= \Big|\int_0^1\int_0^1\frac{\partial^2 g(s,t)}{\partial s\partial t}[\overline{C}(s,t)-\overline{C}^{(m)}_{SSM+}(s,t)]\,dsdt\Big|\\ &\le \frac{2}{m}\int_0^1\int_0^1\Big|\frac{\partial^2 g(s,t)}{\partial s\partial t}\Big|dsdt.\end{aligned}$$

We can argue similarly for $\big|E[g(U,V)]-E[g(U,f_m(U|\varpi^{\#},-\mathbb{1}))]\big|$. The proof of equation (3.8) is complete. If g is a convex function, i.e. $\frac{\partial^2 g(u,v)}{\partial u\partial v}\ge 0$, then by comparing the expressions of $E[g(U,V)]$, $E[[g(U,f_m(U|\varpi^*,\mathbb{1}))]$ and $E[g(U,f_m(U|\varpi^{\#},-\mathbb{1}))]$, equation (3.9) follows naturally. □

Remark 3.4. *Comparing with the discussions of bivariate shuffle of min approximations in [16] and [2], we directly construct the random vector approximations $(U,f_m(U|\varpi,\omega))$ from the original random vector (U,V) by a shuffle of min approximation, which makes it convenient for an application in simulation and finance.*

4 Applications

Value-at-risk (VaR) is a useful measure of risk in financial risk management. For any random variable X with distribution F, given the confidence level $\alpha\in(0,1)$, the $VaR_\alpha(X)$ is defined as

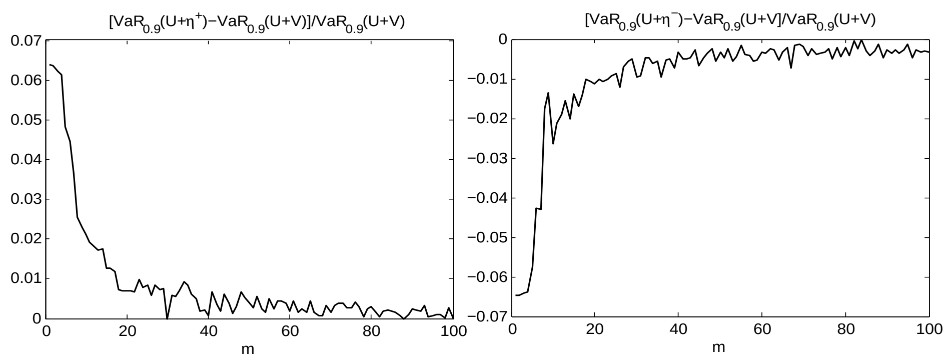
$$VaR_\alpha(X)=\inf\{t\le 0: F(t)\ge\alpha\}.$$

Calculation of the VaR for a portfolio usually involves modeling the dependence of asset returns using copulas (see [3], [5] and [13] etc). Since it is often difficult to have a closed-form expression of the VaR for a portfolio, simulation methods are widely used (see [4]). In the following example, we illustrate the use of the shuffle of min approximations for generating random numbers from a given copula and the application in computation of the VaR of a portfolio.

There are general algorithms available to simulate bivariate Archimedean copulas (as the Clayton copula), see e.g. [14], Chapter 2. Here we use a bivariate Clayton copula to show the approximation effect, from the approximation results we can see that the shuffle of min approximation can be used to compute the approximate VaR and the approximate errors decrease as m gets large. Note, however, that our use of bivariate Clayton copula is for illustration purpose. The proposed method is much more useful in cases when the original copula cannot be easily simulated, e.g., in the case of Archimedean copulas with an alpha-stable mixing distribution (see [14], Chapter 6).

