Dependence Structure between LIBOR Rates by Copula Method

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Abstract

This paper discusses the correlation structure between London Interbank Offered Rates (LIBOR) by using copula function. We start from one simplified model of Brace, Gatarek & Musiela (1997) and find out that the copula function between two LIBOR rates can be expressed as a sum of infinite series, where the main term is a distribution function with Gaussian copula. Partial differential equation (PDE) method is used for deriving the copula expansion. Numerical results show that the copula of the LIBOR rates and Gaussian copula are very close in the central region and differ in the tail, and in the normal situation the Gaussian copula approximation to the copula function between the LIBOR rates provides satisfying results.

Keywords: LIBOR rates, copula function, PDE.

1 Introduction

The London Interbank Offered Rate (LIBOR) is widely used in the financial market as a benchmark to define the interest rates. Suppose that the current time is 0 and consider a set of maturity

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dates $\{T_0, T_1, \ldots, T_N\}$ with $T_0 < T_1 < \cdots < T_N$. The forward LIBOR rate $L_k(t) = L(t, T_{k-1}, T_k)$ is defined by

$$L_k(t) = \frac{1}{\delta_k}\left(\frac{P(t,T_{k-1})}{P(t,T_k)} - 1\right),\ t \le T_{k-1},$$

where the time fraction $\delta_k = T_k - T_{k-1}, 1 \le k \le N$, and $P(t,T)$ is the price of the zero-coupon bond at time t with maturity T. Note that $L_k(t)$ is the annualized forward price at time t for borrowing 1 dollar at a future time T_{k-1} and paying back at T_k. Specially, $L_k(T_{k-1})$ is the market price for borrowing 1 dollar at time T_{k-1}. Denote $\mathbb{Q}^k$ to be the forward measure with $P(t, T_k)$ as its numéraire. With the assumption that the financial market is arbitrage-free, it is known from Musiela & Rutkowski (2005) that the forward measure $\mathbb{Q}^k$ is equivalent to the risk-neutral measure $\mathbb{Q}$ and the Radon-Nikodym derivative exists. Furthermore, the process $\{L_k(t), t \le T_{k-1}\}$ is a martingale under $\mathbb{Q}^k$.

Brace, Gatarek & Musiela (1997) extended the Heath, Jarrow & Morton (HJM) (1992) model, and derived the pricing formulas for caps and swaptions by assuming lognormal volatility structures for LIBOR rates. Under their assumptions, for each $k = 1, \ldots, N$ the LIBOR rate $L_k(t)$ satisfies the following stochastic differential equation (SDE) under the forward measure $\mathbb{Q}^k$,

$$dL_k(t) = L_k(t)\vec{\sigma}_k(t)^T d\vec{W}_k(t),$$

where $\vec{\sigma}_k(t)^T = (\sigma_{k,1}(t), \ldots, \sigma_{k,N}(t))$ is a deterministic vector function, and the component volatility function $\sigma_{k,j}(t),\ j = 1, \ldots, N$ is positive, piecewise continuous and bounded. And $\vec{W}_k^T = (W_{k,1}(t), \ldots,\ W_{k,N}(t))$ is an N-dimensional Wiener process under $\mathbb{Q}^k$ such that each component $W_{k,j}(t),\ j = 1, \ldots, N$ is a standard Wiener process and $d\vec{W}_k d\vec{W}_k^T = \Xi dt$, where Ξ is a constant correlation matrix.

Based on Brace, Gatarek & Musiela (1997)'s work, applying the changing-measure formulas by the Radon-Nikodym derivatives between any two forward measures in Musiela & Rutkowski (2005, page 475), the dynamics of two LIBOR rates $(L_1(t), L_2(t))$ under the T_1-forward measure $\mathbb{Q}^1$ are given by

$$\begin{cases} dL_1(t) = L_1(t)\vec{\sigma}_1(t)^T d\vec{W}_1(t), \\ dL_2(t) = \dfrac{\delta_2 L_2(t)^2}{1+\delta_2 L_2(t)}\vec{\sigma}_2(t)^T \Xi \vec{\sigma}_2(t)dt + L_2(t)\vec{\sigma}_2(t)^T d\vec{W}_1(t), \end{cases} \tag{1.1}$$

where $0 \leq t \leq T \leq T_0$.

In the financial market, some derivatives are written on multiple LIBOR rates, such as swaps and swaptions. The dependence structure between LIBOR rates plays an important role for pricing these products. In empirical studies, some priori assumptions are made to simplify the calibration. A typical way is to parameterize the volatility family, which can be found in Brigo, Mercurio & Morini (2005). Another way is to approximate the drift terms in the dynamics. For example, in Brigo & Mercurio (2001, page 250), the dynamics of $(L_1(t), L_2(t))$ in equation (1.1) are approximated by

$$\begin{cases} dL_1(t) = L_1(t)\vec{\sigma}_1(t)^T d\vec{W}_1(t), \\ dL_2(t) = \dfrac{\delta_2 L_2(0)}{1+\delta_2 L_2(0)} L_2(t)\vec{\sigma}_2(t)^T \Xi \vec{\sigma}_2(t)dt + L_2(t)\vec{\sigma}_2(t)^T d\vec{W}_1(t). \end{cases} \tag{1.2}$$

The same idea is also used in Hull & White (1999) and is referred to as the initial freeze approximation in Henrard (2007) for pricing the rolled deposit options. This approximation method has been tested by Monte-Carlo simulation in Hull & White (1999), but theoretical evidences are not given to our knowledge.

Now copula method is widely used to model the correlation between risks. A copula function is a multivariate distribution function with uniform marginal distributions on $[0, 1]$. The Sklar's theorem shows that a multivariate distribution function can be divided into two parts, its copula function and marginal distributions. More details can be found in Nelsen (2006) and Cherubini, Luciano & Vecchiato (2004). Gaussian copula is one widely used copula function in finance for modeling the correlation between risks. Note that for the model (1.2), the dependence can be captured by Gaussian copula.

In this paper, we will consider one simplified dynamic model of equation (1.1). We will prove that the copula function between LIBOR rates $(L_1(t), L_2(t))$ can be expanded as a sum of infinite series. We will also compare this copula function with the Gaussian copula and the approximation error will be discussed. Numerical results will be provided for measuring the distance between the real copula function and the Gaussian copula function, and we will show that the first two terms of the copula function's expansion can explain most of the information of the dependence structure. In our discussion, PDE method will be used to obtain the joint density function of the LIBOR rates.

The paper is organized as follows. In Section 2 we give one simplified model of (1.1) and the PDEs for the marginal and joint density functions of the LIBOR rates are obtained. In Section 3 we solve the PDE for the joint density and prove that the joint density function can be expressed as a sum of an infinite series. In Section 4 we use the joint density function to obtain the copula expansion, where the main term is a distribution function with Gaussian copula. In Section 5 numerical results are given to show the difference between the real copula and Gaussian copula. Conclusions are given in Section 6. Some proofs are put in the Appendix.

2 Model Setup

In this paper, we consider a simplified model for LIBOR rates $(L_1(t), L_2(t))$ of (1.1). Assuming that $N = 2$, $\vec{\sigma}_1(t)^T = (\sigma_1, 0)$, $\vec{\sigma}_2(t)^T = (0, \sigma_2)$, and $W_i = W_{1,i}$, $i = 1, 2$ in model (1.1), the dynamics of $(L_1(t), L_2(t))$ under $\mathbb{Q}^1$ are modeled by

$$\begin{cases} dL_1(t) = \sigma_1 L_1(t) dW_1(t), \\ dL_2(t) = \dfrac{\delta \sigma_2^2 L_2(t)^2}{1 + \delta L_2(t)} dt + \sigma_2 L_2(t) dW_2(t), \end{cases} \tag{2.1}$$

where $W_i, i = 1, 2$ are the standard Wiener processes under $\mathbb{Q}^1$ with $dW_1(t)dW_2(t) = \rho dt$.

Copula function is a multivariate distribution function with uniform [0,1] margins. By Sklar's theorem (see Nelsen (2006)), for any time $0 < t \le T$, there exists a copula function $C(u_1, u_2; t)$ such that for any $(x_1, x_2) \in \mathbb{R}^2$,

$$\mathbb{Q}^1 \left(L_1(t) \le x_1, L_2(t) \le x_2 \right) = C \left(\mathbb{Q}^1 \left(L_1(t) \le x_1 \right), \mathbb{Q}^1 \left(L_2(t) \le x_2 \right); t \right).$$

The above equation shows the fundamental relationship among the joint distribution, the marginal distributions and the copula function. To find the copula function from the above equation, the marginal distributions and the joint distribution are needed.

Note that the copula function is invariant under strictly monotone transformations. Consider the following transformation

$$X_i(t) = \frac{1}{\sigma_i} \ln \left(\frac{L_i(t)}{L_i(0)} \right) + \frac{1}{2} \sigma_i t, \ i = 1, 2,$$

then $(X_1(t), X_2(t))$ and $(L_1(t), L_2(t))$ have the same copula function $C(u_1, u_2; t)$. From equation (2.1) and Itô's formula, the dynamics of $(X_1(t), X_2(t))$ are given by

$$\begin{cases} dX_1(t) = dW_1(t), \\ dX_2(t) = \theta(X_2(t), t)dt + dW_2(t), \\ X_1(0) = X_2(0) = 0, \end{cases} \tag{2.2}$$

where

$$\theta(x_2, t) = \frac{\sigma_2 e^{\sigma_2 x_2 - \frac{1}{2}\sigma_2^2 t}}{1/(\delta L_2(0)) + e^{\sigma_2 x_2 - \frac{1}{2}\sigma_2^2 t}}. \tag{2.3}$$

The function θ and its partial derivative $\dfrac{\partial \theta}{\partial x_2}$ are bounded on $\mathbb{R} \times (0, T]$, then for fixed $t > 0$ the function $\theta(x_2, t)$ is Lipshitz continuous with respect to x_2. Thus by Øksendal (2000, page 66), we know that the SDE (2.2) has a unique solution .

We will derive the marginal distributions and the joint distribution of $(X_1(t), X_2(t))$, then find the copula function $C(u_1, u_2; t)$. A powerful tool to solve the probability density function of a stochastic process is the Fokker-Planck equation (see Grigoriu (2002, page 481)).

Lemma 2.1 (Fokker-Planck Equation)**.** *Suppose that $\vec{Y}_t$ is a d-dimensional diffusion process satisfying the stochastic differential equation*

$$d\vec{Y}_t = \vec{\mu}(\vec{Y}_t, t)dt + R(\vec{Y}_t, t)d\vec{Z}_t,$$

where $\vec{\mu} = (\mu_1, \ldots, \mu_d)^T$ is a vector function valued in $\mathbb{R}^d$, $R = (R_{ij})_{1\le i,j\le d}$ is a matrix function valued in $\mathbb{R}^{d\times d}$, and $\vec{Z}_t$ is a d-dimensional standard Wiener process with independent components. Let $p(\vec{y}, t)$ be the density of $\vec{Y}_t$ conditional on $\vec{Y}_0 = \vec{y}_0$. If $p(\vec{y}, t)$ satisfies the conditions

$$\lim_{|\vec{y}|\to\infty} [\mu_i(\vec{y}, t)p(\vec{y}, t)] = 0, \ \ i = 1, \ldots, d, \tag{2.4a}$$

$$\lim_{|\vec{y}|\to\infty} \left(R(\vec{y}, t)R(\vec{y}, t)^T\right)_{ij} p(\vec{y}, t) = 0, \ \ i, j = 1, \ldots, d, \tag{2.4b}$$

$$\lim_{|\vec{y}|\to\infty} \partial\left[\left(R(\vec{y}, t)R(\vec{y}, t)^T\right)_{ij} p(\vec{y}, t)\right] \Big/ \partial y_i = 0, \ \ i, j = 1, \ldots, d, \tag{2.4c}$$

then p is the solution of the Fokker-Planck equation

$$\frac{\partial p(\vec{y}, t)}{\partial t} = -\sum_{i=1}^{d} \frac{\partial}{\partial y_i}[\mu_i(\vec{y}, t)p(\vec{y}, t)] + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} \frac{\partial^2}{\partial y_i \partial y_j}\left[\left(R(\vec{y}, t)R(\vec{y}, t)^T\right)_{ij} p(\vec{y}, t)\right].$$

For $i = 1, 2$, denote the density and distribution functions of $X_i(t)$ by $f_i(x;t)$ and $F_i(x;t)$ respectively. The joint density and distribution of $(X_1(t), X_2(t))$ are denoted as $f(x_1, x_2; t)$ and $H(x_1, x_2; t)$.

The following proposition gives the marginal densities and the PDE for the joint density. We use ϕ and Φ to denote the density and the distribution of standard normal distribution respectively, i.e.,

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \ \Phi(x) = \int_{-\infty}^{x} \phi(z)dz, \ x \in \mathbb{R}.$$

Proposition 2.1. *The marginal distributions of $X_1(t)$ and $X_2(t)$ are given by*

$$\begin{aligned} f_1(x_1;t) &= \frac{1}{\sqrt{t}}\phi\left(\frac{x_1}{\sqrt{t}}\right), \ F_1(x_1;t) = \Phi\left(\frac{x_1}{\sqrt{t}}\right), \\ f_2(x_2;t) &= (1-q)\frac{1}{\sqrt{t}}\phi\left(\frac{x_2}{\sqrt{t}}\right) + q\frac{1}{\sqrt{t}}\phi\left(\frac{x_2-\sigma_2 t}{\sqrt{t}}\right), \\ F_2(x_2;t) &= (1-q)\Phi\left(\frac{x_2}{\sqrt{t}}\right) + q\Phi\left(\frac{x_2-\sigma_2 t}{\sqrt{t}}\right), \end{aligned}$$

where

$$q = \frac{\delta L_2(0)}{1+\delta L_2(0)}.$$

And the joint density $f(x_1, x_2; t)$ of $(X_1(t), X_2(t))$ is the solution to the following Cauchy problem

$$\begin{cases} \mathcal{L}p = 0, \ (x_1, x_2, t) \in \mathbb{R}^2 \times (0, T], \\ p(x_1, x_2; 0) = \delta(x_1, x_2), \ (x_1, x_2) \in \mathbb{R}^2, \end{cases} \tag{2.5}$$

where $\delta(x_1, x_2)$ is the 2-dimensional Dirac function, and the parabolic operator $\mathcal{L}$ is defined by

$$\mathcal{L}p = \frac{\partial p}{\partial t} - \frac{1}{2}\frac{\partial^2 p}{\partial x_1^2} - \rho\frac{\partial^2 p}{\partial x_1 \partial x_2} - \frac{1}{2}\frac{\partial^2 p}{\partial x_2^2} + \frac{\partial(\theta p)}{\partial x_2} \tag{2.6}$$

for any function $p(x_1, x_2; t)$ such that the partial derivatives exist, where $\theta = \theta(x_2, t)$ is defined in equation (2.3).

Proof. From Lemma 2.1, the marginal density function f_1 of $X_1(t)$ satisfies the following Cauchy problem

$$\begin{cases} \frac{\partial f_1}{\partial t}(x_1;t) = \frac{1}{2}\frac{\partial^2 f_1}{\partial x_1^2}(x_1;t), \ (x_1, t) \in \mathbb{R} \times (0, T], \\ f_1(x_1; 0) = \delta(x_1), \ x_1 \in \mathbb{R}, \end{cases} \tag{2.7}$$

where the 1-dimensional Dirac function $\delta(\cdot)$ has the property that for any smooth function u with compact support,

$$\int_{\mathbb{R}} u(x)\delta(x)dx = u(0),$$

and the initial value of f_1 comes from the fact that $X_1(0) = 0$. The first equation in (2.7) is the standard heat equation, thus the unique solution is

$$f_1(x_1;t) = \frac{1}{\sqrt{t}}\phi\left(\frac{x_1}{\sqrt{t}}\right).$$

The function f_1 is actually the density of normal distribution and satisfies the conditions (2.4a) to (2.4c).

