Central Limit Theorems for Supercritical Branching Nonsymmetric Markov Processes

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Abstract

In this paper, we establish a spatial central limit theorem for a large class of supercritical branching, not necessarily symmetric, Markov processes with spatially dependent branching mechanisms satisfying a second moment condition. This central limit theorem generalizes and unifies all the central limit theorems obtained recently in $\begin{bmatrix} 123 \\ 23 \end{bmatrix}$ for supercritical branching symmetric Markov processes. To prove our central limit theorem, we have to carefully develop the spectral theory of nonsymmetric strongly continuous semigroups which should be of independent interest.

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1 Introduction

Central limit theorems for supercritical branching processes were initiated by Kesten and Stigum in [11, 12]. In these two papers, they established central limit theorems for supercritical multi-type Galton-Watson processes by using the Jordan canonical form of the expectation matrix M. Then in [4, 5, 6], Ath71 Theorems for supercritical multi-type continuous time branching processes, using the Jordan canonical form and the eigenvectors of the matrix M_t , the mean matrix at time t. Asmussen and Keiding [5] used martingale central limit theorems to prove central limit theorems for supercritical multi-type branching processes. In [2], Asmussen and Hering established spatial central limit theorems for general supercritical branching Markov processes under a certain condition. However, the condition in [2] is not easy to check and essentially the only examples given in [2] of branching Markov processes satisfying this condition are branching diffusions in bounded smooth domains. We note that the limit normal random variables in [2] may be degenerate.

The recent study of spatial central limit theorem for branching Markov processes started with [I]. In this paper, Adamczak and Miłoś proved some central limit theorems for supercritical branching

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Ornstein-Uhlenbeck processes with binary branching mechanism. We note that branching Ornstein-Uhlenbeck processes do not satisfy the condition in [2]. In [20], Miłoś proved some central limit theorems for supercritical super Ornstein-Uhlenbeck processes with branching mechanisms satisfying a fourth moment condition. Similar to the case of [2], the limit normal random variables in [1, 20] may be degenerate. In [22], we established central limit theorems for supercritical super Ornstein-Uhlenbeck processes with branching mechanisms satisfying only a second moment condition. More importantly, the central limit theorems in [22] are more satisfactory since our limit normal random variables are non-degenerate. In [22], we obtained central limit theorems for a large class of general supercritical branching symmetric Markov processes with spatially dependent branching mechanisms satisfying only a second moment condition. In [24], we obtained central limit theorems for a large class of general supercritical supercritical superprocesses with symmetric spatial motions and with spatially dependent branching mechanisms satisfying only a second moment condition. Furthermore, we also obtained the covariance structure of the limit Gaussian field in [22]. Compared with [4, 5, 6, 11, 12], the spatial processes in [1, 20, 22, 23, 24] are assumed to be

Compared with [4, 5, 6, 11, 12], the spatial processes in [1, 20, 22, 23, 24] are assumed to be symmetric. The reason for this assumption is that one of the main tools in [1, 20, 22, 23, 24] is the well-developed spectral theory of self-adjoint operators.

The main purpose of this paper is to establish central limit theorems for general supercritical branching, not necessarily symmetric, Markov processes with spatially dependent branching mechanisms satisfying only a second moment condition. To accomplish this, we need to carefully develop the spectral theory of not necessarily symmetric strongly continuous semigroups. We believe these spectral results are of independent interest and should be very useful in studying non-symmetric Markov processes.

In this paper, \mathbb{R} and \mathbb{C} stand for the sets of real and complex numbers respectively, all vectors in \mathbb{R}^n or \mathbb{C}^n will be understood as column vectors. For any $z \in \mathbb{C}$, we use $\Re(z)$ and $\Im(z)$ to denote real and imaginary parts of z respectively. For a matrix A, we use \overline{A} and A^T to denote the conjugate and transpose of A respectively.

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1.1 Spatial process

In this subsection, we spell out our assumptions on the spatial Markov process. Throughout this paper, E stands for a locally compact separable metric space, m is a σ -finite Borel measure on E with full support and ∂ is a separate point not contained in E. ∂ will be interpreted as the cemetery point. We will use E_{∂} to denote $E \cup \{\partial\}$. Every function f on E is automatically extended to E_{∂} by setting $f(\partial) = 0$. We will assume that $\xi = \{\xi_t, \Pi_x\}$ is a Hunt process on E and $\zeta := \inf\{t > 0 : \xi_t = \partial\}$ is the lifetime of ξ . We will use $\{P_t : t \ge 0\}$ to denote the semigroup of ξ . Our standing assumption on ξ is that there exists a family of continuous strictly positive functions $\{p(t, x, y) : t > 0\}$ on $E \times E$ such that, for any t > 0 and nonnegative function f on E,

$$P_t f(x) = \int_E p(t, x, y) f(y) m(dy).$$

For $p \ge 1$, we define $L^p(E,m;\mathbb{C}) := \{f : E \to \mathbb{C} : \int_E |f(x)|^p m(dx) < \infty\}$ and $L^p(E,m) := \{f \in L^p(E,m;\mathbb{C}) : f \text{ is real}\}$. We also define

$$a_t(x) := \int_E p(t, x, y)^2 m(dy), \qquad \hat{a}_t(x) := \int_E p(t, y, x)^2 m(dy)$$

In this paper, we assume that

Assumption 1 (a) For all t > 0 and $x \in E$, $\int_E p(t, y, x) m(dy) \le 1$.

- (b) For any t > 0, a_t and \hat{a}_t are continuous functions in E and they belong to $L^1(E,m)$.
- (c) There exists $t_0 > 0$ such that $a_{t_0}, \hat{a}_{t_0} \in L^2(E, m)$.

It is easy to see that

$$p(t+s,x,y) = \int_{E} p(t,x,z)p(s,z,y) \, m(dz) \le (a_t(x))^{1/2} (\widehat{a}_s(y))^{1/2}, \tag{1.1}$$

which implies

$$a_{t+s}(x) \le \int_E \widehat{a}_s(y) \, m(dy) a_t(x) \quad \text{and} \quad \widehat{a}_{t+s}(x) \le \int_E a_s(y) \, m(dy) \widehat{a}_t(x). \tag{1.2}$$

So condition (c) above is equivalent to

(c') There exists $t_0 > 0$ such that for all $t \ge t_0$, $a_t, \hat{a}_t \in L^2(E, m)$.

It is well known and easy to check that, for $p \in [1, \infty)$, $\{P_t : t \ge 0\}$ is a strongly continuous contraction semigroup on $L^p(E, m; \mathbb{C})$. We claim that the function $t \to \int_E a_t(x) m(dx)$ is decreasing. In fact, by Fubini's theorem and Hölder's inequality, we get

$$\begin{aligned} a_{t+s}(x) &= \int_E p(t+s,x,y) \int_E p(t,x,z) p(s,z,y) \, m(dz) \, m(dy) \\ &= \int_E p(t,x,z) \int_E p(t+s,x,y) p(s,z,y) \, m(dy) \, m(dz) \\ &\leq a_{t+s}(x)^{1/2} \int_E p(t,x,z) a_s(z)^{1/2} \, m(dz) \end{aligned}$$

which implies

$$a_{t+s}(x) \le \left(\int_E p(t, x, z) a_s(z)^{1/2} m(dz)\right)^2 \le \int_E p(t, x, z) a_s(z) m(dz).$$
(1.3) 8.9

Thus, by Fubini's theorem and condition (a), we get

$$\int_{E} a_{t+s}(x) \, m(dx) \le \int_{E} a_s(z) \int_{E} p(t, x, z) \, m(dx) \, m(dz) \le \int_{E} a_s(z) \, m(dz). \tag{1.4}$$

Therefore, the function $t \to \int_E a_t(x) m(dx)$ is decreasing.

Now we give some examples of non-symmetric Markov processes satisfying the above assumptions. The purpose of these examples is to show that the above assumptions are satisfied by many Markov processes. We will not try to give the most general examples possible. For examples of symmetric Markov processes satisfying the above assumptions, see $\begin{bmatrix} RSZ2\\ Z3 \end{bmatrix}$.

Example 1.1 Suppose that E consists of finitely many points. If $X = \{X_t : t \ge 0\}$ is an irreducible conservative Markov process in E, then X satisfies Assumption 1 for some finite measure m on E with full support.

examp1

examp0

Example 1.2 Suppose that $\alpha \in (0, 2)$ and that $Y = \{Y_t : t \ge 0\}$ is a strictly α -stable process in \mathbb{R}^d . Suppose that, in the case $d \ge 2$, the spherical part η of the Lévy measure μ of Y satisfies the following assumption: there exist a positive function Φ on the unit sphere S in \mathbb{R}^d and $\kappa > 1$ such that

$$\Phi = \frac{d\eta}{d\sigma}$$
 and $\kappa^{-1} \le \Phi(z) \le \kappa$ on S

where σ is the surface measure on S. In the case d = 1, we assume that the Lévy measure of Y is given by

$$\mu(dx) = c_1 x^{-1-\alpha} \mathbf{1}_{\{x>0\}} + c_2 |x|^{-1-\alpha} \mathbf{1}_{\{x<0\}}$$

with $c_1, c_2 > 0$. Suppose that D is an open set in \mathbb{R}^d of finite Lebesgue measure. Let X be the process in D obtained by killing Y upon exiting D. Then X satisfies Assumption 1 with E = D and m being the Lebesgue measure. For details, see [17, Example 4.1].

Example 1.3 Suppose that $\alpha \in (0, 2)$ and that $Z = \{Z_t : t \ge 0\}$ is a truncated strictly α -stable process in \mathbb{R}^d , that is, Z is a Lévy process with Lévy measure given by

$$\widetilde{\mu}(dx) = \mu(dx) \mathbf{1}_{\{|x|<1\}},$$

where μ is the Lévy measure of the process Y in the previous example. Suppose that D is a connected open set in \mathbb{R}^d of finite Lebesgue measure. Let X be the process in D obtained by killing Z upon exiting D. Then X satisfies Assumption 1 with E = D and m being the Lebesgue measure. For details, see [17, Example 4.2 and Proposition 4.4].

examp3

Example 1.4 Suppose $\alpha \in (0, 2)$, $Y = \{Y_t : t \ge 0\}$ is a strictly α -stable process in \mathbb{R}^d satisfying the assumptions in Example 1.2 and that B is an independent Brownian motion in \mathbb{R}^d . Let W be the process defined by $W_t = Y_t + B_t$. Suppose that D is an open set in \mathbb{R}^d of finite Lebesgue measure. Let X be the process in D obtained by killing W upon exiting D. Then X satisfies Assumption 1 with E = D and m being the Lebesgue measure. For details, see [17, Example 4.5] and Lemma 4.6].

Example 1.5 Suppose $\alpha \in (0, 2)$, $Z = \{Z_t : t \ge 0\}$ is a truncated strictly α -stable process in \mathbb{R}^d satisfying the assumptions in Example 1.3 and that B is an independent Brownian motion in \mathbb{R}^d . Let V be the process defined by $V_t = Z_t + B_t$. Suppose that D is a connected open set in \mathbb{R}^d of finite Lebesgue measure. Let X be the process in D obtained by killing V upon exiting D. Then X satisfies Assumption 1 with E = D and m being the Lebesgue measure. For details, see 17, Example 4.7 and Lemma 4.8]. examp5

Example 1.6 Suppose $d \ge 3$ and that $\mu = (\mu^1, \dots, \mu^d)$, where each μ^j is a signed measure on \mathbb{R}^d such that

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{B(x,r)} \frac{|\mu^j| (dy)}{|x - y|^{d - 1}} = 0.$$

Let $Y = \{Y_t : t \ge 0\}$ be a Brownian motion with drift μ in \mathbb{R}^d , see [13]. Suppose that D is a bounded connected open set in \mathbb{R}^d and suppose K > 0 is a constant such that $D \subset B(0, K/2)$. Put B = B(0, K). Let G_B be the Green function of Y in B and define $H(x) := \int_B G_B(x, y) dy$. Then H is a strictly positive continuous function on B. Let X be the process obtained by killing Y upon exiting D. Then X satisfies Assumption 1 with E = D and m being the measure defined by m(dx) = H(x)dx. For details, see [28], Example 4.6] or [14, 16].

Example 1.7 Suppose $d \ge 2$, $\alpha \in (1,2)$, and that $\mu = (\mu^1, \dots, \mu^d)$, where each μ^j is a signed measure on \mathbb{R}^d such that

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{B(x,r)} \frac{|\mu^j| (dy)}{|x - y|^{d - \alpha + 1}} = 0.$$

Let $Y = \{Y_t : t \ge 0\}$ be an α -stable process with drift μ in \mathbb{R}^d , see [18]. Suppose that D is a bounded open set in \mathbb{R}^d and suppose K > 0 is such that $D \subset B(0, K/2)$. Put B = B(0, K). Let G_B be the Green function of Y in B and define $H(x) := \int_B G_B(x, y) dy$. Then H is a strictly positive continuous function on B. Let X be the process obtained by killing Y upon exiting D. Then X satisfies Assumption 1 with E = D and m being the measure defined by m(dx) = H(x)dx. For details, see [28], Example 4.7] or [8].

1.2 Branching Markov Processes

The branching Markov process $\{X_t : t \ge 0\}$ on E we are going to work with is determined by three parameters: a spatial motion $\xi = \{\xi_t, \Pi_x\}$ on E satisfying the assumptions at the beginning of the previous subsection, a branching rate function $\beta(x)$ on E which is a non-negative bounded measurable function and an offspring distribution $\{p_n(x) : n = 0, 1, 2, ...\}$ satisfying the assumption

$$\sup_{x \in E} \sum_{n=0}^{\infty} n^2 p_n(x) < \infty.$$
 (1.5) 1.16

We denote the generating function of the offspring distribution by

$$\varphi(x,z) = \sum_{n=0}^{\infty} p_n(x) z^n, \quad x \in E, \quad |z| \le 1.$$

Consider a branching system on E characterized by the following properties: (i) each individual has a random birth and death time; (ii) given that an individual is born at $x \in E$, the conditional distribution of its path is determined by Π_x ; (iii) given the path ξ of an individual up to time tand given that the particle is alive at time t, its probability of dying in the interval [t, t + dt) is $\beta(\xi_t)dt + o(dt)$; (iv) when an individual dies at $x \in E$, it splits into n individuals all positioned at x, with probability $p_n(x)$; (v) when an individual reaches ∂ , it disappears from the system; (vi) all the individuals, once born, evolve independently.

Let $\mathcal{M}_a(E)$ be the space of finite integer-valued atomic measures on E, and let $\mathcal{B}_b(E)$ be the set of bounded real-valued Borel measurable functions on E. Let $X_t(B)$ be the number of particles alive at time t located in $B \in \mathcal{B}(E)$. Then $X = \{X_t, t \ge 0\}$ is an $\mathcal{M}_a(E)$ -valued Markov process. For any $\nu \in \mathcal{M}_a(E)$, we denote the law of X with initial configuration ν by \mathbb{P}_{ν} . As usual, $\langle f, \nu \rangle := \int_E f(x) \nu(dx)$. For $0 \le f \in \mathcal{B}_b(E)$, let

$$\omega(t,x) := \mathbb{P}_{\delta_x} e^{-\langle f, X_t \rangle},$$

then $\omega(t, x)$ is the unique positive solution to the equation

$$\omega(t,x) = \Pi_x \int_0^t \psi(\xi_s, \omega(t-s,\xi_s)) \, ds + \Pi_x(e^{-f(\xi_t)}), \tag{1.6}$$

where $\psi(x, z) = \beta(x)(\varphi(x, z) - z), x \in E, z \in [0, 1]$, while $\psi(\partial, z) = 0, z \in [0, 1]$. By the branching property, we have

$$\mathbb{P}_{\nu}e^{-\langle f, X_t \rangle} = e^{\langle \log \omega(t, \cdot), \nu \rangle}$$

Define

$$\alpha(x) := \frac{\partial \psi}{\partial z}(x, 1) = \beta(x) \left(\sum_{n=1}^{\infty} n p_n(x) - 1 \right)$$
(1.7) e:alpha

and

$$A(x) := \frac{\partial^2 \psi}{\partial z^2}(x, 1) = \beta(x) \sum_{n=2}^{\infty} (n-1)np_n(x).$$

$$(1.8) \quad \boxed{\mathbf{e}:\mathbf{A}}$$

By (1.16), there exists K > 0, such that

$$\sup_{x \in E} (|\alpha(x)| + A(x)) \le K.$$
(1.9) 1.5

For any $f \in \mathcal{B}_b(E)$ and $(t, x) \in (0, \infty) \times E$, define

$$T_t f(x) := \Pi_x \left[e^{\int_0^t \alpha(\xi_s) \, ds} f(\xi_t) \right]. \tag{1.10}$$

It is well known that $T_t f(x) = \mathbb{P}_{\delta_x} \langle f, X_t \rangle$ for every $x \in E$.

It is elementary to show that, see $\begin{bmatrix} RSZ4\\ 25 \end{bmatrix}$, Lemma 2.1], that there exists a function q(t, x, y) on $(0, \infty) \times E \times E$ which is continuous in (x, y) for each t > 0 such that

$$e^{-Kt}p(t,x,y) \le q(t,x,y) \le e^{Kt}p(t,x,y), \quad (t,x,y) \in (0,\infty) \times E \times E$$
(1.11) comp

and that for any bounded Borel function f on E and $(t, x) \in (0, \infty) \times E$,

$$T_t f(x) = \int_E q(t, x, y) f(y) m(dy).$$

Define

$$b_t(x) := \int_E q(t, x, y)^2 m(dy), \qquad \widehat{b}_t(x) := \int_E q(t, y, x)^2 m(dy).$$

The functions $x \to b_t(x)$ and $x \to \hat{b}_t(x)$ are continuous. In fact, by ([1,1]),

$$q(t, x, y) \le e^{Kt} p(t, x, y) \le e^{Kt} a_{t/2}(x)^{1/2} \widehat{a}_{t/2}(y)^{1/2}.$$
(1.12) 1.4

Since $q(t, \cdot, y)$ and $a_{t/2}$ are continuous, by the dominated convergence theorem, we get b_t is continuous. Similarly, \hat{b}_t is also continuous. Thus, it follows from $(\stackrel{1.4}{1.12})$ and the assumptions (b) and (c') in the previous subsection that b_t and \hat{b}_t enjoy the following properties.

- (i) For any t > 0, we have $b_t \in L^1(E, m)$. Moreover, $b_t(x)$ and $\hat{b}_t(x)$ are continuous in $x \in E$;
- (ii) There exists $t_0 > 0$ such that for all $t \ge t_0$, b_t , $\hat{b}_t \in L^2(E, m)$.

1.3 Preliminaries

For $p \ge 1$, $\{T_t : t \ge 0\}$ is a strongly continuous semigroup on $L^p(E, m; \mathbb{C})$. In fact, by $(\stackrel{\text{comp}}{\text{II.II}})$, we get $|T_t f(x)| \le e^{Kt} P_t |f|(x)$. Thus,

$$||T_t f||_p \le e^{Kt} ||P_t| f||_p \le e^{Kt} ||f||_p.$$
(1.13) Lp

For $f, g \in L^2(E, m; \mathbb{C})$, define

$$\langle f,g\rangle_m := \int_E f(x)\overline{g(x)} \, m(dx).$$

Let $\{\widehat{T}_t, t > 0\}$ be the adjoint semigroup of $\{T_t : t \ge 0\}$ on $L^2(E, m; \mathbb{C})$, that is, for $f, g \in L^2(E, m; \mathbb{C})$,

$$\langle T_t f, g \rangle_m = \langle f, \widehat{T}_t g \rangle_m.$$
 (1.14) adjiont

Thus,

$$\widehat{T}_t g(x) = \int_E q(t, y, x) g(y) \, m(dy).$$

It is well known, see for instance $\mathbb{P}^{\mathbf{a}}_{21}$, Corollary 1.10.6, Lemma 1.10.1], that $\{\widehat{T}_t : t \geq 0\}$ is a strongly continuous semigroup on $L^2(E, m; \mathbb{C})$ and that

$$\|\widehat{T}_t\|_2 = \|T_t\|_2 \le e^{Kt}.$$
(1.15) 1.66

For all t > 0 and $f \in L^2(E, m; \mathbb{C})$, $T_t f$ and $\hat{T}_t f$ are continuous. In fact, since q(t, x, y) is continuous, by $(\stackrel{1.4}{\Pi, \Pi 2})$ and Assumption 1(b), using the dominated convergence theorem, we get $T_t f$ and $\hat{T}_t f$ are continuous.

It follows from (i) above that, for any t > 0, T_t and \widehat{T}_t are compact operators on $L^2(E, m; \mathbb{C})$. Let \mathcal{A} and $\widehat{\mathcal{A}}$ be the infinitesimal generators of $\{T_t : t \ge 0\}$ and $\{\widehat{T}_t : t \ge 0\}$ in $L^2(E, m; \mathbb{C})$ respectively. Let $\sigma(\mathcal{A})$ and $\sigma(\widehat{\mathcal{A}})$ be the spectra of \mathcal{A} and $\widehat{\mathcal{A}}$ respectively. It follows from 2.1, Theorem 2.2.4 and Corollary 2.3.7] that both $\sigma(\mathcal{A})$ and $\sigma(\widehat{\mathcal{A}})$ consist of eigenvalues only, and that \mathcal{A} and $\widehat{\mathcal{A}}$ have the same number, say N, of eigenvalues. Of course N might be finite or infinite. Let $\mathbb{I} = \{1, 2, \ldots, N\}$, when $N < \infty$; otherwise $\mathbb{I} = \{1, 2, \ldots\}$. Under the assumptions of Subsection 1.1, using ($\widehat{\mathbb{I}}$.11) and Jentzsch's theorem ($\widehat{\mathbb{I}}$, Theorem V.6.6 on page 337], we know that the common value $\lambda_1 = \sup \Re(\sigma(\mathcal{A})) = \sup \Re(\sigma(\widehat{\mathcal{A}}))$ is an eigenvalue of multiplicity one for both \mathcal{A} and $\widehat{\mathcal{A}}$, and that an eigenfunction ϕ_1 of \mathcal{A} associated with λ_1 can be chosen to be strictly positive almost everywhere with $\|\phi_1\|_2 = 1$ and an eigenfunction ψ_1 of $\widehat{\mathcal{A}}$ associated with λ_1 can be chosen to be strictly positive almost everywhere with $\langle \phi_1, \psi_1 \rangle_m = 1$. We list the eigenvalues $\{-\lambda_k, k \in \mathbb{I}\}$ of \mathcal{A} in an order so $\lambda_1 < \Re(\lambda_2) \leq \Re(\lambda_3) \leq \cdots$. Then $\{-\overline{\lambda}_k, k \in \mathbb{I}\}$ are the eigenvalues of $\widehat{\mathcal{A}}$. For convenience, we define, for any positive integer k not belong to $\mathbb{I}, \lambda_k = \overline{\lambda}_k = \infty$. For $k \in \mathbb{I}$, we write $\Re_k := \Re(\lambda_k)$ and $\Im_k := \Im(\lambda_k)$. We use the convention $\Re_\infty = \infty$.

Let $\sigma(T_t)$ be the spectrum of T_t in $L^2(E, m; \mathbb{C})$. It follows from [21, Theorem 2.2.4] that $\sigma(T_t) \setminus \{0\} := \{e^{-\lambda_k t} : k \in \mathbb{I}\}$. In particular, $\sigma(T_1) \setminus \{0\} = \{e^{-\lambda_k}, k \in \mathbb{I}\}$.

Tek4 Remark 1.8 It is easy to see that, there exists t^* such that, for any $k \neq j$, $e^{-\lambda_k t^*} \neq e^{-\lambda_j t^*}$. So without lose of generality, we assume that, for $k \neq j$, $e^{-\lambda_k} \neq e^{-\lambda_j}$. Otherwise, we can consider T_{t^*} instead of T_1 in the following arguments.

Now we recall some basic facts about spectral theory, for more details, see [7, Chapter 6]. For any $k \in \mathbb{I}$, we define $\mathcal{N}_{k,0} := \{0\}$ and for $n \geq 1$,

$$\mathcal{N}_{k,n} := \mathcal{N}((e^{-\lambda_k}I - T_1)^n) = \{ f \in L^2(E,m;\mathbb{C}) : (e^{-\lambda_k}I - T_1)^n f = 0 \}$$

and

$$\mathcal{R}_{k,n} := \mathcal{R}((e^{-\lambda_k}I - T_1)^n) = (e^{-\lambda_k}I - T_1)^n (L^2(E, m; \mathbb{C}))$$

For each $k \in \mathbb{I}$, there exists an integer $\nu_k \geq 1$ such that

$$\mathcal{N}_{k,n} \subsetneqq \mathcal{N}_{k,n+1}, \quad n = 0, 1, \cdots, \nu_k - 1; \quad \mathcal{N}_{k,n} = \mathcal{N}_{k,n+1}, \quad n \ge \nu_k$$

and

$$\mathcal{R}_{k,n} \supseteq \mathcal{R}_{k,n+1}, \quad n = 0, 1, \cdots, \nu_k - 1; \quad \mathcal{R}_{k,n} = \mathcal{R}_{k,n+1}, \quad n \ge \nu_k.$$

For all $k \in \mathbb{I}$ and $n \geq 0$, $\mathcal{N}_{k,n}$ is a finite dimensional linear subspace of $L^2(E, m; \mathbb{C})$. $\mathcal{N}_{k,n}$ and $\mathcal{R}_{k,n}$ are invariant subspaces of T_t . In fact, for any $f \in \mathcal{N}_{k,n}$,

$$(e^{-\lambda_k}I - T_1)^n (T_t f) = T_t (e^{-\lambda_k}I - T_1)^n f = 0,$$

which implies that $T_t f \in \mathcal{N}_{k,n}$. If $f = (e^{-\lambda_k}I - T_1)^n g$, then $T_t f = T_t (e^{-\lambda_k}I - T_1)^n g = (e^{-\lambda_k}I - T_1)^n f = T_1)^n T_t g \in \mathcal{R}_{k,n}$. Thus, $\{T_t|_{\mathcal{N}_{k,\nu_k}}, t > 0\}$ is a semigroup on \mathcal{N}_{k,ν_k} . We denote the corresponding infinitesimal generator as \mathcal{A}_k . By [7, Theorem 6.7.4], $\sigma(T_1|_{\mathcal{N}_{k,\nu_k}}) = \{e^{-\lambda_k}\}$. Since $\sigma(\mathcal{A}_k) \subset \sigma(\mathcal{A})$, we have $\sigma(\mathcal{A}_k) = \{-\lambda_k\}$. Define $n_k := \dim(\mathcal{N}_{k,\nu_k})$ and $r_k := \dim(\mathcal{N}_{k,1})$. Then from linear algebra we know that there exists a basis $\{\phi_j^{(k)}, j = 1, 2, \cdots, n_k\}$ of \mathcal{N}_{k,ν_k} such that

$$\mathcal{A}_{k}(\phi_{1}^{(k)},\phi_{2}^{(k)},\cdots,\phi_{n_{k}}^{(k)}) = (\phi_{1}^{(k)},\phi_{2}^{(k)},\cdots,\phi_{n_{k}}^{(k)}) \begin{pmatrix} J_{k,1} & & 0 \\ & J_{k,2} & & \\ & & \ddots & \\ 0 & & & J_{k,r_{k}} \end{pmatrix}$$

$$:= (\phi_1^{(k)}, \phi_2^{(k)}, \cdots, \phi_{n_k}^{(k)}) D_k,$$
(1.16)

where

$$J_{k,j} = \begin{pmatrix} -\lambda_k & 1 & & 0 \\ & -\lambda_k & 1 & & \\ & & \ddots & \ddots & \\ & & & -\lambda_k & 1 \\ 0 & & & & -\lambda_k \end{pmatrix}, \quad a \ d_{k,j} \times d_{k,j} \quad \text{matrix}$$
(1.17) $\boxed{J_kj}$

with $\sum_{j=1}^{r_k} d_{k,j} = n_k$. D_k is uniquely determined by the dimensions of $\mathcal{N}_{k,n}$, $n = 1, 2, \dots, \nu_k$ (see [19, Section 7.8] for more details). Here and in the remainder of this paper we use the convention that when an operator, like \mathcal{A} or \mathcal{A}_k or T_t , acts on a vector-valued function, it acts componentwise. For convenience, we define the following \mathbb{C}^{n_k} -valued functions:

$$\Phi_k(x) := \left(\phi_1^{(k)}(x), \phi_2^{(k)}(x), \cdots, \phi_{n_k}^{(k)}(x)\right)^T.$$

Thus, we have, for a.e. $x \in E$,

$$T_{t}(\Phi_{k})^{T}(x) = e^{-\lambda_{k}t}(\Phi_{k}(x))^{T} \begin{pmatrix} J_{k,1}(t) & & 0 \\ & J_{k,2}(t) & & \\ & & \ddots & \\ 0 & & & J_{k,r_{k}}(t) \end{pmatrix}$$

$$:= e^{-\lambda_{k}t}(\Phi_{k}(x))^{T}D_{k}(t), \qquad (1.18)$$

where $J_{k,j}(t)$ is a $d_{k,j} \times d_{k,j}$ matrix given by

$$J_{k,j}(t) = \begin{pmatrix} 1 & t & t^2/2! & \cdots & t^{d_{k,j}-1}/(d_{k,j}-1)! \\ 0 & 1 & t & t^2/2! & \cdots \\ & \ddots & \ddots & & \\ & & 1 & t & \\ 0 & & & 1 & \end{pmatrix}.$$
(1.19) J_kjt

More details can be found in [19, p. 609]. Under our assumptions, $T_t(\Phi_k)^T(x)$ is continuous. Thus, by $(\overline{1.18})$, we can choose Φ_k to be continuous, which implies $(\overline{1.18})$ holds for all $x \in E$. We note that here the matrix $D_k(t)$ satisfies the semigroup property, that is, for t, s > 0, $D_k(t+s) = D_k(t)D_k(s)$ and $D_k(t)$ is invertible with $D_k(t)^{-1} = D_k(-t)$.

