Sliced space-filling designs with different levels of two-dimensional uniformity

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Abstract. We consider sliced computer experiments where priori knowledge suggests that factors may have different levels of importance, and so some factors need to be paid more attention than others. A new class of sliced space-filling designs are proposed to deal with this type of sliced computer experiments, in which the whole design and each slice may have different levels of two-dimensional uniformity for different factors, besides they all achieve maximum stratification in univariate margins. They are generated by elaborately randomizing a special type of asymmetric orthogonal arrays, called asymmetric balanced sliced orthogonal arrays, which can be partitioned into several slices such that each slice is balanced and becomes an asymmetric orthogonal array after some level-collapsing. Several methods are developed to construct such asymmetric balanced sliced orthogonal arrays.

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1 Introduction

Sliced space-filling designs, proposed by Qian and Wu (2009), are intended for computer experiments with both qualitative and quantitative factors [Qian Wu and Wu (2008); Han et al.(2009)], linking parameters in engineering and cross-validation. For those sliced designs constructed by Qian and Wu (2009) and Qian (2012), each slice can not achieve the univariate and multiple-dimensional uniformity simultaneously. Recently, Xu, Haaland and Qian (2011) constructed Sudoku-based sliced space-filling designs in which the whole design and each slice all achieve maximum stratification in both univariate and bivariate margins. Ai, Jiang and Li (2014) proposed a general approach to constructing sliced space-filling designs by randomizing symmetric balanced sliced orthogonal arrays (BSOAs) so that the whole design

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and each slice can achieve stratification in two- or more-dimensional projections, in addition to achieving maximum stratification in univariate margins.

In this article, we consider sliced computer experiments where priori knowledge suggests that some factors are more important than others, and so need to be paid more attention. To deal with this issue, we are ready to propose a new class of sliced space-filling designs, in which the whole design and each slice can achieve maximum stratification in any univariate margin, but have different levels of two-dimensional uniformity for different factors. The proposed designs are generated by randomizing asymmetric BSOAs, which are a special class of asymmetric orthogonal arrays whose rows can be partitioned into several slices such that each slice is balanced and also becomes an asymmetric orthogonal array after some level-collapsing. Thus, the more important factors can be assigned to the columns with higher level in an asymmetric BSOA.

The remainder of this article will unfold as follows. A formal definition of asymmetric BSOAs is given in Section 2. Section 3 and Section 4 provide the construction of asymmetric BSOAs via the Kronecker sum and the replacement of levels, respectively. The generation of sliced space-filling designs based on asymmetric BSOAs is presented in Section 5. Section 6 concludes this article with some discussions.

2 Definition of asymmetric BSOAs

An orthogonal array (OA), denoted by $OA(n, s_1^{\gamma_1} \cdots s_k^{\gamma_k}, t)$, with n runs, $m = \sum_{i=1}^k \gamma_i$ factors and strength t ($m \geq t \geq 1$), is an $n \times m$ matrix in which the first γ_1 columns have s_1 levels from a set of s_1 elements, the next γ_2 columns have s_2 levels from a set of s_2 elements, and so on, such that every $n \times t$ submatrix contains all possible level combinations as rows with the same frequency. When $s_1 = \cdots = s_k = s$, in particular, this special case is called a symmetric OA and denoted by $OA(n, s^m, t)$; otherwise, it is an asymmetric OA. An array is called *balanced* if it is an OA of strength one. Throughout, we consider only OAs of strength two and drop the strength parameter in $OA(n, s_1^{\gamma_1} \cdots s_k^{\gamma_k}, 2)$.

Let F be a set of s_1 elements and G be a set of s_2 elements with s_2 dividing s_1, denoted by $s_2|s_1$. A level-collapsing *projection* from F to G, say δ, divides the elements of F into s_2 groups, each of size $q = s_1/s_2$, and projects any two elements of F to the same element of G if and only if they belong to the same group. The kernel matrix of δ is an $s_2 \times q$ matrix in which each row consists of the elements of F in the same group [Qian and Wu (2009)]. For a matrix $\boldsymbol{A}$, let $\boldsymbol{A}'$ denote its transpose. If $\boldsymbol{A}$ takes entries from F, denote $\delta(\boldsymbol{A})$ as the array obtained from $\boldsymbol{A}$ after its entries are collapsed according to δ.

The definition of asymmetric BSOAs is given as follows. Let $\boldsymbol{H}$ be an $OA(n_1, s_{11}^{\gamma_1} \cdots s_{k1}^{\gamma_k})$. Suppose the n_1 rows of $\boldsymbol{H}$ can be partitioned into v subarrays each with n_2 rows, denoted by $\boldsymbol{H}_i$, $i = 1, \ldots, v$, and each $\boldsymbol{H}_i$ becomes an $OA(n_2, s_{12}^{\gamma_1} \cdots s_{k2}^{\gamma_k})$ after the s_{j1} levels of the s_{j1}-level factors are collapsed to s_{j2} levels according to some level-collapsing projection δ_j, for $j = 1, \ldots, k$. Then $\boldsymbol{H}$, or more precisely $(\boldsymbol{H}_1', \ldots, \boldsymbol{H}_v')'$, is called a sliced orthogonal array (SOA). For an SOA $\boldsymbol{H}$ in which each slice $\boldsymbol{H}_i$ is balanced, it is called a balanced SOA (BSOA). Provided that the s_{j1}'s are not all the same, it is an asymmetric BSOA.

3 Construction of asymmetric BSOAs via Kronecker sum

Similar to the construction of nested OAs with mixed levels in Qian, Ai and Wu (2009), we propose three methods of constructing asymmetric BSOAs via the Kronecker sum. Throughout this section, we consider only the level-collapsing projection δ from an abelian group F to another abelian group G that has the additivity property, i.e., $\delta(f_1 + f_2) = \delta(f_1) + \delta(f_2)$ for any $f_1, f_2 \in F$.

The Kronecker sum of an $n \times m$ matrix $\boldsymbol{A} = (a_{ij})$ and a $u \times v$ matrix $\boldsymbol{B} = (b_{lk})$ based on the same abelian group with the addition operation '+', is defined to be the $nu \times mv$ matrix $\boldsymbol{A} \oplus \boldsymbol{B} = (a_{ij} + \boldsymbol{B})$, where $a_{ij} + \boldsymbol{B}$ denotes the $u \times v$ matrix with entries $a_{ij} + b_{lk}$, $1 \le l \le u$ and $1 \le k \le v$. Let $D(r, c, g)$ denote a difference matrix (DM), which is an $r \times c$ array based on an abelian group $\mathcal{A}$ of g elements such that every element of $\mathcal{A}$ appears equally often in the vector difference between any two columns of the array. Here we present an obvious conclusion for constructing asymmetric OAs in the following lemma for convenience of later use [Wang and Wu (1991)].