Similarly, the marginal density function f_2 of $X_2(t)$ satisfies the following Cauchy problem

$$\begin{cases} \dfrac{\partial f_2}{\partial t}(x_2;t) = \dfrac{1}{2}\dfrac{\partial^2 f_2}{\partial x_2^2}(x_2;t) - \dfrac{\partial(\theta f_2)}{\partial x_2}(x_2;t), \ (x_2,t) \in \mathbb{R}\times(0,T], \\ f_2(x_2;0) = \delta(x_2), \ x_2 \in \mathbb{R}. \end{cases} \tag{2.8}$$

For solving equation (2.8), denote

$$g(x_2;t) = \frac{f_2(x_2;t)}{\frac{1-q}{q} + e^{\sigma_2 x_2 - \frac{1}{2}\sigma_2^2 t}}. \tag{2.9}$$

Then $g(x_2;t)$ satisfies the following Cauchy problem

$$\begin{cases} \dfrac{\partial g}{\partial t}(x_2;t) = \dfrac{1}{2}\dfrac{\partial^2 g}{\partial x_2^2}(x_2;t), \ (x_2,t) \in \mathbb{R}\times(0,T], \\ g(x_2;0) = q\delta(x_2), \ x_2 \in \mathbb{R}. \end{cases}$$

Again it is a heat equation and the solution is

$$g(x_2;t) = \frac{q}{\sqrt{t}}\phi\left(\frac{x_2}{\sqrt{t}}\right),$$

then (2.9) implies that the density $f_2(x_2;t)$ of $X_2(t)$ has the form

$$f_2(x_2;t) = (1-q)\frac{1}{\sqrt{t}}\phi\left(\frac{x_2}{\sqrt{t}}\right) + q\frac{1}{\sqrt{t}}\phi\left(\frac{x_2-\sigma_2 t}{\sqrt{t}}\right).$$

The function f_2 is actually a mixture of two normal density functions and also satisfies the conditions (2.4a) to (2.4c). For $i = 1,2$ the distribution function F_i can be derived by integrating f_i.

Next we consider the joint density f. To apply Lemma 2.1 on SDE (2.2), we notice that

$$\vec{\mu}(x_1, x_2, t) = \begin{pmatrix} 0 \\ \theta(x_2, t) \end{pmatrix}.$$

Since $dW_1(t)dW_2(t) = \rho dt$, the two Wiener processes $W_1(t)$ and $W_2(t)$ in SDE (2.2) are not independent when $\rho \neq 0$. However, there exists a 2-dimensional standard Wiener process $(\tilde{W}_1(t), \tilde{W}_2(t))$, such that $\tilde{W}_1(t)$ is independent of $\tilde{W}_2(t)$, and

$$\begin{pmatrix} W_1(t) \\ W_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} \tilde{W}_1(t) \\ \tilde{W}_2(t) \end{pmatrix}.$$

Then the SDE (2.2) can be expressed as

$$\begin{pmatrix} dX_1(t) \\ dX_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \theta(X_2(t), t) \end{pmatrix} dt + \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} d\tilde{W}_1(t) \\ d\tilde{W}_2(t) \end{pmatrix}. \tag{2.10}$$

It follows that the matrix function $R(x_1, x_2, t)$ in Lemma 2.1 is

$$R(x_1, x_2, t) = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix},$$

and

$$R(x_1, x_2, t)R(x_1, x_2, t)^T = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

Applying Lemma 2.1 to SDE (2.10), we conclude that

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2 f}{\partial x_1^2} + \rho \frac{\partial^2 f}{\partial x_1 \partial x_2} + \frac{1}{2}\frac{\partial^2 f}{\partial x_2^2} - \frac{\partial(\theta f)}{\partial x_2}.$$

Thus the joint density f is one solution of (2.6). Thus we complete the proof of the proposition.

□

Proposition 2.1 shows that $X_1(t)$ follows a Gaussian distribution having no relationship with q, σ_2 and δ. The distribution of $X_2(t)$ is a mixture of two Gaussian distributions with weight q depending on the initial value of LIBOR rate $L_2(0)$.

The marginal and joint distribution functions can be used for deriving the copula function. From Proposition 2.1 we know that the inverse functions F_i^{-1} of F_i, $i = 1, 2$ can be obtained for

$0 < t \le T$. Recall that the joint distribution function of $(X_1(t), X_2(t))$ is denoted as $H(x_1, x_2; t)$, then the copula function $C(u_1, u_2; t)$ can be expressed as

$$C(u_1, u_2; t) = H\left(F_1^{-1}(u_1; t), F_2^{-1}(u_2; t); t\right), \tag{2.11}$$

and the density $c(u_1, u_2; t)$ of copula C satisfies

$$c(u_1, u_2; t) = \frac{f\left(F_1^{-1}(u_1; t), F_2^{-1}(u_2; t); t\right)}{f_1\left(F_1^{-1}(u_1; t); t\right) f_2\left(F_2^{-1}(u_2; t); t\right)}, \tag{2.12}$$

where the functions f_i and F_i, $i = 1, 2$ are given in Proposition 2.1.

From equation (2.12) we can see that the key problem for getting the copula density function $c(u_1, u_2; t)$ is to find the joint density f. In the next section we will solve equation (2.5) for the joint density f.

3 Solving PDE (2.5) for the Joint Density

In this section we will prove that the density function f is a unique solution to equation (2.5) and can be expressed as a sum of an infinite series.

Denote

$$Z(\vec{x}, t; \vec{\xi}, \tau) = \frac{1}{2\pi\sqrt{1-\rho^2}(t-\tau)} e^{-\frac{(\vec{x}-\vec{\xi})^T A (\vec{x}-\vec{\xi})}{2(t-\tau)}},$$

where $\vec{x} = (x_1, x_2)^T, \vec{\xi} = (\xi_1, \xi_2)^T \in \mathbb{R}^2$, $0 < \tau < t \le T$, and the matrix $A = \frac{1}{1-\rho^2}\begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}$.

The function $Z(\vec{x}, t; \vec{\xi}, \tau)$ satisfies the following parabolic Cauchy problem

$$\begin{cases} \frac{\partial p}{\partial t}(\vec{x}; t) - \frac{1}{2}\frac{\partial^2 p}{\partial x_1^2}(\vec{x}; t) - \rho \frac{\partial^2 p}{\partial x_1 \partial x_2}(\vec{x}; t) - \frac{1}{2}\frac{\partial^2 p}{\partial x_2^2}(\vec{x}; t) = 0, \ (\vec{x}, t) \in \mathbb{R}^2 \times (\tau, T], \\ p(\vec{x}, \tau) = \delta(\vec{x} - \vec{\xi}), \ \vec{x} \in \mathbb{R}^2, \end{cases}$$

where $\delta(\cdot)$ is the Dirac function. Note that the function Z is smooth and shift-invariant, i.e.,

$$Z(\vec{x}, t; \vec{\xi}, \tau) = Z(\vec{x} - \vec{\xi}, t - \tau; \vec{0}, 0),$$

and for any rapidly decreasing function $g(\vec{x})$ (i.e., for any non-negative integers n_1, n_2, m_1 and m_2, the partial derivatives exist and satisfy $\sup_{\vec{x} \in \mathbb{R}^2} \left| x_1^{n_1} x_2^{n_2} \frac{\partial^{m_1+m_2} g(\vec{x})}{\partial x_1^{m_1} \partial x_2^{m_2}} \right| < \infty$),

$$\lim_{t \to \tau+} \iint_{\mathbb{R}^2} Z(\vec{x}, t; \vec{\xi}, \tau) g(\vec{\xi}) d\vec{\xi} = g(\vec{x}).$$

In a special case that $\rho = 0$, the function Z is called the Gaussian kernel, which is discussed in Friedman (1964, page 1). The definition of the fundamental solution of a parabolic operator is given in Friedman (1964, page 3). We will use this concept to find the solution to equation (2.5).

Definition 3.1 (Fundamental Solution)**.** *A function $\Psi(\vec{x}, t; \vec{\xi}, \tau)$, $\vec{x}, \vec{\xi} \in \mathbb{R}^2$, $0 < \tau < t \leq T$ is called the fundamental solution of the parabolic operator $\mathcal{L}$ in equation* (2.6)*, if it satisfies the following properties:*

1. *For fixed $(\vec{\xi}, \tau) \in \mathbb{R}^2 \times (0, T]$, $\mathcal{L}\Psi = 0$ holds for any $(\vec{x}, t) \in \mathbb{R}^2 \times (\tau, T]$;*
2. *For any continuous function $g(\vec{x}), \vec{x} \in \mathbb{R}^2$ satisfying $|g(\vec{x})| \leq Me^{h|\vec{x}|^2}$, where $h < \frac{\lambda}{8T}$, $\lambda = \frac{2}{1+|\rho|}$, and M is some constant,*

$$\lim_{t \to \tau+} \iint_{\mathbb{R}^2} \Psi(\vec{x}, t; \vec{\xi}, \tau) g(\vec{\xi}) d\vec{\xi} = g(\vec{x}).$$

The fundamental solution is a basic concept in partial differential equations. Given the fundamental solution, it is easy to find the solution of the original equation by convolution. Next we will give the solution to equation (2.5) by using the fundamental solution of $\mathcal{L}$. First we need some lemmas.

Lemma 3.1. *The function $\theta(x, t)$, $(x, t) \in \mathbb{R} \times [0, T]$ in equation* (2.3) *is a smooth function. Further, for $n, m \geq 0$ its partial derivative $\dfrac{\partial^{m+n}\theta}{\partial x^m \partial t^n}(x, t)$ is bounded, i.e., there exists some constant $\alpha(m, n) < \infty$ such that*

$$\left| \frac{\partial^{m+n}\theta}{\partial x^m \partial t^n}(x, t) \right| \leq \alpha(m, n), \text{ for all } (x, t) \in \mathbb{R} \times [0, T].$$

Here $\alpha(0, 0)$ and $\alpha(1, 0)$ can be chosen as

$$\alpha(0, 0) = \sigma_2, \ \alpha(1, 0) = \sigma_2^2.$$

Proof. For $m = n = 0$, it holds that

$$0 \leq \theta(x, t) \leq \sigma_2, \text{ for all } (x, t) \in \mathbb{R} \times [0, T].$$

By the chain rule we have the partial derivative of $\theta(x,t)$ for non-negative integers m and n with $m+n \geq 1$,

$$\begin{aligned}&\frac{\partial^{m+n}\theta}{\partial x^m \partial t^n}(x,t)\\&= \frac{(-2)^{-n}\sigma_2^{m+2n+1}}{\left((\delta L_2(0))^{-1} + e^{\sigma_2 x - \frac{1}{2}\sigma_2^2 t}\right)^{m+n+1}} \sum_{k=1}^{m+n} d_{m+n,k}[(\delta L_2(0))^{-1}]^{m+n-k+1}(e^{\sigma_2 x - \frac{1}{2}\sigma_2^2 t})^k,\end{aligned}$$

where the constants $d_{n,k}$, $n \geq 1$, $k = 1, \ldots, n$ are given by

$$\begin{aligned}&d_{n,1} = 1,\ d_{n,n} = (-1)^{n-1},\ n \geq 1,\\&d_{n,k} = k d_{n-1,k} + (k-n-1) d_{n-1,k-1},\ n \geq 3,\ k = 2, \ldots, n-1.\end{aligned}$$

Thus $\dfrac{\partial^{m+n}\theta}{\partial x^m \partial t^n}(x,t)$ is a continuous function which tends to 0 as $x \to \pm\infty$, and is bounded by $\alpha(m,n)$, where

$$\alpha(m,n) = 2^{-n}\sigma_2^{m+2n+1} \sum_{k=1}^{m+n} |d_{m+n,k}|.$$

Specially, $\alpha(0,0) = \sigma_2$ and $\alpha(1,0) = \sigma_2^2 |d_{1,1}| = \sigma_2^2$. Now we finish the proof of the lemma. $\square$

The next lemma can be easily verified by change of variables.

Lemma 3.2. *For $\alpha, \beta < 2$ and $h > 0$,*

$$\begin{aligned}&\int_\tau^t \iint_{\mathbb{R}^2} (t-\sigma)^{-\alpha} e^{-\frac{h|\vec{x}-\vec{\eta}|^2}{4(t-\sigma)}} (\sigma-\tau)^{-\beta} e^{-\frac{h|\vec{\eta}-\vec{\xi}|^2}{4(\sigma-\tau)}} d\vec{\eta} d\sigma\\&= \frac{4\pi}{h} B(2-\alpha, 2-\beta)(t-\tau)^{2-\alpha-\beta} e^{-\frac{h|\vec{x}-\vec{\xi}|^2}{4(t-\tau)}},\end{aligned}$$

where the Beta function $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$ for $m, n > 0$.

Define

$$\begin{cases} (\mathcal{L}Z)_1(\vec{x},t;\vec{\xi},\tau) = -\mathcal{L}Z(\vec{x},t;\vec{\xi},\tau), \\ (\mathcal{L}Z)_n(\vec{x},t;\vec{\xi},\tau) = \displaystyle\int_\tau^t \iint_{\mathbb{R}^2} (\mathcal{L}Z)_1(\vec{x},t;\vec{\eta},\sigma)(\mathcal{L}Z)_{n-1}(\vec{\eta},\sigma;\vec{\xi},\tau) d\vec{\eta} d\sigma,\ n > 1. \end{cases} \tag{3.1}$$

Then it will be proved that the fundamental solution of $\mathcal{L}$ can be constructed by $\{(\mathcal{L}Z)_n, n \geq 1\}$. First we will show some properties of $(\mathcal{L}Z)_n$ in the following lemma.

Lemma 3.3. *For $0 < \tau < t \le T$ and $n \ge 1$, there exists some constant C_0 depending on ρ, T and σ_2 such that*

$$|(\mathcal{L}Z)_n(\vec{x},t;\vec{\xi},\tau)| \le \frac{\lambda C_0^n}{8\pi\Gamma(\frac{n}{2})}(t-\tau)^{\frac{n}{2}-2}e^{-\frac{\lambda|\vec{x}-\vec{\xi}|^2}{8(t-\tau)}} \tag{3.2}$$

and

$$\begin{aligned}&\left|\int_\tau^t \iint_{\mathbb{R}^2} Z(\vec{x},t;\vec{\eta},\sigma)(\mathcal{L}Z)_n(\vec{\eta},\sigma;\vec{\xi},\tau)d\vec{\eta}d\sigma\right| \\ &\le \frac{1}{2\pi\sqrt{1-\rho^2}}\frac{C_0^n}{\Gamma(\frac{n}{2}+1)}(t-\tau)^{\frac{n}{2}-1}e^{-\frac{\lambda|\vec{x}-\vec{\xi}|^2}{8(t-\tau)}}.\end{aligned} \tag{3.3}$$

Here the Gamma function $\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx$ for $s > 0$.