Example 4.1. *Consider the bivariate Clayton copula $C(u,v)$ with parameter $\theta=5$ and let (U,V) be the random vector whose distribution is this copula. Given an integer m, we use (U,η^+) and (U,η^-) to denote the random vectors corresponding to the straight shuffle of min approximations $C^{(m)}_{SSM+}$ under straight order and the flipped shuffle of min approximation $C^{(m)}_{FSM-}$ under flipped* *order of $C(u,v)$. Note that the VaRs of $U+V$, $U+\eta^+$ and $U+\eta^-$ are the quantiles of the corresponding distributions. We can generate random numbers from these distributions and use the empirical quantiles to approximate the theoretical quantiles. As an illustration, we computed the VaRs at the level $\alpha = 0.9$, using $N = 10000$ random numbers from each distribution. Figure 2 shows the relative errors of $\frac{VaR_{0.9}(U+\eta^+)-VaR_{0.9}(U+V)}{VaR_{0.9}(U+V)}$ and $\frac{VaR_{0.9}(U+\eta^-)-VaR_{0.9}(U+V)}{VaR_{0.9}(U+V)}$ as a function of m averaged over 100 simulation runs. We observe from the figure that the shuffle of min approximation can be used to compute the approximate VaR and the relative errors decrease as m gets large.*

Figure 2: The relative errors of the VaR by using the shuffle of min approximations to generate random numbers from the Clayton copula.



Next, we consider the empirical shuffle of min approximation based directly on the data, which is the use of the empirical copula as a basis for an extension to a shuffle of min copula. This comes very close to ideas in the recent paper by [6] using rook and Bernstein copulas for practical applications.

Example 4.2. *For an easy illustration, we generated random numbers $(U_i,V_i), i = 1,\dots,N$, $N = 500$ from the bivariate Clayton copula with parameter $\theta = 6$ as our data. Given $m = 100$, the key to simulate random numbers from the empirical shuffle of min approximation is to find its random variable expressions in equation (3.5). The specific steps are given in the following.*

Algorithm

1. *For the data $(U_i,V_i), i = 1,\dots,N$, compute the cell probabilities as*

$$P^e(A_{i,j}) = \frac{1}{N}\sum_{k=1}^{N} \mathbb{I}_{\left\{\frac{i}{m}\le U_k\le \frac{i+1}{m}, \frac{j}{m}\le V_k\le \frac{j+1}{m}\right\}},\ i,j = 0,1,\dots,m-1.$$

2. *The cell probabilities $P^e(A_{i,j})$ may not satisfy the uniform margin properties, that is, we may have*

$$\sum_{i=0}^{m-1} P^e(A_{i,j}) \neq \frac{1}{m} \ \ \textit{and} \ \ \sum_{j=0}^{m-1} P^e(A_{i,j}) \neq \frac{1}{m}.$$

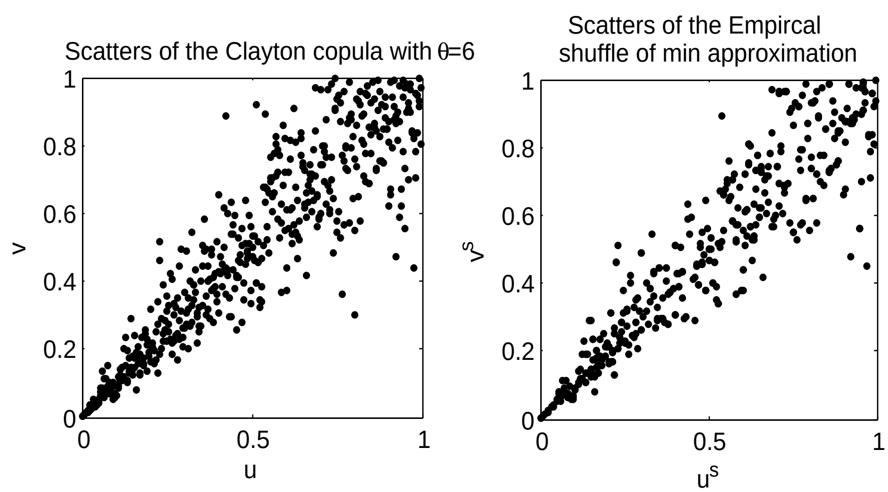
Thus it is not suitable to use $P^e(A_{i,j})$ for partitioning $I_i^{(1)} \times I_j^{(2)}$ directly. In order to construct an empirical shuffle of min approximation, we adjust the cell probabilities by solving the following minimization problem: (see [19]).