Lemma 1. *Suppose $\boldsymbol{A} = (\boldsymbol{A}_1, \ldots, \boldsymbol{A}_k)$ is an $OA(n, s_1^{\gamma_1} \cdots s_k^{\gamma_k})$, where $\boldsymbol{A}_j$ is the subarray corresponding to the s_j-level factors with levels from an abelian group $\mathcal{A}_j$. Let $\boldsymbol{B}_j$ be a $D(r, c_j, s_j)$ based on $\mathcal{A}_j$, for $j = 1, \ldots, k$. Then $\boldsymbol{H} = (\boldsymbol{A}_1 \oplus \boldsymbol{B}_1, \ldots, \boldsymbol{A}_k \oplus \boldsymbol{B}_k)$ is an $OA(nr, s_1^{\gamma_1 c_1} \cdots s_k^{\gamma_k c_k})$.*

3.1 Using sliced orthogonal arrays and difference matrices

This construction makes use of SOAs and difference matrices. For $j = 1, \ldots, k$, let $s_{j1} \ge s_{j2} > 1$ with $s_{j2} | s_{j1}$, F_j be an abelian group of s_{j1} elements, G_j be an abelian group of s_{j2} elements, and δ_j be a level-collapsing projection from F_j to G_j, where the s_{j1}'s are assumed to be all distinct. Suppose $\boldsymbol{A} = (\boldsymbol{A}_1, \ldots, \boldsymbol{A}_k)$ is an SOA with v slices, where $\boldsymbol{A}$ is an $OA(n_1, s_{11}^{\gamma_1} \cdots s_{k1}^{\gamma_k})$, $\boldsymbol{A}_j = (\boldsymbol{A}'_{j1}, \ldots, \boldsymbol{A}'_{jv})'$ is the subarray of $\boldsymbol{A}$ corresponding to the s_{j1}-level factors with levels from F_j, and each slice $(\boldsymbol{A}_{1i}, \ldots, \boldsymbol{A}_{ki})$ becomes an $OA(n_2, s_{12}^{\gamma_1} \cdots s_{k2}^{\gamma_k})$ after the levels of the s_{j1}-level factors are collapsed according to δ_j, for $j = 1, \ldots, k$ and $i = 1, \ldots, v$.

Furthermore, for $j = 1, \ldots, k$, let $\boldsymbol{B}_j$ be a $D(r, c_j, s_{j1})$ based on F_j. Put

$$\boldsymbol{H} = (\boldsymbol{A}_1 \oplus \boldsymbol{B}_1, \ldots, \boldsymbol{A}_k \oplus \boldsymbol{B}_k) \text{ and } \boldsymbol{H}_i = (\boldsymbol{A}_{1i} \oplus \boldsymbol{B}_1, \ldots, \boldsymbol{A}_{ki} \oplus \boldsymbol{B}_k)$$

for $i = 1, \ldots, v$. Note that $\delta_j(\boldsymbol{B}_j)$ is a $D(r, c_j, s_{j2})$, for $j = 1, \ldots, k$. By using the additivity property of δ_j's and Lemma 1, the following theorem is obtained.

Theorem 1. *If each slice of $\boldsymbol{A}_j$ is balanced or $\boldsymbol{B}_j$ is balanced for $j = 1, \ldots, k$, we have*
(i) the matrix $\boldsymbol{H}$ is an $OA(n_1 r, s_{11}^{\gamma_1 c_1} \cdots s_{k1}^{\gamma_k c_k})$;
(ii) each slice $\boldsymbol{H}_i$ is balanced and becomes an $OA(n_2 r, s_{12}^{\gamma_1 c_1} \cdots s_{k2}^{\gamma_k c_k})$ after the levels of the s_{j1}-level factors are collapsed according to δ_j, for $j = 1, \ldots, k$ and $i = 1, \ldots, v$.

Example 1. *Let $\boldsymbol{A} = (\boldsymbol{A}_1, \boldsymbol{A}_2)$ be an $OA(24, 6^1 4^1)$, where $\boldsymbol{A}_1$ corresponds to the six-level factor with levels from the residue ring $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ and $\boldsymbol{A}_2$ corresponds to the four-level factor with levels from $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{00, 01, 10, 11\}$. For $i_1, i_2 = 0, 1$, let $(\boldsymbol{A}_{1,2i_1+i_2}, \boldsymbol{A}_{2,2i_1+i_2})$ be the slice of $\boldsymbol{A}$, where $\boldsymbol{A}_{1,2i_1+i_2}$ takes elements from $\{3i_1, 3i_1+1, 3i_1+2\}$ and $\boldsymbol{A}_{2,2i_1+i_2}$ takes elements from $\{i_2 0, i_2 1\}$. Define the projection δ_1 as $\{0, 3\} \to 0$, $\{1, 4\} \to 1$ and $\{2, 5\} \to 2$, and δ_2 as $\{00, 10\} \to 0$ and $\{01, 11\} \to 1$. Furthermore, let $\boldsymbol{B}_1$ be the balanced $D(12, 5, 6)$ [Johnson, Dulmage and Mendelsohn (1961)], given in transpose by*

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 3 & 0 & 1 & 3 & 5 & 2 & 2 & 5 & 4 & 1 & 4 \\ 0 & 2 & 1 & 5 & 5 & 3 & 3 & 4 & 2 & 1 & 0 & 4 \\ 0 & 4 & 5 & 4 & 2 & 1 & 2 & 0 & 3 & 1 & 3 & 5 \\ 0 & 0 & 2 & 2 & 1 & 1 & 3 & 5 & 4 & 4 & 5 & 3 \end{pmatrix}$$

and $\boldsymbol{B}_2$ be the balanced $D(12, 11, 4)$ [Seberry (1979)], given by

$$\begin{pmatrix} 00 & 00 & 00 & 00 & 00 & 00 & 00 & 00 & 00 & 00 & 00 \\ 00 & 00 & 01 & 01 & 01 & 11 & 11 & 11 & 10 & 10 & 10 \\ 00 & 00 & 11 & 11 & 11 & 10 & 10 & 10 & 01 & 01 & 01 \\ 11 & 01 & 10 & 01 & 11 & 01 & 10 & 00 & 11 & 00 & 10 \\ 11 & 01 & 11 & 10 & 01 & 00 & 01 & 10 & 10 & 11 & 00 \\ 11 & 01 & 01 & 11 & 10 & 10 & 00 & 01 & 00 & 10 & 11 \\ 01 & 10 & 11 & 00 & 10 & 01 & 00 & 11 & 01 & 11 & 10 \\ 01 & 10 & 10 & 11 & 00 & 11 & 01 & 00 & 10 & 01 & 11 \\ 01 & 10 & 00 & 10 & 11 & 00 & 11 & 01 & 11 & 10 & 01 \\ 10 & 11 & 01 & 10 & 00 & 01 & 11 & 10 & 01 & 00 & 11 \\ 10 & 11 & 00 & 01 & 10 & 10 & 01 & 11 & 11 & 01 & 00 \\ 10 & 11 & 10 & 00 & 01 & 11 & 10 & 01 & 00 & 11 & 01 \end{pmatrix}.$$