For any vector $a = (a_1, \dots, a_n)^T \in \mathbb{C}^n$, we define the L^p norm of a by $|a|_p := \left(\sum_{j=1}^n |a_j|^p\right)^{1/p}$ when $1 \le p < \infty$ and $|a|_{\infty} := \max_i(|a_i|)$ when $p = \infty$.

By Hölder's inequality, $|T_t(\phi_j^{(k)})(x)| \leq b_t(x)^{1/2}$. By $(\boxed{\mathbb{I} \cdot 18})$, we get $(\Phi_k)^T = e^{\lambda_k t} T_t(\Phi_k)^T (D_k(t))^{-1}$. Thus,

$$|\Phi_k(x)|_{\infty} \le c(t,k)b_t(x)^{1/2},$$
 (1.20) [phi]

where c(t,k) does not depend on x. When we choose $t = t_0$, we get that $\phi_j^{(k)} \in L^2(E,m;\mathbb{C}) \cap L^4(E,m;\mathbb{C})$.

Now we consider the corresponding formula for \widehat{T}_t . We know that $\sigma(\widehat{T}_1) \setminus \{0\} = \{e^{-\overline{\lambda_k}}, k \in \mathbb{I}\}$. Define

$$\widehat{\mathcal{N}}_{k,n} := \mathcal{N}((e^{-\overline{\lambda_k}}I - \widehat{T}_1)^n) = \{ f \in L^2(E,m;\mathbb{C}) : (e^{-\overline{\lambda_k}}I - \widehat{T}_1)^n f = 0 \}.$$

We have

$$(e^{-\lambda_k}I - T_1)^n = e^{-n\lambda_k}I - \sum_{j=1}^n e^{-(n-j)\lambda_k}T_1^j.$$
(1.21) (7.4)

Since $\sum_{j=1}^{n} e^{-(n-j)\lambda_k} T_1^j$ is also a compact operator, by [7, Theorem 6.6.13], $\widehat{\mathcal{N}}_{k,n}$ is of the same dimension as $\mathcal{N}_{k,n}$. In particular, $\dim(\widehat{\mathcal{N}}_{k,\nu_k}) = \dim(\mathcal{N}_{k,\nu_k}) = n_k$. Thus we have

$$\widehat{\mathcal{N}}_{k,n} \subsetneqq \widehat{\mathcal{N}}_{k,n+1}, \quad n = 0, 1, \cdots, \nu_k - 1; \quad \widehat{\mathcal{N}}_{k,n} = \widehat{\mathcal{N}}_{k,n+1}, \quad n \ge \nu_k.$$

Similarly, we can get, for all $k \in \mathbb{I}$ and $n \geq 0$, $\widehat{\mathcal{N}}_{k,n}$ is an invariant subspace of \widehat{T}_t . Hence, $\{\widehat{T}_t|_{\widehat{\mathcal{N}}_{k,\nu_k}}, t > 0\}$ is a semigroup on $\widehat{\mathcal{N}}_{k,\nu_k}$ with infinitesimal generator $\widehat{\mathcal{A}}_k$.

Let $\{\widehat{\psi}_1^{(k)}, \widehat{\psi}_2^{(k)}, \cdots, \widehat{\psi}_{n_k}^{(k)}\}$ be a basis of $\widehat{\mathcal{N}}_{k,\nu_k}$ such that

$$\widehat{T}_t(\widehat{\psi}_1^{(k)}, \widehat{\psi}_2^{(k)}, \cdots, \widehat{\psi}_{n_k}^{(k)}) = (\widehat{\psi}_1^{(k)}, \widehat{\psi}_2^{(k)}, \cdots, \widehat{\psi}_{n_k}^{(k)}) \widehat{D}_k(t), \qquad (1.22)$$

where $\widehat{D}_k(t)$ is an $n_k \times n_k$ invertible matrix. Since $\widehat{T}_t(\widehat{\psi}_1^{(k)}, \widehat{\psi}_2^{(k)}, \cdots, \widehat{\psi}_{n_k}^{(k)})(x)$ is continuous, we can choose $(\widehat{\psi}_1^{(k)}, \widehat{\psi}_2^{(k)}, \cdots, \widehat{\psi}_{n_k}^{(k)})$ to be continuous. We define an $n_k \times n_k$ matrix \widetilde{A}_k by

$$(\widetilde{A}_k)_{j,l} := \langle \phi_j^{(k)}, \widehat{\psi}_l^{(k)} \rangle_m. \tag{1.23}$$

lemma1.1 Lemma 1.9 For each $k \in \mathbb{I}$,

$$L^{2}(E,m;\mathbb{C}) = \mathcal{N}_{k,\nu_{k}} \oplus (\widehat{\mathcal{N}}_{k,\nu_{k}})^{\perp} = \widehat{\mathcal{N}}_{k,\nu_{k}} \oplus (\mathcal{N}_{k,\nu_{k}})^{\perp}.$$
(1.24) 1.15

Morover, the matrix \widetilde{A}_k defined in $(\overset{\textbf{A}}{\textbf{I}}, \overset{\textbf{k}}{\textbf{2}}3)$ is invertible.

Proof: By [7, Theorem 6.6.7], we have $L^2(E, m; \mathbb{C}) = \mathcal{N}_{k,\nu_k} \oplus \mathcal{R}_{k,\nu_k}$. It follows from [7, Theorem 6.6.14] that $\mathcal{R}_{k,\nu_k} = (\widehat{\mathcal{N}}_{k,\nu_k})^{\perp}$. Thus, $L^2(E, m; \mathbb{C}) = \mathcal{N}_{k,\nu_k} \oplus (\widehat{\mathcal{N}}_{k,\nu_k})^{\perp}$. Similarly, we have $L^2(E, m; \mathbb{C}) = \widehat{\mathcal{N}}_{k,\nu_k} \oplus (\mathcal{N}_{k,\nu_k})^{\perp}$.

For any vector $a = (a_1, \cdots, a_{n_k})^T \in \mathbb{C}^{n_k}$, we have

$$\widetilde{A}_k a = (\langle \phi_1^{(k)}, h \rangle_m, \langle \phi_2^{(k)}, h \rangle_m, \cdots, \langle \phi_{n_k}^{(k)}, h \rangle_m)^T,$$

where $h = (\widehat{\psi}_1^{(k)}, \widehat{\psi}_2^{(k)}, \cdots, \widehat{\psi}_{n_k}^{(k)}) \bar{a} \in \widehat{\mathcal{N}}_{k,\nu_k}.$

If $\widetilde{A}_k a = 0$, then $h \in (\mathcal{N}_{k,\nu_k})^{\perp}$. Since $\widehat{\mathcal{N}}_{k,\nu_k} \cap (\mathcal{N}_{k,\nu_k})^{\perp} = \{0\}$, we have h = 0, which implies a = 0. Therefore, \widetilde{A}_k is invertible.

lemma T* Lemma 1.10 For any $k \in \mathbb{I}$, define

$$(\Psi_k(x))^T := \left(\psi_1^{(k)}(x), \psi_2^{(k)}(x), \cdots, \psi_{n_k}^{(k)}(x)\right) := \left(\widehat{\psi}_1^{(k)}(x), \widehat{\psi}_2^{(k)}(x), \cdots, \widehat{\psi}_{n_k}^{(k)}(x)\right) \overline{\widetilde{A}_k^{-1}}$$

Then $\{\psi_1^{(k)}, \psi_2^{(k)}, \cdots, \psi_{n_k}^{(k)}\}$ is a basis of $\widehat{\mathcal{N}}_{k,\nu_k}$ such that the $n_k \times n_k$ matrix $A_k := (\langle \phi_j^{(k)}, \psi_l^{(k)} \rangle_m)$ satisfies

$$A_k = I \tag{1.25} \quad \blacksquare$$

and for any $x \in \mathbb{E}$,

$$\widehat{T}_t(\Psi_k)(x) = e^{-\overline{\lambda_k}t} D_k(t) \Psi_k(x).$$
(1.26) T^{*}

Moreover, the basis of $\widehat{\mathcal{N}}_{k,\nu_k}$ satisfying ($\mathring{\mathbb{I}}.25$) is unique.

Proof: For any \mathbb{C}^n -valued functions $(f_1(x), f_2(x), \cdots f_n(x))^T$ and $(g_1(x), g_2(x), \cdots g_n(x))^T$, we use $\langle (f_1, f_2, \cdots f_n), (g_1, g_2, \cdots g_n) \rangle_m$ to denote the $n \times n$ matrix $(\langle f_j, g_l \rangle_m)$. Since $\overline{\widetilde{A}_k^{-1}}$ is invertible, $\{\psi_1^{(k)}, \psi_2^{(k)}, \cdots, \psi_{n_k}^{(k)}\}$ is a basis of $\widehat{\mathcal{N}}_{k,\nu_k}$. By $(\overline{\mathbb{I}}$ -Jordan ($\overline{\mathbb{I}}$ -14 ($\overline{\mathbb{I}}$ -22), we get

$$e^{-\lambda_{k}t}(D_{k}(t))^{T}\widetilde{A}_{k} = \left\langle T_{t}\left(\phi_{1}^{(k)},\phi_{2}^{(k)},\cdots,\phi_{n_{k}}^{(k)}\right),\left(\widehat{\psi}_{1}^{(k)},\widehat{\psi}_{2}^{(k)},\cdots,\widehat{\psi}_{n_{k}}^{(k)}\right)\right\rangle_{m}$$
$$= \left\langle \left(\phi_{1}^{(k)},\phi_{2}^{(k)},\cdots,\phi_{n_{k}}^{(k)}\right),\widehat{T}_{t}\left(\widehat{\psi}_{1}^{(k)},\widehat{\psi}_{2}^{(k)},\cdots,\widehat{\psi}_{n_{k}}^{(k)}\right)\right\rangle_{m} = \widetilde{A}_{k}\overline{\widehat{D}_{k}(t)}.$$

Since $D_k(t)$ is a real matrix, we have

$$e^{-\overline{\lambda_k}t}\overline{\widetilde{A}_k^{-1}}(D_k(t))^T = \widehat{D_k}(t)\overline{\widetilde{A}_k^{-1}}.$$
(1.27) 1.17

By (1.22) and (1.27), we have

$$\widehat{T}_t \left(\psi_1^{(k)}, \psi_2^{(k)}, \cdots, \psi_{n_k}^{(k)} \right) = \left(\widehat{\psi}_1^{(k)}, \widehat{\psi}_2^{(k)}, \cdots, \widehat{\psi}_{n_k}^{(k)} \right) \widehat{D}_k(t) \overline{\widetilde{A}_k^{-1}}$$

$$= e^{-\overline{\lambda_k}t} \left(\widehat{\psi}_1^{(k)}, \widehat{\psi}_2^{(k)}, \cdots, \widehat{\psi}_{n_k}^{(k)} \right) \overline{\widetilde{A}_k^{-1}} (D_k(t))^T = e^{-\overline{\lambda_k}t} \left(\psi_1^{(k)}, \psi_2^{(k)}, \cdots, \psi_{n_k}^{(k)} \right) (D_k(t))^T.$$

Assume that there exists another basis $\widetilde{\Psi}_k(x)$ of $\widehat{\mathcal{N}}_{k,\nu_k}$ satisfying ($\mathring{\mathbb{L}}$.25). Then there exists matrix B such that $(\widetilde{\Psi}_k(x))^T = (\Psi_k(x))^T B$. Thus,

$$I = \langle (\Phi_k)^T, (\widetilde{\Psi}_k)^T \rangle_m = \langle (\Phi_k)^T, (\Psi_k)^T \rangle_m \overline{B} = \overline{B},$$

which implies B = I. Thus, we get $\widetilde{\Psi}_k(x) = \Psi_k(x)$. The proof is now complete.

Tek5 Remark 1.11 We know that $T_t(\overline{\Phi_k^T})(x) = e^{-\overline{\lambda_k}t}\overline{\Phi_j^T(x)}D_k(t)$. Thus $e^{-\overline{\lambda_k}t}$ is also a eigenvalue of T_t . Hence there exists a unique k' such that $\lambda_{k'} = \overline{\lambda_k}$. It is obvious that $D_k(t) = D_{k'}(t)$ and we can choose $\Phi_{k'}(x) = \overline{\Phi_k(x)}$. By Lemma $\overline{1.10}$, we have $\Psi_{k'}(x) = \overline{\Psi_k(x)}$. In particular, if λ_k is real, then k' = k.

lemma1.2 Lemma 1.12 For $j, k \in \mathbb{I}$ and $j \neq k$, we have

$$\mathcal{N}_{j,\nu_j} \subset \mathcal{R}_{k,\nu_k} = (\widehat{\mathcal{N}}_{k,\nu_k})^{\perp}.$$
(1.28) 1.18

In particular, $\mathcal{N}_{j,\nu_j} \cap \mathcal{N}_{k,\nu_k} = \{0\}.$

Proof: Assume $f \in \mathcal{N}_{j,\nu_j}$, then $(e^{-\lambda_j}I - T_1)^{\nu_j}f = 0$. Since $\nu_j \ge 1$, we can define $g = (e^{-\lambda_j}I - T_1)^{\nu_j - 1}f$. Thus $e^{-\lambda_j}g = T_1g$. Hence, $(e^{-\lambda_k}I - T_1)g = (e^{-\lambda_k} - e^{-\lambda_j})g$, which implies

$$(e^{-\lambda_k}I - T_1)^{\nu_k}g = (e^{-\lambda_k} - e^{-\lambda_j})^{\nu_k}g.$$

Therefore $g = (e^{-\lambda_k} - e^{-\lambda_j})^{-\nu_k} (e^{-\lambda_k}I - T_1)^{\nu_k} g \in \mathcal{R}_{k,\nu_k}.$

Assume $f = f_1 + f_2$ with $f_1 \in \mathcal{N}_{k,\nu_k}$ and $f_2 \in \mathcal{R}_{k,\nu_k}$. Then $(e^{-\lambda_j}I - T_1)^{\nu_j - 1}f_1 \in \mathcal{N}_{k,\nu_k}$. On the other hand, $(e^{-\lambda_j}I - T_1)^{\nu_j - 1}f_1 = g - (e^{-\lambda_j}I - T_1)^{\nu_j - 1}f_2 \in \mathcal{R}_{k,\nu_k}$. Thus $(e^{-\lambda_j}I - T_1)^{\nu_j - 1}f_1 = 0$.

If $\nu_j = 1$, then $f = g \in \mathcal{R}_{k,\nu_k}$. If $\nu_j > 1$ and $f_1 \neq 0$, then $e^{-\lambda_j} \in \sigma(T_1|_{\mathcal{N}_{k,\nu_k}})$. By [7, Theorem 6.7.4], $\sigma(T_1|_{\mathcal{N}_{k,\nu_k}}) = \{e^{-\lambda_k}\}$. This is a contradiction. Thus, $f_1 = 0$, which implies $f = f_2 \in \mathcal{R}_{k,\nu_k}$. Therefore $\mathcal{N}_{j,\nu_j} \subset \mathcal{R}_{k,\nu_k}$.

By Lemma 1.12, for $k \in \mathbb{I}$, we can define

$$\mathcal{M}_k := \mathcal{N}_{1,\nu_1} \oplus \mathcal{N}_{2,\nu_2} \oplus \cdots \oplus \mathcal{N}_{k,\nu_k} \text{ and } \widehat{\mathcal{M}}_k := \widehat{\mathcal{N}}_{1,\nu_1} \oplus \widehat{\mathcal{N}}_{2,\nu_2} \oplus \cdots \oplus \widehat{\mathcal{N}}_{k,\nu_k}$$

cor2 Corollary 1.13 For any $k \in \mathbb{I}$,

$$L^{2}(E,m;\mathbb{C}) = \mathcal{M}_{k} \oplus (\widehat{\mathcal{M}}_{k})^{\perp} = \widehat{\mathcal{M}}_{k} \oplus (\mathcal{M}_{k})^{\perp}.$$
(1.29)

Proof: By $(\stackrel{1.15}{1.24})$, $(\stackrel{2.1}{1.29})$ holds for k = 1. Assume that $(\stackrel{2.1}{1.29})$ holds for k - 1. Then

$$L^{2}(E,m;\mathbb{C}) = \mathcal{M}_{k-1} \oplus (\widehat{\mathcal{M}}_{k-1})^{\perp}.$$
(1.30) 2.9

For any $f \in (\widehat{\mathcal{M}}_{k-1})^{\perp}$, by $(\stackrel{1.15}{1.24})$, we have $f = f_3 + f_4$, where $f_3 \in \mathcal{N}_{k,\nu_k}$ and $f_4 \in (\widehat{\mathcal{N}}_{k,\nu_k})^{\perp}$. By $(\stackrel{1.18}{1.28})$, $f_3 \in \bigcap_{j=1}^{k-1} (\widehat{\mathcal{N}}_{j,\nu_j})^{\perp} = (\widehat{\mathcal{M}}_{k-1})^{\perp}$, which implies $f_4 = f - f_3 \in (\mathcal{M}_{k-1})^{\perp}$. Thus, we obtain

$$f_4 \in (\widehat{\mathcal{N}}_{k,\nu_k})^{\perp} \cap (\mathcal{M}_{k-1})^{\perp} = (\mathcal{M}_k)^{\perp}.$$

Hence

 $(\mathcal{M}_{k-1})^{\perp} = \mathcal{N}_{k,\nu_k} \oplus (\mathcal{M}_k)^{\perp}.$ Therefore, by induction, the first part of $(\stackrel{[2.1]}{\text{I.29}})$ holds for all $k \in \mathbb{I}$.

The proof of $L^2(E,m;\mathbb{C}) = \widehat{\mathcal{M}}_k \oplus (\mathcal{M}_k)^{\perp}$ is similar.

Tek2 Remark 1.14 Since $-\lambda_1$ is simple, which means $n_1 = r_1 = \nu_1 = 1$, we know that $\Phi_1(x) = \phi_1(x)$ and $\Psi_1(x) = \psi_1(x)$. Moreover, since $T_t\phi_1(x) = e^{-\lambda_1 t}\phi_1(x)$ and $\hat{T}_t\psi_1(x) = e^{-\lambda_1 t}\psi_1(x)$ for every x, ϕ_1 and ψ_1 are continuous and strictly positive. It is easy to see that $D_1(t) \equiv 1$.

By Lemma $\frac{1 \text{ emma 1}, 2}{1.12}, \{\phi_l^{(j)}, j = 1, \cdots, k, l = 1, \cdots, n_j\}$ is a basis of \mathcal{M}_k and $\{\psi_l^{(j)}, j = 1, \cdots, k, l = 1, \cdots, n_j\}$ is a basis of $\widehat{\mathcal{M}}_k$. By $(\frac{1.18}{1.28})$ and $(\stackrel{\texttt{A}}{\texttt{I}}.25)$, we get $\langle\phi_l^{(j)}, \psi_n^{(k)}\rangle_m = 1$, when j = k and l = n; otherwise $\langle\phi_l^{(j)}, \psi_n^{(k)}\rangle_m = 0$.

In this paper, we always assume that the branching Markov process X is supercritical, that is, Assumption 2 $\lambda_1 < 0$.

We will use $\{\mathcal{F}_t : t \geq 0\}$ to denote the filtration of X, that is $\mathcal{F}_t = \sigma(X_s : s \in [0, t])$. Using the expectation formula of $\langle \phi_1, X_t \rangle$ and the Markov property of X, it is easy to show that (see Lemma $\frac{thrm 1}{3.1}$), for any nonzero $\nu \in \mathcal{M}_a(E)$, under \mathbb{P}_{ν} , the process $W_t := e^{\lambda_1 t} \langle \phi_1, X_t \rangle$ is a positive martingale. Therefore it converges:

 $W_t \to W_\infty$, \mathbb{P}_{ν} -a.s. as $t \to \infty$.

Using the assumption $(\stackrel{1.16}{1.5})$ we can show that, as $t \to \infty$, W_t also converges in $L^2(\mathbb{P}_{\nu})$, so W_{∞} is non-degenerate and the second moment is finite. Moreover, we have $\mathbb{P}_{\nu}(W_{\infty}) = \langle \phi_1, \nu \rangle$. Put $\mathcal{E} = \{W_{\infty} = 0\}$, then $\mathbb{P}_{\nu}(\mathcal{E}) < 1$. It is clear that $\mathcal{E}^c \subset \{X_t(E) > 0, \forall t \ge 0\}$.

1.4 Main results

For any $k \in \mathbb{I}$, every function $f \in L^2(E, m; \mathbb{C})$ can be written uniquely as the sum of a function $f_k \in \mathcal{M}_k$ and a function in $(\widehat{\mathcal{M}}_k)^{\perp}$. Similarly, every function $f \in L^2(E, m; \mathbb{C})$ can be written uniquely as the sum of a function $\widehat{f}_k \in \widehat{\mathcal{M}}_k$ and a function in $(\mathcal{M}_k)^{\perp}$. Using $(\overset{\mathbb{A}}{\mathbb{I}}.25)$, we can easily get that

$$f_k(x) = \sum_{j=1}^k (\Phi_j(x))^T \langle f, \Psi_j \rangle_m \in \mathcal{M}_k \quad \text{and} \quad \widehat{f}_k(x) = \sum_{j=1}^k (\Psi_j(x))^T \langle f, \Phi_j \rangle_m \in \widehat{\mathcal{M}}_k, \tag{1.31}$$

where

$$\langle f, \Psi_j \rangle_m := (\langle f, \psi_1^{(j)} \rangle_m, \langle f, \psi_2^{(j)} \rangle_m, \cdots, \langle f, \psi_{n_j}^{(j)} \rangle_m)^T$$

and

$$\langle f, \Phi_j \rangle_m := (\langle f, \phi_1^{(j)} \rangle_m, \langle f, \phi_2^{(j)} \rangle_m, \cdots, \langle f, \phi_{n_j}^{(j)} \rangle_m)^T.$$

For any $f \in L^2(E, m; \mathbb{C})$, we define

$$\gamma(f) := \inf\{j \in \mathbb{I} : \langle f, \Psi_j \rangle_m \neq 0\},\$$

where we use the usual convention that $\inf \emptyset = \infty$. If $\gamma(f) < \infty$, define

$$\zeta(f) := \sup\{j \in \mathbb{I} : \Re_j = \Re_{\gamma(f)}\}.$$

For each $j \in \mathbb{I}$, every component of the function $t :\to D_j(t)\langle f, \Psi_j \rangle_m$ is a polynomial of t. Denote the degree of the *l*-th component of $D_j(t)\langle f, \Psi_j \rangle_m$ by $\tau_{j,l}(f)$. We define

$$\tau(f) := \sup\{\tau_{j,l}(f) : \gamma(f) \le j \le \zeta(f), 1 \le l \le n_j\}.$$

Then for any j with $\Re_j = \Re_{\gamma(f)}$,

$$F_{f,j} := \lim_{t \to \infty} t^{-\tau(f)} D_j(t) \langle f, \Psi_j \rangle_m \tag{1.32}$$

exists and there exists a j such that $F_{f,j} \neq 0$.

Note that if $g \in L^2(E, m)$, then for any $j \in \mathbb{I}$,

$$\overline{\langle g, \Psi_j \rangle_m} = \langle g, \overline{\Psi_j} \rangle_m = \langle g, \Psi_{j'} \rangle_m$$

where j' is defined in Remark $[1,1]^{\text{rek5}}$. For $g(x) = \sum_{k:\lambda_1 \ge 2\Re_k} (\Phi_k(x))^T b_k$, we get $b_k = \langle g, \Psi_j \rangle_m$. Thus, if g(x) is real, we get $\overline{b_k} = b_{k'}$. The following three subsets of $L^2(E,m)$ will be needed in the statement of our main result:

$$\mathcal{C}_{l} := \left\{ g(x) = \sum_{k \in \mathbb{I}: \lambda_{1} > 2\Re_{k}} (\Phi_{k}(x))^{T} b_{k} : b_{k} \in \mathbb{C}^{n_{k}} \text{ with } \overline{b_{k}} = b_{k'} \right\},$$
$$\mathcal{C}_{c} := \left\{ g(x) = \sum_{k \in \mathbb{I}: \lambda_{1} = 2\Re_{k}} (\Phi_{k}(x))^{T} b_{k} : b_{k} \in \mathbb{C}^{n_{k}} \text{ with } \overline{b_{k}} = b_{k'} \right\},$$

and

$$\mathcal{C}_s := \left\{ g \in L^2(E,m) \cap L^4(E,m) : \lambda_1 < 2\Re_{\gamma(g)} \right\}.$$

1.4.1 Some basic law of large numbers

For any $k \in \mathbb{I}$, we define an n_k -dimensional random vector $H_t^{(k)}$ as follows:

$$H_t^{(k)} := e^{\lambda_k t} (\langle \phi_1^{(k)}, X_t \rangle, \cdots, \langle \phi_{n_k}^{(k)}, X_t \rangle) (D_k(t))^{-1}.$$

One can show (see Lemma $\frac{\texttt{thrm1}}{\texttt{5.1 below}}$) that, if $\lambda_1 > 2\Re_k$, then, for any $\nu \in \mathcal{M}_a(E)$ and $b \in \mathbb{C}^{n_k}$, $H_t^{(k)}b$ is a martingale under \mathbb{P}_{ν} and bounded in $L^2(\mathbb{P}_{\nu})$. Thus the limit $H_{\infty}^{(k)} := \lim_{t \to \infty} H_t^{(k)}$ exists \mathbb{P}_{ν} -a.s. and in $L^2(\mathbb{P}_{\nu})$.

thrm2 Theorem 1.15 If $f \in L^2(E,m;\mathbb{C}) \cap L^4(E,m;\mathbb{C})$ with $\lambda_1 > 2\Re_{\gamma(f)}$, then for any nonzero $\nu \in \mathcal{M}_a(E)$, as $t \to \infty$,

$$t^{-\tau(f)}e^{\Re_{\gamma(f)}t}\langle f, X_t\rangle - \sum_{j=\gamma(f)}^{\zeta(f)} e^{-i\Im_j t}H_{\infty}^{(j)}F_{f,j} \to 0, \quad in \ L^2(\mathbb{P}_{\nu}).$$

rem:large

ge Remark 1.16 Suppose $f \in L^2(E,m;\mathbb{C}) \cap L^4(E,m;\mathbb{C})$ with $\gamma(f) = 1$. Then $\zeta(f) = 1$. Since $D_1(t) \equiv 1, \tau(f) = 0$. Thus $H_t^{(1)}$ reduces to W_t and $H_{\infty}^{(1)} = W_{\infty}$. Therefore by Theorem $\frac{\texttt{thrm2}}{1.15}$ and the fact that $F_{f,1} = \langle f, \psi_1 \rangle_m$, we get that for any nonzero $\nu \in \mathcal{M}_a(E)$,

$$e^{\lambda_1 t} \langle f, X_t \rangle \to \langle f, \psi_1 \rangle_m W_\infty, \quad in \ L^2(\mathbb{P}_{\nu}),$$

as $t \to \infty$. It is obvious that the convergence also holds in \mathbb{P}_{ν} -probability.