$$\min\{\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}[x_{i,j}-P^e(A_{i,j})]^2\} \quad \textit{such that} \quad 0 \le x_{i,j} \le 1, \sum_{i=0}^{m-1} x_{i,j} = \frac{1}{m}, \sum_{j=0}^{m-1} x_{i,j} = \frac{1}{m}.$$

From discussions in [19], we make use of their explicit (suboptimal) solution denoted as $x^*_{i,j} =: \frac{x_{i,j}+a}{1+m^2\times a}$, where $a =: -\min\{x_{i,j} | 1 \le i,j \le m\}$ and $x_{i,j} = P^e(A_{i,j}) - \frac{\sum_{i=1}^m P^e(A_{i,j})}{m} - \frac{\sum_{j=1}^m P^e(A_{i,j})}{m} + \frac{2}{m^2}$ (see the detailed discussions in [19]).
3. *According to equation* (3.4) *and the empirical probabilities $x^*_{i,j}$ obtained in step* 2*, we compute the end points of subintervals by the two-step partition under straight orders.*
4. *Simulate a uniform* $[0,1]$ *random number u^s and compute v^s by using* (3.5)*, then the random number pair (u^s, v^s) is from the empirical shuffle of min approximation.*

Using the algorithm above, we generated $N = 500$ random numbers (u_i^s, v_i^s) from the empirical shuffle of min approximation. We also performed the Kolmogorov-Smirnov test and obtained the p-value 0.2000. *The large p-value suggests that $\{(u_i, v_i)\}_{i=1}^N$ and $(u_i^s, v_i^s)_{i=1}^N$ can be statistically considered as coming from the same distribution. Figure 3 shows scatters of the original empirical data (u_i, v_i) and the random numbers (u_i^s, v_i^s) from the empirical shuffle of min approximation.*

Figure 3: scatters of the original data $\{(u_i, v_i), i = 1, 2, \ldots, N\}$ (left) and its empirical shuffle of min approximation random data $\{(u_i^s, v_i^s), i = 1, 2, \ldots, N\}$ (right)



Appendix

We first prove the following lemma before the proof of Theorem 3.1.

Lemma 4.1. *If* $\omega \equiv \mathbb{1}$, *then we have*

$$\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\big(\underline{I}^{(1)}_{i,j}(\sigma)-\underline{I}^{(2)}_{i,j}(\sigma)\big)E[U\mathbb{I}_{\{U\in I^{(1)}_{i,j}\}}]=\frac{1}{2}\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\big(\underline{I}^{(1)}_{i,j}(\sigma)-\underline{I}^{(2)}_{i,j}(\sigma)\big)^2P(A_{i,j}); \tag{A-1}$$

If $\omega \equiv -\mathbb{1}$, *then we have*

$$\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\big(\overline{I}^{(2)}_{i,j}(\sigma)+\underline{I}^{(1)}_{i,j}(\sigma)\big)E[U\mathbb{I}_{\{U\in I^{(1)}_{i,j}\}}]=\frac{1}{2}\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\big(\overline{I}^{(2)}_{i,j}(\sigma)+\underline{I}^{(1)}_{i,j}(\sigma)\big)^2P(A_{i,j}). \tag{A-2}$$

Proof. Based on the fact that $E[U^2]=E[f_m^2(U|\sigma,\omega)]$, it is easy to obtain equations (A-1) and (A-2) from equation (3.5). □

Proof of Theorem 3.1. We only give the proof of the case $\arg\max_\sigma \rho_{SM}(\sigma,\mathbb{1})$. It is similar to prove the case $\arg\min_\sigma \rho_{SM}(\sigma,-\mathbb{1})$. It is equal to show the inverse negative proposition. That is, we will prove that if there exists h,k such that $\hat{\sigma}^{(1)}_h(k)\neq k$, $\hat{\sigma}^{(1)}_i(j)=j$, $\forall i\neq h, j\neq k$ and $\hat{\sigma}^{(2)}_i(j)=j$, $i,j\in\{0,1,\dots,m-1\}$, denoting $\hat{\sigma}=\{\hat{\sigma}^{(1)}_i(j),\hat{\sigma}^{(2)}_i(j),i,j\in\{0,1,\dots,m-1\}$, then $\rho_{SM}(\hat{\sigma},\mathbb{1})$ is not the maximum Spearman's rho. In fact, we will show $\rho_{SM}(\hat{\sigma},\mathbb{1})<\rho_{SM}(\sigma^*,\mathbb{1})$.