Put $\boldsymbol{H} = (\boldsymbol{A}_1 \oplus \boldsymbol{B}_1, \boldsymbol{A}_2 \oplus \boldsymbol{B}_2)$ and $\boldsymbol{H}_i = (\boldsymbol{A}_{1,i} \oplus \boldsymbol{B}_1, \boldsymbol{A}_{2,i} \oplus \boldsymbol{B}_2)$ for $i = 0, 1, 2, 3$. From Theorem 1, we know that $\boldsymbol{H}$ is an $OA(288, 6^5 4^{11})$, each slice $\boldsymbol{H}_i$ is balanced and becomes an $OA(72, 3^5 2^{11})$ after the levels of the six-level factors are collapsed according to δ_1 and the levels of the four-level factors are collapsed according to δ_2, for $i = 0, 1, 2, 3$.

3.2 Using orthogonal arrays and sliced difference matrices

This construction makes use of OAs and sliced difference matrices (SDMs). For $j = 1, \ldots, k$, let $s_{j1} \geq s_{j2} > 1$ with $s_{j2} | s_{j1}$, F_j be an abelian group of s_{j1} elements, G_j be an abelian group of s_{j2} elements, and δ_j be a level-collapsing projection from F_j to G_j, where the s_{j1}'s are assumed to be all distinct. Suppose $\boldsymbol{A} = (\boldsymbol{A}_1, \ldots, \boldsymbol{A}_k)$ is an $OA(n, s_{11}^{\gamma_1} \cdots s_{k1}^{\gamma_k})$, where $\boldsymbol{A}_j$ is the subarray of $\boldsymbol{A}$ corresponding to the s_{j1}-level factors with levels from F_j, for $j = 1, \ldots, k$.

Furthermore, for $j = 1, \ldots, k$, let $\boldsymbol{B}_j = (\boldsymbol{B}'_{j1}, \ldots, \boldsymbol{B}'_{jv})'$ be an SDM with v slices, i.e., $\boldsymbol{B}_j$ is a $D(r_1, c_j, s_{j1})$ based on F_j, each $\boldsymbol{B}_{ji}$ is a submatrix of $\boldsymbol{B}_j$ and $\delta_j(\boldsymbol{B}_{ji})$ is a $D(r_2, c_j, s_{j2})$, for $i = 1, \ldots, v$. Put

$$\boldsymbol{H} = (\boldsymbol{A}_1 \oplus \boldsymbol{B}_1, \ldots, \boldsymbol{A}_k \oplus \boldsymbol{B}_k) \text{ and } \boldsymbol{H}_i = (\boldsymbol{A}_1 \oplus \boldsymbol{B}_{1i}, \ldots, \boldsymbol{A}_k \oplus \boldsymbol{B}_{ki})$$

for $i = 1, \ldots, v$. Note that $(\delta_1(\boldsymbol{A}_1), \ldots, \delta_k(\boldsymbol{A}_k))$ is an $OA(n, s_{12}^{\gamma_1} \cdots s_{k2}^{\gamma_k})$. By using the additivity property of δ_j's and Lemma 1, the following theorem is obtained.

Theorem 2. *For the matrix $\boldsymbol{H}$ constructed above, we have*
(i) the matrix $\boldsymbol{H}$ is an $OA(nr_1, s_{11}^{\gamma_1 c_1} \cdots s_{k1}^{\gamma_k c_k})$;
(ii) each slice $\boldsymbol{H}_i$ is balanced and becomes an $OA(nr_2, s_{12}^{\gamma_1 c_1} \cdots s_{k2}^{\gamma_k c_k})$ after the levels of the s_{j1}-level factors are collapsed according to δ_j, for $j = 1, \ldots, k$ and $i = 1, \ldots, v$.

Galois fields are widely used to construct OAs and DMs in virtue of their attractive algebraic structures. For any prime p and integer $u \geq 1$, there exists a Galois field $GF(p^u)$ of order p^u with a primitive irreducible polynomial in x of degree u. Any element $f(x)$ of $GF(p^u)$ has the general expression $f(x) = a_0 + a_1 x + \cdots + a_{u-1} x^{u-1}$, where $a_i \in GF(p)$, $0 \leq i \leq u-1$ and $GF(p) = \{0, 1, \ldots, p-1\}$ is the residue field modulo p. Taking advantage of Galois fields, we now give an example that implements the method in Theorem 2.

Example 2. *According to the construction method in Theorem 6.6 of Hedayat, Sloane and Stufken (1999), we obtain a $D(8,8,4)$ based on $GF(4)$, denoted by $\boldsymbol{D}_1$. Let $\boldsymbol{c} = (0, 1, x, x+1)'$ based on $GF(4)$ and $\boldsymbol{a} = (0, 1, x, x+1, x^2, x^2+1, x^2+x, x^2+x+1)'$ based on $GF(8)$. Put $\boldsymbol{A}_1 = \boldsymbol{c} \oplus \boldsymbol{D}_1$, $\boldsymbol{A}_2 = (\boldsymbol{a}', \boldsymbol{a}', \boldsymbol{a}', \boldsymbol{a}')'$ and $\boldsymbol{A} = (\boldsymbol{A}_1, \boldsymbol{A}_2)$. Following Theorem 9.15 of Hedayat, Sloane and Stufken (1999), we know that $\boldsymbol{A}$ is an $OA(32, 4^8 8^1)$. Let $\boldsymbol{B}_{11}$ be the multiplication table of $GF(4)$ and $\boldsymbol{B}_{12} = \boldsymbol{B}_{11} + 1$. It can be verified that $\boldsymbol{B}_1 = (\boldsymbol{B}_{11}', \boldsymbol{B}_{12}')'$ is a $D(8,4,4)$ and each $\delta_1(\boldsymbol{B}_{1i})$ is an $D(4,4,2)$ based on $GF(2)$ for $i = 1, 2$. Here δ_1 is defined as $\{0, x\} \to 0$ and $\{1, x+1\} \to 1$. Furthermore, by using the method in Lemma 4 of Ai, Jiang and Li (2014), we can obtain an SDM $\boldsymbol{B}_2 = (\boldsymbol{B}_{21}', \boldsymbol{B}_{22}')'$, where $\boldsymbol{B}_2$ is a $D(8,4,8)$ based on $GF(8)$ and each $\delta_2(\boldsymbol{B}_{2i})$ is a $D(4,4,4)$ based on $GF(4)$ for $i = 1, 2$. Here δ_2 is defined as $\{0, x^2+x+1\} \to 0$, $\{1, x^2+x\} \to 1$, $\{x, x^2+1\} \to x$ and $\{x+1, x^2\} \to x+1$. From Theorem 2, we know that $\boldsymbol{H}$ is an $OA(256, 4^{32} 8^4)$, each slice $\boldsymbol{H}_i$ is balanced and becomes an $OA(128, 2^{32} 4^4)$ after the levels of the four-level factors are collapsed according to δ_1 and the levels of the eight-level factors are collapsed according to δ_2, for $i = 1, 2$.*