Proof. We will prove the lemma by mathematical induction. From Lemma 3.1 we have $|\theta(x_2,t)| \le \sigma_2$ and $\left|\frac{\partial\theta}{\partial x_2}(x_2,t)\right| \le \sigma_2^2$. Then

$$\begin{aligned}|(\mathcal{L}Z)_1(\vec{x},t;\vec{\xi},\tau)| &= \left|\frac{\partial}{\partial x_2}\Big(\theta(x_2,t)Z(\vec{x},t;\vec{\xi},\tau)\Big)\right| \\ &\le \sigma_2\left|\frac{\partial Z}{\partial x_2}(\vec{x},t;\vec{\xi},\tau)\right| + \sigma_2^2\left|Z(\vec{x},t;\vec{\xi},\tau)\right| \\ &= \frac{|\rho(x_1-\xi_1)-(x_2-\xi_2)|}{(1-\rho^2)(t-\tau)}\frac{\sigma_2}{2\pi\sqrt{1-\rho^2}(t-\tau)}e^{-\frac{(\vec{x}-\vec{\xi})^TA(\vec{x}-\vec{\xi})}{2(t-\tau)}} \\ &\quad + \frac{\sigma_2^2}{2\pi\sqrt{1-\rho^2}(t-\tau)}e^{-\frac{(\vec{x}-\vec{\xi})^TA(\vec{x}-\vec{\xi})}{2(t-\tau)}}.\end{aligned}$$

Using the fact that

$$(\vec{x}-\vec{\xi})^TA(\vec{x}-\vec{\xi}) \ge \frac{\lambda}{2}|\vec{x}-\vec{\xi}|^2 \tag{3.4}$$

and for all $z \ge 0$

$$ze^{-\frac{\lambda z^2}{8}} \le \frac{2}{\sqrt{\lambda e}},$$

from the above equation we can get the following estimation

$$\begin{aligned}&|(\mathcal{L}Z)_1(\vec{x},t;\vec{\xi},\tau)| \\ \le\ & \frac{\sigma_2}{2\pi(1-|\rho|)^{\frac{3}{2}}(t-\tau)^{\frac{3}{2}}}\frac{|\vec{x}-\vec{\xi}|}{\sqrt{t-\tau}}e^{-\frac{\lambda|\vec{x}-\vec{\xi}|^2}{8(t-\tau)}}e^{-\frac{\lambda|\vec{x}-\vec{\xi}|^2}{8(t-\tau)}} + \frac{\sigma_2^2}{2\pi\sqrt{1-\rho^2}(t-\tau)}e^{-\frac{\lambda|\vec{x}-\vec{\xi}|^2}{8(t-\tau)}} \\ \le\ & \frac{\sigma_2}{2\pi(1-|\rho|)^{\frac{3}{2}}(t-\tau)^{\frac{3}{2}}}\frac{2}{\sqrt{\lambda e}}e^{-\frac{\lambda|\vec{x}-\vec{\xi}|^2}{8(t-\tau)}} + \frac{\sigma_2^2}{2\pi\sqrt{1-\rho^2}(t-\tau)}e^{-\frac{\lambda|\vec{x}-\vec{\xi}|^2}{8(t-\tau)}} \\ \le\ & \left(\frac{\sigma_2}{\sqrt{\lambda e}\pi(1-|\rho|)^{\frac{3}{2}}} + \frac{\sigma_2^2\sqrt{T}}{2\pi\sqrt{1-\rho^2}}\right)(t-\tau)^{-\frac{3}{2}}e^{-\frac{\lambda|\vec{x}-\vec{\xi}|^2}{8(t-\tau)}}.\end{aligned}$$

Denote

$$C_0 = \frac{8\pi\Gamma(\frac{1}{2})}{\lambda}\left(\frac{\sigma_2}{\sqrt{\lambda e}\pi(1-|\rho|)^{\frac{3}{2}}} + \frac{\sigma_2^2\sqrt{T}}{2\pi\sqrt{1-\rho^2}}\right),$$

then the inequality (3.2) is proved for $n=1$. Now we assume that (3.2) is true for $(\mathcal{L}Z)_1, \ldots, (\mathcal{L}Z)_n$, $n \geq 1$. Choosing $\alpha = \frac{3}{2}$, $\beta = 2 - \frac{n}{2}$, $h = \frac{\lambda}{2}$ in Lemma 3.2, we can get that

$$\begin{aligned}
& \left|(\mathcal{L}Z)_{n+1}(\vec{x},t;\vec{\xi},\tau)\right| \\
= & \left|\int_\tau^t \iint_{\mathbb{R}^2} (\mathcal{L}Z)_1(\vec{x},t;\vec{\eta},\sigma)(\mathcal{L}Z)_n(\vec{\eta},\sigma;\vec{\xi},\tau)d\vec{\eta}d\sigma\right| \\
\leq & \int_\tau^t \iint_{\mathbb{R}^2} \frac{\lambda C_0}{8\pi\Gamma(\frac{1}{2})}(t-\sigma)^{-\frac{3}{2}} e^{-\frac{\lambda|\vec{x}-\vec{\eta}|^2}{8(t-\sigma)}} \frac{\lambda C_0^n}{8\pi\Gamma(\frac{n}{2})}(\sigma-\tau)^{\frac{n}{2}-2} e^{-\frac{\lambda|\vec{\eta}-\vec{\xi}|^2}{8(\sigma-\tau)}} d\vec{\eta}d\sigma \\
= & \frac{\lambda C_0}{8\pi\Gamma(\frac{1}{2})}\frac{\lambda C_0^n}{8\pi\Gamma(\frac{n}{2})}\frac{8\pi}{\lambda} B\left(\frac{n}{2},\frac{1}{2}\right)(t-\tau)^{\frac{n+1}{2}-2} e^{-\frac{\lambda|\vec{x}-\vec{\xi}|^2}{8(t-\tau)}} \\
= & \frac{\lambda C_0^{n+1}}{8\pi\Gamma(\frac{n+1}{2})}(t-\tau)^{\frac{n+1}{2}-2} e^{-\frac{\lambda|\vec{x}-\vec{\xi}|^2}{8(t-\tau)}},
\end{aligned}$$

which implies that (3.2) is true for $n+1$. Thus (3.2) holds for all $n \geq 1$.

Again from inequality (3.4),

$$|Z(\vec{x},t;\vec{\xi},\tau)| \leq \frac{1}{2\pi\sqrt{1-\rho^2}(t-\tau)} e^{-\frac{\lambda|\vec{x}-\vec{\xi}|^2}{8(t-\tau)}}.$$

Combining the above inequality with (3.2) and choosing $\alpha = 1$, $\beta = 2 - \frac{n}{2}$, $h = \frac{\lambda}{2}$ in Lemma 3.2, we can get (3.3). Thus the proof is completed. □

By (3.3), the function

$$\Psi(\vec{x},t;\vec{\xi},\tau) = Z(\vec{x},t;\vec{\xi},\tau) + \sum_{n=1}^{\infty}\int_\tau^t \iint_{\mathbb{R}^2} Z(\vec{x},t;\vec{\eta},\sigma)(\mathcal{L}Z)_n(\vec{\eta},\sigma;\vec{\xi},\tau)d\vec{\eta}d\sigma \tag{3.5}$$

is well-defined. Note that

$$\Psi(\vec{x},t;\vec{0},0) = Z(\vec{x},t;\vec{0},0) + \sum_{n=1}^{\infty}\int_0^t \iint_{\mathbb{R}^2} Z(\vec{x},t;\vec{\eta},\sigma)(\mathcal{L}Z)_n(\vec{\eta},\sigma;\vec{0},0)d\vec{\eta}d\sigma.$$

Theorem 3.1 (Fundamental Solution)**.** *The function* Ψ *in equation* (3.5) *is the fundamental solution of equation* (2.5)*.*

Proof. Define

$$G(\vec{x},t;\vec{\xi},\tau)=\sum_{n=1}^{\infty}(\mathcal{L}Z)_n(\vec{x},t;\vec{\xi},\tau).$$

Referring to Friedman (1964, pages 7–13) and Lemma 3.3, the function G is well-defined, differentiable for $t>\tau$ and satisfies

$$G(\vec{x},t;\vec{\xi},\tau)=(\mathcal{L}Z)_1(\vec{x},t;\vec{\xi},\tau)+\int_{\tau}^{t}\iint_{\mathbb{R}^2}(\mathcal{L}Z)_1(\vec{x},t;\vec{\eta},\sigma)G(\vec{\eta},\sigma;\vec{\xi},\tau)d\vec{\eta}d\sigma.$$

Notice that the function $\Psi(\vec{x},t;\vec{\xi},\tau)$ satisfies the following integration equation

$$\Psi(\vec{x},t;\vec{\xi},\tau)=Z(\vec{x},t;\vec{\xi},\tau)+\int_{\tau}^{t}\iint_{\mathbb{R}^2}Z(\vec{x},t;\vec{\eta},\sigma)G(\vec{\eta},\sigma;\vec{\xi},\tau)d\vec{\eta}d\sigma.$$

Then from Friedman (1964, pages 14–20), the function $\Psi(\vec{x},t;\vec{\xi},\tau)$ is a fundamental solution of $\mathcal{L}$. □

In the next theorem we will use the fundamental solution Ψ in equation (3.5) to prove that the equation (2.5) has a unique solution, which is the joint density of $(X_1(t),X_2(t))$. First we denote

$$v_n(\vec{x};t)=\int_{0}^{t}\iint_{\mathbb{R}^2}Z(\vec{x},t;\vec{\eta},\sigma)(\mathcal{L}Z)_n(\vec{\eta},\sigma;\vec{0},0)d\vec{\eta}d\sigma. \tag{3.6}$$

Then the function $\Psi(\vec{x},t;\vec{0},0)$ can be expressed as

$$\Psi(\vec{x},t;\vec{0},0)=Z(\vec{x},t;\vec{0},0)+\sum_{n=1}^{\infty}v_n(\vec{x};t).$$

Theorem 3.2. *The function $\Psi(\vec{x},t;\vec{0},0)$ is the joint density of $(X_1(t),X_2(t))$, i.e.,*

$$f(\vec{x};t)=\Psi(\vec{x},t;\vec{0},0).$$

Proof. We will prove that $\Psi(\vec{x},t;\vec{0},0)$ is the unique solution to equation (2.5) under the condition that

$$\sup_{t\in(0,T]}\iint_{\mathbb{R}^2}|p(\vec{x};t)|d\vec{x}<\infty,$$

and we will also show that $\iint_{\mathbb{R}^2}\Psi(\vec{x},t;\vec{0},0)d\vec{x}=1$ for any $t\in(0,T]$. From Proposition 2.1, we know that the density function $f(\vec{x};t)$ is a solution to equation (2.5), thus from the above proved results we get that $f(\vec{x};t)=\Psi(\vec{x},t;\vec{0},0)$ and the theorem is proved.

For $(\vec{x},t)\in\mathbb{R}^2\times(0,T]$, the function $\Psi(\vec{x},t;\vec{0},0)$ satisfies $\mathcal{L}\Psi(\vec{x},t;\vec{0},0)=0$. We only need to check the initial condition in equation (2.5). Notice that for any continuous function $g(\vec{x})$ defined on $\mathbb{R}^2$ satisfying $|g(\vec{x})|\le Me^{h|\vec{x}|^2}$, $h<\frac{\lambda}{8T}$ and M is some constant,

$$\begin{aligned}
&\lim_{t\to0+}\iint_{\mathbb{R}^2}\Psi(\vec{x},t;\vec{0},0)g(\vec{x})d\vec{x}\\
&=\lim_{t\to0+}\iint_{\mathbb{R}^2}Z(\vec{x},t;\vec{0},0)g(\vec{x})d\vec{x}\\
&\quad+\lim_{t\to0+}\sum_{n=1}^{\infty}\iint_{\mathbb{R}^2}\left[\int_0^t\iint_{\mathbb{R}^2}Z(\vec{x},t;\vec{\eta},\sigma)(\mathcal{L}Z)_n(\vec{\eta},\sigma;\vec{0},0)g(\vec{x})d\vec{\eta}d\sigma\right]d\vec{x}. \qquad (3.7)
\end{aligned}$$

Then we will estimate the two terms of the right-hand side of equation (3.7) separately. From Lemma 3.3, we have

$$\begin{aligned}
&\left|\iint_{\mathbb{R}^2}\left[\int_0^t\iint_{\mathbb{R}^2}Z(\vec{x},t;\vec{\eta},\sigma)(\mathcal{L}Z)_n(\vec{\eta},\sigma;\vec{0},0)g(\vec{x})d\vec{\eta}d\sigma\right]d\vec{x}\right|\\
&\le\frac{MC_0^n t^{\frac{n}{2}-1}}{2\pi\sqrt{1-\rho^2}\Gamma(\frac{n}{2}+1)}\iint_{\mathbb{R}^2}e^{-(\frac{\lambda}{8t}-h)|\vec{x}|^2}d\vec{x}\\
&\le\frac{M_1C_0^n t^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}\le\frac{M_1C_0^nT^{\frac{n-1}{2}}}{\Gamma(\frac{n}{2}+1)}t^{\frac{1}{2}}
\end{aligned}$$

with $M_1=\dfrac{4M}{\sqrt{1-\rho^2}(\lambda-8hT)}$ and the sum $\displaystyle\sum_{n\ge1}\frac{M_1C_0^nT^{\frac{n-1}{2}}}{\Gamma(\frac{n}{2}+1)}<\infty$ for positive and finite T. Then it follows that the sum of the infinite series in equation (3.7) tends to 0 uniformly as $t\to0+$. On the other hand, $\displaystyle\lim_{t\to0+}\iint_{\mathbb{R}^2}Z(\vec{x},t;\vec{0},0)g(\vec{x})d\vec{x}=g(\vec{0})$, which leads to

$$\lim_{t\to0+}\iint_{\mathbb{R}^2}\Psi(\vec{x},t;\vec{0},0)g(\vec{x})d\vec{x}=g(\vec{0}).$$

Therefore, $\Psi(\vec{x},t;\vec{0},0)$ is one solution to equation (2.5). We leave the proof of uniqueness in Appendix A. Thus we complete the proof of the theorem. □

Theorem 3.2 shows that the joint density $f(\vec{x};t)$ can be expressed as a sum of an infinite series, and for each $n\ge1$ the term $v_n(\vec{x};t)$ can be expressed as (3.6). Actually, the function $v_n(\vec{x};t)$ is the solution to the following parabolic equation

$$\begin{cases}\dfrac{\partial v_n}{\partial t}=\dfrac{1}{2}\dfrac{\partial^2v_n}{\partial x_1^2}+\rho\dfrac{\partial^2v_n}{\partial x_1\partial x_2}+\dfrac{1}{2}\dfrac{\partial^2v_n}{\partial x_2^2}+(\mathcal{L}Z)_n(\vec{x},t;\vec{0},0),\ (\vec{x},t)\in\mathbb{R}^2\times(0,T],\\ v_n(\vec{x};0)=0,\ \vec{x}\in\mathbb{R}^2.\end{cases}$$

Thus numerical method can be applied for computing $v_n(\vec{x};t)$.