In particular, if f is non-zero and non-negative, then $\langle f, \psi_1 \rangle_m \neq 0$ which implies $\gamma(f) = 1$. \Box

1.4.2 Main result

For $f \in \mathcal{C}_s$, define

$$\sigma_f^2 := \int_0^\infty e^{\lambda_1 s} \langle A | T_s f |^2, \psi_1 \rangle_m \, ds + \langle |f|^2, \psi_1 \rangle_m. \tag{1.33} \quad \texttt{e:sigma}$$

For $h = \sum_{k:\lambda_1=2\Re_k} (\Phi_k(x))^T b_k \in \mathcal{C}_c$, define

$$\rho_h^2 := (1 + 2\tau(h))^{-1} \langle AF_h, \psi_1 \rangle_m, \qquad (1.34) \quad \text{e:rho}$$

where $F_h(x) := \sum_{k:\lambda_1=2\Re_k} \left| (\Phi_k(x))^T F_{h,k} \right|^2$. For $g(x) = \sum_{k:\lambda_1>2\Re_k} (\Phi_k(x))^T b_k \in \mathcal{C}_l$, define

$$I_{s}g(x) := \sum_{k:\lambda_{1}>2\Re_{k}} e^{\lambda_{k}s} \Phi_{k}(x)^{T} D_{k}(s)^{-1} b_{k}, \quad \beta_{g}^{2} := \int_{0}^{\infty} e^{-\lambda_{1}u} \langle A | I_{u}g |^{2}, \psi_{1} \rangle_{m} \, du - \langle g^{2}, \psi_{1} \rangle_{m}$$

and

$$E_t(g) := \sum_{k:\lambda_1 > 2\Re_k} \left(e^{-\lambda_k t} H_{\infty}^{(k)} D_k(t) b_k \right).$$

The:1.3 Theorem 1.17 If $f \in C_s$, $h \in C_c$ and $g \in C_l$, then σ_f^2 , ρ_h^2 and β_g^2 all belong to $(0, \infty)$. Furthermore, it holds that, under $\mathbb{P}_{\nu}(\cdot | \mathcal{E}^c)$, as $t \to \infty$,

$$\begin{pmatrix} e^{\lambda_1 t} \langle \phi_1, X_t \rangle, \quad \frac{\langle g, X_t \rangle - E_t(g)}{\sqrt{\langle \phi_1, X_t \rangle}}, \quad \frac{\langle h, X_t \rangle}{\sqrt{t^{1+2\tau(h)} \langle \phi_1, X_t \rangle}}, \quad \frac{\langle f, X_t \rangle}{\sqrt{\langle \phi_1, X_t \rangle}} \end{pmatrix}
\stackrel{d}{\to} (W^*, G_3(g), G_2(h), \quad G_1(f)),$$
(1.35)

where W^* has the same distribution as W_{∞} conditioned on \mathcal{E}^c , $G_3(g) \sim \mathcal{N}(0, \beta_g^2)$, $G_2(h) \sim \mathcal{N}(0, \rho_h^2)$ and $G_1(f) \sim \mathcal{N}(0, \sigma_f^2)$. Moreover, W^* , $G_3(g)$, $G_2(h)$ and $G_1(f)$ are independent.

Whenever $f \in \mathcal{C}_s$, we will use $G_1(f)$ to denote a normal random variable $\mathcal{N}(0, \sigma_f^2)$. For $f_1, f_2 \in \mathcal{C}_s$, define

$$\sigma(f_1, f_2) := \int_0^\infty e^{\lambda_1 s} \langle A(T_s f_1)(T_s f_2), \psi_1 \rangle_m \, ds + \langle f_1 f_2, \psi_1 \rangle_m \, ds$$

Cor:1 Corollary 1.18 If $f_1, f_2 \in \mathcal{C}_s$, then, under $\mathbb{P}_{\nu}(\cdot \mid \mathcal{E}^c)$,

$$\left(\frac{\langle f_1, X_t \rangle}{\sqrt{\langle \phi_1, X_t \rangle}}, \frac{\langle f_2, X_t \rangle}{\sqrt{\langle \phi_1, X_t \rangle}}\right) \stackrel{d}{\to} (G_1(f_1), G_1(f_2)), \quad t \to \infty,$$

and $(G_1(f_1), G_1(f_2))$ is a bivariate normal random variable with covariance

$$Cov(G_1(f_1), G_1(f_2)) = \sigma(f_1, f_2).$$
 (1.36) sigma(fg)

Proof: Using the convergence of the fourth component in Theorem $\begin{bmatrix} The: 1.3 \\ 1.17 \end{bmatrix}$, we get

$$\mathbb{P}_{\nu}\left(\exp\left\{i\theta_{1}\frac{\langle f_{1}, X_{t}\rangle}{\sqrt{\langle\phi_{1}, X_{t}\rangle}} + i\theta_{2}\frac{\langle f_{2}, X_{t}\rangle}{\sqrt{\langle\phi_{1}, X_{t}\rangle}}\right\} \mid \mathcal{E}^{c}\right)$$

$$= \mathbb{P}_{\nu}\left(\exp\left\{i\frac{\langle\theta_{1}f_{1}+\theta_{2}f_{2},X_{t}\rangle}{\sqrt{\langle\phi_{1},X_{t}\rangle}}\right\} \mid \mathcal{E}^{c}\right)$$

$$\to \exp\left\{-\frac{1}{2}\sigma^{2}_{(\theta_{1}f_{1}+\theta_{2}f_{2})}\right\}, \text{ as } t \to \infty,$$

where

$$\begin{aligned} \sigma_{(\theta_1 f_1 + \theta_2 f_2)}^2 &= \int_0^\infty e^{\lambda_1 s} \langle A(T_s(\theta_1 f_1 + \theta_2 f_2))^2, \psi_1 \rangle_m \, ds + \langle (\theta_1 f_1 + \theta_2 f_2)^2, \psi_1 \rangle_m \\ &= \theta_1^2 \sigma_{f_1}^2 + 2\theta_1 \theta_2 \sigma(f_1, f_2) + \theta_2^2 \sigma_{f_2}^2. \end{aligned}$$

Now (1.36) follows immediately.

Whenever $h \in \mathcal{C}_c$, we will use $G_2(h)$ to denote a normal random variable $\mathcal{N}(0, \rho_h^2)$. For $h_1, h_2 \in \mathcal{C}_c$, define

$$\rho(h_1, h_2) := (1 + \tau(h_1) + \tau(h_2))^{-1} \langle AF_{h_1, h_2}, \psi_1 \rangle_m, \qquad (1.37) \quad \text{rho2}$$

where

$$F_{h_1,h_2}(x) := \sum_{j:\lambda_1=2\Re_j} \Phi_j(x)^T F_{h_1,j} \Phi_{j'}(x)^T F_{h_2,j'} = \sum_{j:\lambda_1=2\Re_j} \Phi_j(x)^T F_{h_1,j} \overline{\Phi_j(x)^T F_{h_2,j}}.$$
 (1.38) e:Ffg

cor:2 Corollary 1.19 If $h_1, h_2 \in C_c$, then we have, under $\mathbb{P}_{\nu}(\cdot \mid \mathcal{E}^c)$,

$$\left(\frac{\langle h_1, X_t \rangle}{\sqrt{t^{1+2\tau(h_1)}\langle \phi_1, X_t \rangle}}, \frac{\langle h_2, X_t \rangle}{\sqrt{t^{1+2\tau(h_2)}\langle \phi_1, X_t \rangle}}\right) \stackrel{d}{\to} (G_2(h_1), G_2(h_2)), \quad t \to \infty,$$

and $(G_2(h_1), G_2(h_2))$ is a bivariate normal random variable with covariance

$$Cov(G_2(h_1), G_2(h_2)) = \rho(h_1, h_2).$$

Whenever $g \in C_l$, we will use $G_3(g)$ to denote a normal random variable $\mathcal{N}(0, \beta_g^2)$. For $g_1(x), g_2(x) \in C_l$, define

$$\beta(g_1,g_2) := \int_0^\infty e^{-\lambda_1 s} \langle A(I_s g_1)(I_s g_2), \psi_1 \rangle_m \, ds - \langle g_1 g_2, \psi_1 \rangle_m \, ds$$

Using the convergence of the second component in Theorem $\frac{\text{The: 1.3}}{1.17 \text{ and }}$ an argument similar to that in the proof of Corollary $\frac{\text{Cor: 1}}{1.18}$, we get

cor:3 Corollary 1.20 If $g_1(x), g_2(x) \in \mathcal{C}_l$, then we have, under $\mathbb{P}_{\nu}(\cdot | \mathcal{E}^c)$,

$$\left(\frac{\langle g_1, X_t \rangle - E_t(g_1)}{\sqrt{\langle \phi_1, X_t \rangle}}, \frac{\langle g_2, X_t \rangle - E_t(g_2)}{\sqrt{\langle \phi_1, X_t \rangle}}\right) \stackrel{d}{\to} (G_3(g_1), G_3(g_2)),$$

and $(G_3(g_1), G_3(g_2))$ is a bivariate normal random variable with covariance

$$Cov(G_3(g_1), G_3(g_2)) = \beta(g_1, g_2).$$

For any $f \in L^2(E,m) \cap L^4(E,m)$, define

$$\begin{split} f_{(s)}(x) &:= \sum_{j:2\Re_j < \lambda_1} (\Phi_j(x))^T \langle f, \Psi_j \rangle_m, \\ f_{(c)}(x) &:= \sum_{j:2\Re_j = \lambda_1} (\Phi_j(x))^T \langle f, \Psi_j \rangle_m, \\ f_{(l)}(x) &:= f(x) - f_{(s)}(x) - f_{(l)}(x). \end{split}$$

Then $f_{(s)} \in \mathcal{C}_l, f_{(c)} \in \mathcal{C}_c$ and $f_{(l)} \in \mathcal{C}_s$.

Remark 1.21 If $f \in L^2(E,m) \cap L^4(E,m)$ with $\lambda_1 = 2\Re_{\gamma(f)}$, then $f = f_{(c)} + f_{(l)}$. Using the convergence of the fourth component in Theorem Theorem I. If $f_{(l)}$, it holds under $\mathbb{P}_{\nu}(\cdot | \mathcal{E}^c)$ that

$$\frac{\langle f_{(l)}, X_t \rangle}{\sqrt{t^{1+2\tau(f)} \langle \phi_1, X_t \rangle}} \xrightarrow{d} 0, \quad t \to \infty.$$

Thus using the convergence of the first and third components in Theorem $\frac{\text{The: } 1.3}{1.17}$, we get, under $\mathbb{P}_{\nu}(\cdot \mid \mathcal{E}^{c})$,

$$\left(e^{\lambda_1 t} \langle \phi_1, X_t \rangle, \ \frac{\langle f, X_t \rangle}{\sqrt{t^{1+2\tau(f)} \langle \phi_1, X_t \rangle}}\right) \stackrel{d}{\to} (W^*, \ G_2(f_{(c)})), \quad t \to \infty$$

where W^* has the same distribution as W_{∞} conditioned on \mathcal{E}^c and $G_2(f_{(c)}) \sim \mathcal{N}(0, \rho_{f_{(c)}}^2)$. Moreover, W^* and $G_2(f_{(c)})$ are independent.

Remark 1.22 Assume $f \in L^2(E,m) \cap L^4(E,m)$ satisfies $\lambda_1 > 2\Re_{\gamma(f)}$.

If $f_{(c)} = 0$, then $f = f_{(l)} + f_{(s)}$. Using the convergence of the first, second and fourth components in Theorem 1.17, we get for any nonzero $\nu \in \mathcal{M}_a(E)$, it holds under $\mathbb{P}_{\nu}(\cdot | \mathcal{E}^c)$ that, as $t \to \infty$,

$$\left(e^{\lambda_1 t} \langle \phi_1, X_t \rangle, \ \frac{\left(\langle f, X_t \rangle - \sum_{k:2\Re_k < \lambda_1} e^{-\lambda_k t} H_\infty^{(k)} D_k(t) \langle f, \Psi_k \rangle_m\right)}{\langle \phi_1, X_t \rangle^{1/2}}\right) \stackrel{d}{\to} (W^*, \ G_1(f_{(l)}) + G_3(f_{(s)})),$$

where W^* , $G_3(f_{(s)})$ and $G_1(f_{(l)})$ are the same as those in Theorem [1.17]. Since $G_3(f_{(s)})$ and $G_1(f_{(l)})$ are independent, $G_1(f_{(l)}) + G_3(f_{(s)}) \sim \mathcal{N}\left(0, \sigma_{f_{(l)}}^2 + \beta_{f_{(s)}}^2\right)$.

If $f_{(c)} \neq 0$, then as $t \to \infty$,

large

$$\frac{\left(\langle f_{(l)} + f_{(s)}, X_t \rangle - \sum_{k:2\Re_k < \lambda_1} e^{-\lambda_k t} H_{\infty}^{(k)} \langle f, \Psi_k \rangle_m\right)}{\sqrt{t^{1+2\tau(f)} \langle \phi_1, X_t \rangle}} \stackrel{d}{\to} 0.$$

Then using the convergence of the first and third components in Theorem $\begin{bmatrix} \text{The: } 1.3\\ 1.17, we \end{bmatrix}$ get

$$\left(e^{\lambda_1 t}\langle\phi_1, X_t\rangle, \frac{\left(\langle f, X_t\rangle - \sum_{k:2\Re_k < \lambda_1} e^{-\lambda_k t} H_\infty^{(k)} D_k(t)\langle f, \Psi_k\rangle_m\right)}{\sqrt{t^{1+2\tau(f)}\langle\phi_1, X_t\rangle}}\right) \stackrel{d}{\to} (W^*, \ G_2(f_{(c)})),$$

where W^* and $G_2(f_{(c)})$ are the same as those in Remark $\frac{\texttt{r:critical}}{1.21.}$ Thus $\frac{\texttt{RSZ}}{[22]}$, Theorem 1.13] is a consequence of Theorem $\frac{\texttt{The:1.3}}{1.17.}$

2 Estimates on the moments of X

In the remainder of this paper we will use the following notation: for two positive functions f(t, x)and $g(t,x), f(t,x) \leq g(t,x)$ means that there exists a constant c > 0 such that $f(t,x) \leq cg(t,x)$ for all t, x.

Estimates on the first moment of X $\mathbf{2.1}$

Lemma 2.1 For each $k \in \mathbb{I}$, if $a < \Re_{k+1}$, there exists a constant c(k, a) > 0 such that for all t > 0, lemma2.1

$$\|T_t|_{(\widehat{\mathcal{M}}_k)^{\perp}}\|_2 \le c(k,a)e^{-at} \quad and \quad \|\widehat{T}_t|_{(\mathcal{M}_k)^{\perp}}\|_2 \le c(k,a)e^{-at}.$$
(2.1) 2.4

Proof: Since $(\mathcal{M}_k)^{\perp}$ is invariant for \widehat{T}_t , $\{\widehat{T}_t|_{(\mathcal{M}_k)^{\perp}} : t > 0\}$ is a semigroup on $(\mathcal{M}_k)^{\perp}$. By [7, 7]Theorem 6.7.5], we have $\sigma(\widehat{T}_1|_{(\mathcal{M}_k)^{\perp}}) = \{e^{-\lambda_j}, k+1 \leq j \in \mathbb{I}\} \cup \{0\}$. Thus, if $k+1 \in \mathbb{I}$, the spectral radius of $\widehat{T}_1|_{(\mathcal{M}_k)^{\perp}}$ is $r(\widehat{T}_t|_{(\mathcal{M}_k)^{\perp}}) = e^{-\Re_{k+1}} < e^{-a}$. If k+1 does not belong to \mathbb{I} , then $r(\widehat{T}_t|_{(\mathcal{M}_k)^{\perp}}) = 0 < e^{-a}.$

By $[7, \text{ Theorem 6.3.10}], r(\widehat{T}_1|_{(\mathcal{M}_k)^{\perp}}) = \lim_{n \to \infty} (\|\widehat{T}_n|_{(\mathcal{M}_k)^{\perp}}\|_2)^{1/n}$, thus there exists a constant n_1 , such that

$$\|\widehat{T}_{n_1}|_{(\mathcal{M}_k)^{\perp}}\|_2 \le e^{-an_1}.$$
 (2.2) 2.3

By $(\frac{1.66}{1.15})$, we have

$$\sup_{0 \le t \le n_1} \|\widehat{T}_t|_{(\mathcal{M}_k)^{\perp}}\|_2 \le \sup_{0 \le t \le n_1} \|\widehat{T}_t\|_2 \le e^{Kn_1}.$$
(2.3) 2.2

For any t > 0, there exist $l \in \mathbb{N}$ and $r \in [0, n_1)$ such that $t = n_1 l + r$. By $(\stackrel{2.3}{2.2})$ and $(\stackrel{2.2}{2.3})$, we have

$$\|\widehat{T}_t|_{(\mathcal{M}_k)^{\perp}}\|_2 \le \|\widehat{T}_{n_1}|_{(\mathcal{M}_k)^{\perp}}\|_2^l \|\widehat{T}_r|_{(\mathcal{M}_k)^{\perp}}\|_2 \le e^{-an_1l}e^{Kn_1} \le e^{Kn_1} \left(\sup_{0\le r\le n_1} e^{ar}\right)e^{-at}.$$

Thus we can find c(k,a) > 1 such that $\|\widehat{T}_t|_{(\mathcal{M}_k)^{\perp}}\|_2 \leq c(k,a)e^{-at}$. Similarly, we can show that $||T_t|_{(\widehat{\mathcal{M}}_k)^{\perp}}||_2 \le c(k,a)e^{-at}.$

Lemma 2.2 For each $k \in \mathbb{I}$ and $t_1 > 0$, if $a < \Re_{k+1}$, there exists a constant $c(k, a, t_1) > 0$ such that for all $(t, x, y) \in (2t_1, \infty) \times E \times E$,

$$\left| q(t,x,y) - \sum_{j=1}^{k} e^{-\lambda_j t} (\Phi_j(x))^T D_j(t) \overline{\Psi_j(y)} \right| \le c e^{-at} b_{t_1}(x)^{1/2} \widehat{b}_{t_1}(y)^{1/2}.$$
(2.4) density

Proof: Recall that for any $f \in L^2(E,m;\mathbb{C})$ and $k \in \mathbb{I}$, \widehat{f}_k is defined in the paragraph containing $|\hat{\mathbf{f}}_{k}^{\mathbf{k}}|_{2} \leq c_{1}(k)||f||_{2}$. Since $|\langle f, \phi_{l}^{(j)} \rangle_{m}| \leq ||f||_{2}$, we have $|\hat{f}_{k}(x)| \leq ||f||_{2} \sum_{j=1}^{k} \sum_{l=1}^{n_{j}} |\psi_{l}^{(j)}(x)|$. Thus, we get $||\hat{f}_{k}||_{2} \leq c_{1}(k)||f||_{2}$. By Lemma 2.1 for any $a < \Re_{k+1}$, there exists a constant $c_{2} = c_{2}(k, a) > 0$ such that for all t > 0,

$$\|\widehat{T}_t(f - \widehat{f}_k)\|_2 \le c_2 e^{-at} \|f - \widehat{f}_k\|_2 \le c_3 e^{-at} \|f\|_2, \qquad (2.5) \quad \boxed{2.7}$$

lemma2.2

where $c_3 = c_2(1 + c_1(k))$. For $t > t_1$, we have

$$q(t, x, y) = \int_E q(t_1, x, z)q(t - t_1, z, y) m(dz) = \widehat{T}_{t-t_1}(h_x)(y),$$

where $h_x(z) = q(t_1, x, z) \in L^2(E, m)$. It is easy to see that

$$\langle h_x, \phi_l^{(j)} \rangle_m = \int_E q(t_1, x, z) \overline{\phi_l^{(j)}}(z) \, m(dz) = \overline{T_{t_1}(\phi_l^{(j)})(x)}.$$

Let

$$h_{x,k}(z) := \sum_{j=1}^{k} \sum_{l=1}^{n_j} \langle h_x, \phi_l^{(j)} \rangle_m \psi_l^{(j)}(z) = \sum_{j=1}^{k} \overline{T_{t_1}((\Phi_j)^T)(x)} \Psi_j(z).$$

By $(\underbrace{I-Jordan}_{I.18})$ and $(\underbrace{I.26})$, we have

$$\begin{aligned} \widehat{T}_{t-t_1}(h_{x,k})(y) &= \sum_{j=1}^k \overline{T_{t_1}(\Phi_j)^T(x)} \widehat{T}_{t-t_1}(\Psi_j)(y) = \sum_{j=1}^k e^{-\overline{\lambda_j}t} (\overline{\Phi_j(x)})^T D_j(t_1) D_j(t-t_1) \Psi_j(y) \\ &= \sum_{j=1}^k e^{-\overline{\lambda_j}t} (\overline{\Phi_j(x)})^T D_j(t) \Psi_j(y). \end{aligned}$$

Thus, by (2.5), we have

$$\int_{E} |q(t,x,y) - \sum_{j=1}^{k} e^{-\overline{\lambda_{j}}t} (\overline{\Phi_{j}(x)})^{T} D_{j}(t) \Psi_{j}(y)|^{2} m(dy) \le (c_{3})^{2} e^{-2a(t-t_{1})} ||h_{x}||_{2}^{2} = c_{4} e^{-2at} b_{t_{1}}(x),$$

where $c_4 = c_4(k, a, t_1) = c_3^2 e^{-2at_1}$. Since q(t, x, y) is a real-valued function, we have, for $t > t_1$,

$$\int_{E} |q(t,x,y) - \sum_{j=1}^{k} e^{-\lambda_{j}t} (\Phi_{j}(x))^{T} D_{j}(t) \overline{\Psi_{j}(y)}|^{2} m(dy) \le c_{4} e^{-2at} b_{t_{1}}(x).$$
(2.6) 2.11

Repeating the above argument with T_t , we get that there exists $c_5 = c_5(k, a, t_1) > 0$ such that for $t > t_1$,

$$\int_{E} |q(t,z,y) - \sum_{j=1}^{k} e^{-\lambda_j t} (\Phi_j(z))^T D_j(t) \overline{\Psi_j(y)}|^2 m(dz) \le c_5 e^{-2at} \widehat{b}_{t_1}(y).$$
(2.7) 2.15

Since $D_j(t) = D_j(t/2)D_j(t/2)$, we get

$$e^{-\lambda_j t} (\Phi_j(x))^T D_j(t) \overline{\Psi_j(y)} = e^{-\lambda_j t/2} \int_E q(t/2, x, z) (\Phi_j(z))^T D_j(t/2) \overline{\Psi_j(y)} \, m(dz), \qquad (2.8)$$

$$e^{-\lambda_j t} (\Phi_j(x))^T D_j(t) \overline{\Psi_j(y)} = e^{-\lambda_j t/2} \int_E q(t/2, z, y) (\Phi_j(x))^T D_j(t/2) \overline{\Psi_j(z)} \, m(dz), \qquad (2.9) \quad \boxed{2.13}$$

and by $(\overset{\texttt{A}}{\amalg}.25)$, we have

$$\int_E \left(\sum_{j=1}^k e^{-\lambda_j t/2} (\Phi_j(x))^T D_j(t/2) \overline{\Psi_j(z)} \right) \left(\sum_{j=1}^k e^{-\lambda_j t/2} (\Phi_j(z))^T D_j(t/2) \overline{\Psi_j(y)} \right) m(dz)$$

$$= \sum_{j=1}^{k} e^{-\lambda_j t} (\Phi_j(x))^T D_j(t/2) D_j(t/2) \overline{\Psi_j(y)} = \sum_{j=1}^{k} e^{-\lambda_j t} (\Phi_j(x))^T D_j(t) \overline{\Psi_j(y)}.$$
 (2.10)

Thus, by the semigroup property of T_t and (2.12) (2.14), we obtain

$$\begin{aligned} q(t,x,y) &- \sum_{j=1}^{k} e^{-\lambda_{j}t} (\Phi_{j}(x))^{T} D_{j}(t) \overline{\Psi_{j}(y)} \\ &= \int_{E} q(t/2,x,z) q(t/2,z,y) \, m(dz) - \sum_{j=1}^{k} e^{-\lambda_{j}t/2} \int_{E} q(t/2,x,z) (\Phi_{j}(z))^{T} D_{j}(t/2) \overline{\Psi_{j}(y)} \, m(dz) \\ &- \sum_{j=1}^{k} e^{-\lambda_{j}t/2} \int_{E} q(t/2,z,y) (\Phi_{j}(x))^{T} D_{j}(t/2) \overline{\Psi_{j}(z)} \, m(dz) \\ &+ \int_{E} \left(\sum_{j=1}^{k} e^{-\lambda_{j}t/2} (\Phi_{j}(x))^{T} D_{j}(t/2) \overline{\Psi_{j}(z)} \right) \left(\sum_{j=1}^{k} e^{-\lambda_{j}t/2} (\Phi_{j}(z))^{T} D_{j}(t/2) \overline{\Psi_{j}(y)} \right) \, m(dz) \\ &= \int_{E} \left(q(t/2,x,z) - \left(\sum_{j=1}^{k} e^{-\lambda_{j}t/2} (\Phi_{j}(x))^{T} D_{j}(t/2) \overline{\Psi_{j}(z)} \right) \right) \\ &\left(q(t/2,z,y) - \left(\sum_{j=1}^{k} e^{-\lambda_{j}t/2} (\Phi_{j}(z))^{T} D_{j}(t/2) \overline{\Psi_{j}(y)} \right) \right) \, m(dz). \end{aligned}$$