3.3 Using a special two-tuple column and difference matrices

Wang (1996) proposed an approach to constructing asymmetric OAs with more flexible runs by using a special two-tuple column and mixed DMs. Here we embed balanced and sliced structures in it to construct asymmetric BSOAs.

For $j = 1, 2$, let $s_{j1} \geq s_{j2} > 1$ with $s_{j2} | s_{j1}$, F_j be an abelian group of s_{j1} elements, G_j be an abelian group of s_{j2} elements, δ_j be a level-collapsing projection from F_j to G_j, and $\boldsymbol{\Gamma}_j$ be the $s_{j2} \times q_j$ kernel matrix of δ_j with $q_j = s_{j1}/s_{j2}$. Denote by $F_1 \times F_2$ the set $\{(f_1, f_2) | f_1 \in F_1, f_2 \in F_2\}$ and $G_1 \times G_2$ is similarly defined. Let δ_0 be the level-collapsing projection from $F_1 \times F_2$ to $G_1 \times G_2$, defined by $\delta_0((f_1, f_2)) = (\delta_1(f_1), \delta_2(f_2))$. Furthermore, for $j = 1, 2$, define $\sigma_j((a_1, a_2)) = a_j$ for any two-tuple (a_1, a_2) and $\sigma_j(\boldsymbol{A})$ to be the matrix obtained by putting the operation σ_j on all two-tuple entries of a matrix $\boldsymbol{A}$.

Let $\boldsymbol{a}$ be the column consisting of all two-tuples in $F_1 \times F_2$. For $l_1 = 1, \ldots, q_1$ and $l_2 = 1, \ldots, q_2$, take $i = (l_1 - 1) q_2 + l_2$ and let $\boldsymbol{a}_i$ be the slice consisting of all two-tuples with the first element from the l_1-th column of $\mathbf{\Gamma}_1$ and the second one from the l_2-th column of $\mathbf{\Gamma}_2$. It can be seen that each $\delta_0(\boldsymbol{a}_i)$ is simply the column consisting of all two-tuples in $G_1 \times G_2$, for $i = 1, \ldots, q_1 q_2$.

Furthermore, for $j = 1, 2$, let $\boldsymbol{B}_j$ be a $D(r, k_j, s_{j1})$ based on F_j. Suppose we can further construct a $D(r, k_0, s_{11}s_{21})$ based on $F_1 \times F_2$, denoted by $\boldsymbol{B}_0$. Put

$$\boldsymbol{H} = (\boldsymbol{a} \oplus \boldsymbol{B}_0, \sigma_1(\boldsymbol{a}) \oplus \boldsymbol{B}_1, \sigma_2(\boldsymbol{a}) \oplus \boldsymbol{B}_2) \text{ and } \boldsymbol{H}_i = (\boldsymbol{a}_i \oplus \boldsymbol{B}_0, \sigma_1(\boldsymbol{a}_i) \oplus \boldsymbol{B}_1, \sigma_2(\boldsymbol{a}_i) \oplus \boldsymbol{B}_2)$$

for $i = 1, \ldots, q_1 q_2$. The following result is obtained.

Theorem 3. *If the three $\boldsymbol{B}_j$'s are all balanced, and both $(\sigma_1(\boldsymbol{B}_0), \boldsymbol{B}_1)$ and $(\sigma_2(\boldsymbol{B}_0), \boldsymbol{B}_2)$ are DMs, we have*
(i) the matrix $\boldsymbol{H}$ is an $OA(rs_{11}s_{21}, (s_{11}s_{21})^{k_0} s_{11}^{k_1} s_{21}^{k_2})$;
(ii) each slice $\boldsymbol{H}_i$ is balanced and becomes an $OA(rs_{12}s_{22}, (s_{12}s_{22})^{k_0} s_{12}^{k_1} s_{22}^{k_2})$ after the levels of the $s_{11}s_{21}$-level factors are collapsed according to δ_0 and the levels of the s_{j1}-level factors are collapsed according to δ_j, for $j = 1, 2$ and $i = 1, \ldots, q_1 q_2$.

Proof. Note that $\boldsymbol{a}$ is the column consisting of all two-tuples in $F_1 \times F_2$. Under the conditions of Theorem 3, the unique theorem in Wang (1996) shows that the matrix $\boldsymbol{H}$ constructed above is an $OA(rs_{11}s_{21}, (s_{11}s_{21})^{k_0} s_{11}^{k_1} s_{21}^{k_2})$. The part (i) of this theorem follows.

Now we are ready to prove the part (ii). First, because the three $\boldsymbol{B}_j$'s are all balanced, we know that each slice $\boldsymbol{H}_i$ is also balanced. Next, recall that each $\delta_0(\boldsymbol{a}_i)$ is exactly the column consisting of all two-tuples in $G_1 \times G_2$ for $i = 1, \ldots, q_1 q_2$. Moreover, it is easy to verify that $\delta_j(\sigma_j(\boldsymbol{a}_i)) = \sigma_j(\delta_0(\boldsymbol{a}_i))$ for $j = 1, 2$. Finally, by using the additivity property of δ_j's, we know that the three $\boldsymbol{B}_j$'s, $(\sigma_1(\boldsymbol{B}_0), \boldsymbol{B}_1)$ and $(\sigma_2(\boldsymbol{B}_0), \boldsymbol{B}_2)$ are all still DMs after the above level-collapsings are performed. So the part (ii) of this theorem follows again by applying the theorem in Wang (1996). □