Combining equation (3.7) with Lemma 4.1, we have

$$\rho_{SM}(\sigma,\mathbb{1})=4-6\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\big[\big(\underline{I}^{(1)}_{i,j}(\sigma)-\underline{I}^{(2)}_{i,j}(\sigma)\big)^2P(A_{i,j})\big].$$

Then

$$\rho_{SM}(\sigma^*,\mathbb{1})-\rho_{SM}(\hat{\sigma},\mathbb{1})=6\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\big\{\big[\big(\underline{I}^{(1)}_{i,j}(\hat{\sigma})-\underline{I}^{(2)}_{i,j}(\hat{\sigma})\big)^2-\big(\underline{I}^{(1)}_{i,j}(\sigma^*)-\underline{I}^{(2)}_{i,j}(\sigma^*)\big)^2\big]P(A_{i,j})\big\}, \tag{A-3}$$

Suppose $\hat{\sigma}^{(1)}_h(k)=l\leq k$. By comparing σ^* with $\hat{\sigma}$, there are only $\sigma^{(1)}_h(k)\neq\sigma^{(1)*}_h(k)$ and $\sigma^{(1)}_h(l)\neq\sigma^{(1)*}_h(l)$. Thus, equation (A-3) becomes

$$\begin{aligned}\rho_{SM}(\sigma^*,\mathbb{1})-\rho_{SM}(\hat{\sigma},\mathbb{1})=6\big\{&\big[\big(\underline{I}^{(1)}_{h,k}(\hat{\sigma})-\underline{I}^{(2)}_{h,k}(\hat{\sigma})\big)^2-\big(\underline{I}^{(1)}_{h,k}(\sigma^*)-\underline{I}^{(2)}_{h,k}(\sigma^*)\big)^2\big]P(A_{h,k})\\&+\big[\big(\underline{I}^{(1)}_{h,l}(\hat{\sigma})-\underline{I}^{(2)}_{h,l}(\hat{\sigma})\big)^2-\big(\underline{I}^{(1)}_{h,l}(\sigma^*)-\underline{I}^{(2)}_{h,l}(\sigma^*)\big)^2\big]P(A_{h,l})\big\}\end{aligned}$$

Based on the end points in equation (3.4), it is easy to compute that

$$
\begin{aligned}
&\rho_{SM}(\sigma^*,\mathbb{1})-\rho_{SM}(\hat{\sigma},\mathbb{1})\\
&=6P(A_{h,k})P(A_{h,l})[\frac{2(h-k)}{m}-2\sum_{q=0}^{h-1}P(A_{q,k})+P(A_{h,l})+D_1]\\
&\quad-6P(A_{h,k})P(A_{h,l})[\frac{2(h-l)}{m}-2\sum_{q=0}^{h-1}P(A_{q,l})+P(A_{h,k})+D_1]\\
&=6P(A_{h,k})P(A_{h,l})[\frac{2(l-k-1)}{m}+2P(U\le\frac{h}{m},\frac{l}{m}\le V\le\frac{l+1}{m})\\
&\quad+2P(U\ge\frac{h+1}{m},\frac{k}{m}\le V\le\frac{l}{m})+P(A_{h,k})+P(A_{h,l})]\\
&\ge 0.
\end{aligned}
$$

where $D_1=\sum_{q=k+1}^{l-1}P(A_{h,q})+2\sum_{q=0}^{k-1}P(A_{h,q})$. Thus $\rho_{SM}(\hat{\sigma},1)\le\rho_{SM}(\sigma^*,1)$. The conclusion comes. $\square$

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References

[1] Carley, H. and Taylor, M. D. (2002). A new proof of Sklar's theorem. In Proceedings of the Conference on Distributions with Given Marginals and Statistical Modelling (Barcelona, 2000), Kluwer Academic Publishers, Dordrecht, pages 29-34.