4 The Copula Function between LIBOR Rates

In this section we study the properties of copula function in equation (2.11). From Theorem 3.2, the joint density $f(\vec{x};t)$ of $(X_1(t), X_2(t))$ can be expressed as

$$f(\vec{x};t) = Z(\vec{x},t;\vec{0},0) + \sum_{n\geq 1}\int_0^t \iint_{\mathbb{R}^2} Z(\vec{x},t;\vec{\xi},\tau)(\mathcal{L}Z)_n(\vec{\xi},\tau;\vec{0},0)d\vec{\xi}d\tau. \tag{4.1}$$

Denote Φ_ρ to be the distribution function of the 2-dimensional normal distribution with standard normal margins and correlation index ρ. Notice that

$$\Phi_\rho(x_1,x_2) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} Z(\vec{\xi},1;\vec{0},0)d\xi_1 d\xi_2.$$

Then the joint distribution function $H(x_1,x_2;t)$ is given by integrating f in equation (4.1)

$$H(x_1,x_2;t) = \Phi_\rho\left(\frac{\vec{x}}{\sqrt{t}}\right) + \sum_{n\geq 1}\int_0^t \iint_{\mathbb{R}^2} \Phi_\rho\left(\frac{\vec{x}-\vec{\xi}}{\sqrt{t-\tau}}\right)(\mathcal{L}Z)_n(\vec{\xi},\tau;\vec{0},0)d\vec{\xi}d\tau. \tag{4.2}$$

In the above expansion we denote

$$V_n(\vec{x};t) = \int_0^t \iint_{\mathbb{R}^2} \Phi_\rho\left(\frac{\vec{x}-\vec{\xi}}{\sqrt{t-\tau}}\right)(\mathcal{L}Z)_n(\vec{\xi},\tau;\vec{0},0)d\vec{\xi}d\tau,\ n\geq 1.$$

To find the copula function between LIBOR rates $(L_1(t), L_2(t))$, first we need the following lemma.

Lemma 4.1. *There exists a series of positive constants $M_{n,i}$, $0\leq i\leq n$, $n=1,2,\ldots$, independent of t, such that for $0<\tau<t\leq T$ and $\vec{x},\vec{\xi}\in\mathbb{R}^2$*

$$\left|(\mathcal{L}Z)_n(\vec{\xi},t;\vec{\eta},\tau)\right| \leq \sum_{i=0}^n M_{n,i}(t-\tau)^{\frac{n-2-i}{2}}|\vec{\xi}-\vec{\eta}|^i Z(\vec{\xi},t;\vec{\eta},\tau).$$

We put the proof of Lemma 4.1 in Appendix B. Note that we have obtained one estimation of $(\mathcal{L}Z)_n$ in Lemma 3.3. The two estimations are for different purposes.

Theorem 4.1. *(a) The copula function $C(u_1,u_2;t)$ between $(L_1(t), L_2(t))$ has the following expression*

$$\begin{aligned}C(u_1,u_2;t) &= C_g\left(u_1,(F_1\circ F_2^{-1})(u_2;t);\rho\right)\\ &+\sum_{n\geq 1}\int_0^t \iint_{\mathbb{R}^2} \Phi_\rho\left(\frac{F_1^{-1}(u_1,t)-\xi_1}{\sqrt{t-\tau}},\frac{F_2^{-1}(u_2,t)-\xi_2}{\sqrt{t-\tau}}\right)(\mathcal{L}Z)_n(\vec{\xi},\tau;\vec{0},0)d\vec{\xi}d\tau,\end{aligned} \tag{4.3}$$

where $(\mathcal{L}Z)_n$ *is defined in equation* (3.1) *and* $C_g(u_1, u_2; \rho)$ *is the Gaussian copula with parameter* ρ,

$$C_g(u_1, u_2; \rho) = \Phi_\rho\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2)\right).$$

(b) *For* $n \geq 1$, *we denote*

$$V_n^c(u_1, u_2; t) = \int_0^t \iint_{\mathbb{R}^2} \Phi_\rho\left(\frac{F_1^{-1}(u_1, t) - \xi_1}{\sqrt{t-\tau}}, \frac{F_2^{-1}(u_2, t) - \xi_2}{\sqrt{t-\tau}}\right)(\mathcal{L}Z)_n(\vec{\xi}, \tau; \vec{0}, 0)d\vec{\xi}d\tau$$

in equation (4.3). *Then there exists a series of constants* $D_{n,i}$, $0 \leq i \leq n$, $n = 1, 2, \ldots$, *independent of* t, *such that* $V_n^c(u_1, u_2; t)$ *can be bounded by*

$$|V_n^c(u_1, u_2; t)| \leq \int_{-\infty}^{F_1^{-1}(u_1;t)} \int_{-\infty}^{F_2^{-1}(u_2;t)} \left(\sum_{i=0}^n D_{n,i} t^{\frac{n-i}{2}} |\vec{\xi}|^i\right) Z(\vec{\xi}, t; \vec{0}, 0)d\vec{\xi}. \tag{4.4}$$

Furthermore, the term $V_n^c(u_1, u_2; t)$ *satisfies*

$$\lim_{u \to 0+} \frac{V_n^c(u, u; t)}{u} = \lim_{u \to 1-} \frac{1 - 2u + V_n^c(u, u; t)}{1-u} = 0.$$

Proof. (a) From the definition of copula $C(u_1, u_2; t)$ in equation (2.11), by combining the expression of the joint distribution function H in equation (4.2) and the marginal distributions F_i, $i = 1, 2$ in Proposition 2.1 we can get the result.

(b) From Lemma 4.1, the function $v_n(\vec{x}; t)$ in equation (3.6) satisfies the property that

$$\begin{aligned}
|v_n(\vec{x}; t)| &= \left|\int_0^t \iint_{\mathbb{R}^2} Z(\vec{x}, t; \vec{\xi}, \tau)(\mathcal{L}Z)_n(\vec{\xi}, \tau; \vec{0}, 0)d\vec{\xi}d\tau\right| \\
&\leq \sum_{i=0}^n M_{n,i} \int_0^t \iint_{\mathbb{R}^2} \tau^{\frac{n-2-i}{2}} |\vec{\xi}|^i Z(\vec{x}, t; \vec{\xi}, \tau)Z(\vec{\xi}, \tau; \vec{0}, 0)d\vec{\xi}d\tau.
\end{aligned} \tag{4.5}$$

Noticing that

$$\begin{aligned}
& Z(\vec{x}, t; \vec{\xi}, \tau)Z(\vec{\xi}, \tau; \vec{0}, 0) \\
= & \frac{1}{2\pi\sqrt{1-\rho^2}(t-\tau)} \frac{1}{2\pi\sqrt{1-\rho^2}\tau} \\
& \times \exp\left\{-\frac{t}{2(t-\tau)\tau}\left(\vec{\xi} - \frac{\tau}{t}\vec{x}\right)^T A \left(\vec{\xi} - \frac{\tau}{t}\vec{x}\right) - \frac{1}{2t}\vec{x}^T A \vec{x}\right\} \\
= & Z(\vec{x}, t; \vec{0}, 0) Z\left(\vec{\xi}, \frac{(t-\tau)\tau}{t}; \frac{\tau}{t}\vec{x}, 0\right),
\end{aligned}$$

and for $i = 0, 1, \dots,$

$$\iint_{\mathbb{R}^2} |\vec{x}|^i Z(\vec{x}, 1; \vec{0}, 0) d\vec{x} \le \iint_{\mathbb{R}^2} \left[2(x^2+y^2)\right]^{\frac{i}{2}} \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} dxdy = 2^i \Gamma\left(1+\frac{i}{2}\right), \qquad (4.6)$$

by making the linear transformation $\vec{\xi} = \sqrt{\frac{(t-\tau)\tau}{t}}\vec{\eta} + \frac{\tau}{t}\vec{x}$ in (4.5) we can get that

$$\begin{aligned}
|v_n(\vec{x};t)| &\le \sum_{i=0}^{n} M_{n,i} \int_0^t \iint_{\mathbb{R}^2} \tau^{\frac{n-2-i}{2}} |\vec{\xi}|^i Z(\vec{x}, t; \vec{\xi}, \tau) Z(\vec{\xi}, \tau; \vec{0}, 0) d\vec{\xi} d\tau \\
&\le \sum_{i=0}^{n} M_{n,i} Z(\vec{x}, t; \vec{0}, 0) \int_0^t \iint_{\mathbb{R}^2} \tau^{\frac{n-2-i}{2}} Z(\vec{\eta}, 1; \vec{0}, 0) \left| \sqrt{\frac{(t-\tau)\tau}{t}}\vec{\eta} + \frac{\tau}{t}\vec{x} \right|^i d\vec{\eta} d\tau \\
&\le \sum_{i=0}^{n} 2^i M_{n,i} Z(\vec{x}, t; \vec{0}, 0) \\
&\quad \times \int_0^t \iint_{\mathbb{R}^2} \tau^{\frac{n-2-i}{2}} Z(\vec{\eta}, 1; \vec{0}, 0) \left\{ \left[\frac{(t-\tau)\tau}{t}\right]^{\frac{i}{2}} |\vec{\eta}|^i + \left[\frac{\tau}{t}\right]^i |\vec{x}|^i \right\} d\vec{\eta} d\tau \\
&\le \sum_{i=0}^{n} 2^i M_{n,i} Z(\vec{x}, t; \vec{0}, 0) \\
&\quad \times \left[2^i \Gamma\left(1+\frac{i}{2}\right) B\left(1+\frac{i}{2}, \frac{n}{2}\right) t^{\frac{n}{2}} + |\vec{x}|^i B\left(1, \frac{n+i}{2}\right) t^{\frac{n-i}{2}} \right]. \qquad (4.7)
\end{aligned}$$

Writing

$$\begin{aligned}
D_{n,0} &= M_{n,0} B\left(1, \frac{n}{2}\right) + \sum_{i=0}^{n} 2^{2i} M_{n,i} \Gamma\left(1+\frac{i}{2}\right) B\left(1+\frac{i}{2}, \frac{n}{2}\right), \\
D_{n,i} &= 2^i M_{n,i} B\left(1, \frac{n+i}{2}\right), \ 1 \le i \le n,
\end{aligned}$$

from (4.7) we know that

$$|v_n(\vec{x};t)| \le \sum_{i=0}^{n} D_{n,i} t^{\frac{n-i}{2}} |\vec{x}|^i Z(\vec{x}, t; \vec{0}, 0).$$

Substituting the above inequality to the definition of $V_n^c(u_1, u_2; t)$ we can get

$$\begin{aligned}
|V_n^c(u_1, u_2; t)| &= \left| \int_{-\infty}^{F_1^{-1}(u_1;t)} \int_{-\infty}^{F_2^{-1}(u_1;t)} v_n(\vec{\xi}; t) d\vec{\xi} \right| \\
&\le \int_{-\infty}^{F_1^{-1}(u_1;t)} \int_{-\infty}^{F_2^{-1}(u_1;t)} \left| v_n(\vec{\xi}; t) \right| d\vec{\xi} \\
&\le \int_{-\infty}^{F_1^{-1}(u_1;t)} \int_{-\infty}^{F_2^{-1}(u_2;t)} \left(\sum_{i=0}^{n} D_{n,i} t^{\frac{n-i}{2}} |\vec{\xi}|^i \right) Z(\vec{\xi}, t; \vec{0}, 0) d\vec{\xi},
\end{aligned}$$

then the boundary in (4.4) can be obtained.

To prove that $\lim_{u\to 0+} \frac{V_n^c(u,u;t)}{u} = 0$, it is sufficient to prove that for $0 \le i \le n$,

$$\lim_{u\to 0+} \frac{1}{u} \int_{-\infty}^{F_1^{-1}(u;t)} \int_{-\infty}^{F_2^{-1}(u;t)} |\vec{\xi}|^i Z(\vec{\xi},t;\vec{0},0)d\vec{\xi} = 0. \tag{4.8}$$

Since $F_1^{-1}(u;t) \le F_2^{-1}(u;t)$ for any $u \in (0,1)$,

$$\begin{aligned}
0 &\le \lim_{u\to 0+} \frac{1}{u} \int_{-\infty}^{F_1^{-1}(u;t)} \int_{-\infty}^{F_2^{-1}(u;t)} |\vec{\xi}|^i Z(\vec{\xi},t;\vec{0},0)d\vec{\xi} \\
&\le \lim_{u\to 0+} \frac{1}{u} \int_{-\infty}^{F_2^{-1}(u;t)} \int_{-\infty}^{F_2^{-1}(u;t)} |\vec{\xi}|^i Z(\vec{\xi},t;\vec{0},0)d\vec{\xi} \\
&= \lim_{x\to -\infty} \frac{1}{F_2(x;t)} \int_{-\infty}^{x} \int_{-\infty}^{x} |\vec{\xi}|^i Z(\vec{\xi},t;\vec{0},0)d\vec{\xi}.
\end{aligned} \tag{4.9}$$

Applying the L'Hôpital's Rule,

$$\begin{aligned}
& \lim_{x\to -\infty} \frac{1}{F_2(x;t)} \int_{-\infty}^{x} \int_{-\infty}^{x} |\vec{\xi}|^i Z(\vec{\xi},t;\vec{0},0)d\vec{\xi} \\
= & \lim_{x\to -\infty} \frac{2}{f_2(x;t)} \int_{-\infty}^{x} (x^2+\xi_2^2)^{\frac{i}{2}} Z((x,\xi_2),t;\vec{0},0)d\xi_2 \\
= & \lim_{x\to -\infty} \frac{2f_1(x;t)}{f_2(x;t)} \int_{-\infty}^{\sqrt{\frac{1-\rho}{1+\rho}}x} \left[x^2+(\rho x+\sqrt{1-\rho^2}y)^2\right]^{\frac{i}{2}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} dy \\
= & \ 0,
\end{aligned} \tag{4.10}$$

where the last equality is from

$$\lim_{x\to -\infty} \frac{f_1(x;t)}{f_2(x;t)} = \frac{1}{1-q}$$

and

$$\lim_{x\to -\infty} \int_{-\infty}^{\sqrt{\frac{1-\rho}{1+\rho}}x} \left[x^2+(\rho x+\sqrt{1-\rho^2}y)^2\right]^{\frac{i}{2}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} dy = 0.$$

Thus combining (4.9) and (4.10) we conclude that (4.8) is true. Similarly it can be proved that

$$\lim_{u\to 1-} \frac{1-2u+V_n^c(u,u;t)}{1-u} = 0.$$

Thus the proof of the theorem is completed.

□

By Theorem 4.1, the copula function $C(u_1, u_2; t)$ between $(L_1(t), L_2(t))$ can be expanded as a sum of infinite series, where the first term $V_0^c(u_1, u_2; t) = C_g\left(u_1, (F_1 \circ F_2^{-1})(u_2; t); \rho\right)$ is the main term of the expansion and the second term $V_1^c(u_1, u_2; t)$ is an adjustment to the main term. The corresponding expansion of the copula density $c(u_1, u_2; t)$ is

$$\begin{aligned} c(u_1, u_2; t) &= c_g\left(u_1, (F_1 \circ F_2^{-1})(u_2; t); \rho\right) \frac{f_1(F_2^{-1}(u_2; t); t)}{f_2(F_2^{-1}(u_2; t); t)} \\ &+ \sum_{n \geq 1} \int_0^t \iint_{\mathbb{R}^2} \frac{Z\left((F_1^{-1}(u_1, t), F_2^{-1}(u_2, t)), t; \vec{\xi}, \tau\right)}{f_1(F_1^{-1}(u_2; t); t) f_2(F_2^{-1}(u_2; t); t)} (\mathcal{L}Z)_n(\vec{\xi}, \tau; \vec{0}, 0) d\vec{\xi} d\tau, \end{aligned} \tag{4.11}$$

where $c_g(u_1, u_2; \rho)$ is the Gaussian copula density with correlation parameter ρ. In the above expansion of the copula density we denote

$$v_0^c(u_1, u_2; t) = c_g\left(u_1, (F_1 \circ F_2^{-1})(u_2; t); \rho\right) \frac{f_1(F_2^{-1}(u_2; t); t)}{f_2(F_2^{-1}(u_2; t); t)},$$

and

$$v_n^c(u_1, u_2; t) = \int_0^t \iint_{\mathbb{R}^2} \frac{Z\left((F_1^{-1}(u_1, t), F_2^{-1}(u_2, t)), t; \vec{\xi}, \tau\right)}{f_1(F_1^{-1}(u_2; t); t) f_2(F_2^{-1}(u_2; t); t)} (\mathcal{L}Z)_n(\vec{\xi}, \tau; \vec{0}, 0) d\vec{\xi} d\tau, \; n \geq 1.$$

In the copula density's expansion, the second term $v_1^c(u_1, u_2; t)$ is an adjustment term to the first term $v_0^c(u_1, u_2; t)$. In the next section we can see from the numerical results that the first and second terms in the copula expansion can explain most of the dependence information and are important in the series expression.