Now we present a method modified from that in Lemma 7 of Qian, Ai and Wu (2009) to construct a class of the foregoing matrices $\boldsymbol{B}_1, \boldsymbol{B}_2$ and $\boldsymbol{B}_0$. For $j = 1, 2$, let $\tilde{\boldsymbol{B}}_j = (\tilde{\boldsymbol{B}}_{j0}, \tilde{\boldsymbol{B}}_{j1})$ be a balanced $D(r_j, c_j, s_j)$ based on F_j, where $\tilde{\boldsymbol{B}}_{j0}$ has c_0 columns for some $1 \leq c_0 < \min(c_1, c_2)$. Denote $\mathbf{0}_n$ as an $n \times 1$ vector of zeros. Put $\boldsymbol{B}_1 = \tilde{\boldsymbol{B}}_{11} \oplus \mathbf{0}_{r_2}$ and $\boldsymbol{B}_2 = \mathbf{0}_{r_1} \oplus \tilde{\boldsymbol{B}}_{21}$. Finally, obtain an $r_1 r_2 \times c_0$ matrix $\boldsymbol{B}_0$ whose $((i-1)r_2 + j, k)$-th entry is the two-tuple with the first element being the (i, k)-th entry of $\tilde{\boldsymbol{B}}_{10}$ and the second one being the (j, k)-th entry of $\tilde{\boldsymbol{B}}_{20}$, for $1 \leq i \leq r_1, 1 \leq j \leq r_2$ and $1 \leq k \leq c_0$. Then it can be verified that $\boldsymbol{B}_0$ is a balanced $D(r_1 r_2, c_0, s_1 s_2)$, $\boldsymbol{B}_j$ is a balanced $D(r_1 r_2, c_j - c_0, s_j)$ and $(\sigma_j(\boldsymbol{B}_0), \boldsymbol{B}_j)$ is a $D(r_1 r_2, c_j, s_j)$, for $j = 1, 2$.

Example 3. *Let $\boldsymbol{a}$ be the column consisting of all two-tuples in $GF(4) \times GF(3)$. The transpose of $(\boldsymbol{a}, \sigma_1(\boldsymbol{a}), \sigma_2(\boldsymbol{a}))$ is given by*

$$\begin{pmatrix} (0,0) & (0,1) & (0,2) & (1,0) & (1,1) & (1,2) & (x,0) & (x,1) & (x,2) & (x+1,0) & (x+1,1) & (x+1,2) \\ 0 & 0 & 0 & 1 & 1 & 1 & x & x & x & x+1 & x+1 & x+1 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \end{pmatrix}.$$

Define the projection δ_1 *as* $\{0, x\} \to 0$ *and* $\{1, x+1\} \to 1$*, and* δ_2 *to project any element to itself. Let* $\boldsymbol{a}_1$ *be the first six elements of* $\boldsymbol{a}$ *and* $\boldsymbol{a}_2$ *be the remaining six elements. Furthermore, let*

$$\tilde{\boldsymbol{B}}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & x & x+1 \\ x & x+1 & 1 \\ x+1 & 1 & x \end{pmatrix} \quad \text{and} \quad \tilde{\boldsymbol{B}}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

It is obvious that $\tilde{\boldsymbol{B}}_1$ *is a balanced* $D(4,3,4)$ *based on* $GF(4)$ *and* $\tilde{\boldsymbol{B}}_2$ *is a balanced* $D(3,2,3)$ *based on* $GF(3)$*. By taking* $c_0 = 1$ *in the preceding method, we can construct* $(\boldsymbol{B}_0, \boldsymbol{B}_1, \boldsymbol{B}_2)$ *as given in transpose by*

$$\begin{pmatrix} (0,0) & (0,1) & (0,2) & (1,0) & (1,1) & (1,2) & (x,0) & (x,1) & (x,2) & (x+1,0) & (x+1,1) & (x+1,2) \\ 0 & 0 & 0 & x & x & x & x+1 & x+1 & x+1 & 1 & 1 & 1 \\ 0 & 0 & 0 & x+1 & x+1 & x+1 & 1 & 1 & 1 & x & x & x \\ 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 \end{pmatrix}.$$

Put $\boldsymbol{H} = (\boldsymbol{a} \oplus \boldsymbol{B}_0, \sigma_1(\boldsymbol{a}) \oplus \boldsymbol{B}_1, \sigma_2(\boldsymbol{a}) \oplus \boldsymbol{B}_2)$ *and* $\boldsymbol{H}_i = (\boldsymbol{a}_i \oplus \boldsymbol{B}_0, \sigma_1(\boldsymbol{a}_i) \oplus \boldsymbol{B}_1, \sigma_2(\boldsymbol{a}_i) \oplus \boldsymbol{B}_2)$ *for* $i = 1, 2$*. From Theorem 3,* $\boldsymbol{H}$ *is an* $OA(144, 12^1 4^2 3^1)$*, each slice* $\boldsymbol{H}_i$ *is balanced and becomes an* $OA(72, 6^1 2^2 3^1)$ *after the levels of the 12-level factor are collapsed according to* δ_0 *and the levels of the four-level factors are collapsed according to* δ_1*, for* $i = 1, 2$*.*

4 Construction of asymmetric BSOAs via the replacement of levels

In this section, we propose another approach to constructing asymmetric BSOAs from symmetric BSOAs by applying the replacement of levels [Hedayat, Sloane and Stufken (1999)]. Suppose $\boldsymbol{A} = (\boldsymbol{A}'_1, \ldots, \boldsymbol{A}'_v)'$ is a symmetric BSOA, where $\boldsymbol{A}$ is an $OA(n_1, s_1^{m_1})$, each $\boldsymbol{A}_i$ is balanced and there is a level-collapsing projection δ_1 such that $\delta_1(\boldsymbol{A}_i)$ is an $OA(n_2, s_2^{m_1})$, for $i = 1, \ldots, v$. Furthermore, suppose $\boldsymbol{B} = (\boldsymbol{B}'_1, \ldots, \boldsymbol{B}'_q)'$ is a symmetric SOA, where $q = s_1/s_2$, $\boldsymbol{B}$ is an $OA(s_1, r_1^{m_2})$ and there is a level-collapsing projection δ_2 such that each $\delta_2(\boldsymbol{B}_i)$ is the same $OA(s_2, r_2^{m_2})$, for $i = 1, \ldots, q$. Let $\boldsymbol{\Gamma}_1$ be the $s_2 \times q$ kernel matrix of δ_1. For each of the first $k (\leq m_1)$ factors in $\boldsymbol{A}$, take the following two steps.