Therefore, by Hölder's inequality, $\binom{2.11}{2.6}$ and $\binom{2.15}{2.7}$, we get, for $t > 2t_1$,

$$\left| q(t,x,y) - \sum_{j=1}^{k} e^{-\lambda_j t} (\Phi_j(x))^T D_j(t) \overline{\Psi_j(y)} \right| \le \sqrt{c_4 c_5} e^{-at} b_{t_1}(x)^{1/2} \widehat{b}_{t_1}(y)^{1/2}.$$

lemma2.3

Corollary 2.3 Assume $f \in L^2(E,m;\mathbb{C})$. If $\gamma(f) < \infty$, then, for any $t_1 > 0$, there exists a constant $c(f,t_1) > 0$ such that for all $(t,x) \in (2t_1,\infty) \times E$,

$$\left| t^{-\tau(f)} e^{\Re_{\gamma(f)} t} T_t f(x) - \sum_{j=\gamma(f)}^{\zeta(f)} e^{-i\Im_j t} (\Phi_j(x))^T F_{f,j} \right| \le c(f,t_1) t^{-1} b_{t_1}(x)^{1/2}.$$
(2.11) 1.31

Moreover, we have, for $(t, x) \in (2t_1, \infty) \times E$,

$$|T_t f(x)| \lesssim t^{\tau(f)} e^{-\Re_{\gamma(f)} t} b_{t_1}(x)^{1/2}.$$
(2.12) 1.23

If $\gamma(f) = \infty$, for any $t_1 > 0$, we have, for $(t, x) \in (2t_1, \infty) \times E$,

$$|T_t f(x)| \lesssim b_{t_1}(x)^{1/2}.$$
 (2.13) 1.23'

Proof: First, we consider the case $\gamma(f) < \infty$, which implies $\gamma(f) \in \mathbb{I}$. By the definition of $\zeta(f)$, we have $\Re_{\gamma(f)} < \Re_{\zeta(f)+1}$. Since $\langle f, (\hat{b}_{t_1})^{1/2} \rangle_m \leq \|\hat{b}_{t_1}^{1/2}\|_2 \|f\|_2$, applying Lemma 2.2 with $k = \zeta(f)$

and a fixed a with $\Re_{\gamma(f)} < a < \Re_{\zeta(f)+1}$, we get that there exists $c_1 = c_1(f, t_1) > 0$ such that for $(t, x) \in (2t_1, \infty) \times E$,

$$\left| T_t f(x) - e^{-\Re_{\gamma(f)}t} \sum_{j=\gamma(f)}^{\zeta(f)} e^{-i\Im_j t} (\Phi_j(x))^T D_j(t) \langle f, \Psi_j \rangle_m \right| \le c_1 e^{-at} b_{t_1}(x)^{1/2}.$$
 (2.14) 1.21

If $\tau(f) \geq 1$, the degree of each component of $D_j(t)\langle f, \Psi_j \rangle_m - t^{\tau(f)}F_{f,j}$ is no larger than $\tau(f) - 1$. Thus, for $t > 2t_1$,

$$|D_{j}(t)\langle f, \Psi_{j}\rangle_{m} - t^{\tau(f)}F_{f,j}|_{\infty} \lesssim t^{\tau(f)-1}.$$
(2.15) 1.22

If $\tau(f) = 0$, $D_j(t)\langle f, \Psi_j \rangle_m - t^{\tau(f)}F_{f,j} = 0$. By $(\stackrel{\text{phi}}{1.20})$, we get, for $(t, x) \in (2t_1, \infty) \times E$,

$$\left| \sum_{j=\gamma(f)}^{\zeta(f)} e^{-i\Im_{j}t} (\Phi_{j}(x))^{T} D_{j}(t) \langle f, \Psi_{j}(y) \rangle_{m} - t^{\tau(f)} \sum_{j=\gamma(f)}^{\zeta(f)} e^{-i\Im_{j}t} (\Phi_{j}(x))^{T} F_{f,j} \right|$$

$$\lesssim t^{\tau(f)-1} |\Phi_{j}(x)|_{\infty} \lesssim t^{\tau(f)-1} b_{t_{1}}(x)^{1/2}.$$
(2.16)

Now $(\stackrel{1.31}{2.11})$ follows easily from $(\stackrel{1.21}{2.14})$ and $(\stackrel{1.24}{2.16})$. By $(\stackrel{1.31}{2.11})$ and $(\stackrel{\text{phi}}{1.20})$, we get $(\stackrel{1.23}{2.12})$ immediately.

Now, we deal with the case $\gamma(f) = \infty$. Let $k_0 := \sup\{j : \Re_j \leq 0\}$. Thus, we have $k_0 \in \mathbb{I}$ and $\Re_{k_0+1} > 0$. Since $\gamma(f) = \infty$, so for any $k \in \mathbb{I}$, we have $\langle f, \Psi_k \rangle_m = 0$. Now, applying Lemma $\frac{12mma2.2}{2.2}$ with $k = k_0$ and a = 0, we get $(\frac{12.23}{2.13})$ immediately.

Tek3 Remark 2.4 Since $D_1(t) \equiv 1$, using $\binom{\texttt{density}}{2.4}$ with k = 1 and $\lambda_1 < a < \Re_2$, we get that, for any $t_1 > 0$, there exists $c_1(t_1, a) > 0$ such that for any $f \in L^2(E, m)$ and $(t, x) \in (2t_1, \infty) \times E$,

$$|e^{\lambda_1 t} T_t f(x) - \langle f, \psi_1 \rangle_m \phi_1(x)| \le c_1(t_1, a) e^{-(a-\lambda_1)t} ||f||_2 b_{t_1}(x)^{1/2}, \qquad (2.17) \quad 2.6$$

and hence there exists $c_2(t_1, a) > 0$ such that

$$e^{\lambda_1 t} |T_t f(x)| \le c_2 ||f||_2 b_{t_1}(x)^{1/2}.$$
 (2.18) 2.8

2.2 Estimates on the second moment of X

We first recall the formula for the second moment of the branching Markov process $\{X_t : t \ge 0\}$ (see, for example, [27, Lemma 3.3]): for $f \in \mathcal{B}_b(E)$, we have for any $(t, x) \in (0, \infty) \times E$,

$$\mathbb{P}_{\delta_x}\langle f, X_t \rangle^2 = \int_0^t T_s[A|T_{t-s}f|^2](x) \, ds + T_t(f^2)(x). \tag{2.19}$$

For any $f \in L^2(E,m) \cap L^4(E,m)$ and $x \in E$, since $(T_{t-s}f)^2(x) \leq e^{K(t-s)}T_{t-s}(f^2)(x)$, we have

$$\int_0^t T_s[A(T_{t-s}f)^2](x) \, ds \le K e^{Kt} T_t(f^2)(x) < \infty,$$

which implies

$$\int_0^t T_s[A(T_{t-s}f)^2](x)\,ds + T_t(f^2)(x) \le (1 + Ke^{Kt})T_t(f^2)(x) < \infty.$$
(2.20) 2.27

Thus, using a routine limit argument, one can easily check that $(\stackrel{1.19}{(2.19)})$ also holds for $f \in L^2(E,m) \cap L^4(E,m)$. Thus, for $f \in L^2(E,m;\mathbb{C}) \cap L^4(E,m;\mathbb{C})$, we have

$$\mathbb{P}_{\delta_x}|\langle f, X_t \rangle|^2 = \mathbb{P}_{\delta_x}\langle \Re(f), X_t \rangle^2 + \mathbb{P}_{\delta_x}\langle \Im(f), X_t \rangle^2 = \int_0^t T_s[A|T_{t-s}f|^2](x)\,ds + T_t(|f|^2)(x). \quad (2.21)$$

Let $\mathbb{V}ar_{\nu}$ be the variance under \mathbb{P}_{ν} . Then by the branching property, we have $\mathbb{V}ar_{\nu}\langle f, X_t \rangle = \langle \mathbb{V}ar_{\delta}\langle f, X_t \rangle, \nu \rangle$. By $(\stackrel{2.27}{2.20})$ and $(\stackrel{2.8}{2.18})$, we get, for $t > 2t_0$,

$$\begin{aligned} \operatorname{Var}_{\delta_{x}}\langle f, X_{t} \rangle &\leq \operatorname{\mathbb{P}}_{\delta_{x}} |\langle f, X_{t} \rangle|^{2} \leq (1 + Ke^{Kt}) T_{t}(|f|^{2})(x) \\ &\leq (1 + Ke^{Kt})e^{-\lambda_{1}t}b_{t}(x)^{1/2} |||f|^{2} ||_{2} \in L^{2}(E, m) \cap L^{4}(E, m). \end{aligned}$$
(2.22)

Recall that t_0 is the constant in Assumption 1(c).

Lemma 2.5 Assume that $f \in L^2(E,m;\mathbb{C}) \cap L^4(E,m;\mathbb{C})$. If $\lambda_1 > 2\Re_{\gamma(f)}$, then for any $(t,x) \in (10t_0,\infty) \times E$ we have,

$$\sup_{t>10t_0} t^{-2\tau(f)} e^{2\Re_{\gamma(f)}t} \mathbb{P}_{\delta_x} |\langle f, X_t \rangle|^2 \lesssim b_{t_0}(x)^{1/2}.$$
(2.23) 1.51

Proof: In this proof, we always assume $t > 10t_0$. For $s \le 2t_0$, we have $T_{t-s}[A|T_s f|^2](x) \le Ke^{Ks}T_t(|f|^2)(x) \le T_t(|f|^2)(x)$. Thus, by (2.12), we have for $t > 10t_0$,

$$\int_{0}^{2t_0} T_{t-s}[A|T_s f|^2](x) \, ds \lesssim T_t(|f|^2)(x) \lesssim e^{-\lambda_1 t} b_{t_0}(x)^{1/2}. \tag{2.24}$$

It follows from $(\stackrel{1.23}{(2.12)}$ again that, for $(s, x) \in (8t_0, \infty) \times E$, $|T_s f(x)| \lesssim s^{\tau(f)} e^{-\Re_{\gamma(f)} s} b_{4t_0}(x)^{1/2}$. Thus, for $(t, x) \in (10t_0, \infty) \times E$,

$$\int_{t-2t_0}^{t} T_{t-s}[A|T_s f|^2](x) \, ds \lesssim t^{2\tau(f)} \int_{t-2t_0}^{t} e^{-2\Re_{\gamma(f)}s} T_{t-s}(b_{4t_0})(x) \, ds$$
$$= t^{2\tau(f)} e^{-2\Re_{\gamma(f)}t} \int_{0}^{2t_0} e^{2\Re_{\gamma(f)}s} T_s(b_{4t_0})(x) \, ds \lesssim t^{2\tau(f)} e^{-2\Re_{\gamma(f)}t} \int_{0}^{2t_0} T_s(b_{4t_0})(x) \, ds. \quad (2.25)$$

We now show that for any $x \in E$, $\int_0^{2t_0} T_s(b_{4t_0})(x) ds < \infty$. By (B.9), we get

$$b_{4t_0}(x) \le e^{8Kt_0} a_{4t_0}(x) \le e^{10Kt_0} T_{2t_0}(a_{2t_0})(x).$$

Thus, by (2.18), we have

$$\int_{0}^{2t_0} T_s(b_{4t_0})(x) \, ds \le e^{10Kt_0} \int_{0}^{2t_0} T_{s+2t_0}(a_{2t_0})(x) \, ds \lesssim \int_{0}^{2t_0} e^{-\lambda_1(s+2t_0)} \, ds b_{t_0}(x)^{1/2} \lesssim b_{t_0}(x)^{1/2}.$$
(2.26) 1.37

By (2.25) and (2.25), we get

$$\int_{t-2t_0}^t T_{t-s}[A|T_s f|^2](x) \, ds \lesssim t^{2\tau(f)} e^{-2\Re_{\gamma(f)}t} b_{t_0}(x)^{1/2}. \tag{2.27}$$

For $s \in [2t_0, t - 2t_0]$, by $(\stackrel{1.23}{2.12})$, we have $|T_s f(x)|^2 \lesssim s^{2\tau(f)} e^{-2\Re_{\gamma(f)}s} b_{t_0}(x)$. By $(\stackrel{2.8}{2.18})$, we get $T_{t-s}[A(T_s f)^2](x) \lesssim s^{2\tau(f)} e^{-2\Re_{\gamma(f)}s} e^{-\lambda_1(t-s)} b_{t_0}(x)^{1/2}$. So, for $(t,x) \in (10t_0,\infty) \times E$,

$$\int_{2t_0}^{t-2t_0} T_{t-s}[A|T_s f|^2](x) \, ds \lesssim t^{2\tau(f)} e^{-\lambda_1 t} \int_0^t e^{(\lambda_1 - 2\Re_{\gamma(f)})s} \, ds b_{t_0}(x)^{1/2}$$

$$\lesssim t^{2\tau(f)} e^{-2\Re_{\gamma(f)}t} b_{t_0}(x)^{1/2}.$$
(2.29)

Combining $(\stackrel{|\!\!| 1.52}{2.24})$, $(\stackrel{|\!\!| 5.2}{2.27})$ and $(\stackrel{|\!\!| 1.54}{2.29})$, when $\lambda_1 > 2\Re_{\gamma(f)}$, we get

$$\int_{0}^{t} T_{t-s}[A|T_{s}f|^{2}](x) \, ds \lesssim t^{2\tau(f)} e^{-2\Re_{\gamma(f)}t} b_{t_{0}}(x)^{1/2}.$$

Since $\lambda_1 > 2\Re_{\gamma(f)}$, by (2.8), we have, for $(t, x) \in (10t_0, \infty) \times E$,

$$T_t(|f|^2)(x) \lesssim e^{-\lambda_1 t} b_{t_0}(x)^{1/2} \lesssim t^{2\tau(f)} e^{-2\Re_{\gamma(f)} t} b_{t_0}(x)^{1/2}.$$

Now $(\frac{1.51}{2.23})$ follows easily.

Lemma 2.6 Assume that $f \in L^2(E,m) \cap L^4(E,m)$. If $\lambda_1 < 2\Re_{\gamma(f)}$, then for $(t,x) \in (10t_0,\infty) \times E$,

$$\left| e^{\lambda_1 t} \mathbb{V}\mathrm{ar}_{\delta_x} \langle f, X_t \rangle - \sigma_f^2 \phi_1(x) \right| \lesssim c_t (b_{t_0}(x)^{1/2} + b_{t_0}(x)), \tag{2.30}$$
 small

where c_t is independent of x with $\lim_{t\to\infty} c_t = 0$ and σ_f^2 is defined in (1.33).

Proof: First, we consider the case $\gamma(f) < \infty$. In this proof, we always assume $t > 10t_0$ and $f \in L^2(E,m) \cap L^4(E,m)$. By (2.12), we have

$$e^{\lambda_1 t/2} |\mathbb{P}_{\delta_x}\langle f, X_t \rangle| \lesssim t^{\tau(f)} e^{-(2\Re_{\gamma(f)} - \lambda_1)t/2} b_{t_0}(x)^{1/2}.$$
(2.31) 1.6

We first show that $\sigma_f^2 < \infty$. For $s \leq 2t_0$, by $(\stackrel{\text{Lp}}{1.13})$, we have

$$||A|T_s f|^2||_2 \le K ||T_s f||_4^2 \le K e^{2Ks} ||f||_4^2.$$
(2.32) 1.32

For $s > 2t_0$, by $(\frac{1.23}{2.12})$, $|T_s f(x)| \lesssim e^{-\Re_{\gamma(f)}s} s^{\tau(f)} b_{t_0}(x)^{1/2}$. Thus, we have

$$\int_{0}^{\infty} e^{\lambda_{1}s} \langle A|T_{s}f|^{2}, \psi_{1} \rangle_{m} \, ds \leq K \|\psi_{1}\|_{2} \int_{0}^{\infty} e^{\lambda_{1}s} \||T_{s}f|^{2}\|_{2} \, ds$$

$$\lesssim \int_{0}^{2t_{0}} e^{\lambda_{1}s} \, ds + \int_{2t_{0}}^{\infty} e^{(\lambda_{1}-2\Re_{\gamma(f)})s} s^{2\tau(f)} \, ds < \infty, \qquad (2.33)$$

from which we easily see that $\sigma_f^2 < \infty$. By $(\stackrel{|1.13}{2.21})$, we have

$$\begin{aligned} & \left| e^{\lambda_{1}t} \mathbb{P}_{\delta_{x}} \langle f, X_{t} \rangle^{2} - \sigma_{f}^{2} \phi_{1}(x) \right| \\ \leq & e^{\lambda_{1}t} \int_{0}^{t-2t_{0}} \left| T_{t-s}[A|T_{s}f|^{2}](x) - e^{-\lambda_{1}(t-s)} \langle A|T_{s}f|^{2}, \psi_{1} \rangle_{m} \phi_{1}(x) \right| \, ds \\ & + e^{\lambda_{1}t} \int_{t-2t_{0}}^{t} T_{t-s}[A|T_{s}f|^{2}](x) \, ds + \int_{t-2t_{0}}^{\infty} e^{\lambda_{1}s} \langle A|T_{s}f|^{2}, \psi_{1} \rangle_{m} \, ds \phi_{1}(x) \end{aligned}$$

$$+|e^{\lambda_{1}t}T_{t}(|f|^{2})(x) - \langle |f|^{2}, \psi_{1}\rangle_{m}\phi_{1}(x)|$$

=: $V_{1}(t,x) + V_{2}(t,x) + V_{3}(t,x) + V_{4}(t,x).$ (2.34)

First, we consider $V_1(t,x)$. By (2.6) for $t-s > 2t_0$, there exists $a \in (\lambda_1, \Re_2)$ such that

$$\left| T_{t-s}[A|T_sf|^2](x) - e^{-\lambda_1(t-s)} \langle A|T_sf|^2, \psi_1 \rangle_m \phi_1(x) \right| \lesssim e^{-a(t-s)} \|A(T_sf)^2\|_2 b_{t_0}(x)^{1/2} \| \|A(T_sf)\|_2 \| \|A(T_sf$$

Therefore, by (2.12) and (2.32), we have

$$V_{1}(t,x) \lesssim e^{\lambda_{1}t}t^{2\tau(f)} \int_{2t_{0}}^{t-2t_{0}} e^{-a(t-s)}e^{-2\Re_{\gamma(f)}s} ds \, b_{t_{0}}(x)^{1/2} + e^{\lambda_{1}t} \int_{0}^{2t_{0}} e^{-a(t-s)} ds \, b_{t_{0}}(x)^{1/2}$$

$$\lesssim e^{-(a-\lambda_{1})t}t^{2\tau(f)} \int_{0}^{t} e^{(a-2\Re_{\gamma(f)})s} ds \, b_{t_{0}}(x)^{1/2} + e^{-(a-\lambda_{1})t} b_{t_{0}}(x)^{1/2}$$

$$\lesssim t^{2\tau(f)} \left(e^{(\lambda_{1}-2\Re_{\gamma(f)})t} + e^{-(a-\lambda_{1})t} \right) b_{t_{0}}(x)^{1/2}. \qquad (2.35)$$

Now we deal with $V_2(t, x)$. By (2.27), we have

$$V_2(t,x) \lesssim t^{2\tau(f)} e^{(\lambda_1 - 2\Re_{\gamma(f)})t} b_{t_0}(x)^{1/2}.$$
 (2.36) V2

For $V_3(t,x)$, by $(\stackrel{1.34}{(2.33)}$, we get $\int_{t-2t_0}^{\infty} e^{\lambda_1 s} \langle A|T_s f|^2, \psi_1 \rangle_m \, ds \to 0$, as $t \to \infty$. By $(\stackrel{\text{phi}}{(1.20)})$, we have $\phi_1(x) \lesssim b_{t_0}(x)^{1/2}$.

Finally, we consider $V_4(t, x)$. By (2.17), we have

$$V_4(t,x) \lesssim e^{-(a-\lambda_1)t} b_{t_0}(x)^{1/2}.$$
 (2.37) V4

Thus, by $({{\underline{V}}_{2.35}})-({{\underline{V}}_{2.37}})$, we have, for $(t,x) \in (10t_0,\infty) \times E$,

$$\left| e^{\lambda_1 t} \mathbb{P}_{\delta_x} \langle f, X_t \rangle^2 - \sigma_f^2 \phi_1(x) \right| \lesssim c_t b_{t_0}(x)^{1/2}, \tag{2.38}$$

with $\lim_{t\to\infty} c_t = 0$. Now $(\frac{2.22}{2.30})$ follows immediately from $(\frac{1.6}{2.31})$ and $(\frac{2.22}{2.38})$.

Now, we consider the case $\gamma(f) = \infty$. The proof is similar to that of the case $\gamma(f) < \infty$, the only difference being that we now use $\binom{1.23}{2.13}$ instead of $\binom{1.23}{2.12}$.

critical Lemma 2.7 Assume that $f, h \in L^2(E, m) \cap L^4(E, m)$. If $\lambda_1 = 2\Re_{\gamma(f)} = 2\Re_{\gamma(h)}$, then for $(t, x) \in (10t_0, \infty) \times E$,

$$\left| t^{-(1+\tau(f)+\tau(h))} e^{\lambda_1 t} \mathbb{C}_{\text{ov}_{\delta_x}}(\langle f, X_t \rangle, \langle h, X_t \rangle) - \rho(f, h) \phi_1(x) \right| \lesssim t^{-1} \left(b_{t_0}(x)^{1/2} + b_{t_0}(x) \right), \quad (2.39) \quad \boxed{7.49}$$

where $\mathbb{C}ov_{\delta_x}$ is the covariance under \mathbb{P}_{δ_x} and $\rho(f,h)$ is defined by $(\overbrace{1.37}^{\text{EO2}})$ with f and h in place of h_1 and h_2 respectively. In particular, we have, for $(t,x) \in (10t_0,\infty) \times E$,

$$\left| t^{-(1+2\tau(f))} e^{\lambda_1 t} \mathbb{V} \mathrm{ar}_{\delta_x} \langle f, X_t \rangle - \rho_f^2 \phi_1(x) \right| \lesssim t^{-1} \left(b_{t_0}(x)^{1/2} + b_{t_0}(x) \right), \tag{2.40}$$

where ρ_f^2 is defined by $(\overbrace{1.34}^{\texttt{erno}})$. Moreover, we have, for $(t, x) \in (10t_0, \infty) \times E$,

$$t^{-(1+2\tau(f))}e^{\lambda_1 t} \mathbb{V}\mathrm{ar}_{\delta_x}\langle f, X_t \rangle \lesssim \left(b_{t_0}(x)^{1/2} + b_{t_0}(x) \right).$$
(2.41) (2.41)

Proof: In this proof we always assume $t > 10t_0$ and $f, h \in L^2(E, m) \cap L^4(E, m)$. By $(\stackrel{1.13}{2.21})$, we have

$$\mathbb{C}\operatorname{ov}_{\delta_{x}}(\langle f, X_{t} \rangle, \langle h, X_{t} \rangle) \\
= \frac{1}{4} \left(\mathbb{V}\operatorname{ar}_{\delta_{x}} \langle (f+h), X_{t} \rangle - \mathbb{V}\operatorname{ar}_{\delta_{x}} \langle (f-h), X_{t} \rangle \right) \\
= \int_{0}^{t} T_{t-s} \left[A(T_{s}f)(T_{s}h) \right](x) \, ds + T_{t}(fh)(x) - T_{t}(f)(x)T_{t}(h)(x). \quad (2.42)$$

Let

$$C_{f}(s,x) := \sum_{j:\lambda_{1}=2\Re_{j}} \left(e^{-i\Im_{j}s} (\Phi_{j}(x))^{T} F_{f,j} \right), \quad C_{h}(s,x) := \sum_{j:\lambda_{1}=2\Re_{j}} \left(e^{-i\Im_{j}s} (\Phi_{j}(x))^{T} F_{h,j} \right).$$

Define

$$V_{5}(t,x) := e^{\lambda_{1}t} \int_{2t_{0}}^{t-2t_{0}} T_{t-s}[A(T_{s}f)(T_{s}h))](x) ds,$$

$$V_{6}(t,x) := e^{\lambda_{1}t} \int_{2t_{0}}^{t-2t_{0}} s^{\tau(f)+\tau(h)} e^{-\lambda_{1}s} T_{t-s}[AC_{f}(s,\cdot)C_{h}(s,\cdot)](x) ds,$$

$$V_{7}(t,x) := \int_{2t_{0}}^{t-2t_{0}} s^{\tau(f)+\tau(h)} \langle AC_{f}(s,\cdot)C_{h}(s,\cdot),\psi_{1} \rangle_{m} ds\phi_{1}(x)$$

and

$$V_8(t,x) := \int_{2t_0}^{t-2t_0} s^{\tau(f)+\tau(h)} \langle AF_{f,h}, \psi_1 \rangle_m \, ds \phi_1(x),$$

where $F_{f,h}$ is defined in $(\stackrel{[e:Ffg]}{1.38})$ with f and h in place of h_1 and h_2 respectively. It is easy to see from the definition of $\rho(f,h)$ that

$$\rho(f,h) = t^{-(1+\tau(f)+\tau(h))} \int_0^t s^{\tau(f)+\tau(h)} \langle AF_{f,h}, \psi_1 \rangle_m \, ds.$$

Thus we have

$$\left| e^{\lambda_{1}t} \int_{0}^{t} T_{t-s}[A(T_{s}f)(T_{s}h)](x) \, ds - t^{1+\tau(f)+\tau(h)}\rho(f,h)\phi_{1}(x) \right| \\
\leq e^{\lambda_{1}t} \left(\int_{0}^{2t_{0}} + \int_{t-2t_{0}}^{t} \right) T_{t-s}[A|T_{s}f||T_{s}h|](x) \, ds + |V_{5}(t,x) - V_{6}(t,x)| + |V_{6}(t,x) - V_{7}(t,x)| \\
+ |V_{7}(t,x) - V_{8}(t,x)| + \left(\int_{0}^{2t_{0}} + \int_{t-2t_{0}}^{t} \right) s^{\tau(f)+\tau(h)} \, ds \langle AF_{f,h}, \psi_{1} \rangle_{m} \phi_{1}(x). \tag{2.43}$$

By $(\frac{2.8}{2.18})$, for $s \le t - 2t_0$, we have

$$T_{t-s}[A|T_sf||T_sh|](x) \lesssim e^{-\lambda_1(t-s)} \|A|T_sf||T_sh|\|_2 (b_{t_0}(x))^{1/2}.$$

By $(\underline{\mathbb{Lp}}_{1.13})$, it is easy to see that $||A|T_sf||T_sh||_2 \le K||T_sf||_4||T_sh||_4 \le Ke^{2Ks}||f||_4||h||_4$. Thus,

$$e^{\lambda_1 t} \int_0^{2t_0} T_{t-s}[A|T_s f||T_s h|](x) \, ds \lesssim \int_0^{2t_0} e^{\lambda_1 s} \, ds (b_{t_0}(x))^{1/2} \lesssim (b_{t_0}(x))^{1/2}. \tag{2.44}$$

For $s > t - 2t_0$, using arguments similar to those leading to (2.27), we get

$$e^{\lambda_{1}t} \int_{t-2t_{0}}^{t} T_{t-s} \left[A|T_{s}f||T_{s}h| \right](x) \, ds \lesssim t^{\tau(f)+\tau(h)} e^{\lambda_{1}t} e^{-(\Re_{\gamma}(h)+\Re_{\gamma(f)})t} (b_{t_{0}}(x))^{1/2} \qquad (2.45)$$

$$t^{\tau(f)+\tau(h)} (b_{t_{0}}(x))^{1/2}. \qquad (2.46)$$

$$= t^{\tau(f)+\tau(h)} (b_{t_0}(x))^{1/2}.$$
(2.4)