Next we will show some properties of the copula function $C(u_1, u_2; t)$. When t is small, the distance between the copula function $C(u_1, u_2; t)$ and the Gaussian copula is small, which is mathematically presented in the following proposition.

Proposition 4.1. *There exists some $t_0 > 0$ and $B < \infty$, such that for all $0 < t < t_0$,*

$$|C(u_1, u_2; t) - C_g(u_1, u_2; \rho)| \leq B\sqrt{t}.$$

Proof. Theorem 4.1 shows that

$$C(u_1, u_2; t) - C_g(u_1, u_2; \rho) = [V_0^c(u_1, u_2; t) - C_g(u_1, u_2; \rho)] + \sum_{n=1}^{\infty} V_n^c(u_1, u_2; t). \tag{4.12}$$

Then we will estimate each term of the right-hand side of equation (4.12) separately.

First recall that

$$\Phi\left(\frac{x-\sigma_2 t}{\sqrt{t}}\right) \le F_2(x;t) \le \Phi\left(\frac{x}{\sqrt{t}}\right),$$

then

$$\sqrt{t}\Phi^{-1}(u_2) \le F_2^{-1}(u_2;t) \le \sqrt{t}\Phi^{-1}(u_2) + \sigma_2 t. \tag{4.13}$$

From Theorem 4.1 and the definition of Gaussian copula, the first term $V_0^c(u_1,u_2;t)$ and Gaussian copula function $C_g(u_1,u_2;\rho)$ satisfy that

$$\begin{aligned} V_0^c(u_1,u_2;t) &= \int_{-\infty}^{F_1^{-1}(u_1;t)} \int_{-\infty}^{F_2^{-1}(u_2;t)} Z(\vec{\xi},t;\vec{0},0)d\vec{\xi} \\ &= \int_{-\infty}^{\frac{F_1^{-1}(u_1;t)}{\sqrt{t}}} \int_{-\infty}^{\frac{F_2^{-1}(u_2;t)}{\sqrt{t}}} Z(\vec{\eta},1;\vec{0},0)d\vec{\eta}, \end{aligned} \tag{4.14}$$

and

$$C_g(u_1,u_2;\rho) = \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} Z(\vec{\eta},1;\vec{0},0)d\vec{\eta}. \tag{4.15}$$

Thus from equations (4.13) to (4.15) we have

$$\begin{aligned} 0 &\le V_0^c(u_1,u_2;t) - C_g(u_1,u_2;\rho) \\ &= \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{\Phi^{-1}(u_2)}^{\frac{F_2^{-1}(u_2;t)}{\sqrt{t}}} Z(\vec{\eta},1;\vec{0},0)d\vec{\eta} \\ &\le \left(\frac{F_2^{-1}(u_2;t)}{\sqrt{t}} - \Phi^{-1}(u_2)\right) \sup_{\eta_2\in\mathbb{R}} \left\{\int_{-\infty}^{\Phi^{-1}(u_1)} Z(\vec{\eta},1;\vec{0},0)d\eta_1\right\} \\ &\le \frac{F_2^{-1}(u_2;t)}{\sqrt{t}} - \Phi^{-1}(u_2) \\ &\le \sigma_2\sqrt{t}. \end{aligned}$$

To estimate the other terms in equation (4.12), Lemma 3.3 shows that for $n \ge 1$ there exists $C_n = \dfrac{1}{2\pi\sqrt{1-\rho^2}}\dfrac{C_0^n}{\Gamma(\frac{n}{2}+1)}$ such that

$$|v_n(\vec{x};t)| \le C_n t^{\frac{n}{2}-1} \exp\left(-\frac{\lambda(x_1^2+x_2^2)}{8t}\right).$$

Then

$$|V_n^c(u_1,u_2;t)| \le \iint_{\mathbb{R}^2} C_n t^{\frac{n}{2}-1} \exp\left(-\frac{\lambda(x_1^2+x_2^2)}{8t}\right) dx_1 dx_2 = \frac{8\pi C_n}{\lambda} t^{\frac{n}{2}}.$$

Notice that

$$\sum_{n\geq 1} C_n < \infty.$$

Then for sufficiently small $0 < t_0 < 1$ equation (4.12) can be estimated by

$$\begin{aligned}|C(u_1,u_2;t) - C_g(u_1,u_2;\rho)| &\leq \sigma_2\sqrt{t} + \sum_{n\geq 1}\frac{8\pi C_n}{\lambda}t^{\frac{n}{2}} \\ &\leq \sqrt{t}\left[\sigma_2 + \sum_{n\geq 1}\frac{8\pi C_n}{\lambda}t_0^{\frac{n-1}{2}}\right] = B\sqrt{t}.\end{aligned}$$

Here

$$B = \sigma_2 + \sum_{n\geq 1}\frac{8\pi C_n}{\lambda}t_0^{\frac{n-1}{2}}$$

is finite and positive. Thus we complete the proof of the proposition. □

In the copula expansion, the change of the first term $V_0^c(u_1,u_2;t)$ with respect to time t is bounded. The result will be given in the following proposition.

Proposition 4.2. *For $0 < t < t' < T$ the first term V_0^c satisfies*

$$\left|V_0^c(u_1,u_2;t') - V_0^c(u_1,u_2;t)\right| \leq \frac{\sigma_2}{\sqrt{2\pi}}\sqrt{t'-t}.$$

Proof. Denote

$$z' = \frac{F_2^{-1}(u_2;t')}{\sqrt{t'}},\ z = \frac{F_2^{-1}(u_2;t)}{\sqrt{t}}.$$

First we will estimate $|z - z'|$. From the expression of $F_2(x_2;t)$ in Proposition 2.1,

$$(1-q)\Phi(z) + q\Phi(z - \sigma_2\sqrt{t}) = u_2 = (1-q)\Phi(z') + q\Phi(z' - \sigma_2\sqrt{t'}).$$

It follows that

$$(1-q)\left(\Phi(z) - \Phi(z')\right) = q\left(\Phi(z' - \sigma_2\sqrt{t'}) - \Phi(z - \sigma_2\sqrt{t})\right). \tag{4.16}$$

In equation (4.16) we notice that if $z > z'$,

$$z' - \sigma_2\sqrt{t'} > z - \sigma_2\sqrt{t},$$

which leads to a contradiction since $t' > t$. It implies that $z' \geq z$ and

$$z' - \sigma_2\sqrt{t'} \leq z - \sigma_2\sqrt{t}.$$

In conclusion, for any $0 < t < t' < T$ and $u_2 \in (0,1)$ equation (4.16) implies that

$$0 \leq \frac{F_2^{-1}(u_2;t')}{\sqrt{t'}} - \frac{F_2^{-1}(u_2;t)}{\sqrt{t}} \leq \sigma_2\sqrt{t'-t}.$$

Then

$$\begin{aligned}
\left|V_0^c(u_1,u_2;t') - V_0^c(u_1,u_2;t)\right| &\leq \sup_{v\in[0,1]}\left|\frac{\partial C_g}{\partial v}(u_1,v;\rho)\right|\left|F_1\circ F_2^{-1}(u_2;t') - F_1\circ F_2^{-1}(u_2;t)\right| \\
&\leq \left|\Phi\left(\frac{F_2^{-1}(u_2;t')}{\sqrt{t'}}\right) - \Phi\left(\frac{F_2^{-1}(u_2;t)}{\sqrt{t}}\right)\right| \\
&\leq \frac{1}{\sqrt{2\pi}}\left|\frac{F_2^{-1}(u_2;t')}{\sqrt{t'}} - \frac{F_2^{-1}(u_2;t)}{\sqrt{t}}\right| \\
&\leq \frac{\sigma_2}{\sqrt{2\pi}}\sqrt{t'-t}.
\end{aligned}$$

Thus the proposition is proved. □

In this section we find that the copula function $C(u_1,u_2;t)$ between $(L_1(t), L_2(t))$ can be expressed as a sum of an infinite series. In the next section we will provide numerical analysis for measuring the approximation error of the first two terms of the series and compare it with the Gaussian approximation.

5 Numerical Analysis

In this section we provide numerical results for the copula function $C(u_1,u_2;t)$. We consider the following two cases in our numerical analysis:

1. Case 1: $L_1(0) = 0.05$, $L_2(0) = 0.06$, $\sigma_1 = 15\%$, $\sigma_2 = 20\%$, $\rho = 0.5$, $\delta = 0.5$.

2. Case 2: $L_1(0) = 0.05$, $L_2(0) = 0.10$, $\sigma_1 = 15\%$, $\sigma_2 = 35\%$, $\rho = 0.5$, $\delta = 0.5$.

The parameter set in Case 1 describes the normal situation in the financial market such that the interest rate and the volatility are low, which can be regarded as the basic case. In Case

2, the market faces great challenge and the volatility of the LIBOR rate is higher than that in Case 1, which can be regarded as an extreme case. From the expression of the copula function $C(u_1, u_2; t)$ we know that it is independent of the initial value and the volatility of $L_1(t)$, so we do not make changes of σ_1 and $L_1(0)$ in these two cases. The assumption for $\rho > 0$ is reasonable because interest rates with close maturities are positive correlated in the sense that when one rate goes up, the other is more likely to increase. This assumption is supported by the calibration results in Brigo & Mercurio (2001, Chapter 7).

In this section we will first give the features of copula function between $(L_1(t), L_2(t))$ under the above two parameter sets. Then we will analyze the influence of the first two terms in the series of the copula expansion. Finally we will study the difference between the real copula $C(u_1, u_2; t)$ and Gaussian copula $C_g(u_1, u_2; \rho)$. During our numerical calculating, the Crank-Nicolson scheme (see Crank & Nicolson (1947)) is used to solve the joint density function f in PDE (2.5), and linear interpolation method is used to calculate the copula density $c(u_1, u_2; t)$ in (2.12).

5.1 General results

Figure 1 presents the shapes of the copula function $C(u_1, u_2; t)$ and its density function $c(u_1, u_2; t)$ between $(L_1(t), L_2(t))$ for Case 1 and Case 2 at time $t = 1.0$. It can be found that the copula $C(u_1, u_2; 1)$ has similar appearance with the Gaussian family for that it has high density value near $(0, 0)$ and $(1, 1)$ (for $\rho > 0$) and tends to 0 near $(0, 1)$ and $(1, 0)$.

We are also interested in the Spearman's rho and Kendall's tau, which are both indexes for measuring the global correlation in the copula function. Spearman's rho can be calculated by

$$\rho_S = 12 \iint uv dC(u, v) - 3,$$

which measures the distance between the copula function C and the independent copula $\Pi(u, v) = uv$. Another measure of the correlation in copula is Kendall's tau, which is given by

$$\tau_K = 4 \iint C(u, v) dC(u, v) - 1.$$

For a Gaussian copula with parameter $\rho = 0.5$, these two measures are $\rho_S = 0.4826$ and

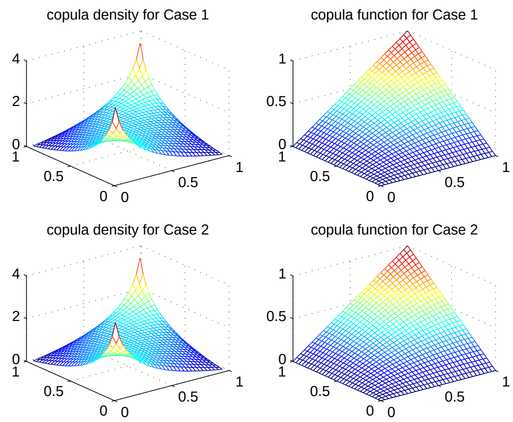

Figure 1: Numerical results for the copula function $C(u_1, u_2; t)$ and density $c(u_1, u_2; t)$ for Case 1 and Case 2 at $t = 1.0$. Top: density and copula function in Case 1; Bottom: density and copula function in Case 2.

$\tau_K = 0.3333$. More properties about ρ_S and τ_K can be found in Cherubini, Luciano & Vecchiato (2004, Chapter 1).

The two measures ρ_S and τ_K can be numerically computed by the copula function $C(u_1, u_2; t)$ and the results are given in Table 1 for $t = 1.0$. Both of them are smaller than the Gaussian copula with the correlation index $\rho = 0.5$. It may suggest that the time-varying drift terms of the LIBOR processes can bring down the correlation between the LIBOR Rates.

time	Case 1		Case 2	
	ρ_S	τ_K	ρ_S	τ_K
$t = 0.5$	0.4650428	0.3274752	0.4650357	0.3274682
$t = 1.0$	0.4650454	0.3274796	0.4650364	0.3274718
$t = 1.5$	0.4650461	0.3274773	0.4650372	0.3274693
$t = 2.0$	0.4650465	0.3273998	0.4650360	0.3273925

Table 1: Dynamic Spearman's rho and Kendall's tau for the copula $C(u_1, u_2; 1)$ in Case 1 and Case 2.

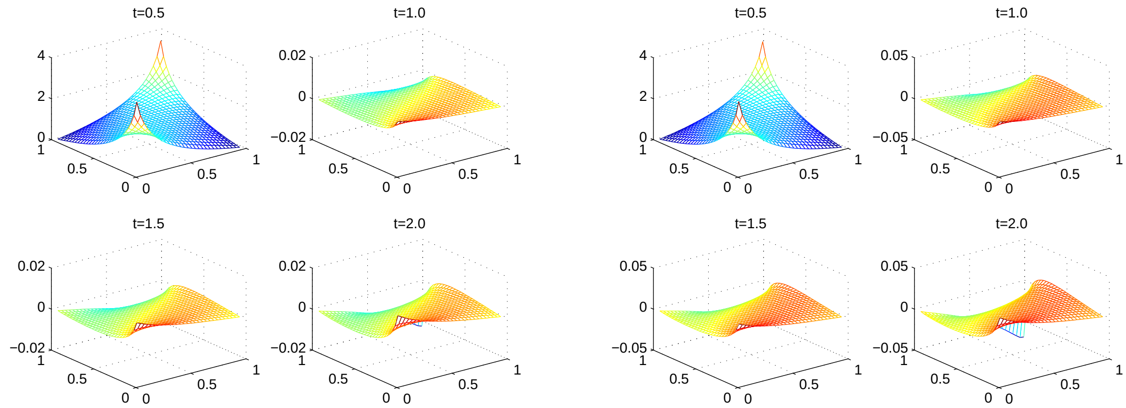

Figure 2: Numerical results of the first term $v_0^c(u_1, u_2; t)$. In each sub-figure, left top: $v_0^c(u_1, u_2; 0.5)$; right top to right bottom: the change of the first term to the case $t = 0.5$, i.e., $v_0^c(u_1, u_2; t) - v_0^c(u_1, u_2; 0.5)$ for $t = 1.0, 1.5, 2.0$. Left: Case 1. Right: Case 2.