Labeling: Arbitrarily label the s_2 rows of $\boldsymbol{\Gamma}_1$ by $1, 2, \ldots, s_2$. Then label the i-th rows of $\boldsymbol{B}_1, \ldots, \boldsymbol{B}_q$ by a random permutation of the q levels in the row of $\boldsymbol{\Gamma}_1$ labeled by i, for $i = 1, \ldots, s_2$.

Replacing: Replace each level of the factor in $\boldsymbol{A}$ with the row of $\boldsymbol{B}$ labeled by the level.

Let $\boldsymbol{H} = (\boldsymbol{H}'_1, \ldots, \boldsymbol{H}'_v)'$ denote the resulting matrix after the above process is successively carried out for each of the first k factors in $\boldsymbol{A}$. The following result is obtained.

Theorem 4. *For the matrix* $\boldsymbol{H}$ *constructed above, we have*
(i) the matrix $\boldsymbol{H}$ *is an* $OA(n_1, r_1^{km_2} s_1^{m_1-k})$*;*

(ii) each slice $\boldsymbol{H}_i$ is balanced and becomes an $OA(n_2, r_2^{km_2} s_2^{m_1-k})$ after the levels of the r_1-level factors are collapsed according to δ_2 and the levels of the s_1-level factors are collapsed according to δ_1, for $i = 1, \dots, v$.

Proof. By noting that the s_1 levels of each of the first k factors in $\boldsymbol{A}$ are replaced by the s_1 rows of $\boldsymbol{B}$ one by one, part (i) of Theorem 4 is easy to verify.

Each $\boldsymbol{A}_i$ is balanced, so each $\boldsymbol{H}_i$ is also balanced for $i = 1, \dots, v$. Since the levels in the same row of $\boldsymbol{\Gamma}_1$ are collapsed to the same one and $\delta_2(\boldsymbol{B}_i)$ is the same $OA(s_2, r_2^{m_2})$, it can be seen that the s_2 levels of each of the first k factors in $\delta_1(\boldsymbol{A}_i)$ are actually replaced by the s_2 rows of the $OA(s_2, r_2^{m_2})$ one by one, and thus part (ii) of Theorem 4 follows. □

Example 4. *By using the construction method in Theorem 5 of Ai, Jiang and Li (2014), we can obtain a BSOA $\boldsymbol{A} = (\boldsymbol{A}_1', \dots, \boldsymbol{A}_{16}')'$, where $\boldsymbol{A}$ is an $OA(256, 16^5)$ with levels from $GF(16)$ and each $\delta_1(\boldsymbol{A}_i)$ is an $OA(16, 4^5)$ with levels from $GF(4)$, for $i = 1, \dots, 16$. Here δ_1 is defined as $\{0, x^2, x^3+x+1, x^3+x^2+x+1\} \to 0$,$\{1, x^2+1, x^3+x, x^3+x^2+x\} \to 1$,$\{x, x^2+x, x^3+1, x^3+x^2+1\} \to x^2+x$ and $\{x+1, x^2+x+1, x^3, x^3+x^2\} \to x^2+x+1$. We can also obtain a BSOA $\boldsymbol{B} = (\boldsymbol{B}_1', \dots, \boldsymbol{B}_4')'$, where $\boldsymbol{B}$ is an $OA(16, 4^3)$ with levels from $GF(4)$ and each $\delta_2(\boldsymbol{B}_i)$ is the same $OA(4, 2^3)$ with levels from $GF(2)$, for $i = 1, \dots, 4$. Here δ_2 is defined as $\{0, x+1\} \to 0$ and $\{1, x\} \to 1$. For example, $\boldsymbol{B}$ in transpose is given by*

$$\begin{pmatrix} 0 & 1 & x & x+1 & 0 & 1 & x & x+1 & x+1 & x & 1 & 0 & x+1 & x & 1 & 0 \\ 0 & x & x+1 & 1 & x+1 & 1 & 0 & x & 0 & x & x+1 & 1 & x+1 & 1 & 0 & x \\ 0 & x+1 & 1 & x & x+1 & 0 & x & 1 & x+1 & 0 & x & 1 & 0 & x+1 & 1 & x \end{pmatrix}.$$

According to the labeling step, we label the 16 rows of $\boldsymbol{B}$ in order by $0, 1, x, x+1, x^2, x^2+1, x^2+x, x^2+x+1, x^3+x+1, x^3+x, x^3+1, x^3, x^3+x^2+x+1, x^3+x^2+x, x^3+x^2+1$ and x^3+x^2. Then replace each level of the first factor in $\boldsymbol{A}$ with the row of $\boldsymbol{B}$ labeled by the level. From Theorem 4, we know that $\boldsymbol{H}$ is an $OA(256, 4^3 16^4)$, each $\boldsymbol{H}_i$ is balanced and becomes an $OA(16, 2^3 4^4)$ after the levels of the four-level factors are collapsed according to δ_2 and the levels of the 16-level factors are collapsed according to δ_1, for $i = 1, \dots, 16$.

5 Generation of sliced space-filling designs based on asymmetric BSOAs

In this section, we generate sliced space-filling designs by randomizing the asymmetric BSOAs obtained in the previous sections. The randomization approach is a generalization of that in Ai, Jiang and Li (2014) and covers both symmetric and asymmetric BSOAs. We assume that each of the quantitative factors takes values in the interval $[0, 1]$. Suppose $\boldsymbol{H} = (\boldsymbol{H}_1', \dots, \boldsymbol{H}_v')'$ is a BSOA, where $\boldsymbol{H}$ is an $OA(n_1,\, s_{11}^{\gamma_1} \cdots s_{k1}^{\gamma_k})$ with $m = \sum_{j=1}^{k} \gamma_j$, each $\boldsymbol{H}_i$ is balanced and becomes an $OA(n_2, s_{12}^{\gamma_1} \cdots s_{k2}^{\gamma_k})$ after the levels of the s_{j1}-level factors are collapsed to s_{j2} levels according to some level-collapsing projection δ_j, for $j = 1, \dots, k$ and $i = 1, \dots, v$. The randomization approach is described as follows.

For $j = 1, \dots, k$, let $\boldsymbol{\Gamma}_j$ denote the $s_{j2} \times q_j$ kernel matrix of δ_j, where $q_j = s_{j1}/s_{j2}$. Arbitrarily label the s_{j2} rows of $\boldsymbol{\Gamma}_j$ by $1, 2, \dots, s_{j2}$, and then relabel the q_j levels in the row labeled by l as a random permutation of $\{(l-1)q_j+1,\dots,(l-1)q_j+q_j\}$ for $l=1,\dots,s_{j2}$. Now the levels of the s_{j1}-level factors in $\boldsymbol{H}$ are $1,\dots,s_{j1}$.