By (1.20), it is easy to see that

$$\left(\int_{0}^{2t_{0}} + \int_{t-2t_{0}}^{t}\right) s^{\tau(f)+\tau(h)} ds \langle AF_{f,h}, \psi_{1} \rangle_{m} \phi_{1}(x) \lesssim t^{\tau(f)+\tau(h)} b_{t_{0}}(x)^{1/2}.$$
(2.47) 1.42

Next we consider $|V_5(t,x) - V_6(t,x)|$. By (2.11), we have, for $(s,x) \in (2t_0,\infty) \times E$,

$$|T_s f(x) - s^{\tau(f)} e^{-\lambda_1 s/2} C_f(s, x)| \lesssim s^{\tau(f) - 1} e^{-\lambda_1 s/2} b_{t_0}(x)^{1/2}$$

The same is also true for h. Thus by $(\stackrel{1.23}{[2.12]})$ and $(\stackrel{\text{phi}}{[1.20]})$, we get, for $(s, x) \in (2t_0, \infty) \times E$,

$$\begin{aligned} \left| |T_{s}f(x)T_{s}h(x)| - s^{\tau(f)+\tau(h)}e^{-\lambda_{1}s}C_{f}(s,x)C_{h}(s,x)| \right| \\ \lesssim \left| T_{s}f(x) - s^{\tau(f)}e^{-\lambda_{1}s/2}C_{f}(s,x) \right| \left| T_{s}h(x) - s^{\tau(h)}e^{-\lambda_{1}s/2}C_{h}(s,x) \right| \\ + s^{\tau(h)}e^{-\lambda_{1}s/2} \left| T_{s}f(x) - s^{\tau(f)}e^{-\lambda_{1}s/2}C_{f}(s,x) \right| \left| C_{h}(s,x) \right| \\ + s^{\tau(f)}e^{-\lambda_{1}s/2} \left| T_{s}h(x) - s^{\tau(h)}e^{-\lambda_{1}s/2}C_{h}(s,x) \right| \left| C_{f}(s,x) \right| \\ \lesssim s^{\tau(f)+\tau(h)-1}e^{-\lambda_{1}s}b_{t_{0}}(x). \end{aligned}$$

$$(2.48)$$

Therefore, by (2.8), we have, for $(t, x) \in (10t_0, \infty) \times E$,

$$|V_{5}(t,x) - V_{6}(t,x)| \lesssim \int_{2t_{0}}^{t-2t_{0}} s^{\tau(f)+\tau(h)-1} e^{\lambda_{1}(t-s)} T_{t-s}(b_{t_{0}})(x) \, ds$$

$$\lesssim \int_{2t_{0}}^{t-2t_{0}} s^{\tau(f)+\tau(h)-1} \, ds b_{t_{0}}(x)^{1/2} \lesssim t^{\tau(f)+\tau(h)} b_{t_{0}}(x)^{1/2}. \tag{2.49}$$

For $|V_6(t,x) - V_7(t,x)|$, by (2.17), there exists $\lambda_1 < a < \Re_2$, such that, for $t - s > 2t_0$,

$$\left| e^{\lambda_1(t-s)} T_{t-s} \left[A C_f(s, \cdot) C_h(s, \cdot) \right](x) - \langle A C_f(s, \cdot) C_h(s, \cdot), \psi_1 \rangle_m \phi_1(x) \right|$$

$$\lesssim e^{-(a-\lambda_1)(t-s)} \| C_f(s, \cdot) C_h(s, \cdot) \|_2 b_{t_0}(x)^{1/2}.$$

By $(\stackrel{\text{phi}}{1.20})$, we get, for $s > 2t_0$, $|C_f(s, x)C_h(s, x)| \leq b_{t_0}(x)$. Thus, we get

$$|V_{6}(t,x) - V_{7}(t,x)| \lesssim \int_{2t_{0}}^{t-2t_{0}} s^{\tau(f)+\tau(h)} e^{-(a-\lambda_{1})(t-s)} ds b_{t_{0}}(x)^{1/2}$$

$$\lesssim t^{\tau(f)+\tau(h)} \int_{2t_{0}}^{t-2t_{0}} e^{-(a-\lambda_{1})(t-s)} ds b_{t_{0}}(x)^{1/2} \lesssim t^{\tau(f)+\tau(h)} b_{t_{0}}(x)^{1/2}.$$
 (2.50)

Now we deal with $|V_7(t,x) - V_8(t,x)|$. We can check that $C_h(s,x)$ is real. In fact, for each j with $\lambda_1 = 2\Re_j$, we also have $\lambda_1 = 2\Re_{j'}$ and $e^{-i\Im_{j'}s}(\Phi_{j'}(x))^T F_{h,j'} = \overline{e^{-i\Im_j s}(\Phi_j(x))^T F_{h,j}}$. Thus, we have $C_h(s,x) = \overline{C_h(s,x)} = \sum_{j:\lambda_1=2\Re_j} \left(e^{i\Im_j s} \overline{(\Phi_j(x))^T F_{h,j}}\right)$. Therefore,

$$C_f(s,x)C_h(s,x) = \sum_{j:\lambda_1=2\Re_j} \Phi_j(x))^T F_{f,j} \overline{(\Phi_j(x))^T F_{h,j}}$$

+
$$\sum_{\gamma(f) \le j \ne l \le \zeta(f)} \left(e^{-i(\Im_j - \Im_l)s} (\Phi_j(x))^T F_{f,j}(\overline{\Phi_l(x)})^T F_{h,l} \right)$$

When $j \neq l$, since $\lambda_j \neq \lambda_l$ and $\Re_j = \Re_l$, we have $\Im_j \neq \Im_l$.

We claim that for any non-zero $\theta \in \mathbb{R}$ and $n \geq 0$, we have for $t > 2t_0$,

$$\left| \int_{2t_0}^{t-2t_0} s^n e^{i\theta s} \, ds \right| \lesssim t^n. \tag{2.51}$$

Then, using (1.38), we get

$$|V_{7}(t,x) - V_{8}(t,x)| \leq \sum_{\gamma(f) \leq j \neq l \leq \zeta(f)} \left| \int_{2t_{0}}^{t-2t_{0}} s^{\tau(f)+\tau(h)} e^{-i(\Im_{j}-\Im_{l})s} \, ds \right| \left| \langle (\Phi_{j}(x))^{T} F_{f,j}(\overline{\Phi_{l}(x)})^{T} \overline{F_{h,l}}, \psi_{1} \rangle_{m} \right| \phi_{1}(x) \leq t^{\tau(f)+\tau(h)} b_{t_{0}}(x)^{1/2}.$$

$$(2.52)$$

Now we prove (2.51). Using integration by parts, for $n \ge 1$, we get

$$\int_{2t_0}^{t-2t_0} s^n e^{i\theta s} \, ds \bigg| = \left| \frac{s^n e^{i\theta s} \big|_{2t_0}^{t-2t_0} - \int_{2t_0}^{t-2t_0} n s^{n-1} e^{i\theta s} \, ds}{i\theta} \right| \lesssim t^n + \int_{2t_0}^{t-2t_0} s^{n-1} \, ds \lesssim t^n.$$

For n = 0, we have

$$\left|\int_{2t_0}^{t-2t_0} e^{i\theta s} \, ds\right| = \left|\frac{e^{i\theta(t-2t_0)} - e^{i2\theta t_0}}{i\theta}\right| \le 2/|\theta|.$$

Therefore, $(\stackrel{[2.26]}{2.51})$ follows immediately. Combining $(\stackrel{[7.3]}{2.44})$, $(\stackrel{[1.42]}{2.46})$, $(\stackrel{[1.42]}{2.47})$, $(\stackrel{[1.38]}{2.49})$, $(\stackrel{[1.38]}{2.50})$ and $(\stackrel{[1.41]}{2.52})$, we get $(t, x) \in (10t_0, \infty) \times E$,

$$\left| e^{\lambda_1 t} \int_0^t T_{t-s} [A(T_s f)(T_s h)](x) \, ds - t^{1+\tau(f)+\tau(h)} \rho(f,h) \phi_1(x) \right| \lesssim t^{\tau(f)+\tau(h)} b_{t_0}(x)^{1/2}. \tag{2.53}$$
By (2.8) , we have, for $(t,x) \in (10t_0,\infty) \times E$,

$$e^{\lambda_1 t} T_t(|fh|)(x) \lesssim b_{t_0}(x)^{1/2}$$

And by (2.17) and $\lambda_1 = 2\Re_{\gamma(f)}$,

$$e^{\lambda_1 t} |T_t f(x)| |T_t h(x)| \lesssim t^{\tau(f) + \tau(h)} b_{t_0}(x).$$

Now $(\underline{7.49}_{2.39})$ follows immediately.

Ind critical Lemma 2.8 Assume that $f \in L^2(E,m) \cap L^4(E,m)$ with $\lambda_1 < 2\Re_{\gamma(f)}$ and $h \in L^2(E,m) \cap L^4(E,m)$ with $\lambda_1 = 2\Re_{\gamma(h)}$. Then, for any $(t,x) \in (10t_0,\infty) \times E$,

$$e^{\lambda_1 t} \mathbb{C}\operatorname{ov}_{\delta_x}(\langle f, X_t \rangle, \langle h, X_t \rangle) \lesssim ((b_{t_0}(x))^{1/2} + b_{t_0}(x)).$$

$$(2.54) \quad \text{[cov:sc]}$$

Proof: In this proof, we always assume that $t > 10t_0$, $f \in L^2(E, m) \cap L^4(E, m)$ with $\lambda_1 < 2\Re_{\gamma(f)}$ and $h \in L^2(E, m) \cap L^4(E, m)$ with $\lambda_1 = 2\Re_{\gamma(h)}$. First, we assume $\gamma(f) < \infty$. By $(\overline{\underline{L}.42})$, we have

$$\mathbb{C}\operatorname{ov}_{\delta_x}(\langle f, X_t \rangle, \langle h, X_t \rangle) = \int_0^t T_{t-s} \left[A(T_s f)(T_s h) \right](x) \, ds + T_t(fh)(x) - T_t(f)(x) T_t(h)(x).$$

By $(\frac{l\cdot 3}{2.44})$ and $(\frac{l\cdot 5}{2.45})$, we have, for $(t,x) \in (10t_0,\infty) \times E$,

$$e^{\lambda_{1}t} \left(\int_{0}^{2t_{0}} + \int_{t-2t_{0}}^{t} \right) T_{t-s} \left[A|T_{s}f||T_{s}h| \right](x) \, ds$$

$$\lesssim \quad b_{t_{0}}(x)^{1/2} + t^{\tau(f)+\tau(h)} e^{(\lambda_{1}/2 - \Re_{\gamma(f)})t} (b_{t_{0}}(x))^{1/2} \lesssim (b_{t_{0}}(x))^{1/2}$$

By (2.12), we have

$$e^{\lambda_{1}t} \int_{2t_{0}}^{t-2t_{0}} T_{t-s} \left[A|T_{s}f||T_{s}h|\right](x) ds \lesssim e^{\lambda_{1}t} \int_{2t_{0}}^{t-2t_{0}} s^{\tau(f)+\tau(h)} e^{-(\lambda_{1}/2+\Re_{\gamma(f)})s} T_{t-s}(b_{t_{0}})(x) ds$$
$$\lesssim \left(\int_{2t_{0}}^{t-2t_{0}} s^{\tau(f)+\tau(h)} e^{(\lambda_{1}/2-\Re_{\gamma(f)})s} ds\right) b_{t_{0}}(x)^{1/2} \lesssim b_{t_{0}}(x)^{1/2}.$$
(2.55)

Thus, we have

$$e^{\lambda_1 t} \left| \int_0^t T_{t-s} \left[A(T_s f)(T_s h) \right](x) \, ds \right| \lesssim (b_{t_0}(x))^{1/2}.$$
 (2.56) 1.61

By $(\frac{2.8}{2.18})$, we get

 $e^{\lambda_1 t} |T_t(fh)(x)| \le e^{\lambda_1 t} T_t(|fh|)(x) \lesssim b_{t_0}(x)^{1/2}.$

By $(\stackrel{1.23}{2.12})$, for $(t, x) \in (10t_0, \infty) \times E$, we have

$$e^{\lambda_1 t} |T_t f(x) T_t h(x)| \lesssim t^{\tau(f) + \tau(h)} e^{(\lambda_1/2 - \Re_{\gamma(f)})t} b_{t_0}(x) \lesssim b_{t_0}(x).$$

Now (2.54) follows immediately.

Repeating the proof above by using $(\frac{1.23}{2.13})$ instead of $(\frac{1.23}{2.12})$, we get $(\frac{\text{cov:sc}}{2.54})$ also holds when $\gamma(f) = \infty$.

3 Proofs of Main Results

In this section, we will prove the main results of this paper. When referring to individuals in X we will use the classical Ulam-Harris notation so that every individual in X has a unique label, see [10]. For each individual $u \in \mathcal{T}$ we shall write b_u and d_u for its birth and death times respectively and $\{z_u(r) : r \in [b_u, d_u]\}$ for its spatial trajectory. Define

$$\mathcal{L}_t = \{ u \in \mathcal{T}, b_u \le t < d_u \}, \quad t \ge 0.$$

Thus, X_{s+t} has the following decomposition:

$$X_{s+t} = \sum_{u \in \mathcal{L}_t} X_s^{u,t}, \tag{3.1}$$

where given $\mathcal{F}_t, X_s^{u,t}, u \in \mathcal{L}_t$, are independent and $X_s^{u,t}$ has the same law as X_s under $\mathbb{P}_{\delta_{z_u(t)}}$.

3.1 A basic law of large numbers

Recall that

thrm1

$$H_t^{(k)} := e^{\lambda_k t} (\langle \phi_1^{(k)}, X_t \rangle, \cdots, \langle \phi_{n_k}^{(k)}, X_t \rangle) (D_k(t))^{-1}.$$

Lemma 3.1 Assume that b is an n_k -dimensional vector. If $\lambda_1 > 2\Re_k$, then, for any $\nu \in \mathcal{M}_a(E)$, $H_t^{(k)}b$ is a martingale under \mathbb{P}_{ν} . Moreover, the limit

$$H_{\infty}^{(k)} := \lim_{t \to \infty} H_t^{(k)} \tag{3.2}$$

exists \mathbb{P}_{ν} -a.s. and in $L^2(\mathbb{P}_{\nu})$.

Proof: By the branching property, it suffices to prove the lemma for $\nu = \delta_x$ with $x \in E$. By $(\stackrel{[I-Jordan]}{(I.18)}$, we have

$$\mathbb{P}_{\delta_x} H_t^{(k)} b = e^{\lambda_k t} T_t((\Phi_k)^T)(x) (D_k(t))^{-1} b = (\Phi_k(x))^T b.$$

Thus, by the Markov property, we get that $H_t^{(k)}b$ is a martingale under \mathbb{P}_{δ_x} . We claim that, for $(t,x) \in (2t_0,\infty) \times E$,

$$\mathbb{P}_{\delta_x} |H_t^{(k)}b|^2 \lesssim |b|_{\infty}^2 b_{t_0}(x)^{1/2}, \tag{3.3}$$

from which $\begin{pmatrix} 1.48\\ 3.2 \end{pmatrix}$ follows immediately.

Now we prove the claim. Let $f_t(x) = e^{\lambda_k t} b^T (D_k(t)^{-1})^T \Phi_k(x)$. Then $H_t^{(k)} b = \langle f_t, X_t \rangle$, and by $(\underline{\mathbb{I}}$ -Jordan t < t, we have

$$T_s(f_t)(x) = e^{\lambda_k(t-s)}b^T (D_k(t-s)^{-1})^T \Phi_k = f_{t-s}(x).$$

By $(\frac{1.13}{2.21})$, we have

$$\mathbb{P}_{\delta_x}|H_t^{(k)}b|^2 = \mathbb{P}_{\delta_x}|\langle f_t, X_t\rangle|^2 = \int_0^t T_s[A|f_s|^2](x)\,ds + T_t(|f_t|^2)(x). \tag{3.4}$$

Since each component of $D_k(s)^{-1} = D_k(-s)$ is a polynomial of s with degree no larger than ν_k , we get $|D_k(s)^{-1}|_{\infty} \leq (1 + s^{\nu_k})$. Thus, for all s > 0, we have

$$|f_s| \lesssim e^{\Re_k s} |b|_{\infty} |D_k(s)|_{\infty} |\Phi_k(x)|_{\infty} \lesssim |b|_{\infty} (1 + s^{\nu_k}) e^{\Re_k s} b_{4t_0}(x)^{1/2}.$$
(3.5) 7.8

By (2.8), we have, for $(s, x) \in (2t_0, \infty) \times E$,

$$T_s(|f_s|^2)(x) \lesssim e^{-\lambda_1 s} |||f_s|^2 ||_2 b_{t_0}(x)^{1/2} \lesssim |b|_{\infty}^2 (1 + s^{2\nu_k}) e^{-(\lambda_1 - 2\Re_k)s} b_{t_0}(x)^{1/2}.$$
(3.6) 1.56

Thus, we have

$$\int_{2t_0}^t T_s[A|f_s|^2](x) \, ds \lesssim |b|_\infty^2 b_{t_0}(x)^{1/2}. \tag{3.7}$$

By $(\frac{7.8}{3.5})$ and $(\frac{1.37}{2.26})$, we get

$$\int_{0}^{2t_0} T_s[A|f_s|^2](x) \, ds \lesssim |b|_{\infty}^2 \int_{0}^{2t_0} T_s b_{4t_0}(x) \, ds \lesssim |b|_{\infty}^2 b_{t_0}(x)^{1/2}. \tag{3.8}$$

Thus, by $(\overbrace{B.7}{\textbf{7.7}})$ and $(\overbrace{B.8}{\textbf{1.59}})$, we have

$$\int_0^t T_s[A|f_s|^2](x) \, ds \lesssim |b|_\infty^2 b_{t_0}(x)^{1/2}. \tag{3.9}$$

Since $\lambda_1 > 2\Re_k$, we have $\sup_{s>2t_0}(1+s^{2\nu_k})e^{-(\lambda_1-2\Re_k)s} < \infty$. Thus, by $(\overset{1.56}{(3.6)}$, we get

$$T_t(|f_t|^2)(x) \lesssim |b|_{\infty}^2 b_{t_0}(x)^{1/2}$$

from which $(\frac{1.47}{3.3})$ follows immediately.

Now, we present the proof of Theorem 1.15.

Proof of Theorem 1.15: By the branching property, it suffices to prove the theorem for $\nu = \delta_x$ with $x \in E$. Put

$$f^*(x) := \sum_{j=\gamma(f)}^{\zeta(f)} \Phi_j(x)^T \langle f, \Psi_j \rangle, \quad \widetilde{f}(x) := f(x) - f^*(x)$$

and $f_t(x) := \sum_{j=\gamma(f)}^{\zeta(f)} \Phi_j(x)^T D_j(t)^{-1} F_{f,j}$. Then

$$t^{-\tau(f)}f^*(x) - f_t(x) = \sum_{j=\gamma(f)}^{\zeta(f)} \Phi_j(x)^T D_j(t)^{-1} \left(t^{-\tau(f)} D_j(t) \langle f, \Psi_j \rangle - F_{f,j} \right).$$

By $(\underline{1.47}, \underline{1.47}, \underline{1.47})$ and $(\underline{1.22}, \underline{1.15})$, we have, for $(t, x) \in (2t_0, \infty) \times E$,

$$\mathbb{P}_{\delta_{x}} \left| t^{-\tau(f)} e^{\Re_{\gamma(f)} t} \langle f^{*}, X_{t} \rangle - e^{\Re_{\gamma(f)} t} \langle f_{t}, X_{t} \rangle \right|^{2} \\
\lesssim \sum_{j=\gamma(f)}^{\zeta(f)} |t^{-\tau(f)} D_{j}(t) \langle f, \Psi_{j} \rangle - F_{f,j}|_{\infty}^{2} b_{t_{0}}(x)^{1/2} \lesssim t^{-2} b_{t_{0}}(x)^{1/2}.$$
(3.10)

By the definition of $H_t^{(j)}$ and $(\stackrel{1.48}{5.2})$, we have, as $t \to \infty$,

$$e^{\Re_{\gamma(f)}t}\langle f_t, X_t \rangle - \sum_{j=\gamma(f)}^{\zeta(f)} \left(e^{-i\Im_j t} H_{\infty}^{(j)} F_{f,j} \right) = \sum_{j=\gamma(f)}^{\zeta(f)} \left(e^{-i\Im_j t} (H_t^{(j)} - H_{\infty}^{(j)}) F_{f,j} \right) \to 0, \quad (3.11) \quad \boxed{2.16}$$

in $L^2(\mathbb{P}_{\delta_x})$. Thus, by $(\underline{3.10}^{2.17})$ and $(\underline{3.11}^{2.16})$, we obtain that, as $t \to \infty$,

$$t^{-\tau(f)}e^{\Re_{\gamma(f)}t}\langle f^*, X_t\rangle - \sum_{j=\gamma(f)}^{\zeta(f)} \left(e^{-i\Im_j t}H_{\infty}^{(j)}F_{f,j}\right) \to 0, \quad \text{in } L^2(\mathbb{P}_{\delta_x}).$$

$$(3.12) \quad \boxed{2.18}$$

Now, to complete the proof, we only need to show that, as $t \to \infty$,

$$t^{-2\tau(f)} e^{2\Re_{\gamma(f)} t} \mathbb{P}_{\delta_x} |\langle \widetilde{f}, X_t \rangle|^2 \to 0.$$
(3.13) 2.21

(1) If $\lambda_1 > 2\Re_{\gamma(\widetilde{f})}$, then by (2.23), we get, for $(t, x) \in (2t_0, \infty) \times E$, as $t \to \infty$,

$$t^{-2\tau(f)}e^{2\Re_{\gamma(f)}t}\mathbb{P}_{\delta_x}|\langle \widetilde{f}, X_t \rangle|^2 \lesssim t^{-2\tau(f)}t^{2\tau(\widetilde{f})}e^{2(\Re_{\gamma(f)}-\Re_{\gamma(\widetilde{f})})t}b_{t_0}(x)^{1/2} \to 0$$

(2) If $\lambda_1 = 2\Re_{\gamma(\tilde{f})}$, then by $(\stackrel{1.49}{2.40})$, we get, as $t \to \infty$,

$$\begin{split} t^{-2\tau(f)} e^{2\Re_{\gamma(f)}t} \mathbb{P}_{\delta_x} |\langle \widetilde{f}, X_t \rangle|^2 &= t^{-2\tau(f)} t^{(1+2\tau(\widetilde{f}))t} e^{(2\Re_{\gamma(f)}-\lambda_1)t} t^{-(1+2\tau(widef))t} e^{\lambda_1 t} \mathbb{P}_{\delta_x} |\langle \widetilde{f}, X_t \rangle|^2 \to 0. \\ (3) \text{ If } \lambda_1 < 2\Re_{\gamma(\widetilde{f})}, \text{ then by } (\frac{\texttt{small}}{2.30} \text{, we get, as } t \to \infty, \\ t^{-2\tau(f)} e^{2\Re_{\gamma(f)}t} \mathbb{P}_{\delta_x} |\langle \widetilde{f}, X_t \rangle|^2 &= t^{-2\tau(f)} e^{(2\Re_{\gamma(f)}-\lambda_1)t} e^{\lambda_1 t} \mathbb{P}_{\delta_x} |\langle \widetilde{f}, X_t \rangle|^2 \to 0. \end{split}$$

Combining the three cases above, we get $\begin{pmatrix} 2.21\\ 3.13 \end{pmatrix}$. The proof is now complete.

3.2 Proof of the main theorem

s:3

First, we recall a metric on the space of distributions on \mathbb{R}^d . For $f : \mathbb{R}^d \to \mathbb{R}$, define

$$||f||_{BL} := ||f||_{\infty} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$

For any distributions ν_1 and ν_2 on \mathbb{R}^d , define

$$\beta(\nu_1, \nu_2) := \sup\left\{ \left| \int f \, d\nu_1 - \int f \, d\nu_2 \right| : \|f\|_{BL} \le 1 \right\}.$$
Dudley

Then β is a metric. It follows from [9, Theorem 11.3.3] that the topology generated by this metric is equivalent to the weak convergence topology. From the definition, we can easily see that, if ν_1 and ν_2 are the distributions of two \mathbb{R}^d -valued random variables X and Y respectively, then

$$\beta(\nu_1, \nu_2) \le \mathbb{E} \|X - Y\| \le \sqrt{\mathbb{E} \|X - Y\|^2}.$$
 (3.14) 5.20

Lemma 3.2 If $f \in \mathcal{C}_s$, then $\sigma_f^2 \in (0, \infty)$ and, for any nonzero $\nu \in \mathcal{M}_a(E)$, it holds under \mathbb{P}_{ν} that

$$\left(e^{\lambda_1 t}\langle \phi_1, X_t\rangle, \ e^{\lambda_1 t/2}\langle f, X_t\rangle\right) \xrightarrow{d} \left(W_{\infty}, \ G_1(f)\sqrt{W_{\infty}}\right), \quad t \to \infty$$

where $G_1(f) \sim \mathcal{N}(0, \sigma_f^2)$. Moreover, W_{∞} and $G_1(f)$ are independent.

Proof: The proof is similar that of [23, Theorem 1.8]. We define an \mathbb{R}^2 -valued random variable $U_1(t)$ by

$$U_1(t) := \left(e^{\lambda_1 t} \langle \phi_1, X_t \rangle, e^{\lambda_1 t/2} \langle f, X_t \rangle \right).$$
(3.15) 6.4

To prove this lemma, it suffices to show that, for any $x \in E$, under \mathbb{P}_{δ_x} ,

$$U_1(t) \stackrel{d}{\to} \left(W_{\infty}, \sqrt{W_{\infty}} G_1(f) \right), \tag{3.16}$$

where $G_1(f) \sim \mathcal{N}(0, \sigma_f^2)$ is independent of W_{∞} . In fact, if $\nu = \sum_{j=1}^n \delta_{x_j}, n = 1, 2, \dots, \{x_j; j = 1, \dots, n\} \subset E$, then

$$X_t = \sum_{j=1}^n X_t^j,$$

where X_t^j is a branching Markov process starting from δ_{x_j} , $j = 1, \ldots, n$, and X^j , $j = 1, \cdots, n$, are independent. If $(\underline{\beta}.\underline{5})$ is valid, we put $W_{\infty}^j := \lim_{t \to \infty} e^{\lambda_1 t} \langle \phi_1, X_t^j \rangle$. Then we easily get that, under $\mathbb{P}_{\nu}, W_{\infty} = \sum_{j=1}^n W_{\infty}^j$. Since $\lambda_1 < 2\Re_{\gamma(f)}$,

$$\mathbb{P}_{\nu} \exp\left\{i\theta_{1}e^{\lambda_{1}t}\langle\phi_{1}, X_{t}\rangle + i\theta_{2}e^{(\lambda_{1}/2)t}\langle f, X_{t}\rangle\right)$$

$$= \prod_{j=1}^{n} \mathbb{P}_{\delta_{x_{j}}} \exp\left\{i\theta_{1}e^{\lambda_{1}t}\langle\phi_{1}, X_{t}^{j}\rangle + i\theta_{2}e^{(\lambda_{1}/2)t}\langle f, X_{t}^{j}\rangle\right)$$

$$\rightarrow \prod_{j=1}^{n} \mathbb{P}_{\delta_{x_{j}}} \exp\left\{i\theta_{1}W_{\infty}^{j} - \frac{1}{2}\theta_{2}^{2}\sigma_{f}^{2}W_{\infty}^{j}\right)$$

$$= \mathbb{P}_{\nu} \exp\left\{i\theta_{1}W_{\infty} - \frac{1}{2}\theta_{2}^{2}\sigma_{f}^{2}W_{\infty}\right),$$

which implies that $\begin{pmatrix} 6.5\\ (3.16) \end{pmatrix}$ is valid for \mathbb{P}_{ν} .