5.2 On the Series Expansion about the Copula Function $C(u_1, u_2; t)$

As shown in Theorem 4.1, the copula function $C(u_1, u_2; t)$ can be expanded as a sum of an infinite series, and the first two terms in the copula density expansion are denoted by $v_i^c(u_1, u_2; t)$, $i = 0, 1$, which is given in equation (4.11). Figure 2 implies that the first term $v_0^c(u_1, u_2; t)$ can be regarded as a distortion from the Gaussian copula. The distortion effect becomes apparent as time goes on. Comparing with the distortion effect in the left part of Figure 2, the increase in volatility enlarges the effect, as shown in the right part. In Proposition 2.1, increasing σ_2 and $L_2(0)$ effects the first term by enlarging the parameter q. When q increases, the difference between the marginal distribution $F_2(x_2; t)$ and Gaussian distribution becomes larger, thus the distortion effect increases.

Unlike the first term $v_0^c(u_1, u_2; t)$, the second term $v_1^c(u_1, u_2; t)$ is an adjustment term to the first term. Figure 3 shows the shapes of $v_1^c(u_1, u_2; t)$ for the two parameter cases, respectively. The adjustment effect is small in the central region of the unit square, and apparent near $(0, 0)$ and $(1, 1)$. It is also more obvious for Case 2 when the volatility is in a higher level.

Figure 4 gives the error of approximating the series of the copula with only the first term,

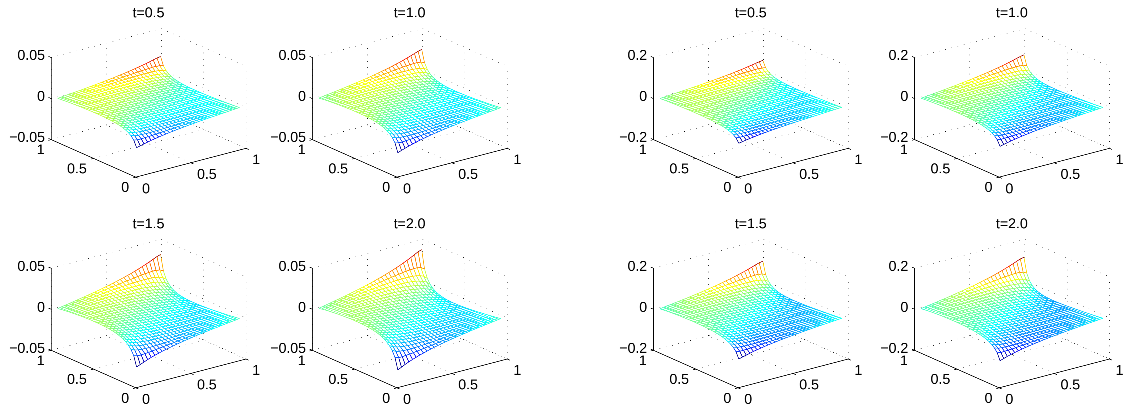

Figure 3: Numerical results of the second term $v_1^c(u_1, u_2; t)$ for $t = 0.5, 1.0, 1.5, 2.0$. Left: Case 1. Right: Case 2.

which is measured by $c(u_1, u_2; t) - v_0^c(u_1, u_2; t)$. The approximation error is small in the central region but the difference in the tail is not ignorable. We also find that the difference between the copula and the first term becomes larger when t increases, which means that the distortion impact of the remaining terms becomes larger as time goes on. The approximation error is much reduced by adding the adjustment term $v_1^c(u_1, u_2; t)$, as shown in Figure 5, and the result is more acceptable for Case 1 than Case 2.

5.3 Difference between the Real Copula and the Gaussian Copula

In this subsection we will study the error of the Gaussian approximation to the real copula function $C(u_1, u_2; t)$. Recall that $c_g(u_1, u_2; \rho)$ is the Gaussian copula density with correlation parameter ρ. Then numerical results for the approximation error which is measured by $c(u_1, u_2; t) - c_g(u_1, u_2; \rho)$ are shown in Figure 6. The traditional Gaussian approximation is not good in the tail, which implies that when the interest rates are too high or too low, the Gaussian correlation structure is not applicable. Comparing with the approximation using the first two terms in Figure 5, Gaussian approximation seems worse than series approximation when volatility is low.

It is useful to calculate the exact difference between the real copula and the Gaussian copula

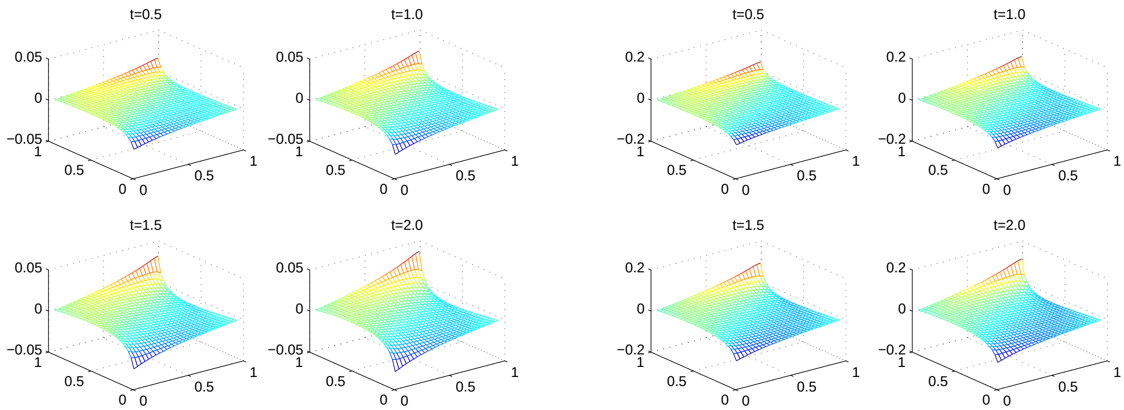

Figure 4: Numerical results of the difference between copula density and the first term $c(u_1, u_2; t) - v_0^c(u_1, u_2; t)$, for $t = 0.5, 1.0, 1.5, 2.0$. Left: Case 1. Right: Case 2.

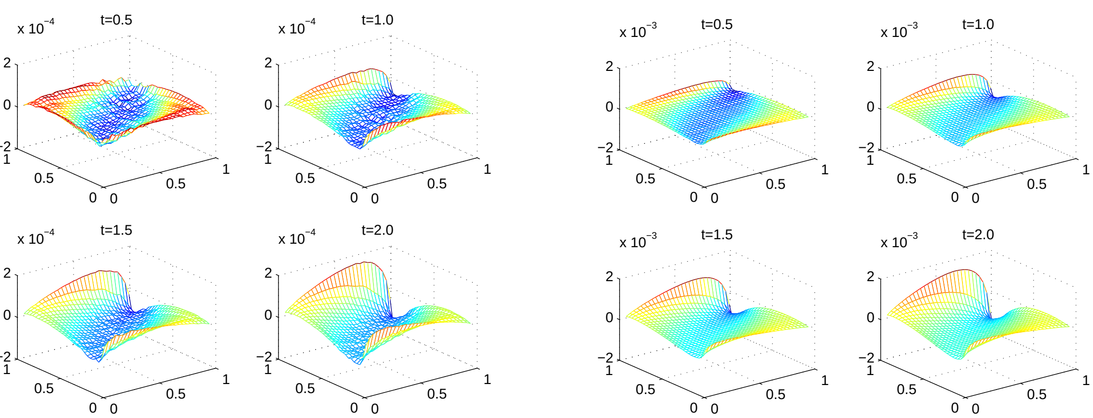

Figure 5: Numerical results of the difference between copula density and the first two terms $c(u_1, u_2; t) - v_0^c(u_1, u_2; t) - v_1^c(u_1, u_2; t)$, for $t = 0.5, 1.0, 1.5, 2.0$. Left: Case 1. Right: Case 2.

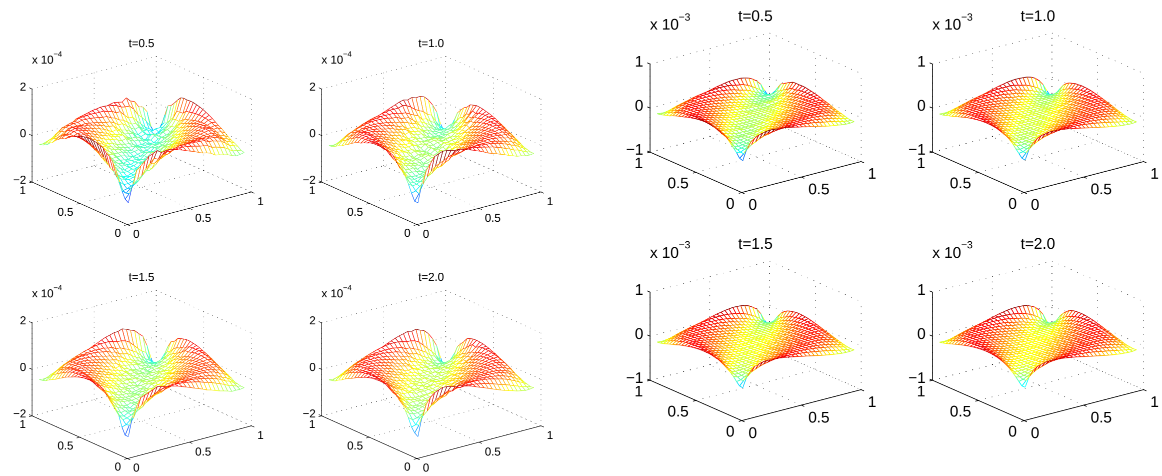

Figure 6: Numerical results of the difference between copula density and Gaussian density $c(u_1, u_2; t) - c_g(u_1, u_2; \rho)$, for $t = 0.5, 1.0, 1.5, 2.0$. Left: Case 1. Right: Case 2.

and we will choose several points in the unit square to see how the two copula functions differ at these points. Since the density functions of the real copula function $C(u_1, u_2; t)$ and the approximated Gaussian copula $C_g(u_1, u_2; \rho)$ mainly differ in the tail, we choose several points near $(1, 1)$. Table 2 presents the values of the differences. From Table 2 we can see that neither of the two densities is dominated by the other, which means that at some points the density for the real copula is larger than the Gaussian density and at the other points the opposite is true. And the difference between the real density $c(u_1, u_2; t)$ and the Gaussian density is larger for Case 2, which implies that the Gaussian approximation is worse when market volatility is higher.

Next we will show the difference between the copula and Gaussian copula by calculating conditional tail expectations of LIBOR-related variables. Suppose that (L_1, L_2) are LIBOR rates at $T = 2.0$, and (U_1, U_2) are the associated transformed variables with $U_i = F_i(L_i; T)$, $i = 1, 2$. Table 3 and Table 4 examine the expected shortfall of two rates L_1 and L_2 when both of them are above some thresholds K_1 and K_2 using the copula function $C(u_1, u_2; t)$ and the Gaussian copula $C_g(u_1, u_2; \rho)$ for the two parameter cases. The difference is hard to observe since the density function tends to 0 rapidly when the interest rates are large.

Point (u_1, u_2)	Case 1 ($\times 10^{-3}$) $c(u_1, u_2; t) - c_g(u_1, u_2; \rho)$	Case 2 ($\times 10^{-3}$) $c(u_1, u_2; t) - c_g(u_1, u_2; \rho)$
(0.995 , 0.995)	-0.5171	-2.2171
(0.995 , 0.990)	-0.5308	-1.8808
(0.995 , 0.950)	-0.0252	-0.1752
(0.990 , 0.995)	-0.2208	-1.0708
(0.990 , 0.990)	-0.4025	-1.3125
(0.990 , 0.950)	-0.1539	-0.4339
(0.950 , 0.995)	0.2948	0.6848
(0.950 , 0.990)	0.0461	0.1461
(0.950 , 0.950)	-0.1279	-0.3779

Table 2: Difference between the density functions for copula function $C(u_1, u_2; t)$ and the approximated Gaussian copula at several points.

Table 5 and Table 6 present the expected shortfall of the transformed variables (U_1, U_2) under two parameter sets. In the real copula case, $(U_1, U_2) \sim C(u_1, u_2; T)$ and in the Gaussian copula case, $(U_1, U_2) \sim C_g(u_1, u_2; \rho)$. Contrary to the numerical results in Table 3 and Table 4, the expected shortfall of the real copula is larger than that of the Gaussian copula.

		$E[L_1 \vert L_1 \geq K_1, L_2 \geq K_2]$		$E[L_2 \vert L_1 \geq K_1, L_2 \geq K_2]$	
K_1	K_2	Real copula	Gaussian	Real copula	Gaussian
0.0801	0.1113	0.088003	0.088004	0.126541	0.126550
0.0693	0.0918	0.077700	0.077770	0.107276	0.107278
0.0642	0.0828	0.072893	0.072893	0.098569	0.098570
0.1000	0.1200	0.105829	0.105835	0.137100	0.137124

Table 3: Case 1, $\rho = 0.5$.

K_1	K_2	$E[L_1 \mid L_1 \geq K_1, L_2 \geq K_2]$		$E[L_2 \mid L_1 \geq K_1, L_2 \geq K_2]$	
		Real copula	Gaussian	Real copula	Gaussian
0.0801	0.2798	0.087958	0.087963	0.354558	0.354772
0.0693	0.1997	0.077661	0.077663	0.266752	0.266807
0.0642	0.1668	0.072858	0.072859	0.230590	0.230617
0.1000	0.1200	0.105394	0.105400	0.302881	0.303237

Table 4: Case 2, $\rho = 0.5$.

K_1	K_2	$E[U_1 \mid U_1 \geq K_1, U_2 \geq K_2]$		$E[U_2 \mid U_1 \geq K_1, U_2 \geq K_2]$	
		Real copula	Gaussian	Real copula	Gaussian
0.99	0.95	0.994495	0.994317	0.979003	0.976499
0.99	0.90	0.994405	0.994266	0.961534	0.955124
0.95	0.90	0.976616	0.975267	0.957644	0.953186

Table 5: Case 1, $\rho = 0.5$.

5.4 Conclusion on the Numerical Results

In this section, we provide some numerical results for the copula between LIBOR rates and conclude that:

(1) The shapes of the copula function and its density are close to those of the Gaussian copula. The density functions are both high in the upper and lower tails, and flat in the central region for positive correlation parameter.

(2) The copula between LIBOR rates can be expressed as a sum of an infinite series. The first term can explain most information of the copula. The other terms are adjustments to the first term. When volatility becomes larger, the distortion effect of the first term and the impact of adjustment of the second term both increase.

(3) When the market volatility is low, the series approximation result using the first two terms is acceptable. But when market volatility is high, more terms should be considered to approximate the copula.

K_1	K_2	$E[U_1\|U_1 \geq K_1, U_2 \geq K_2]$ Real copula	$E[U_1\|U_1 \geq K_1, U_2 \geq K_2]$ Gaussian	$E[U_2\|U_1 \geq K_1, U_2 \geq K_2]$ Real copula	$E[U_2\|U_1 \geq K_1, U_2 \geq K_2]$ Gaussian
0.99	0.95	0.994495	0.994317	0.979002	0.976499
0.99	0.90	0.994405	0.994266	0.961531	0.955124
0.95	0.90	0.976616	0.975267	0.957644	0.953186

Table 6: Case 2, $\rho = 0.5$.