For $j=1,\dots,k$, let $w_j=n_1/s_{j1}$ and $e_j=n_2/s_{j1}$. For $l=1,\dots,s_{j1}$, let $\boldsymbol{M}_{jl}$ be the $e_j\times v$ matrix given by

$$\begin{pmatrix} (l-1)w_j+1 & (l-1)w_j+2 & \cdots & (l-1)w_j+v \\ (l-1)w_j+v+1 & (l-1)w_j+v+2 & \cdots & (l-1)w_j+2v \\ \vdots & \vdots & & \vdots \\ (l-1)w_j+(e_j-1)v+1 & (l-1)w_j+(e_j-1)v+2 & \cdots & (l-1)w_j+w_j \end{pmatrix}.$$

For $r=1,\dots,\gamma_j$, obtain a new matrix $\boldsymbol{M}_{jlr}$ by randomly shuffling the entries in each row of $\boldsymbol{M}_{jl}$. For $i=1,\dots,v$, replace the e_j entries of level l in the r-th s_{j1}-level factors in $\boldsymbol{H}_i$ by a random permutation of the e_j elements in the i-th column of $\boldsymbol{M}_{jlr}$. Denote by $\boldsymbol{L}=(\boldsymbol{L}_1',\dots,\boldsymbol{L}_v')'$ the resulting matrix after such replacement is done for all the columns of $\boldsymbol{H}$, where $\boldsymbol{L}_i$ is the submatrix of $\boldsymbol{L}$ corresponding to $\boldsymbol{H}_i$.

Finally, generate an $n_1\times m$ matrix $\boldsymbol{D}$ whose (i,j)-th entry is $(l_{ij}-u_{ij})/n_1$, where l_{ij} is the (i,j)-th entry of $\boldsymbol{L}$ and u_{ij}'s are independent random variables with uniform distributions on $(0,1]$, for $i=1,\dots,n_1$ and $j=1,\dots,m$. Denote by $\boldsymbol{D}_i$ the submatrix of $\boldsymbol{D}$ corresponding to $\boldsymbol{L}_i$.

It is easy to see that $\boldsymbol{L}$ is obtained by replacing the w_j entries of level l in each of the s_{j1}-level factors in $\boldsymbol{H}$ with a permutation of $\{(l-1)w_j+1,\dots,(l-1)w_j+w_j\}$, for $l=1,\dots,s_{j1}$ and $j=1,\dots,k$. Thus, $\boldsymbol{L}$ is a Latin hypercube based on $OA(n_1,s_{11}^{\gamma_1}\cdots s_{k1}^{\gamma_k})$ [Tang (1993)]. Similarly, it can be shown that each $\boldsymbol{L}_i$ becomes a Latin hypercube based on $OA(n_2,s_{12}^{\gamma_1}\cdots s_{k2}^{\gamma_k})$ after the level z of $\boldsymbol{L}$ is collapsed to $\lceil z/v\rceil$ for $z=1,\dots,n_1$ and $i=1,\dots,v$, where $\lceil a\rceil$ is the smallest integer not less than a. According to McKay, Beckman and Conover (1979), the final matrix $\boldsymbol{D}$ obtained from $\boldsymbol{L}$ is a Latin hypercube design in the unit cube $[0,1]^m$. Note that each factor in $\boldsymbol{D}$ now has n_1 levels. For $j=1,\dots,m$, we use b_{j1} to denote the number of levels of the j-th factor in the BSOA $\boldsymbol{H}$ and b_{j2} to denote that in its each projected slice. Then the following theorem is obtained.

Theorem 5. *For the design $\boldsymbol{D}=(\boldsymbol{D}_1',\dots,\boldsymbol{D}_v')'$ obtained above, we have*
(i) the design $\boldsymbol{D}$ and each slice $\boldsymbol{D}_i$ achieve maximum stratification in any one-dimensional projection;
(ii) when projected onto the two dimensions of the j_1-th and j_2-th factors ($j_1\neq j_2$), the design $\boldsymbol{D}$ achieves the stratification on the $b_{j_11}\times b_{j_21}$ grids and each slice $\boldsymbol{D}_i$ achieves the stratification on the $b_{j_12}\times b_{j_22}$ grids, for $i=1,\dots,v$.

Example 5. *Consider the asymmetric BSOA $\boldsymbol{H}=(\boldsymbol{H}_1',\boldsymbol{H}_2')'$ obtained in Example 3, where $\boldsymbol{H}$ is an $OA(144,12^14^23^1)$, each slice $\boldsymbol{H}_i$ is balanced and collapsed into an $OA(72,6^12^23^1)$ for $i=1,2$. The four columns of $\boldsymbol{H}$ are represented by the four factors x_1,x_2,x_3 and x_4, respectively. Following the randomization approach, we first relabel the levels for each factor in $\boldsymbol{H}$. Table 1 presents the first slice $\boldsymbol{H}_1$ after the relabeling is carried out. Next, we use the new $\boldsymbol{H}$ to construct a Latin hypercube design $\boldsymbol{D}=(\boldsymbol{D}_1',\boldsymbol{D}_2')'$ accordingly. From Theorem*

Table 1: The matrix $\boldsymbol{H}_1$ in Example 5

Run#	x_1	x_2	x_3	x_4	Run#	x_1	x_2	x_3	x_4	Run#	x_1	x_2	x_3	x_4
1	1	1	1	1	25	9	1	1	3	49	7	3	3	2
2	5	1	1	3	26	1	1	1	2	50	11	3	3	1
3	9	1	1	2	27	5	1	1	1	51	3	3	3	3
4	3	2	4	1	28	11	2	4	3	52	5	4	2	2
5	7	2	4	3	29	3	2	4	2	53	9	4	2	1
6	11	2	4	2	30	7	2	4	1	54	1	4	2	3
7	2	4	3	1	31	10	4	3	3	55	8	2	1	2
8	6	4	3	3	32	2	4	3	2	56	12	2	1	1
9	10	4	3	2	33	6	4	3	1	57	4	2	1	3
10	4	3	2	1	34	12	3	2	3	58	6	1	4	2
11	8	3	2	3	35	4	3	2	2	59	10	1	4	1
12	12	3	2	2	36	8	3	2	1	60	2	1	4	3
13	5	1	1	2	37	3	3	3	1	61	11	3	3	3
14	9	1	1	1	38	7	3	3	3	62	3	3	3	2
15	1	1	1	3	39	11	3	3	2	63	7	3	3	1
16	7	2	4	2	40	1	4	2	1	64	9	4	2	3
17	11	2	4	1	41	5	4	2	3	65	1	4	2	2
18	3	2	4	3	42	9	4	2	2	66	5	4	2	1
19	6	4	3	2	43	4	2	1	1	67	12	2	1	3
20	10	4	3	1	44	8	2	1	3	68	4	2	1	2
21	2	4	3	3	45	12	2	1	2	69	8	2	1	1
22	8	3	2	2	46	2	1	4	1	70	10	1	4	3
23	12	3	2	1	47	6	1	4	3	71	2	1	4	2
24	4	3	2	3	48	10	1	4	2	72	6	1	4	1