Now we show that $(\underline{6.5}, \underline{16})$ is valid. In the remainder of this proof, we assume $s, t > 10t_0$ and write

$$U_1(s+t) = \left(e^{\lambda_1(s+t)} \langle \phi_1, X_{t+s} \rangle, e^{(\lambda_1/2)(s+t)} \langle f, X_{s+t} \rangle\right).$$

Recall the decomposition of X_{s+t} in $(\underline{B.1})$. Define

$$Y_1^{u,t}(s) := e^{\lambda_1 s/2} \langle f, X_s^{u,t} \rangle \quad \text{and} \quad y_1^{u,t}(s) := \mathbb{P}_{\delta_x}(Y_1^{u,t}(s)|\mathcal{F}_t). \tag{3.17} \quad \boxed{\texttt{e:new}}$$

Given $\mathcal{F}_t, Y_1^{u,t}(s)$ has the same law as $Y_1(s) := e^{\lambda_1 s/2} \langle f, X_s \rangle$ under $\mathbb{P}_{\delta_{z_u(t)}}$. Then we have

$$e^{(\lambda_{1}/2)(s+t)}\langle f, X_{s+t} \rangle = e^{(\lambda_{1}/2)t} \sum_{u \in \mathcal{L}_{t}} Y_{1}^{u,t}(s)$$

= $e^{(\lambda_{1}/2)t} \sum_{u \in \mathcal{L}_{t}} (Y_{1}^{u,t}(s) - y_{s}^{u,t}) + e^{(\lambda_{1}/2)(t+s)} \mathbb{P}_{\delta_{x}}(\langle f, X_{s+t} \rangle | \mathcal{F}_{t})$
=: $J_{1}(s,t) + J_{2}(s,t).$ (3.18)

We first consider $J_2(s,t)$. By the Markov property, we have

$$J_2(s,t) = e^{(\lambda_1/2)(s+t)} \langle T_s f, X_t \rangle.$$

By $(\stackrel{1.13}{2.21})$, we get

$$\mathbb{P}_{\delta_x} \langle T_s f, X_t \rangle^2 = \int_0^t T_{t-u} [A(T_{u+s}(f))^2](x) \, du + T_t (T_s f)^2(x).$$

First, we consider the case $\gamma(f) < \infty$. Since $u + s \ge s > 10t_0$, by (2.12), we get

$$|T_{u+s}f(x)|^2 \lesssim (u+s)^{2\tau(f)} e^{-2\Re_{\gamma(f)}(u+s)} b_{4t_0}(x).$$
(3.19) 9.1

Thus, for $t > 10t_0$, we have

$$\int_0^{t-2t_0} T_{t-u} [A(T_{s+u}f)^2](x) \, du$$

$$\lesssim e^{-2s\Re_{\gamma(f)}} \int_{0}^{t-2t_{0}} (u+s)^{2\tau(f)} e^{-2\Re_{\gamma(f)}u} T_{t-u}(b_{4t_{0}})(x) du \lesssim e^{-2s\Re_{\gamma(f)}} \int_{0}^{t-2t_{0}} (u+s)^{2\tau(f)} e^{-2\Re_{\gamma(f)}u} e^{-\lambda_{1}(t-u)} du b_{t_{0}}(x)^{1/2}.$$

$$\lesssim e^{-\lambda_{1}t} e^{-2\Re_{\gamma(f)}s} \left(\int_{0}^{t-2t_{0}} u^{2\tau(f)} e^{(\lambda_{1}-2\Re_{\gamma(f)})u} du + s^{2\tau(f)} \int_{0}^{t-2t_{0}} e^{(\lambda_{1}-2\Re_{\gamma(f)})u} du \right) b_{t_{0}}(x)^{1/2} \lesssim s^{2\tau(f)} e^{-\lambda_{1}t} e^{-2\Re_{\gamma(f)}s} b_{t_{0}}(x)^{1/2}.$$

$$(3.20)$$

The second inequality above follows from $\begin{pmatrix} 2.8\\ 2.18 \end{pmatrix}$. And by $\begin{pmatrix} 9.1\\ 3.19 \end{pmatrix}$ and $\begin{pmatrix} 1.37\\ 2.26 \end{pmatrix}$, we have

$$\int_{t-2t_0}^{t} T_{t-u} [A(T_{s+u}f)^2](x) \, du$$

$$\lesssim \quad (t+s)^{2\tau(f)} e^{-2\Re_{\gamma(f)}(t+s-2t_0)} \int_{t-2t_0}^{t} T_{t-u}(b_{4t_0})(x) \, du$$

$$\lesssim \quad (t+s)^{2\tau(f)} e^{-2\Re_{\gamma(f)}(t+s)} b_{t_0}(x)^{1/2}. \tag{3.21}$$

By $(\stackrel{1.23}{\overline{2.12}})$, we get that $|T_s f(x)|^2 \lesssim s^{2\tau(f)} e^{-2\Re_{\gamma(f)}s} b_{t_0}(x)$. Thus, we have

$$T_t(T_s f)^2(x) \lesssim s^{2\tau(f)} e^{-\lambda_1 t} e^{-2\Re_{\gamma(f)} s} b_{t_0}(x)^{1/2}.$$
(3.22) 4.3

Consequently, we have

$$\mathbb{P}_{\delta_x} \langle T_s f, X_t \rangle^2 \lesssim (t+s)^{2\tau(f)} e^{-2\Re_{\gamma(f)}(t+s)} b_{t_0}(x)^{1/2} + s^{2\tau(f)} e^{-\lambda_1 t} e^{-2\Re_{\gamma(f)} s} b_{t_0}(x)^{1/2}.$$
(3.23) 4.49

Therefore, we have

$$\limsup_{t \to \infty} \mathbb{P}_{\delta_x} J_2(s,t)^2 = \limsup_{t \to \infty} e^{\lambda_1(t+s)} \mathbb{P}_{\delta_x} \langle T_s f, X_t \rangle^2 \lesssim s^{2\tau(f)} e^{(\lambda_1 - 2\Re_{\gamma(f)})s} b_{t_0}(x)^{1/2}.$$
(3.24) 6.7

Similarly, for the case $\gamma(f) = \infty$, we have

$$\mathbb{P}_{\delta_x} \langle T_s f, X_t \rangle^2 \lesssim b_{t_0}(x)^{1/2} + e^{-\lambda_1 t} b_{t_0}(x)^{1/2}.$$
(3.25) 4.49'

Thus,

$$\limsup_{t \to \infty} \mathbb{P}_{\delta_x} J_2(s,t)^2 = \limsup_{t \to \infty} e^{\lambda_1(t+s)} \mathbb{P}_{\delta_x} \langle T_s f, X_t \rangle^2 \lesssim e^{\lambda_1 s} b_{t_0}(x)^{1/2}.$$

Combining $(\frac{6.7}{3.24})$ and $(\frac{6.7}{3.26})$, we get

$$\limsup_{s \to \infty} \limsup_{t \to \infty} \mathbb{P}_{\delta_x} J_2(s, t)^2 = 0.$$
(3.27) 6.7''

(3.26) 6.7'

Next we consider $J_1(s,t)$. We define an \mathbb{R}^2 -valued random variable $U_2(s,t)$ by

$$U_2(s,t) := \left(e^{\lambda_1 t} \langle \phi_1, X_t \rangle, J_1(s,t) \right).$$

Let $V_s(x) := \mathbb{V}ar_{\delta_x}Y_1(s)$. We claim that, for any $x \in E$, under \mathbb{P}_{δ_x} ,

$$U_2(s,t) \xrightarrow{d} \left(W_{\infty}, \sqrt{W_{\infty}} G_1(s) \right), \quad \text{as } t \to \infty,$$
 (3.28) 6.1

where $G_1(s) \sim \mathcal{N}(0, \sigma_f^2(s))$ is independent of W_∞ and $\sigma_f^2(s) = \langle V_s, \phi_1 \rangle$. Denote the characteristic function of $U_2(s, t)$ under \mathbb{P}_{δ_x} by $\kappa(\theta_1, \theta_2, s, t)$:

$$\kappa(\theta_1, \theta_2, s, t) = \mathbb{P}_{\delta_x} \left(\exp\left\{ i\theta_1 e^{\lambda_1 t} \langle \phi_1, X_t \rangle + i\theta_2 e^{(\lambda_1/2)t} \sum_{u \in \mathcal{L}_t} (Y_1^{u,t}(s) - y_1^{u,t}(s)) \right\} \right) \\ = \mathbb{P}_{\delta_x} \left(\exp\{ i\theta_1 e^{\lambda_1 t} \langle \phi_1, X_t \rangle\} \prod_{u \in \mathcal{L}_t} h_s(z_u(t), e^{(\lambda_1/2)t} \theta_2) \right),$$
(3.29)

where

$$h_s(x,\theta) := \mathbb{P}_{\delta_x} e^{i\theta(Y_1(s) - \mathbb{P}_{\delta_x}Y_1(s))}$$

Let $t_k, m_k \to \infty$, as $k \to \infty$, and $a_{k,j} \in E, j = 1, 2, \cdots, m_k$. Now we consider

$$S_k := e^{\lambda_1 t_k/2} \sum_{j=1}^{m_k} (Y_{k,j} - y_{k,j}), \qquad (3.30) \quad \boxed{6.16}$$

where $Y_{k,j}$ has the same law as $Y_1(s)$ under $\mathbb{P}_{\delta_{a_{k,j}}}$ and $y_{k,j} = \mathbb{P}_{\delta_{a_{k,j}}}Y_1(s)$. Further, $Y_{k,j}, j = 1, 2, ...$ are independent. Suppose the following Lindeberg conditions hold:

$$e^{\lambda_1 t_k} \sum_{j=1}^{m_k} \mathbb{E}(Y_{k,j} - y_{k,j})^2 = e^{\lambda_1 t_k} \sum_{j=1}^{m_k} V_s(a_{k,j}) \to \sigma^2;$$

(ii) for any $\epsilon > 0$,

(i) as $k \to \infty$,

$$e^{\lambda_{1}t_{k}} \sum_{j=1}^{m_{k}} \mathbb{E}\left(|Y_{k,j} - y_{k,j}|^{2}, |Y_{k,j} - y_{k,j}| > \epsilon e^{-\lambda_{1}t_{k}/2}\right)$$
$$= e^{\lambda_{1}t_{k}} \sum_{j=1}^{m_{k}} g(a_{k,j}, s, t_{k}) \to 0, \quad \text{as } k \to \infty$$

where $g(x, s, t) = \mathbb{P}_{\delta_x} \left(|Y_1(s) - \mathbb{P}_{\delta_x} Y_1(s)|^2, |Y_1(s) - \mathbb{P}_{\delta_x} Y_1(s)| > \epsilon e^{-\lambda_1 t/2} \right).$

Then using the Lindeberg-Feller theorem, we have $S_k \xrightarrow{d} \mathcal{N}(0, \sigma^2)$, which implies

$$\prod_{j=1}^{m_k} h_s(a_{k,j}, e^{\lambda_1 t_k/2}\theta) \to e^{-\frac{1}{2}\sigma^2\theta^2}.$$
(3.31) 6.17

By $(\underline{\mathtt{var}})$, we get $V_s \in L^2(E,m) \cap L^4(E,m)$. So using Remark $\underline{\mathtt{l.16}}$, we have

$$e^{\lambda_1 t} \sum_{u \in \mathcal{L}_t} V_s(z_u(t)) = e^{\lambda_1 t} \langle V_s, X_t \rangle \to \langle V_s, \phi_1 \rangle W_{\infty}, \quad \text{in probability, as } t \to \infty.$$
(3.32) 6.18

We note that $g(x, s, t) \downarrow 0$ as $t \uparrow \infty$ and $g(x, s, t) \leq V_s(x)$ for any $x \in E$. Thus by $\binom{2.8}{2.18}$ we have for any $x \in E$,

$$e^{\lambda_1 t} \mathbb{P}_{\delta_x} \langle g(\cdot, s, t), X_t \rangle \lesssim \|g(\cdot, s, t)\|_2 b_{t_0}(x)^{1/2} \to 0, \quad \text{as} \quad t \to \infty,$$

which implies

$$e^{\lambda_1 t} \sum_{u \in \mathcal{L}_t} g(z_u(t), s, t) \to 0, \quad \text{as} \quad t \to \infty,$$
(3.33) 6.19

in \mathbb{P}_{δ_x} -probability. Therefore, for any sequence $s_k \to \infty$, there exists a subsequence s'_k such that, if we let $t_k = s'_k$, $m_k = |X_{s'_k}|$ and $\{a_{k,j}, j = 1, 2 \cdots m_k\} = \{z_u(s'_k), u \in \mathcal{L}_{s'_k}\}$, then the Lindeberg conditions hold \mathbb{P}_{δ_x} -a.s. for any $x \in E$, which implies

$$\lim_{k \to \infty} \prod_{u \in \mathcal{L}_{s'_k}} h_s(z_u(s'_k), e^{(\lambda_1/2)s'_k}\theta_2) = \exp\left\{-\frac{1}{2}\theta_2^2 \langle V_s, \phi_1 \rangle W_\infty\right\}, \quad \mathbb{P}_{\delta_x}\text{-a.s.}$$
(3.34) 6.25

Consequently, we have

$$\lim_{t \to \infty} \prod_{u \in \mathcal{L}_t} h_s(z_u(t), e^{(\lambda_1/2)t}\theta_2) = \exp\left\{-\frac{1}{2}\theta_2^2 \langle V_s, \phi_1 \rangle W_\infty\right\}, \quad \text{in probability.}$$
(3.35) 6.20

Hence by the dominated convergence theorem, we get

$$\lim_{t \to \infty} \kappa(\theta_1, \theta_2, s, t) = \mathbb{P}_{\delta_x} \exp\left\{i\theta_1 W_\infty\right\} \exp\left\{-\frac{1}{2}\theta_2^2 \langle V_s, \psi_1 \rangle_m W_\infty\right\},\tag{3.36}$$

which implies our claim ($\overline{3.28}$). Thus, we easily get that, for any $x \in E$, under \mathbb{P}_{δ_x} ,

$$U_3(s,t) := \left(e^{\lambda_1(t+s)} \langle \phi_1, X_{t+s} \rangle, J_1(s,t) \right) \xrightarrow{d} \left(W_\infty, \sqrt{W_\infty} G_1(s) \right), \quad \text{as } t \to \infty.$$

By $(\stackrel{\texttt{small}}{2.30})$, we have $\lim_{s\to\infty} \langle V_s, \psi_1 \rangle_m = \sigma_f^2$. Let $G_1(f)$ be a $\mathcal{N}(0, \sigma_f^2)$ random variable independent of W_{∞} . Then

$$\lim_{s \to \infty} \beta(G_1(s), G_1(f)) = 0.$$
(3.37) 6.22

Let $\mathcal{D}(s+t)$ and $\widetilde{\mathcal{D}}(s,t)$ be the distributions of $U_1(s+t)$ and $U_3(s,t)$ respectively, and let $\mathcal{D}(s)$ and \mathcal{D} be the distributions of $(W_{\infty}, \sqrt{W_{\infty}}G_1(s))$ and $(W_{\infty}, \sqrt{W_{\infty}}G_1(f))$ respectively. Then, using (5.20)(5.14), we have

$$\limsup_{t \to \infty} \beta(\mathcal{D}(s+t), \mathcal{D}) \leq \limsup_{t \to \infty} [\beta(\mathcal{D}(s+t), \widetilde{\mathcal{D}}(s,t)) + \beta(\widetilde{\mathcal{D}}(s,t), \mathcal{D}(s)) + \beta(\mathcal{D}(s), \mathcal{D})]$$

$$\leq \limsup_{t \to \infty} (\mathbb{P}_{\delta_x} J_2(s,t)^2)^{1/2} + 0 + \beta(\mathcal{D}(s), \mathcal{D}).$$
(3.38)

Using this and the definition of $\limsup_{t\to\infty}$, we easily get that

$$\limsup_{t \to \infty} \beta(\mathcal{D}(t), \mathcal{D}) = \limsup_{t \to \infty} \beta(\mathcal{D}(s+t), \mathcal{D}) \le \limsup_{t \to \infty} (\mathbb{P}_{\delta_x} J_2(s, t)^2)^{1/2} + \beta(\mathcal{D}(s), \mathcal{D}).$$

Letting $s \to \infty$, we get $\limsup_{t\to\infty} \beta(\mathcal{D}(t), \mathcal{D}) = 0$. The proof is now complete.

Lemma 3.3 Assume $f(x) = \sum_{j:\lambda_1=2\Re_j} (\Phi_j(x))^T b_j \in \mathcal{C}_c$, where $b_j \in \mathbb{C}^{n_j}$. Define

$$S_t f(x) := t^{-(1+2\tau(f))/2} e^{(\lambda_1/2)t} (\langle f, X_t \rangle - T_t f(x)), \qquad (t, x) \in (0, \infty) \times E.$$

Then for any c > 0, $\delta > 0$ and $x \in E$, we have

$$\lim_{t \to \infty} \mathbb{P}_{\delta_x} \left(|S_t f(x)|^2; |S_t f(x)| > c e^{\delta t} \right) = 0.$$

$$(3.39) \quad \boxed{4.5}$$

Proof: In this proof, we always assume $t > 10t_0$. For each j, define

$$S_{j,t}(x) := t^{-(1+2\tau(f))/2} e^{\lambda_1 t/2} \left(\langle \Phi_j^T, X_t \rangle - e^{-\lambda_j t} (\Phi_j(x))^T D_j(t) \right).$$

Thus, $S_t f(x) = \sum_{j:\lambda_1=2\Re_j} S_{j,t}(x) b_j$. Using the fact that for every $n \ge 1$,

$$\left|\sum_{l=1}^{n} x_{l}\right|^{2} \mathbf{1}_{|\sum_{l=1}^{n} x_{l}|^{2} > M} \le n \sum_{l=1}^{n} |x_{l}|^{2} \mathbf{1}_{|x_{l}|^{2} > M/n},$$
(3.40) e:elem

we see that, to prove $(\frac{4.5}{3.39})$, it suffices to show that, as $t \to \infty$,

$$F(t,x,b_j) := \mathbb{P}_{\delta_x}\left(|S_{j,t}(x)b_j|^2; |S_{j,t}(x)b_j| > ce^{\delta t}\right) \to 0.$$

Choose an integer $n_0 > 2t_0$. We write $t = l_t n_0 + \epsilon_t$, where $l_t \in \mathbb{N}$ and $0 \le \epsilon_t < n_0$. By $(\overline{[1.18)}, we$ easily get $T_u(\Phi_j^T)(x) = e^{-\lambda_j u} \Phi_j(x)^T D_j(t)$. Since $\lambda_1 = 2\Re_j$, for any $(t, x) \in (0, \infty) \times E$, we have

$$S_{j,t+n_{0}}(x) = \left(\frac{1}{t+n_{0}}\right)^{1/2+\tau(f)} e^{\lambda_{1}(t+n_{0})/2} \left(\langle \Phi_{j}^{T}, X_{t+n_{0}} \rangle - \langle e^{-\lambda_{j}n_{0}} \Phi_{j}^{T}, X_{t} \rangle D_{j}(n_{0})\right) \\ + \left(\frac{1}{t+n_{0}}\right)^{1/2+\tau(f)} e^{-i\Im_{j}n_{0}} e^{\lambda_{1}t/2} \left(\langle \Phi_{j}^{T}, X_{t} \rangle - e^{-\lambda_{j}t} (\Phi_{j}(x))^{T} D_{j}(t)\right) D_{j}(n_{0}) \\ = \left(\frac{1}{t+n_{0}}\right)^{1/2+\tau(f)} R_{j}(t) + e^{-i\Im_{j}n_{0}} \left(\frac{t}{t+n_{0}}\right)^{1/2+\tau(f)} S_{j,t}(x) D_{j}(n_{0}), \quad (3.41)$$

where

$$R_j(t) := e^{(\lambda_1/2)(t+n_0)} \left(\langle \Phi_j^T, X_{t+n_0} \rangle - \langle e^{-\lambda_j} \Phi_j^T, X_t \rangle D_j(n_0) \right).$$

Hence, for any $(t, x) \in (0, \infty) \times E$, we have

$$F(t + n_0, x, b_j)$$

$$\leq \mathbb{P}_{\delta_x} \left(|S_{j,t+n_0}(x)b_j|^2; |S_{j,t}(x)D_j(n_0)b_j| > ce^{\delta t} \right)$$

$$+ \mathbb{P}_{\delta_x} \left(|S_{j,t+n_0}(x)b_j|^2; |S_{j,t}(x)D_j(n_0)b_j| \le ce^{\delta t}, |S_{j,t+n_0}(x)b_j| > ce^{\delta(t+n_0)} \right)$$

$$=: M_1(t, x) + M_2(t, x).$$

Put

$$A_1(t, x, b_j) := \{ |S_{j,t}(x)D_j(n_0)b_j| > ce^{\delta t} \}, A_2(t, x, b_j) := \{ |S_{j,t}(x)D_j(n_0)b_j| \le ce^{\delta t}, |S_{j,t+n_0}(x)b_j| > ce^{\delta(t+n_0)} \}$$

and

$$A(t, x, b_j) := A_1(t, x, b_j) \cup A_2(t, x, b_j).$$

Since $A_1(t, x, b_j) \in \mathcal{F}_t$ and $\mathbb{P}_{\delta_x}(R_j(t)|\mathcal{F}_t) = 0$ for any $(t, x) \in (0, \infty) \times E$, we have by $(\overset{\textbf{4.8}}{\textbf{5.41}})$ that

$$M_1(t,x) = \left(\frac{1}{t+n_0}\right)^{1+2\tau(f)} \mathbb{P}_{\delta_x}\left(|R_j(t)b_j|^2; A_1(t,x,b_j)\right) + \left(\frac{t}{t+n_0}\right)^{1+2\tau(f)} F(t,x,D_j(n_0)b_j)$$

and

$$M_{2}(t,x) \leq 2\left(\frac{1}{t+n_{0}}\right)^{1+2\tau(f)} \mathbb{P}_{\delta_{x}}\left(|R_{j}(t)b_{j}|^{2}; A_{2}(t,x,b_{j})\right) \\ +2\left(\frac{t}{t+n_{0}}\right)^{1+2\tau(f)} \mathbb{P}_{\delta_{x}}\left(|S_{j,t}(x)D_{j}(n_{0})b_{j}|^{2}; A_{2}(t,x,b_{j})\right)$$

Thus, for any $(t, x) \in (0, \infty) \times E$, we have

$$F(t+n_0, x, b_j) \leq \left(\frac{t}{t+n_0}\right)^{1+2\tau(f)} F(t, x, D_j(n_0)b_j) + \left(\frac{1}{t+n_0}\right)^{1+2\tau(f)} (F_1(t, x, b_j) + F_2(t, x, b_j)),$$
(3.42)

•

where

$$F_1(t, x, b_j) := 2\mathbb{P}_{\delta_x} \left(|R_j(t)b_j|^2; A_1(t, x, b_j) \cup A_2(t, x, b_j) \right),$$

$$F_2(t, x, b_j) := 2t^{1+2\tau(f)} \mathbb{P}_{\delta_x} \left(|S_{j,t}(x)D_j(n_0)b_j|^2; A_2(t, x, b_j) \right).$$

Iterating $(\overset{\textbf{4.9}}{5.42})$, we get for t large enough,

$$F(t+n_{0},x,b_{j}) \leq \left(\frac{1}{t+n_{0}}\right)^{1+2\tau(f)} \sum_{m=5}^{l_{t}} \left(F_{1}(mn_{0}+\epsilon_{t},x,D_{j}((l_{t}-m)n_{0})b_{j})\right) \\ \left(\frac{1}{t+n_{0}}\right)^{1+2\tau(f)} \sum_{m=5}^{l_{t}} \left(F_{2}(mn_{0}+\epsilon_{t},x,D_{j}((l_{t}-m)n_{0})b_{j})\right) \\ + \left(\frac{5n_{0}+\epsilon_{t}}{t+n_{0}}\right)^{1+2\tau(f)} F(5n_{0}+\epsilon_{t},x,D_{j}((l_{t}-4)n_{0})b_{j}) \\ =: L_{1}(t,x) + L_{2}(t,x) + \left(\frac{5n_{0}+\epsilon_{t}}{t+1}\right)^{1+2\tau(f)} F(5n_{0}+\epsilon_{t},x,D_{j}((l_{t}-4)n_{0})b_{j}).$$
(3.43)

First, we consider $L_1(t, x)$. By the definition of $\tau(f)$, we have for s > 0,

$$|D_j(s)b_j|_2 \lesssim |D_j(s)b_j|_\infty \lesssim 1 + s^{\tau(f)}.$$
 (3.44) 2.33

Thus, we have for $0 \le s \le t$ and $t \ge 2t_0$,

$$|R_j(s)D_j(t-s)b_j|^2 \le |R_j(s)|_2^2 |D_j(t-s)b_j|_2^2 \lesssim t^{2\tau(f)} |R_j(s)|_2^2.$$
(3.45) R2

It follows that for any $\epsilon \in (0, 1)$,

$$L_{1}(t,x) \leq \frac{2}{t+n_{0}} \sum_{5 \leq m \leq \epsilon l_{t}} \mathbb{P}_{\delta_{x}} \left(|R_{j}(mn_{0}+\epsilon_{t})|_{2}^{2} \right) \\ + \frac{2}{t+n_{0}} \sum_{l_{t} \epsilon \leq m \leq l_{t}} \mathbb{P}_{\delta_{x}} \left(|R_{j}(mn_{0}+\epsilon_{t})|_{2}^{2}; A(mn_{0}+\epsilon_{t},x,D_{j}((l_{t}-m)n_{0})b_{j}) \right)$$

$$=: L_{1,1}(t,x) + L_{1,2}(t,x).$$
(3.46)

By the definition of $R_j(s)$, we have

$$|R_j(s)|_2^2 = e^{\lambda_1(s+n_0)} \sum_{l=1}^{n_j} |\langle \phi_l^{(j)}, X_{s+n_0} \rangle - \langle T_{n_0}(\phi_l^{(j)}), X_s \rangle|^2.$$
(3.47) 2.34

Note that

$$\begin{aligned} |\langle \phi_l^{(j)}, X_{s+n_0} \rangle - \langle T_{n_0}(\phi_l^{(j)}), X_s \rangle|^2 &= |\langle \Re(\phi_l^{(j)}), X_{s+n_0} \rangle - \langle T_{n_0}(\Re(\phi_l^{(j)})), X_s \rangle|^2 \\ &+ |\langle \Im(\phi_l^{(j)}), X_{s+n_0} \rangle - \langle T_{n_0}(\Im(\phi_l^{(j)})), X_s \rangle|^2 \end{aligned}$$

Thus, we have

$$\mathbb{P}_{\delta_x} |\langle \phi_l^{(j)}, X_{s+n_0} \rangle - \langle T_{n_0}(\phi_l^{(j)}), X_s \rangle|^2 = T_s(\mathbb{V}\mathrm{ar}_{\delta_{\cdot}} \langle \Re(\phi_l^{(j)}), X_{n_0} \rangle)(x) + T_s(\mathbb{V}\mathrm{ar}_{\delta_{\cdot}} \langle \Im(\phi_l^{(j)}), X_{n_0} \rangle)(x).$$

$$(3.48) \quad \boxed{2.35}$$

Hence, by (2.18), we get, for $s \ge 5n_0 > 2t_0$,

$$\mathbb{P}_{\delta_x}|R_j(s)|_2^2 = e^{\lambda_1(s+n_0)} \sum_{l=1}^{n_j} \mathbb{P}_{\delta_x}|\langle \phi_l^{(j)}, X_{t+n_0}\rangle - \langle T_{n_0}(\phi_l^{(j)}), X_t\rangle|^2 \lesssim b_{t_0}(x)^{1/2}.$$
(3.49) 2.37

Therefore, we have, for $(t, x) \in (5n_0, \infty) \times E$,

$$L_{1,1}(t,x) \lesssim \epsilon b_{t_0}(x)^{1/2}.$$
 (3.50) L11

We claim that, for any $x \in E$,

(i)

$$\lim_{M \to \infty} \limsup_{s \to \infty} \mathbb{P}_{\delta_x}(|R_j(s)|_2^2; |R_j(s)|_2^2 > M) = 0, \text{ and}$$
(3.51) 2.44

(ii)

$$\sup_{t\epsilon \le s \le t} \mathbb{P}_{\delta_x}(A_1(s, x, D_j(t-s)b_j) \cup A_2(s, x, D_j(t-s)b_j)) \to 0.$$
(3.52) Atoo

Using these two claims we get that, as $t \to \infty$,

$$\begin{aligned}
& L_{1,2}(t,x) \\
&\leq \frac{2}{t+n_0} \sum_{\epsilon l_t \leq m \leq l_t} \left(\mathbb{P}_{\delta_x} \left(|R_j(mn_0 + \epsilon_t)|_2^2; |R_j(mn_0 + \epsilon_t)|_2^2 > M \right) \\
& + M \mathbb{P}_{\delta_x} \left(A(mn_0 + \epsilon_t, x, D_j((l_t - m)n_0)b_j)) \right) \\
&\lesssim \sup_{s \geq t\epsilon} \mathbb{P}_{\delta_x}(|R_j(s)|_2^2; |R_j(s)|_2^2 > M) + M \sup_{t\epsilon \leq s \leq t} \mathbb{P}_{\delta_x}(A(s, x, D_j(t - s)b_j)) \\
&\to \limsup_{s \to \infty} \mathbb{P}_{\delta_x}(|R_j(s)|_2^2; |R_j(s)|_2^2 > M).
\end{aligned} \tag{3.53}$$

Letting $M \to \infty$, we get

$$\lim_{t \to \infty} L_{1,2}(t,x) = 0. \tag{3.54}$$

Now we prove the two claims.