(4) When the market volatility and the interest rate levels are low, Gaussian approximation works well. But when the interest rate and volatility levels are high, their performances differ in the tail.

6 Conclusion

This paper starts from the simplified BGM model and finds the expansion of the copula function between the LIBOR rates by using PDE method. The copula function between two LIBOR rates can be expressed as a sum of an infinite series, where the first term is a distribution function with Gaussian copula and the other terms of the series are adjustments to the first term. The numerical results are provided to present and compare the performances of the approximations using the first two terms of the series and the Gaussian copula. The numerical results suggest that the Gaussian approximation is acceptable when market volatility and interest rate are low, and when the volatility increases, the approximated Gaussian copula and the original copula differ in the tail.

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Appendix A Proof of Theorem 3.2: the Uniqueness Part

In this part we will prove that $\Psi(\vec{x},t;\vec{0},0)$ is the unique solution to equation (2.5) under the condition that

$$\sup_{t\in(0,T]}\iint_{\mathbb{R}^2}|p(\vec{x};t)|d\vec{x}<\infty.$$

Further we will prove that for any $t\in(0,T]$, the function $\Psi(\vec{x},t;\vec{0},0)$ satisfies

$$\iint_{\mathbb{R}^2}\Psi(\vec{x},t;\vec{0},0)d\vec{x}=1,$$

thus complete the proof that $f(\vec{x};t)=\Psi(\vec{x},t;\vec{0},0)$.

First we will verify that the function $\Psi(\vec{x},t;\vec{0},0)$ satisfies

$$\sup_{t\in(0,T]}\iint_{\mathbb{R}^2}|\Psi(\vec{x},t;\vec{0},0)|d\vec{x}<\infty.$$

In fact, by Theorem 11 in Chapter 2 of Friedman (1964) the function $\Psi(\vec{x},t;\vec{0},0)$ is positive. Then

$$\iint_{\mathbb{R}^2}|\Psi(\vec{x},t;\vec{0},0)|d\vec{x}=\iint_{\mathbb{R}^2}\Psi(\vec{x},t;\vec{0},0)d\vec{x}=\iint_{\mathbb{R}^2}Z(\vec{x},t;0,0)d\vec{x}+\sum_{n=1}^{\infty}\iint_{\mathbb{R}^2}v_n(\vec{x};t)d\vec{x}. \quad \text{(A.1)}$$

It holds that

$$\iint_{\mathbb{R}^2}(\mathcal{L}Z)_1(\vec{\xi},\tau;\vec{\eta},\sigma)d\vec{\xi}=\int_{\mathbb{R}}\left[\int_{\mathbb{R}}\frac{\partial}{\partial\xi_2}\Big(\theta(\xi_2,\tau)Z(\vec{\xi},\tau;\vec{\eta},\sigma)\Big)\,d\xi_2\right]d\xi_1=0,$$

which is followed by

$$\begin{aligned}
&\iint_{\mathbb{R}^2}(\mathcal{L}Z)_n(\vec{\xi},\tau;\vec{0},0)d\vec{\xi}\\
&=\iint_{\mathbb{R}^2}\left[\int_0^{\tau}\iint_{\mathbb{R}^2}(\mathcal{L}Z)_1(\vec{\xi},\tau;\vec{\eta},\sigma)(\mathcal{L}Z)_{n-1}(\vec{\eta},\sigma;\vec{0},0)d\vec{\eta}d\sigma\right]d\vec{\xi}\\
&=0
\end{aligned}$$

for $n\geq 2$. Then from the above equation we get

$$\begin{aligned}
&\iint_{\mathbb{R}^2}v_n(\vec{x};t)d\vec{x}\\
&=\iint_{\mathbb{R}^2}\left[\int_0^{t}\iint_{\mathbb{R}^2}Z(\vec{x},t;\vec{\xi},\tau)(\mathcal{L}Z)_n(\vec{\xi},\tau;\vec{0},0)d\vec{\xi}d\tau\right]d\vec{x}\\
&=\int_0^t\iint_{\mathbb{R}^2}(\mathcal{L}Z)_n(\vec{\xi},\tau;\vec{0},0)d\vec{\xi}d\tau\\
&=0.
\end{aligned}$$

Thus from equation (A.1) we conclude that

$$\iint_{\mathbb{R}^2} \Psi(\vec{x}, t; \vec{0}, 0) = 1.$$

Next we will prove the uniqueness part of the theorem. Suppose that $K_1(\vec{x}, t)$, $K_2(\vec{x}, t)$ both solve equation (2.5), i.e., $\mathcal{L}K_1 = \mathcal{L}K_2 = 0$, for all $(\vec{x}, t) \in \mathbb{R}^2 \times (0, T]$, and for any continuous function $g(\vec{x})$, $\vec{x} \in \mathbb{R}^2$ satisfying $|g(\vec{x})| \leq Me^{h|\vec{x}|^2}$ for some constant M,

$$\lim_{t\to 0+} \iint_{\mathbb{R}^2} K_1(\vec{x}, t) g(\vec{x}) d\vec{x} = \lim_{t\to 0+} \iint_{\mathbb{R}^2} K_2(\vec{x}, t) g(\vec{x}) d\vec{x} = g(0).$$

Denote $u = K_1 - K_2$. It follows that $\mathcal{L}u = 0$ and

$$\lim_{t\to 0+} \iint_{\mathbb{R}^2} u(\vec{x}, t) g(\vec{x}) d\vec{x} = 0. \tag{A.2}$$

In order to show that $u(\vec{x}, t) = 0$, for all $(\vec{x}, t) \in \mathbb{R}^2 \times (0, T]$, define $\mathcal{L}^*$ to be the adjoint operator of $\mathcal{L}$ by

$$\mathcal{L}^* p = -\frac{\partial p}{\partial t} - \frac{1}{2}\frac{\partial^2 p}{\partial x_1^2} - \rho \frac{\partial^2 p}{\partial x_1 \partial x_2} - \frac{1}{2}\frac{\partial^2 p}{\partial x_2^2} - \theta \frac{\partial p}{\partial x_2}$$

for any function $p(\vec{x}, t)$ such that the derivatives exist. Then by Theorem 15 in Friedman (1964, page 28) the fundamental solution Ψ^* of $\mathcal{L}^* p = 0$ exists and

$$\Psi^*(\vec{\xi}, \tau; \vec{x}, t) = \Psi(\vec{x}, t; \vec{\xi}, \tau),\ \ 0 < \tau < t \leq T. \tag{A.3}$$

Next we will define the function $\varphi_R(\vec{x})$. Fix $(\vec{x}_0, t_0) \in \mathbb{R}^2 \times (0, T]$. Denote
$B_R = \left\{\vec{y} \in \mathbb{R}^2 : |\vec{y} - \vec{x}_0| < R\right\}$ for $R > 1$. Then there exists a function $\varphi_R(\vec{\xi})$ such that

(a) $0 \leq \varphi_R(\vec{\xi}) \leq 1$. Moreover, $\varphi_R(\vec{\xi}) = 1$ for $\vec{\xi} \in B_R$, and $\varphi_R(\vec{\xi}) = 0$ for $\vec{\xi} \in \mathbb{R}^2 \setminus B_{R+1}$;

(b) The first and second partial derivatives of φ_R exist and are continuous. Furthermore, $\displaystyle\sum_{i=1}^{2}\left|\frac{\partial \varphi_R(\vec{\xi})}{\partial \xi_i}\right| + \sum_{i=1}^{2}\sum_{j=1}^{2}\left|\frac{\partial^2 \varphi_R(\vec{\xi})}{\partial \xi_i \partial \xi_j}\right|$ has a upper bound which is independent of R.

Now we begin to prove that for fixed $(\vec{x}_0, t_0) \in \mathbb{R}^2 \times (0, T]$, $u(\vec{x}_0, t_0) = 0$. For $u = u(\vec{\xi}, \tau), v = \varphi_R(\vec{\xi})\Psi(\vec{x}_0, t_0; \vec{\xi}, \tau)$, we have

$$\begin{aligned} v\mathcal{L}u - u\mathcal{L}^* v \ \ &= \ \ \frac{\partial (uv)}{\partial t} + \frac{\partial(\theta uv)}{\partial \xi_2} - \frac{1}{2}\frac{\partial}{\partial \xi_1}\left(v\frac{\partial u}{\partial \xi_1} - u\frac{\partial v}{\partial \xi_1}\right) \\ &\quad - \rho\left(\frac{\partial}{\partial \xi_1}\left(v\frac{\partial u}{\partial \xi_2}\right) - \frac{\partial}{\partial \xi_2}\left(u\frac{\partial v}{\partial \xi_1}\right)\right) - \frac{1}{2}\frac{\partial}{\partial \xi_2}\left(v\frac{\partial u}{\partial \xi_2} - u\frac{\partial v}{\partial \xi_2}\right). \end{aligned}$$

Noticing that $\varphi_R(\xi_1,\xi_2)$ and its first partial derivatives are 0 on the edge of B_{R+1} and integrating the above equation on both sides for $(\vec{\xi},\tau)\in B_{R+1}\times(\varepsilon_1,t_0-\varepsilon_2)$, where ε_1 and ε_2 are small positive real numbers satisfying $\varepsilon_1<t_0-\varepsilon_2$, we have

$$\begin{aligned}&-\int_{\varepsilon_1}^{t_0-\varepsilon_2}\iint_{B_{R+1}}u\mathcal{L}^*v\,d\vec{\xi}d\tau \qquad \text{(A.4)}\\ &=\iint_{B_{R+1}}\varphi_R(\vec{\xi})\Psi(\vec{x}_0,t_0;\vec{\xi},t_0-\varepsilon_2)u(\vec{\xi},t_0-\varepsilon_2)d\vec{\xi}-\iint_{B_{R+1}}\varphi_R(\vec{\xi})\Psi(\vec{x}_0,t_0;\vec{\xi},\varepsilon_1)u(\vec{\xi},\varepsilon_1)d\vec{\xi}.\end{aligned}$$

By (A.3), the first term of the right-hand side of equation (A.4) satisfies

$$\iint_{B_{R+1}}\varphi_R(\vec{\xi})\Psi(\vec{x}_0,t_0;\vec{\xi},t_0-\varepsilon_2)u(\vec{\xi},t_0-\varepsilon_2)d\vec{\xi}\to\varphi_R(\vec{x}_0)u(\vec{x}_0,t_0)=u(\vec{x}_0,t_0)$$

as $\varepsilon_2\to 0+$, $R\to+\infty$. The second term of the right-hand side of equation (A.4) tends to $-\iint_{\mathbb{R}^2}\Psi(\vec{x}_0,t_0;\vec{\xi},\varepsilon_1)u(\vec{\xi},\varepsilon_1)d\vec{\xi}$ as $R\to+\infty$. When ε_1 is small and positive, $\Psi(\vec{x}_0,t_0;\vec{\xi},\varepsilon_1)$ is uniformly continuous with respect to ε_1, then

$$\lim_{\varepsilon_1\to 0}\iint_{\mathbb{R}^2}\Psi(\vec{x}_0,t_0;\vec{\xi},\varepsilon_1)u(\vec{\xi},\varepsilon_1)d\vec{\xi}=\lim_{\varepsilon_1\to 0}\iint_{\mathbb{R}^2}\Psi(\vec{x}_0,t_0;\vec{\xi},0)u(\vec{\xi},\varepsilon_1)d\vec{\xi}=0.$$

The above limit follows from the initial condition of u in equation (A.2). Thus we conclude from equation (A.4) that

$$\int_{\varepsilon_1}^{t_0-\varepsilon_2}\iint_{B_{R+1}}u\mathcal{L}^*v\,d\vec{\xi}d\tau\to u(\vec{x}_0,t_0),\ \text{as } \varepsilon_1\to 0+,\ \varepsilon_2\to 0+,\ R\to+\infty. \qquad \text{(A.5)}$$

On the other hand, similar to the proof with Lemma 3.3, for $n\geq 1$, $i=1,2$, it holds that

$$\left|\frac{\partial(\mathcal{L}Z)_n}{\partial\xi_i}(\vec{x}_0,t_0;\vec{\xi},\tau)\right|\leq\frac{C_0^{n-1}D_0}{\Gamma\left(\frac{n-1}{2}\right)}(t_0-\tau)^{\frac{n-5}{2}}e^{-\frac{\lambda|\vec{x}_0-\vec{\xi}|^2}{8(t_0-\tau)}},$$

where the constant C_0 is defined in Lemma 3.3 and

$$D_0=\frac{4\sigma_2}{\lambda e\pi(1-|\rho|)^{5/2}}+\frac{\sigma_2}{2\pi(1-|\rho|)^{3/2}}+\frac{\sigma_2^2\sqrt{T}}{\sqrt{\lambda e}\pi(1-|\rho|)^{3/2}}.$$

It follows that

$$
\begin{aligned}
& |\Psi(\vec{x}_0, t_0; \vec{\xi}, \tau)| + \sum_{i=1}^{2} \left| \frac{\partial}{\partial \xi_i} \Psi(\vec{x}_0, t_0; \vec{\xi}, \tau) \right| \\
\leq \;& \frac{1}{2\pi\sqrt{1-\rho^2}(t_0-\tau)} e^{-\frac{\lambda|\vec{x}_0-\vec{\xi}|^2}{4(t_0-\tau)}} + \sum_{i=1}^{2} \left| \frac{\partial Z}{\partial \xi_i}(\vec{x}_0, t_0; \vec{\xi}, \tau) \right| \\
& + \sum_{i=1}^{2} \sum_{n=1}^{\infty} \int_{\tau}^{t} \iint_{\mathbb{R}^2} Z(\vec{x}_0, t_0; \vec{\eta}, \sigma) \left| \frac{\partial (\mathcal{L}Z)_n}{\partial \xi_i}(\vec{\eta}, \sigma; \vec{\xi}, \tau) \right| d\vec{\eta} d\sigma \\
\leq \;& \frac{\sqrt{T}}{2\pi\sqrt{1-\rho^2}(t_0-\tau)^{3/2}} e^{-\frac{\lambda|\vec{x}_0-\vec{\xi}|^2}{8(t_0-\tau)}} + \sum_{i=1}^{2} \frac{|\vec{x}-\vec{\xi}|}{(1-|\rho|)(t-\tau)} Z(\vec{x}_0, t_0; \vec{\xi}, \tau) \\
& + \sum_{i=1}^{2} \sum_{n=1}^{\infty} \int_{\tau}^{t} \iint_{\mathbb{R}^2} \frac{1}{2\pi\sqrt{1-\rho^2}(t_0-\sigma)} e^{-\frac{\lambda|\vec{x}_0-\vec{\eta}|^2}{8(t_0-\sigma)}} \frac{C_0^{n-1} D_0}{\Gamma\left(\frac{n-1}{2}\right)} (\sigma-\tau)^{\frac{n-5}{2}} e^{-\frac{\lambda|\vec{\eta}-\vec{\xi}|^2}{8(\sigma-\tau)}} d\vec{\eta} d\sigma \\
\leq \;& \frac{\sqrt{T}}{2\pi\sqrt{1-\rho^2}(t_0-\tau)^{3/2}} e^{-\frac{\lambda|\vec{x}_0-\vec{\xi}|^2}{8(t_0-\tau)}} + \frac{2}{\sqrt{\lambda e}\pi(1-|\rho|)^{3/2}(t_0-\tau)^{3/2}} e^{-\frac{\lambda|\vec{x}_0-\vec{\xi}|^2}{8(t_0-\tau)}} \\
& + \sum_{n=1}^{\infty} \frac{4C_0^{n-1} D_0}{\sqrt{1-\rho^2}\Gamma\left(\frac{n+1}{2}\right)} (t_0-\tau)^{\frac{n-3}{2}} e^{-\frac{\lambda|\vec{x}_0-\vec{\xi}|^2}{8(t_0-\tau)}} \\
\leq \;& C_1 (t_0-\tau)^{3/2} e^{-\frac{\lambda|\vec{x}_0-\vec{\xi}|^2}{8(t_0-\tau)}}
\end{aligned}
$$