5, we know that $\boldsymbol{D}$ and each slice $\boldsymbol{D}_i$ achieve maximum stratification in univariate margins, but have different levels of two-dimensional uniformity for different factors. Figure 1 depicts the two-dimensional projections of $\boldsymbol{D}_1$. It is shown that $\boldsymbol{D}_1$ achieves maximum stratification in any univariate margin. Furthermore, in bivariate margins, the design points of $\boldsymbol{D}_1$ are evenly scattered on the 6×2 grids in the dimensions of x_1 and x_2 or x_1 and x_3, on the 6×3 grids in those of x_1 and x_4, on the 2×2 grids in those of x_2 and x_3, and on the 2×3 grids in those of x_2 and x_4 or x_3 and x_4.

6 Discussions

In this article, we propose a new class of sliced space-filling designs in which both the whole design and each slice can achieve maximum stratification in univariate margins, but have different levels of two-dimensional uniformity for different factors. They are useful for

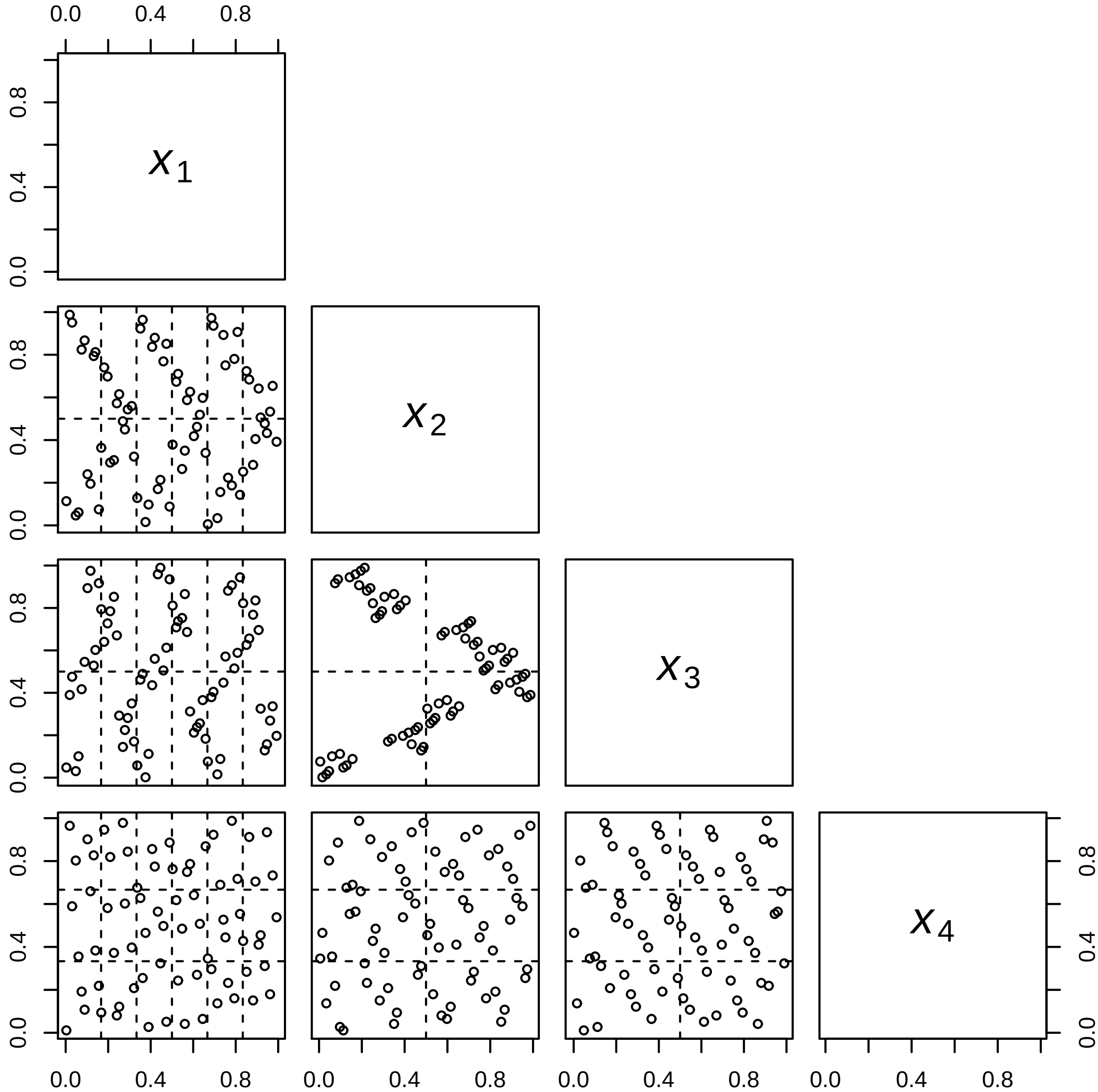

Figure 1: Bivariate projections of $\boldsymbol{D}_1$ in Example 5.

designing sliced computer experiments with qualitative and quantitative factors or multiple models, linking parameters in engineering and cross-validation where priori knowledge suggests that some factors are more important than others and should be paid more attention. Besides, these designs can also be applied to design nested computer experiments with two codes of different levels of accuracy, as the whole design and each slice constitute a nested design [Qian, Ai and Wu (2009); Qian and Ai (2010)].

These new sliced designs are generated by elaborately randomizing asymmetric balanced sliced orthogonal arrays (BSOAs) such that not only the whole design but each slice is an asymmetric OA-based Latin hypercube design. Several methods to construct asymmetric BSOAs are developed. Since the asymmetric BSOAs constructed in this article have only strength two, the corresponding sliced space-filling designs are guaranteed to achieve stratification in two-dimensional projections. If one is interested in sliced designs with better space-filling properties in higher dimensions, it is worth to develop methods to construct asymmetric BSOAs with strength three or higher. Finally, as a great many sliced space-filling designs can be generated based on a given BSOA, we can search for the optimal ones by using the maximin distance [Johnson, Moore and Ylvisaker (1990)], discrepancy measures [Fang, Li and Sudjianto (2006)], or other optimality criteria.

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