[2] Carley, H. (2002). Maximum and minimum extensions of finite subcopulas. Communications in Statistics-Theorey and Methods, 31(12), pages 2151-2166.

[3] Chen, Yi-Hsuan and Tu, H. A. (2013). Estimating hedged portfolio value-at-risk using the conditional copula: An illustration of model risk. International Review of Economics & Finance, 27, pages 35-60.

[4] Christoffersen, P. F. (2012). 2-historical simulation, value-at-risk and expected shortfall. In Elements of Finance Risk Management (Second Edition), Elsevier, pages 21-38.

[5] Christoffersen, P. F. (2013). Forecasting liquidity-adjusted intraday value-at-risk with vine copula. Journal of Banking & Finance, 37(9), pages 3334-3350.

[6] Cottin, C. and Pfeifer, D. (2014). From Bernstein polynomials to Bernstein copulas. J. Appl. Functional Analysis 9, pages 277-288.

[7] Dhaene, J. and Denuit, M. (1999). The safest dependence structure among risks. Insurance: Mathematics and Economics, 25(1), pages 11-21.

[8] Dhaene, J., Denuit, M., Kass, R. G. and Vynche, D. (2002a). The concept of comonotonicity in actuarial science and finance: Theory. Insurance: Mathematics and Economics, 31(1), pages 3-33.

[9] Durante, F., Sánchez, J. F. (2012). On the approximation of copulas via shuffle of min. Statistics & Probability Letters, 82(10), pages 1761-1767.

[10] Durante, F., Sánchez, J. F. (2010). Multivariate shuffles and approximation of copulas. Statistics & Probability Letters, 80(23-24), pages 1827-1834.

[11] Durante, F., Sarkoci, P. and Sempi, C. (2009). Shuffle of copulas. Journal of Mathematical Analysis and Applications, 353, pages 914-921.

[12] Durante, F., Sempi, C. (2010). Copula theory: An introduction. In: Jaworski, P., Durante, F., Härdle, W., and Rychilk, T., editors, Copula Theory and Its Applications, pages 3-31. Springer, Dordrecht.

[13] Huang, J. J., Lee, K. J., Liang, H. and Lin, W. F. (2009). Estimating value at risk of portfolio by conditional copula-garch method. Insurance: Mathematics and Economics, 45(3), pages 315-324.

[14] Mai, J. F. and Scherer, M. (2012). Simulating Copulas. Stochastic Models, Sampling algorithms, and Applications. Imp. College Press, London.

[15] Mikusinski, P., Sherwood, H. and Taylor, M. D. (1992). Shuffles of min. Stochastica, 13, pages 61-74.

[16] Mikusinski, P. and Taylor, M. D. (2010). Some approximations of n-copulas. Metrika, 72(3), pages 385-414.

[17] Nelsen, R. B. (2006). An introduction to Copulas. 2^{nd} ed., Springer, N.Y.

[18] Peacock, J. A. (1983). Two-dimensional goodness-of-fit testing in astronomy. Monthly Notices Royal Astronomy Society 202 615-627.

[19] Pfeifer, D., Strassburger, D. and Philipps, J. (2009). Modelling and simulation of dependence structures in nonlife insurance with Bernstein copulas. International ASTIN-Colloquium, June 1-4, 2009, Helsinki.

[20] Yang, J. P., Cheng, H. S. and Zhang, L. (2006). Bivariate copula decomposition in terms of comonotonicty, countermonotonicity and independence. Insurance: Mathematics and Economics, 39, pages 267-284.

[21] Zheng, Y. T., Yang, J. P. and Huang, J. H. (2011). Approximation of bivariate copulas by patched bivariate Fréchet copulas. Insurance: Mathematics and Economics, 48, pages 246-256.