(i) For $l = 1, 2, \cdots, n_j$, define

$$R_{j,l,1}(s) := e^{\lambda_1(s+n_0)/2} \langle \Re(\phi_l^j), X_{s+n_0} \rangle - \langle T_{n_0}(\Re(\phi_l^j)), X_s \rangle$$

and

$$R_{j,l,2}(s) := e^{\lambda_1(s+n_0)/2} \langle \Im(\phi_l^j), X_{s+n_0} \rangle - \langle T_{n_0}(\Im(\phi_l^j)), X_s \rangle.$$

Using $(\overset{\textbf{e:elem}}{\textbf{3.40}})$ and $(\overset{\textbf{2.34}}{\textbf{3.47}})$, we easily see that, to prove $(\overset{\textbf{2.44}}{\textbf{5.51}})$, we only need to show that, for k = 1, 2, 3

$$\lim_{M \to \infty} \limsup_{s \to \infty} \mathbb{P}_{\delta_x}(|R_{j,l,k}(s)|^2, |R_{j,l,k}(s)|^2 > M) = 0.$$
(3.55) to-prove

Repeating the proof of $(\underline{\textbf{5.28}})$ with $s = n_0$, we see that $(\underline{\textbf{5.28}})$ is valid for $f \in L^2(E,m) \cap L^4(E,m)$. Thus, for $l = 1, 2, \dots, n_j$, as $s \to \infty$,

$$R_{j,l,1}(s) \stackrel{d}{\to} \sqrt{W_{\infty}}G,$$

where $G \sim \mathcal{N}(0, e^{\lambda_1 n_0} \langle \mathbb{V}ar_{\delta}, \langle \Re(\phi_l^j), X_{n_0} \rangle, \psi_1 \rangle_m$. And by (2.17), we get, as $s \to \infty$,

$$\mathbb{P}_{\delta_x}(|R_{j,l,1}(s)|^2) = e^{\lambda_1(s+n_0)} T_s(\mathbb{V}\mathrm{ar}_{\delta_{-}}\langle \Re(\phi_l^j), X_{n_0} \rangle)(x) \to e^{\lambda_1 n_0} \langle (\mathbb{V}\mathrm{ar}_{\delta_{-}}\langle \Re(\phi_l^j), X_{n_0} \rangle, \psi_1 \rangle_m \phi_1(x).$$
(3.56)

(3.56) (3.56) (2.39) Let $h_M(r) = r$ on [0, M-1], $h_M(r) = 0$ on $[M, \infty]$, and let h_M be linear on [M-1, M]. By (3.56), we have that for any $x \in E$,

$$\begin{split} &\limsup_{s \to \infty} \mathbb{P}_{\delta_x}(|R_{j,l,1}(s)|^2, |R_{j,l,1}(s)|^2 > M) \le \limsup_{t \to \infty} \mathbb{P}_{\delta_x}(|R_{j,l,1}(s)|^2) - \mathbb{P}_{\delta_x}(h_M(|R_{j,l,1}(s)|^2)) \\ &= e^{\lambda_1 n_0} \langle (\mathbb{V}\mathrm{ar}_{\delta_-} \langle \Re(\phi_l^j), X_{n_0} \rangle, \psi_1 \rangle_m \phi_1(x) - \mathbb{P}_{\delta_x}(h_M(W_\infty G^2)). \end{split}$$

By the monotone convergence theorem, we have that for any $x \in E$,

$$\lim_{M \to \infty} \mathbb{P}_{\delta_x}(h_M(W_{\infty}G^2)) = \mathbb{P}_{\delta_x}(W_{\infty}G^2) = \mathbb{P}_{\delta_x}(W_{\infty})\mathbb{P}_{\delta_x}(G^2) = e^{\lambda_1 n_0} \langle (\mathbb{V}\mathrm{ar}_{\delta_*}\langle \Re(\phi_l^j), X_{n_0} \rangle, \psi_1 \rangle_m \phi_1(x), \psi_1 \rangle_m \langle (W_{\infty}G^2) \rangle_{\mathcal{H}_{\delta_x}}(W_{\infty}G^2) = e^{\lambda_1 n_0} \langle (\mathbb{V}\mathrm{ar}_{\delta_*}\langle \Re(\phi_l^j), X_{n_0} \rangle, \psi_1 \rangle_m \phi_1(x), \psi_1 \rangle_m \langle (W_{\infty}G^2) \rangle_{\mathcal{H}_{\delta_x}}(W_{\infty}G^2) = e^{\lambda_1 n_0} \langle (\mathbb{V}\mathrm{ar}_{\delta_*}\langle \Re(\phi_l^j), X_{n_0} \rangle, \psi_1 \rangle_m \phi_1(x), \psi_1 \rangle_m \langle (W_{\infty}G^2) \rangle_{\mathcal{H}_{\delta_x}}(W_{\infty}G^2) = e^{\lambda_1 n_0} \langle (\mathbb{V}\mathrm{ar}_{\delta_*}\langle \Re(\phi_l^j), X_{n_0} \rangle, \psi_1 \rangle_m \phi_1(x), \psi_1 \rangle_m \langle (W_{\infty}G^2) \rangle_{\mathcal{H}_{\delta_x}}(W_{\infty}G^2) \rangle_{\mathcal{H}_{\delta_x}}(W_{\infty}G^2) = e^{\lambda_1 n_0} \langle (\mathbb{V}\mathrm{ar}_{\delta_*}\langle \Re(\phi_l^j), X_{n_0} \rangle, \psi_1 \rangle_m \phi_1(x), \psi_1 \rangle_m \langle (W_{\infty}G^2) \rangle_{\mathcal{H}_{\delta_x}}(W_{\infty}G^2) \rangle_{\mathcal{H}_{\delta_x}}(W_{\infty}G^2) = e^{\lambda_1 n_0} \langle (\mathbb{V}\mathrm{ar}_{\delta_*}\langle \Re(\phi_l^j), X_{n_0} \rangle, \psi_1 \rangle_m \phi_1(x), \psi_1 \rangle_m \langle (W_{\infty}G^2) \rangle_{\mathcal{H}_{\delta_x}}(W_{\infty}G^2) \rangle_{\mathcal{H}_{\delta_x}}(W_{\infty}G^2) = e^{\lambda_1 n_0} \langle (\mathbb{V}\mathrm{ar}_{\delta_*}\langle \Re(\phi_l^j), X_{n_0} \rangle, \psi_1 \rangle_m \phi_1(x), \psi_1 \rangle_m \langle \mathbb{V}\mathrm{ar}_{\delta_x}(W_{\infty}G^2) \rangle_{\mathcal{H}_{\delta_x}}(W_{\infty}G^2) \rangle_{\mathcal{H}_{\delta_x}}(W_{\infty}G^2) = e^{\lambda_1 n_0} \langle (\mathbb{V}\mathrm{ar}_{\delta_x}(W_{\infty}G^2), \psi_1 \rangle_m \phi_1(x), \psi_1 \rangle_m \langle \mathbb{V}\mathrm{ar}_{\delta_x}(W_{\infty}G^2) \rangle_{\mathcal{H}_{\delta_x}}(W_{\infty}G^2) \rangle_{\mathcal{H}_{\delta_x}}(W_{\infty}$$

which implies

$$\lim_{M \to \infty} \limsup_{s \to \infty} \mathbb{P}_{\delta_x}(|R_{j,l,1}(s)|^2, |R_{j,l,1}(s)|^2 > M) = 0,$$

which says $(\overset{to-prove}{B.55})$ holds for k = 1. Using similar arguments, we get $(\overset{to-prove}{B.55})$ holds for k = 2. (ii) Since $\tau(\phi_l^j) \le \nu_j$, by $(\overset{\textbf{3.33}}{\textbf{2.41}})$, we get for $10t_0 \le s$,

$$\mathbb{P}_{\delta_x}|S_{j,s}(x)|_2^2 \lesssim s^{1+2\nu_j}s^{-(1+2\tau(f))} \le s^{2\nu_j}.$$
(3.57) vars

By $(\frac{2.33}{5.44})$, we get, for $10t_0 \le s \le t$,

$$\mathbb{P}_{\delta_x}|S_{j,s}(x)D_j(t+1-s)b_j|^2 \lesssim s^{2\nu_j}(1+t^{2\tau(f)}).$$
(3.58) 3.34

By Chebyshev's inequality and (3.34), we have that, for any $x \in E$, as $t \to \infty$

$$\sup_{t \in \le s \le t} \mathbb{P}_{\delta_x}(A_1(s, x, D_j(t-s))) \le \sup_{t \in \le s \le t} c^{-2} e^{-2\delta s} \mathbb{P}_{\delta_x} |S_{j,s}(x) D_j(t+1-s) b_j|^2$$

$$\lesssim e^{-2\delta\epsilon t} t^{2\nu_j} (1 + t^{2\tau(f)}) \to 0.$$

It is easy to see that, under \mathbb{P}_{δ_x} , for any t > 0,

$$A_2(s, x, D_j(t-s)b_j) \subset \left\{ |R_j(s)D_j(t-s)b_j| > ce^{\delta s} \left(e^{\delta n_0} - 1\right) s^{(2\tau(f)+1)/2} \right\}.$$
(3.59) 4.24

By $(\stackrel{\mathbb{R}2}{\text{B.45}})$ and $(\stackrel{\mathbb{2.37}}{\text{B.49}})$, we get

$$\mathbb{P}_{\delta_x} |R_j(s) D_j(t-s) b_j|^2 \lesssim t^{2\tau(f)} b_{t_0}(x)^{1/2}.$$

Similarly, by Chebyshev's inequality, we have that, for any $x \in E$, as $t \to \infty$,

$$\sup_{t\epsilon \le s \le t} \mathbb{P}_{\delta_x} A_2(s, x, D_j(t-s)b_j)$$

$$\le \sup_{t\epsilon \le s \le t} c^{-2} (e^{\delta n_0} - 1)^{-2} e^{-2\delta s} s^{-(1+2\tau(f))} \mathbb{P}_{\delta_x} |R_j(s) D_j(t-s)b_j|^2$$

$$\lesssim e^{-2\delta \epsilon t} (t\epsilon)^{-(1+2\tau(f))} t^{2\tau(f)} \to 0.$$

Thus we have finished proving the two claims. Therefore, by $(\overset{\mathbb{L}11}{\mathbb{B}.50})$ and $(\overset{\mathbb{L}12}{\mathbb{B}.54})$, we get

$$\limsup_{t \to \infty} L_1(t, x) \lesssim \epsilon b_{t_0}(x)^{1/2}.$$

Letting $\epsilon \to 0$, we get

$$\lim_{t \to \infty} L_1(t, x) = 0.$$
(3.60) L1to0
), we have that for any $x \in E$,

$$F_{2}(s, x, D_{j}(t-s)b_{j}) = 2s^{(1+2\tau(f))}\mathbb{P}_{\delta_{x}}\left(|S_{j,s}(x)D_{j}(t+n_{0}-s)b_{j}|^{2}; A_{2}(s, x, D_{j}(t-s)b_{j})\right) \\ \leq 2s^{(1+2\tau(f))}ce^{\delta s}\mathbb{P}_{\delta_{x}}\left(|S_{j,s}(x)D_{j}(t+n_{0}-s)b_{j}|; |R_{j}(s)D_{j}(t-s)b_{j}| > ce^{\delta s}(e^{\delta n_{0}}-1)s^{(2\tau(f)+1)/2}\right) \\ \leq 2c^{-1}(e^{\delta n_{0}}-1)e^{-\delta s}\mathbb{P}_{\delta_{x}}\left(|S_{j,s}(x)D_{j}(t+n_{0}-s)b_{j}| \cdot |R_{j}(s)D_{j}(t-s)b_{j}|^{2}\right) \\ \lesssim e^{-\delta s}e^{\lambda_{1}(s+n_{0})}t^{\tau(f)}\mathbb{P}_{\delta_{x}}\left(|S_{j,s}(x)|_{2}\langle \mathbb{V}ar_{\delta_{\cdot}}(\langle \Phi_{j}^{T}D_{j}(t-s)b_{j}, X_{n_{0}}\rangle), X_{s}\rangle\right) \\ \lesssim e^{-\delta s}t^{\tau(f)}\sqrt{\mathbb{P}_{\delta_{x}}|S_{j,s}(x)|_{2}^{2}}\sqrt{e^{2\lambda_{1}s}\mathbb{P}_{\delta_{x}}\left(\langle \mathbb{V}ar_{\delta_{\cdot}}(\langle \Phi_{j}^{T}D_{j}(t-s)b_{j}, X_{n_{0}}\rangle), X_{s}\rangle^{2}\right)}.$$

By (2.41) and (1.20), we get for $s \le t$,

Now we consider $L_2(t, x)$. By $(\begin{array}{c} 4.24 \\ B.59 \end{array}$

$$\operatorname{Var}_{\delta_x}(\langle \Phi_j^T D_j(t-s)b_j, X_{n_0} \rangle \leq \mathbb{P}_{\delta_x} |\langle \Phi_j^T D_j(t-s)b_j, X_{n_0} \rangle|^2 \lesssim t^{2\tau(f)} \mathbb{P}_{\delta_x} \langle b_{t_0}(x)^{1/2}, X_{n_0} \rangle^2.$$

Thus by $(\underline{B.57})$ and $(\underline{B.57})$, we have for $5n_0 \le s \le t$

$$F_2(s, x, D_j(t-s)b_j) \lesssim e^{-\delta s} t^{2\tau(f)} s^{\nu_j} \sqrt{e^{2\lambda_1 s} \mathbb{P}_{\delta_x} \left(\langle b_{t_0}(x)^{1/2}, X_s \rangle^2 \right)} \lesssim e^{-\delta s} t^{2\tau(f)} s^{\nu_j}.$$

Thus, we get, as $t \to \infty$,

$$L_2(t,x) \lesssim \frac{1}{t+n_0} \sum_{m=5}^{l_t} e^{-\delta(mn_0+\epsilon_t)} (mn_0+\epsilon_t)^{(1+2\nu_j)/2} \le \frac{1}{t+n_0} \sum_{m=5}^{l_t} e^{-\delta mn_0} ((m+1)n_0)^{(1+2\nu_j)/2} \to 0.$$
(3.61) L2to0

To finish the proof, we only need to show that for any $x \in E$,

$$\lim_{t \to \infty} \left(\frac{5n_0 + \epsilon_t}{t + n_0} \right)^{1 + 2\tau(f)} F(5n_0 + \epsilon_t, x, D_j((l_t - 4)n_0)b_j) = 0.$$
(3.62) I3

By $(\overset{\textbf{2.33}}{\textbf{B.44}})$ and $(\overset{\textbf{vars}}{\textbf{B.57}})$, we get that for any $x \in E$,

$$(5n_0 + \epsilon_t)^{1+2\tau(f)} F(5n_0 + \epsilon_t, x, D_j((l_t - 4)n_0)b_j)$$

$$\leq (6n_0)^{1+2\tau(f)} \sup_{5n_0 \le s \le 6n_0} \mathbb{P}_{\delta_x} |S_{j,s}(x)D_j((l_t - 4)n_0)b_j|^2 \lesssim t^{2\tau(f)}(6n_0)^{2\nu_j}$$

which implies $(\underline{\textbf{3.62}})$.

The proof is now complete.

lem:5.6 Lemma 3.4 Assume that $f \in C_s$ and $h \in C_c$. Define

$$Y_1(t) := e^{\lambda_1 t/2} \left(\langle f, X_t \rangle - T_t f(x) \right), \quad Y_2(t) := t^{-(1+\tau(h)/2)} e^{\lambda_1 t/2} \left(\langle h, X_t \rangle - T_t h(x) \right), \quad t > 0$$

and $Y_t := Y_1(t) + Y_2(t), t > 0$. Then for any $c > 0, \delta > 0$ and $x \in E$, we have

$$\lim_{t \to \infty} \mathbb{P}_{\delta_x} \left(|Y_t|^2; |Y_t| > ce^{\delta t} \right) = 0.$$
(3.63) 4.6

Proof: By $(\underline{B.40})$ and Lemma $\underline{B.3}$, it suffices to show that

$$\lim_{t \to \infty} \mathbb{P}_{\delta_x} \left(|Y_1(t)|^2; |Y_1(t)| > ce^{\delta t} \right) = 0.$$
(3.64) 3.55

If $\gamma(f) < \infty$, by $(\stackrel{1.23}{2.12})$, we get, as $t \to \infty$,

 $e^{\lambda_1 t/2} |T_t f(x)| \lesssim t^{\tau(f)} e^{(\lambda_1/2 - \Re_{\gamma(f)})t} b_{t_0}(x)^{1/2} \to 0.$

If $\gamma(f) = \infty$, by $(\stackrel{[1.23]}{\textcircled{2.13}}$, we get, as $t \to \infty$, $e^{\lambda_1 t/2} |T_t f(x)| \lesssim e^{\lambda_1 t/2} b_{t_0}(x)^{1/2} \to 0$. Thus, by Lemma $\stackrel{[lem:small}{\textcircled{3.2}, Y_1(t)} \xrightarrow{d} \sqrt{W_{\infty}} G_1(f)$. By Lemma $\stackrel{[lem:2.2]}{\textcircled{2.6}, we}$ have

$$\lim_{t \to \infty} \mathbb{P}_{\delta_x} \left(|Y_1(t)|^2 \right) = \sigma_f^2 \phi_1(x).$$

Thus, for any M > 0, we have

$$\mathbb{P}_{\delta_x} \left(|Y_1(t)|^2; |Y_1(t)| > ce^{\delta t} \right) \leq \mathbb{P}_{\delta_x} \left(|Y_1(t)|^2; |Y_1(t)| > M \right) + M^2 \mathbb{P}_{\delta_x} \left(|Y_1(t)| > ce^{\delta t} \right)$$

=: $I_1(t, x, M) + I_2(t, x, M).$

Let $h_M(r) = r$ on [0, M-1], $h_M(r) = 0$ on $[M, \infty]$, and let h_M be linear on [M-1, M]. Then

 $\limsup_{t \to \infty} I_1(t, x, M) \le \limsup_{t \to \infty} \mathbb{P}_{\delta_x} \left(|Y_1(t)|^2 \right) - \mathbb{P}_{\delta_x}(h_M(|Y_1(t)|)^2) = \sigma_f^2 \phi_1(x) - \mathbb{P}_{\delta_x}(h_M(|G_1(f)\sqrt{W_\infty}|)^2).$

By Chebyshev's inequality, we have, as $t \to \infty$,

$$I_2(t, x, M) \le M^2 c^{-2} e^{-2\delta t} \mathbb{P}_{\delta_x} \left(|Y_1(t)|^2 \right) \to 0.$$

Thus, we have

$$\limsup_{t \to \infty} \mathbb{P}_{\delta_x}\left(|Y_1(t)|^2; |Y_1(t)| > ce^{\delta t}\right) \le \sigma_f^2 \phi_1(x) - \mathbb{P}_{\delta_x}(h_M(|G_1(f)\sqrt{W_\infty}|)^2.$$

Letting $M \to \infty$, by the monotone convergence theorem, we have that for any $x \in E$,

$$\lim_{M \to \infty} \mathbb{P}_{\delta_x}(h_M(|G_1(f)\sqrt{W_\infty}|)^2 = \mathbb{P}_{\delta_x}(G_1(f)^2W_\infty) = \sigma_f^2\phi_1(x),$$

which implies $(\frac{3.55}{3.64})$. The proof is now complete.

lem:cs

Lemma 3.5 Assume that
$$f \in C_s$$
 and $h \in C_c$. Then

$$\left(e^{\lambda_1 t} \langle \phi_1, X_t \rangle, t^{-(1+2\tau(h))/2} e^{\lambda_1 t/2} \langle h, X_t \rangle, e^{\lambda_1 t/2} \langle f, X_t \rangle \right) \xrightarrow{d} \left(W_{\infty}, \sqrt{W_{\infty}} G_2(h), \sqrt{W_{\infty}} G_1(f) \right),$$

$$(3.65) \quad \text{[CS]}$$

where $G_2(h) \sim \mathcal{N}(0, \rho_h^2)$ and $G_1(f) \sim \mathcal{N}(0, \sigma_f^2)$. Moreover, W_{∞} , $G_2(h)$ and $G_1(f)$ are independent.

Proof: In this proof, we always assume $t > 10t_0$, $f \in C_s$ and $h \in C_c$. We define an \mathbb{R}^3 -valued random variable by

$$U_1(t) := \left(e^{\lambda_1 t} \langle \phi_1, X_t \rangle, t^{-(1+2\tau(h))/2} e^{\lambda_1 t/2} \langle h, X_t \rangle, e^{\lambda_1 t/2} \langle f, X_t \rangle \right).$$

For n > 2, we define

$$U_1(nt) = \left(e^{\lambda_1 nt} \langle \phi_1, X_{nt} \rangle, (nt)^{-(1+2\tau(h))/2} e^{\lambda_1 nt/2} \langle h, X_{nt} \rangle, e^{\lambda_1 nt/2} \langle f, X_{nt} \rangle \right).$$

Now we define another \mathbb{R}^3 -valued random variable $U_2(n,t)$ by

$$U_{2}(n,t) = \left(e^{\lambda_{1}t} \langle \phi_{1}, X_{t} \rangle, \frac{e^{\lambda_{1}nt/2}(\langle h, X_{nt} \rangle - \langle T_{(n-1)t}h, X_{t} \rangle)}{((n-1)t)^{(1+2\tau(h))/2}}, e^{\lambda_{1}nt/2}(\langle f, X_{nt} \rangle - \langle T_{(n-1)t}f, X_{t} \rangle) \right)$$

We claim that

$$U_2(n,t) \xrightarrow{d} \left(W_{\infty}, \sqrt{W_{\infty}} G_2(h), \sqrt{W_{\infty}} G_1(f) \right), \quad \text{as } t \to \infty.$$
(3.66) 9.5

Denote the characteristic function of $U_2(n,t)$ under \mathbb{P}_{μ} by $\kappa_2(\theta_1,\theta_2,\theta_3,n,t)$. Define

$$Y_1^{u,t}(s) := e^{\lambda_1 s/2} \langle f, X_s^{u,t} \rangle, \quad Y_2^{u,t}(s) := s^{-(1+2\tau(h))/2} e^{\lambda_1 s/2} \langle h, X_s^{u,t} \rangle, \quad s, t > 0.$$

We also define

$$Y_1(s) := e^{\lambda_1 s/2} \langle f, X_s \rangle, \quad Y_2(s) := s^{-(1+2\tau(h))/2} e^{\lambda_1 s/2} \langle h, X_s \rangle$$

and

$$Y_s(\theta_2, \theta_3) := \theta_2 Y_2(s) + \theta_3 Y_1(s).$$

Given \mathcal{F}_t , for $k = 1, 2, Y_k^{u,t}(s)$ has the same distribution as $Y_k(s)$ under $\mathbb{P}_{\delta_{z_u(t)}}$. Thus, for k = 1, 2,

$$y_k^{u,t}(s) := \mathbb{P}_{\delta_x}(Y_k^{u,t}(s)|\mathcal{F}_t) = \mathbb{P}_{\delta_{z_u(t)}}Y_k(s).$$

Thus, by $(\underline{\overset{\textbf{3.22}}{\textbf{5.1}}}$, we have

$$U_{2}(n,t) = \left(e^{\lambda_{1}t} \langle \phi_{1}, X_{t} \rangle, e^{\lambda_{1}t/2} \sum_{u \in \mathcal{L}_{t}} (Y_{2}^{u,t}((n-1)t) - y_{2}^{u,t}((n-1)t)), \\ e^{\lambda_{1}t/2} \sum_{u \in \mathcal{L}_{t}} (Y_{1}^{u,t}((n-1)t) - y_{1}^{u,t}((n-1)t))\right) \qquad (3.67)$$

Let $h(s, x, \theta, \theta_2, \theta_3) = \mathbb{P}_{\delta_x}(\exp\{i\theta(Y_s(\theta_2, \theta_3) - \mathbb{P}_{\delta_x}Y_s(\theta_2, \theta_3))\})$. Thus, we get

$$\kappa_2(\theta_1, \theta_2, \theta_3, n, t) = \mathbb{P}_{\delta_x}\left(\exp\{i\theta_1 e^{\lambda_1 t} \langle \phi_1, X_t \rangle\} \prod_{u \in \mathcal{L}_t} h\left((n-1)t, z_u(t), e^{\lambda_1 t/2}, \theta_2, \theta_3\right)\right).$$
(3.68)

Let $t_k, m_k \to \infty$, as $k \to \infty$. Now we consider

$$S_k := e^{\lambda_1 t_k/2} \sum_{j=1}^{m_k} (Y_{k,j} - y_{k,j}), \qquad (3.69) \quad \boxed{3.16}$$

where $Y_{k,j}$ has the same law as $Y_{(n-1)t_k}(\theta_2, \theta_3)$ under $\mathbb{P}_{\delta_{a_{k,j}}}$ and $y_{k,j} = \mathbb{P}_{\delta_{a_{k,j}}}Y_{(n-1)t_k}(\theta_2, \theta_3)$ with $a_{k,j} \in E$. Further, for each positive integer $k, Y_{k,j}, j = 1, 2, \ldots$ are independent. Denote $V_t^n(x) := \mathbb{V}ar_{\delta_x}Y_{(n-1)t}(\theta_2, \theta_3)$. Suppose the following Lindeberg conditions hold:

(i) as $k \to \infty$,

$$e^{\lambda_1 t_k} \sum_{j=1}^{m_k} \mathbb{E}(Y_{k,j} - y_{k,j})^2 = e^{\lambda_1 t_k} \sum_{j=1}^{m_k} V_{t_k}^n(a_{k,j}) \to \sigma^2;$$

(ii) for every c > 0,

$$e^{\lambda_{1}t_{k}}\sum_{j=1}^{m_{k}}\mathbb{E}\left(|Y_{k,j}-y_{k,j}|^{2},|Y_{k,j}-y_{k,j}|>ce^{-\lambda_{1}t_{k}/2}\right)$$
$$= e^{\lambda_{1}t_{k}}\sum_{j=1}^{m_{k}}g_{(n-1)t_{k}}(a_{k,j},\theta_{2},\theta_{3})\to 0, \quad k\to\infty,$$

where

$$g_s(x,\theta_2,\theta_3) = \mathbb{P}_{\delta_x} \left(|Y_s(\theta_2,\theta_3) - \mathbb{P}_{\delta_x} Y_s(\theta_2,\theta_3)|^2, |Y_s(\theta_2,\theta_3) - \mathbb{P}_{\delta_x} Y_s(\theta_2,\theta_3)| > ce^{-\lambda_1 s/(2(n-1))} \right)$$

Then $S_k \xrightarrow{d} \mathcal{N}(0, \sigma^2)$, which implies

$$\prod_{j=1}^{m_k} h((n-1)t_k, a_{k,j}, e^{\lambda_1 t_k/2}, \theta_2, \theta_3) \to e^{-\frac{1}{2}\sigma^2 \theta^2}, \quad \text{as } k \to \infty.$$
(3.70) 3.17

By the definition of Y_s , we get

$$V_t^n(x) := \mathbb{V}ar_{\delta_x}Y_{(n-1)t}(\theta_2, \theta_3) = \theta_2^2 \mathbb{V}ar_{\delta_x}Y_2((n-1)t) + \theta_3^2 \mathbb{V}ar_{\delta_x}Y_1((n-1)t) + 2\theta_2\theta_3((n-1)t)^{-(1+2\tau(h))/2}e^{\lambda_1(n-1)t}\mathbb{C}ov_{\delta_x}(\langle f, X_{(n-1)t} \rangle, \langle h, X_{(n-1)t} \rangle).$$

Thus, by $(\stackrel{\texttt{[smal]}}{\texttt{2.30}}, (\stackrel{\texttt{[1.49]}}{\texttt{2.40}})$ and $(\stackrel{\texttt{cov:sc}}{\texttt{2.54}},$ we easily get

$$\left| V_t^n(x) - (\theta_2^2 \rho_h^2 + \theta_3^2 \sigma_f^2) \phi_1(x) \right| \lesssim \left(c_{(n-1)t} + t^{-1} + t^{-(1+2\tau(h))/2} \right) \left(b_{t_0}(x)^{1/2} + b_{t_0}(x) \right), (3.71)$$

where $c_t \to 0$ as $t \to \infty$. By (2.18), we get, as $t \to \infty$,

$$e^{\lambda_{1}t}T_{t}\left|V_{t}^{n}(x) - (\theta_{2}^{2}\rho_{h}^{2} + \theta_{3}^{2}\sigma_{f}^{2})\phi_{1}(x)\right|(x) \lesssim \left(c_{(n-1)t} + t^{-1} + t^{-(1+2\tau(h))/2}\right)e^{\lambda_{1}t}T_{t}(\sqrt{b_{t_{0}}} + b_{t_{0}})(x) \to 0$$

which implies

$$\lim_{t \to \infty} e^{\lambda_1 t} \sum_{u \in \mathcal{L}_t} V_t^n(z_u(t)) = \lim_{t \to \infty} e^{\lambda_1 t} (\theta_2^2 \rho_h^2 + \theta_3^2 \sigma_f^2) \langle \phi_1, X_t \rangle = (\theta_2^2 \rho_h^2 + \theta_3^2 \sigma_f^2) W_{\infty}, \tag{3.72}$$

in probability.

By Lemma $\frac{\text{lem:5.6}}{\text{5.4, we get, as } s \to \infty, g_s(x, \theta_2, \theta_3) \to 0.$ Since

$$g_{(n-1)t}(x,\theta_2,\theta_3) \le V_t^n(x) \lesssim b_{t_0}(x)^{1/2} + b_{t_0}(x) \in L^2(E,m),$$

by the dominated convergence theorem, we have that for any $x \in E$,

$$\lim_{t \to \infty} \|g_{(n-1)t}(x, \theta_2, \theta_3)\|_2 = 0.$$

By Lemma 2.8, we have that for any $x \in E$,

$$e^{\lambda_1 t} \mathbb{P}_{\delta_x} \langle g_{(n-1)t}(\cdot, \theta_2, \theta_3), X_t \rangle \lesssim \|g_{(n-1)t}(\cdot, \theta_2, \theta_3)\|_2 b_{t_0}(x)^{1/2} \to 0, \quad \text{as} \quad t \to \infty,$$

which implies

$$e^{\lambda_1 t} \sum_{u \in \mathcal{L}_t} g_{(n-1)t}(z_u(t), \theta_2, \theta_3) = e^{\lambda_1 t} \langle g_{(n-1)t}(x, \theta_2, \theta_3), X_t \rangle \to 0,$$
(3.73) 3.21

in probability. Thus, for any sequence $s_k \to \infty$, there exists a subsequence s'_k such that, if we let $t_k = s'_k$, $m_k = |X_{t_k}|$ and $\{a_{k,j}, j = 1, \ldots, m_k\} = \{z_u(t_k), u \in \mathcal{L}_{t_k}\}$, then the Lindeberg conditions hold \mathbb{P}_{δ_x} -a.s. Therefore, by (B.70), we have

$$\lim_{t \to \infty} \prod_{u \in \mathcal{L}_t} h\left((n-1)t, z_u(t), e^{\lambda_1 t/2}, \theta_2, \theta_3 \right) = \exp\left\{ -\frac{1}{2} \left(\theta_2^2 \rho_h^2 + \theta_3^2 \sigma_f^2 \right) W_\infty \right\}, \quad \text{in probability.}$$

$$(3.74) \quad \boxed{3.24}$$

Hence by the dominated convergence theorem, we get

$$\lim_{t \to \infty} \kappa_2(\theta_1, \theta_2, \theta_3, n, t) = \mathbb{P}_{\delta_x} \left(\exp\left\{ i\theta_1 W_\infty \right\} \exp\left\{ -\frac{1}{2} \left(\theta_2^2 \rho_h^2 + \theta_3^2 \sigma_f^2 \right) W_\infty \right\} \right), \tag{3.75}$$

which implies our claim $(\frac{3.3}{3.66})$.

By $(\overset{\textbf{g.5}}{\textbf{b.06}})$ and the fact that $e^{\lambda_1 n t} \langle \phi_1, X_{nt} \rangle - e^{\lambda_1 t} \langle \phi_1, X_t \rangle \to 0$, in probability, as $t \to \infty$, we easily get that

$$U_3(n,t)$$

$$:= \left(e^{\lambda_1 n t} \langle \phi_1, X_{nt} \rangle, \frac{e^{\lambda_1 n t/2} (\langle h, X_{nt} \rangle - \langle T_{(n-1)t} h, X_t \rangle)}{(n t)^{(1+2\tau(h))/2}}, e^{\lambda_1 n t/2} (\langle f, X_{nt} \rangle - \langle T_{(n-1)t} f, X_t \rangle) \right)$$

$$\stackrel{d}{\to} \left(W_{\infty}, \left(\frac{n-1}{n} \right)^{(1+2\tau(h))/2} \sqrt{W_{\infty}} G_2(h), \sqrt{W_{\infty}} G_1(f) \right).$$

Using $(\frac{4.49}{5.23})$ with s = (n-1)t, we get that, if $\gamma(f) < \infty$,

$$\mathbb{P}_{\delta_x} \langle T_{(n-1)t} f, X_t \rangle^2 \lesssim (nt)^{2\tau(f)} e^{-2nt\Re_{\gamma(f)}} b_{t_0}(x)^{1/2} + ((n-1)t)^{2\tau(f)} e^{-\lambda_1 t} e^{-2\Re_{\gamma(f)}(n-1)t} b_{t_0}(x)^{1/2}$$

If $\gamma(f) = \infty$, using $(\frac{4.49}{5.25})$ with s = (n-1)t, we get

$$\mathbb{P}_{\delta_x} \langle T_{(n-1)t} f, X_t \rangle^2 \lesssim b_{t_0}(x)^{1/2} + e^{-\lambda_1 t} b_{t_0}(x)^{1/2}.$$

Therefore, we have

$$\lim_{t \to \infty} e^{\lambda_1 n t} \mathbb{P}_{\delta_x} \langle T_{(n-1)t} f, X_t \rangle^2 = 0.$$
(3.76) 4.7

By $(\frac{4\cdot 4}{5\cdot 20})$, when $\lambda_1 = 2\Re_{\gamma(h)}$, we get

$$\int_{0}^{t-2t_{0}} T_{t-u} [A(T_{u+(n-1)t}h)^{2}](x) du$$

$$\lesssim e^{-\lambda_{1}nt} \int_{0}^{t-2t_{0}} (u+(n-1)t)^{2\tau(h)} dub_{t_{0}}(x)^{1/2} \lesssim n^{2\tau(h)} t^{1+2\tau(h)} e^{-\lambda_{1}nt} b_{t_{0}}(x)^{1/2}.$$
(3.77)

By $(\underline{9.3}, \underline{77})$, $(\underline{9.7}, \underline{15.21})$ and $(\underline{4.3}, \underline{5.22})$, when $\lambda_1 = 2\Re_{\gamma(h)}$, we have

$$\mathbb{P}_{\delta_x} \langle T_{(n-1)t}h, X_t \rangle^2 \lesssim n^{2\tau(h)} t^{1+2\tau(f)} e^{-\lambda_1 n t} b_{t_0}(x)^{1/2} + (nt)^{2\tau(h)} e^{-\lambda_1 n t} b_{t_0}(x)^{1/2}.$$

Therefore, we have

$$\lim_{n \to \infty} \limsup_{t \to \infty} (nt)^{-(1+2\tau(h))} e^{\lambda_1 n t} \mathbb{P}_{\delta_x} \langle T_{(n-1)t} h, X_t \rangle^2 = 0.$$
(3.78) 4.2

Let $\mathcal{D}(nt)$ and $\widetilde{\mathcal{D}}^n(t)$ be the distributions of $U_1(nt)$ and $U_3(n,t)$ respectively, and let \mathcal{D}^n and \mathcal{D} be those of $\left(W_{\infty}, \left(\frac{n-1}{n}\right)^{(1+2\tau(h))/2} \sqrt{W_{\infty}}G_2(h), \sqrt{W_{\infty}}G_1(f)\right)$ and $\left(W_{\infty}, \sqrt{W_{\infty}}G_2(h), \sqrt{W_{\infty}}G_1(f)\right)$ respectively. Then, using $(\underline{\mathbf{5.20}}, \mathbf{14})$, we have

$$\limsup_{t \to \infty} \beta(\mathcal{D}(nt), \mathcal{D}) \leq \limsup_{t \to \infty} [\beta(\mathcal{D}(nt), \widetilde{\mathcal{D}}^{n}(t)) + \beta(\widetilde{\mathcal{D}}^{n}(t), \mathcal{D}^{n}) + \beta(\mathcal{D}^{n}, \mathcal{D})]$$

$$\leq \limsup_{t \to \infty} \left((nt)^{-(1+2\tau(h))} e^{\lambda_{1}nt} \mathbb{P}_{\mu} \langle T_{(n-1)t}h, X_{t} \rangle^{2} + e^{\lambda_{1}nt} \mathbb{P}_{\mu} \langle T_{(n-1)t}f, X_{t} \rangle^{2} \right)^{1/2} + 0 + \beta(\mathcal{D}^{n}, \mathcal{D}).$$
(3.79)

Using the definition of $\limsup_{t\to\infty}$, $(\frac{4.7}{5.76})$ and $(\frac{4.2}{5.78})$, we easily get that

$$\limsup_{t \to \infty} \beta(\mathcal{D}(t), \mathcal{D}) = \limsup_{t \to \infty} \beta(\mathcal{D}(nt), \mathcal{D})$$

$$\leq \limsup_{t \to \infty} (nt)^{-(1+2\tau(h))} e^{nt\lambda_1 t} \mathbb{P}_{\delta_x} \langle T_{(n-1)t}h, X_t \rangle^2 + \beta(\mathcal{D}^n, \mathcal{D}).$$

Letting $n \to \infty$, we get $\limsup_{t\to\infty} \beta(\mathcal{D}(t), \mathcal{D}) = 0$. The proof is now complete. **Proof of Corollary** 1.19: Define

$$Y_1(s) := s^{-(1+2\tau(h_1))/2} e^{\lambda_1 s/2} \langle h_1, X_s \rangle, \quad Y_2(s) := s^{-(1+2\tau(h_2))/2} e^{\lambda_1 s/2} \langle h_2, X_s \rangle$$

and

$$Y_s(\theta_2, \theta_3) := \theta_2 Y_1(s) + \theta_3 Y_2(s)$$

Thus, we have

$$\mathbb{V}ar_{\delta_{x}}Y_{(n-1)t}(\theta_{2},\theta_{3}) = \theta_{2}^{2}\mathbb{V}ar_{\delta_{x}}Y_{1}((n-1)t) + \theta_{3}^{2}\mathbb{V}ar_{\delta_{x}}Y_{2}((n-1)t) + 2\theta_{2}\theta_{3}\mathbb{C}ov_{\delta_{x}}(Y_{1}((n-1)t),Y_{2}((n-1)t)).$$
(3.80)

By $(\frac{7.49}{2.39})$ and $(\frac{1.49}{2.40})$, we get

$$\left| \mathbb{V} \mathrm{ar}_{\delta_x} Y_{(n-1)t}(\theta_2, \theta_3) - (\theta_2^2 \rho_{h_1}^2 + \theta_3^2 \rho_{h_2}^2 + 2\theta_2 \theta_3 \rho(h_1, h_2)) \phi_1(x) \right| \lesssim t^{-1} \left(b_{t_0}(x)^{1/2} + b_{t_0}(x) \right).$$

Using arguments similar to those leading to Lemma $\frac{|lem:cs|}{3.5}$, we get

$$\lim_{t \to \infty} \mathbb{P}_{\delta_x} \exp\left\{i\theta_1 e^{\lambda_1 t} \langle \phi_1, X_t \rangle + i\theta_2 Y_1(t) + i\theta_3 Y_2(t)\right\}$$

= $\mathbb{P}_{\delta_x} \exp\left\{i\theta_1 W_\infty - \frac{1}{2} \left(\theta_2^2 \rho_{h_1}^2 + \theta_3^2 \rho_{h_2}^2 + 2\theta_2 \theta_3 \rho(h_1, h_2)\right) W_\infty\right\}.$ (3.81)

The proof of Corollary 1.19 is now complete.

Recall that

$$g(x) = \sum_{k:\lambda_1 > 2\Re_k} \Phi_k(x)^T b_k \in \mathcal{C}_c \quad \text{and} \quad I_s g(x) = \sum_{k:\lambda_1 > 2\Re_k} e^{\lambda_k s} \Phi_k(x)^T D_k(s)^{-1} b_k.$$

We can show that $I_s g$ is real. In fact, for k with $\lambda_1 > 2\Re_k$, we have $\lambda_1 > 2\Re_{k'}$. And

$$e^{\lambda_{k'}s}\Phi_{k'}(x)^T D_{k'}(s)^{-1}b_{k'} = e^{\overline{\lambda_k}s}\overline{\Phi_k(x)^T} D_k(s)^{-1}\overline{b_k} = \overline{e^{\lambda_k}s}\overline{\Phi_k(x)^T} D_k(s)^{-1}b_k,$$

which implies that $I_s g(x)$ is real. Define

$$H_{\infty} := \sum_{k:\lambda_1 > 2\Re_k} H_{\infty}^{(k)} b_k.$$

By Lemma $\frac{\texttt{thrm1}}{3.1}$, we have, as $s \to \infty$

 $\langle I_s g, X_s \rangle \to H_{\infty}, \quad \mathbb{P}_{\delta_r}$ -a.s. and in $L^2(\mathbb{P}_{\delta_r})$.

Since $\mathbb{P}_{\delta_x}\langle I_s g, X_s \rangle = g(x)$, we get

$$\mathbb{P}_{\delta_x}(H_\infty) = g(x). \tag{3.82}$$

By (2.21), we have

$$\mathbb{P}_{\delta_x} \langle I_s g, X_s \rangle^2 = \int_0^s T_u \left[A \left| I_u g \right|^2 \right] (x) \, du + T_s [(I_s g)^2](x). \tag{3.83}$$

Η

It is easy to see that,

$$|I_s g(x)|^2 \lesssim \sum_{k:\lambda_1 > 2\Re_k} e^{2\Re_k s} s^{2\nu_k} b_{4t_0}(x).$$

Thus, by (2.18), we have, for $s > 2t_0$,

$$T_s |I_s g|^2(x) \lesssim \sum_{k:\lambda_1 > 2\Re_k} e^{2\Re_k s} s^{2\nu_k} T_s(b_{t_0})(x) \lesssim \sum_{k:2\Re_k < \lambda_1} s^{2\nu_k} e^{(2\Re_k - \lambda_1)s} b_{t_0}(x)^{1/2}.$$
(3.84) 5.5

By (2.26), we get

$$\int_{0}^{\infty} T_{u} \left[A \left| I_{u} g \right|^{2} \right] (x) \, du$$

$$\lesssim \sum_{k:\lambda_{1} > 2\Re_{k}} \left(\int_{0}^{2t_{0}} e^{2\Re_{k} u} u^{2\nu_{k}} T_{u}(b_{4t_{0}})(x) \, du + \int_{2t_{0}}^{\infty} u^{2\nu_{k}} e^{(2\Re_{k} - \lambda_{1})u} \, dub_{t_{0}}(x)^{1/2} \right)$$

$$\lesssim b_{t_{0}}(x)^{1/2} \in L^{2}(E, m) \cap L^{4}(E, m).$$

Therefore, by $(\underbrace{\mathtt{yar:Iu}}{\mathtt{3.83}}$ and $(\underbrace{\mathtt{5.5}}{\mathtt{3.84}})$, we get

$$\mathbb{P}_{\delta_x}(H_{\infty})^2 = \lim_{s \to \infty} \mathbb{P}_{\delta_x} |\langle I_s g, X_s \rangle|^2 = \int_0^\infty T_u \left[A |I_u g|^2 \right](x) \, du \in L^2(E, m) \cap L^4(E, m). \tag{3.85}$$

Hence, we have

$$\operatorname{Var}_{\delta_x} H_{\infty} = \mathbb{P}_{\delta_x} (H_{\infty})^2 - (\mathbb{P}_{\delta_x} H_{\infty})^2 = \int_0^\infty T_u \left(A \left| I_u g \right|^2 \right) (x) \, du - g(x)^2. \tag{3.86}$$
LH

Proof of Theorem 1.17: Recall that

$$E_t(g) = \left(\sum_{k:2\lambda_k < \lambda_1} e^{-\lambda_k t} H_{\infty}^{(k)} D_k(t) b_k\right)$$

and

$$Y_1(t) := e^{\lambda_1 t/2} \langle f, X_t \rangle, \quad Y_2(t) := t^{-(1+2\tau(h))/2} e^{\lambda_1 t/2} \langle h, X_t \rangle.$$

Consider an \mathbb{R}^4 -valued random variable $U_4(t)$ defined by:

$$\boxed{8.5}U_4(t) := \left(e^{\lambda_1 t} \langle \phi_1, X_t \rangle, \ e^{\lambda_1 t/2} \left(\langle g, X_t \rangle - E_t(g)\right), Y_2(t), Y_1(t)\right).$$

To get the conclusion of Theorem $\frac{\text{The: } 1.3}{1.17, \text{ it}}$ suffices to show that, under \mathbb{P}_{δ_x} ,

$$U_4(t) \xrightarrow{d} \left(W_{\infty}, \sqrt{W_{\infty}}G_3(g), \sqrt{W_{\infty}}G_2(h), \sqrt{W_{\infty}}G_1(f) \right), \qquad (3.87) \quad \boxed{2.5a}$$

where W_{∞} , $G_3(g)$, $G_2(h)$ and $G_1(f)$ are independent. Denote the characteristic function of $U_4(t)$ under \mathbb{P}_{δ_x} by $\kappa_3(\theta_1, \theta_2, \theta_3, \theta_4, t)$. Then, we only need to prove

$$\lim_{t \to \infty} \kappa_3(\theta_1, \theta_2, \theta_3, \theta_4, t) = \mathbb{P}_{\mu} \left(\exp\{i\theta_1 W_\infty\} \exp\left\{ -\frac{1}{2} (\theta_2^2 \beta_g^2 + \theta_3^2 \rho_h^2 + \theta_4^2 \sigma_f^2) W_\infty \right\} \right).$$
(3.88) (3.88)

Note that, by Lemma $\frac{\texttt{thrm1}}{\texttt{B.1}}$, we get

$$E_t(g) = \lim_{s \to \infty} \langle I_s g, X_{t+s} \rangle = \sum_{u \in \mathcal{L}_t} \lim_{s \to \infty} \langle I_s g, X_s^{u,t} \rangle.$$

Since $X_s^{u,t}$ has the same law as X_s under $\mathbb{P}_{\delta_{z_u(t)}}$, $H_{\infty}^{u,t} := \lim_{s \to \infty} \langle I_s g, X_s^{u,t} \rangle$ exists and has the same law as H_{∞} under $\mathbb{P}_{\delta_{z_u(t)}}$. Thus, we get $E_t(g) = \sum_{u \in \mathcal{L}_t} H_{\infty}^{u,t}$. Let $h(x,\theta) = \mathbb{P}_{\delta_x} \exp\{i\theta(H_{\infty} - g(x))\}$. Therefore, we obtain that

$$\kappa_{3}(\theta_{1},\theta_{2},\theta_{3},\theta_{4},t) = \mathbb{P}_{\delta_{x}}\left(\exp\left\{i\theta_{1}e^{\lambda_{1}t}\langle\phi_{1},X_{t}\rangle+i\theta_{3}Y_{2}(t)+i\theta_{4}Y_{1}(t)\right\}\prod_{u\in\mathcal{L}_{t}}h\left(z_{u}(t),-\theta_{2}e^{\lambda_{1}t/2}\right)\right).$$
 (3.89)

Let $V(x) = \mathbb{V}ar_{\delta_x}H_{\infty}$. We claim that

(i) as $t \to \infty$,

$$e^{\lambda_1 t} \sum_{u \in \mathcal{L}_t} \mathbb{P}_{\delta_x} |H_{\infty}^{u,t} - g(z_u(t))|^2 = e^{\lambda_1 t} \langle V, X_t \rangle \to \langle V, \psi_1 \rangle_m W_{\infty}, \text{ in probability;}$$
(3.90) 8.1

(ii) for any $\epsilon > 0$, as $t \to \infty$,

$$e^{\lambda_{1}t} \sum_{u \in \mathcal{L}_{t}} \mathbb{P}_{\delta_{x}}(|H_{\infty}^{u,t} - g(z_{u}(t))|^{2}, |H_{\infty}^{u,t} - g(z_{u}(t))| > \epsilon e^{-\lambda_{1}t/2})$$

= $e^{\lambda_{1}t} \langle k(\cdot,t), X_{t} \rangle \to 0$, in probability, (3.91)

where $k(x,t) := \mathbb{P}_{\delta_x}(|H_{\infty} - g(x)|^2, |H_{\infty} - g(x)| > \epsilon e^{-\lambda_1 t/2}).$

Then using arguments similar to those in the proof Lemma $\frac{\texttt{Lem:cs}}{3.5}$, we have

$$\prod_{u \in \mathcal{L}_t} h\left(z_u(t), -\theta_2 e^{(\lambda_1/2)t}\right) \to \exp\left\{-\frac{1}{2}\theta_2^2 \langle V, \psi_1 \rangle_m W_\infty\right\}, \text{ in probability.}$$
(3.92) 8.4

Now we prove the claims.

(i) By $(\underline{B.85})$, we have $V(x) \in L^2(E,m) \cap L^4(E,m)$. By Remark $\underline{I.16}, (\underline{B.90})$ follows immediately.

(ii) We easily see that $k(x,t) \downarrow 0$ as $t \uparrow \infty$ and $k(x,t) \leq V(x) \in L^2(E,m)$ for any $x \in E$. Thus, $\lim_{t\to\infty} \|k(\cdot,t)\|_2 = 0$. So, by 2.18, we have that for any $x \in E$,

$$e^{\lambda_1 t} \mathbb{P}_{\delta_x} \langle k(\cdot, t), X_t \rangle \lesssim \|k(\cdot, t)\|_2 b_{t_0}(x)^{1/2} \to 0, \quad \text{as } t \to \infty,$$

which implies $\begin{pmatrix} 8.2\\ 3.91 \end{pmatrix}$.

By $(\underline{8.6}, \underline{8.9})$, $(\underline{8.9})$ and the dominated convergence theorem, we get that as $t \to \infty$,

$$\left| \kappa_{3}(\theta_{1},\theta_{2},\theta_{3},\theta_{4},t) - \mathbb{P}_{\delta_{x}} \left(\exp\left\{ (i\theta_{1} - \frac{1}{2}\theta_{2}^{2}\langle V,\psi_{1}\rangle_{m})e^{\lambda_{1}t}\langle\phi_{1},X_{t}\rangle + i\theta_{3}Y_{2}(t) + i\theta_{4}Y_{1}(t) \right\} \right) \right|$$

$$\leq \mathbb{P}_{\delta_{x}} \left| \prod_{u \in \mathcal{L}_{t}} h\left(z_{u}(t), -\theta_{2}e^{(\lambda_{1}/2)t} \right) - \exp\left\{ -\frac{1}{2}\theta_{2}^{2}\langle V,\psi_{1}\rangle_{m}e^{\lambda_{1}t}\langle\phi_{1},X_{t}\rangle \right\} \right| \to 0.$$

$$(3.93)$$

By Lemma 3.5, we get

$$\lim_{t \to \infty} \kappa_3(\theta_1, \theta_2, \theta_3, \theta_4, t)$$

$$= \lim_{t \to \infty} \mathbb{P}_{\delta_x} \left(\exp\left\{ (i\theta_1 - \frac{1}{2}\theta_2^2 \langle V, \psi_1 \rangle_m) e^{\lambda_1 t} \langle \phi_1, X_t \rangle + i\theta_3 Y_1(t) + i\theta_4 Y_2(t) \right\} \right)$$

$$= \mathbb{P}_{\delta_x} \left(\exp\{i\theta_1 W_\infty\} \exp\left\{ -\frac{1}{2} \left(\theta_2^2 \langle V, \psi_1 \rangle_m + \theta_3^2 \rho_h^2 + \theta_4^2 \sigma_f^2 \right) W_\infty \right\} \right).$$

$$\langle V, \psi_1 \rangle_m = \int_0^\infty e^{-\lambda_1 u} \langle A | I_u g |^2, \psi_1 \rangle_m \, du - \langle g^2, \psi_1 \rangle_m$$

The proof is now complete.

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RP

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