for some constant C_1. Since $\mathcal{L}^* v = 0$ for any $\vec{\xi} \in B_R$, the left-hand side of equation (A.4) can be bounded by

$$
\begin{aligned}
& \left| \int_{\varepsilon_1}^{t_0-\varepsilon_2} \iint_{B_{R+1}} u \mathcal{L}^* v d\vec{\xi} d\tau \right| \\
= \;& \left| \int_{\varepsilon_1}^{t_0-\varepsilon_2} \iint_{B_{R+1}\setminus B_R} u \left[2 \left(-\frac{1}{2} \frac{\partial \varphi_R}{\partial \xi_1} \frac{\partial \Psi}{\partial \xi_1} - \frac{\rho}{2} \left(\frac{\partial \varphi_R}{\partial \xi_1} \frac{\partial \Psi}{\partial \xi_2} + \frac{\partial \varphi_R}{\partial \xi_2} \frac{\partial \Psi}{\partial \xi_1} \right) - \frac{1}{2} \frac{\partial \varphi_R}{\partial \xi_2} \frac{\partial \Psi}{\partial \xi_2} \right) \right. \right. \\
& \left. \left. - \left(\frac{1}{2} \frac{\partial^2 \varphi_R}{\partial \xi_1^2} + \rho \frac{\partial^2 \varphi_R}{\partial \xi_1 \partial \xi_2} + \frac{1}{2} \frac{\partial^2 \varphi_R}{\partial \xi_2^2} \right) \Psi - \theta \frac{\partial \varphi_R}{\partial \xi_2} \Psi \right] d\vec{\xi} d\tau \right| \\
\leq \;& C_2 e^{-\frac{\lambda R^2}{16T}} \int_0^{t_0} \iint_{B_{R+1}\setminus B_R} |u(\vec{\xi}, \tau)| d\vec{\xi} d\tau \to 0, \text{ as } R \to +\infty
\end{aligned}
$$

under the condition that

$$
\sup_{t \in (0,T]} \iint_{\mathbb{R}^2} |u(\vec{x}; t)| d\vec{x} < \infty,
$$

where C_2 is some constant. The above inequality holds for any arbitrary small positive ε_1 and

ε_2, which implies that

$$\int_{\varepsilon_1}^{t_0-\varepsilon_2}\iint_{B_{R+1}} u\mathcal{L}^* v d\vec{\xi} d\tau \to 0, \text{ as } \varepsilon_1 \to 0+, \ \varepsilon_2 \to 0+, \ R \to +\infty. \tag{A.6}$$

Combining equations (A.5) and (A.6), we conclude that $u(\vec{x}_0, t_0) = 0$ for any fixed point $(\vec{x}_0, t_0)$. Thus we complete the proof of the theorem.

Appendix B Proof of Lemma 4.1

To prove the lemma, first we need some inequalities. From equation (4.6) we know that

$$\iint_{\mathbb{R}^2} |\vec{x}|^i Z(\vec{x}, 1; \vec{0}, 0) d\vec{x} \leq 2^i \Gamma\left(1 + \frac{i}{2}\right). \tag{B.1}$$

For $n \geq 2$, $0 \leq i \leq n$ we have

$$\begin{aligned}
& \int_\tau^t \iint_{\mathbb{R}^2} (t-\sigma)^{-\frac{1}{2}} (\sigma-\tau)^{\frac{n-3-i}{2}} |\vec{y}-\vec{\eta}|^i Z(\vec{\xi}, t; \vec{y}, \sigma) Z(\vec{y}, \sigma; \vec{\eta}, \tau) d\vec{y} d\sigma \\
= & \int_\tau^t \iint_{\mathbb{R}^2} (t-\sigma)^{-\frac{1}{2}} (\sigma-\tau)^{\frac{n-3-i}{2}} |\vec{y}-\vec{\eta}|^i Z(\vec{\xi}, t; \vec{\eta}, \tau) \\
& \times Z\left(\vec{y}, \frac{(t-\sigma)(\sigma-\tau)}{t-\tau}; \frac{\sigma-\tau}{t-\tau}\vec{\xi} + \frac{t-\sigma}{t-\tau}\vec{\eta}, 0\right) d\vec{y} d\sigma \\
= & \int_\tau^t \iint_{\mathbb{R}^2} (t-\sigma)^{-\frac{1}{2}} (\sigma-\tau)^{\frac{n-3-i}{2}} \left| \sqrt{\frac{(t-\sigma)(\sigma-\tau)}{t-\tau}} \vec{x} + \frac{\sigma-\tau}{t-\tau}(\vec{\xi}-\vec{\eta}) \right|^i \\
& \times Z(\vec{x}, 1; \vec{0}, 0) Z(\vec{\xi}, t; \vec{\eta}, \tau) d\vec{x} d\sigma \\
\leq & \ 2^i Z(\vec{\xi}, t; \vec{\eta}, \tau) \left[\iint_{\mathbb{R}^2} |\vec{x}|^i Z(\vec{x}, 1; \vec{0}, 0) d\vec{x} \int_\tau^t \frac{(t-\sigma)^{\frac{i-1}{2}} (\sigma-\tau)^{\frac{n-3}{2}}}{(t-\tau)^{\frac{i}{2}}} d\sigma \right. \\
& \left. + |\vec{\xi}-\vec{\eta}|^i \int_\tau^t \frac{(t-\sigma)^{-\frac{1}{2}} (\sigma-\tau)^{\frac{n-3+i}{2}}}{(t-\tau)^i} d\sigma \right].
\end{aligned}$$

Then from (B.1) and using the definition of the Beta function, we have

$$\begin{aligned}
& \int_\tau^t \iint_{\mathbb{R}^2} (t-\sigma)^{-\frac{1}{2}} (\sigma-\tau)^{\frac{n-3-i}{2}} |\vec{y}-\vec{\eta}|^i Z(\vec{\xi}, t; \vec{y}, \sigma) Z(\vec{y}, \sigma; \vec{\eta}, \tau) d\vec{y} d\sigma \\
& \leq Z(\vec{\xi}, t; \vec{\eta}, \tau) \left[2^{2i} \Gamma\left(1 + \frac{i}{2}\right) B\left(\frac{i+1}{2}, \frac{n-1}{2}\right) (t-\tau)^{\frac{n-2}{2}} \right. \\
& \quad \left. + 2^i |\vec{\xi}-\vec{\eta}|^i B\left(\frac{1}{2}, \frac{n-1+i}{2}\right) (t-\tau)^{\frac{n-2-i}{2}} \right].
\end{aligned} \tag{B.2}$$

For $n \geq 2$, $0 \leq i \leq n$ we also have

$$
\begin{aligned}
& \int_\tau^t \iint_{\mathbb{R}^2} (t-\sigma)^{-1}(\sigma-\tau)^{\frac{n-3-i}{2}} |\vec{\xi}-\vec{y}||\vec{y}-\vec{\eta}|^i Z(\vec{\xi},t;\vec{y},\sigma)Z(\vec{y},\sigma;\vec{\eta},\tau)d\vec{y}d\sigma \\
= & \int_\tau^t \iint_{\mathbb{R}^2} (t-\sigma)^{-1}(\sigma-\tau)^{\frac{n-3-i}{2}} \left|\frac{t-\sigma}{t-\tau}(\vec{\xi}-\vec{\eta}) - \sqrt{\frac{(t-\sigma)(\sigma-\tau)}{t-\tau}}\vec{x}\right| \\
& \times \left|\sqrt{\frac{(t-\sigma)(\sigma-\tau)}{t-\tau}}\vec{x} + \frac{\sigma-\tau}{t-\tau}(\vec{\xi}-\vec{\eta})\right|^i Z(\vec{x},1;\vec{0},0)Z(\vec{\xi},t;\vec{\eta},\tau)d\vec{x}d\sigma \\
\leq & \; 2^i Z(\vec{\xi},t;\vec{\eta},\tau)\left[\iint_{\mathbb{R}^2} |\vec{x}|^{i+1} Z(\vec{x},1;\vec{0},0)d\vec{x}\int_\tau^t \frac{(t-\sigma)^{\frac{i-1}{2}}(\sigma-\tau)^{\frac{n-2}{2}}}{(t-\tau)^{\frac{i+1}{2}}}d\sigma\right. \\
& + |\vec{\xi}-\vec{\eta}| \iint_{\mathbb{R}^2} |\vec{x}|^i Z(\vec{x},1;\vec{0},0)d\vec{x}\int_\tau^t \frac{(t-\sigma)^{\frac{i}{2}}(\sigma-\tau)^{\frac{n-3}{2}}}{(t-\tau)^{\frac{i}{2}+1}}d\sigma \\
& + |\vec{\xi}-\vec{\eta}|^i \iint_{\mathbb{R}^2} |\vec{x}| Z(\vec{x},1;\vec{0},0)d\vec{x}\int_\tau^t \frac{(t-\sigma)^{-\frac{1}{2}}(\sigma-\tau)^{\frac{n-2+i}{2}}}{(t-\tau)^{\frac{1}{2}+i}}d\sigma \\
& \left. + |\vec{\xi}-\vec{\eta}|^{i+1} \int_\tau^t \frac{(\sigma-\tau)^{\frac{n-3+i}{2}}}{(t-\tau)^{1+i}}d\sigma\right].
\end{aligned}
$$

Similarly we get that

$$
\begin{aligned}
& \int_\tau^t \iint_{\mathbb{R}^2} (t-\sigma)^{-1}(\sigma-\tau)^{\frac{n-3-i}{2}} |\vec{\xi}-\vec{y}||\vec{y}-\vec{\eta}|^i Z(\vec{\xi},t;\vec{y},\sigma)Z(\vec{y},\sigma;\vec{\eta},\tau)d\vec{y}d\sigma \\
& \leq Z(\vec{\xi},t;\vec{\eta},\tau)\left[2^{2i+1}\Gamma\left(\frac{i+3}{2}\right)B\left(\frac{i+1}{2},\frac{n}{2}\right)(t-\tau)^{\frac{n-2}{2}}\right. \\
& \quad + |\vec{\xi}-\vec{\eta}|^{i+1}2^i B\left(1,\frac{n-1+i}{2}\right)(t-\tau)^{\frac{n-3-i}{2}} \\
& \quad + |\vec{\xi}-\vec{\eta}|2^{2i}\Gamma\left(\frac{i+2}{2}\right)B\left(\frac{i+2}{2},\frac{n-1}{2}\right)(t-\tau)^{\frac{n-3}{2}} \\
& \quad \left. + |\vec{\xi}-\vec{\eta}|^i 2^{i+1}\Gamma\left(\frac{3}{2}\right)B\left(\frac{1}{2},\frac{n+i}{2}\right)(t-\tau)^{\frac{n-2-i}{2}}\right]. \qquad \text{(B.3)}
\end{aligned}
$$

Now we begin to prove the lemma. Applying Lemma 3.1 for $n = 1$,

$$
\begin{aligned}
|(\mathcal{L}Z)(\vec{\xi},t;\vec{\eta},\tau)| &= \left|\frac{\partial\theta}{\partial x_2}(x_2,t)Z(\vec{\xi},t;\vec{\eta},\tau) + \theta(x_2,t)\frac{\partial Z}{\partial x_2}(\vec{\xi},t;\vec{\eta},\tau)\right| \\
&\leq \sigma_2^2\left|Z(\vec{\xi},t;\vec{\eta},\tau)\right| + \sigma_2\left|\frac{\partial Z}{\partial x_2}(\vec{\xi},t;\vec{\eta},\tau)\right| \\
&= \left(\sigma_2^2 + \frac{\sigma_2|\rho(\xi_1-\eta_1)-(\xi_2-\eta_2)|}{(1-\rho^2)(t-\tau)}\right)Z(\vec{\xi},t;\vec{\eta},\tau) \\
&\leq \left(\frac{\sigma_2^2\sqrt{T}}{(t-\tau)^{\frac{1}{2}}} + \frac{2\sigma_2|\vec{\xi}-\vec{\eta}|}{(1-\rho^2)(t-\tau)}\right)Z(\vec{\xi},t;\vec{\eta},\tau).
\end{aligned}
$$

Denoting

$$M_{1,0} = \sigma_2^2\sqrt{T},\ \ M_{1,1} = \frac{2\sigma_2}{1-\rho^2},$$

the lemma is true for $n = 1$. For $n \geq 2$, $0 \leq i \leq n$ define $M_{n,i}$ by the following formulas

$$\begin{aligned}
M_{n,0} &= M_{1,0}M_{n-1,0}B\left(\frac{1}{2},\frac{n-1}{2}\right) + 2M_{1,1}M_{n-1,0}\Gamma\left(\frac{3}{2}\right)B\left(\frac{1}{2},\frac{n}{2}\right) \\
&\quad + \sum_{i=0}^{n-1} 2^{2i}M_{n-1,i}\left[M_{1,0}\Gamma\left(1+\frac{i}{2}\right)B\left(\frac{i+1}{2},\frac{n-1}{2}\right)\right. \\
&\quad \left. +2M_{1,1}\Gamma\left(\frac{i+3}{2}\right)B\left(\frac{i+1}{2},\frac{n}{2}\right)\right], \\
M_{n,1} &= 2M_{1,0}M_{n-1,1}B\left(\frac{1}{2},\frac{n}{2}\right) + 4M_{1,1}M_{n-1,1}\Gamma\left(\frac{3}{2}\right)B\left(\frac{1}{2},\frac{n+1}{2}\right) \\
&\quad + M_{1,1}M_{n-1,0}B\left(1,\frac{n-1}{2}\right) + \sum_{i=0}^{n-1} M_{1,1}M_{n-1,i}2^{2i}\Gamma\left(\frac{i+2}{2}\right)B\left(\frac{i+2}{2},\frac{n-1}{2}\right), \\
M_{n,n} &= M_{1,1}M_{n-1,n-1}2^{n-1}B(1,n-1),
\end{aligned}$$

and for $n \geq 3$, $2 \leq i \leq n-1$,

$$\begin{aligned}
M_{n,i} &= M_{1,0}M_{n-1,i}2^{i}B\left(\frac{1}{2},\frac{n-1+i}{2}\right) + M_{1,1}M_{n-1,i}2^{i+1}\Gamma\left(\frac{3}{2}\right)B\left(\frac{1}{2},\frac{n+i}{2}\right) \\
&\quad + M_{1,1}M_{n-1,i-1}2^{i-1}B\left(1,\frac{n-2+i}{2}\right).
\end{aligned}$$

Now assume that the lemma is true for $1, 2, \ldots, n-1$. Then

$$\begin{aligned}
\left|(\mathcal{L}Z)_n(\vec{\xi},t;\vec{\eta},\tau)\right| &= \left|\int_\tau^t \iint_{\mathbb{R}^2} (\mathcal{L}Z)(\vec{\xi},t;\vec{y},\sigma)(\mathcal{L}Z)_{n-1}(\vec{y},\sigma;\vec{\eta},\tau)d\vec{y}d\sigma\right| \\
&\leq \int_\tau^t \iint_{\mathbb{R}^2} \left(M_{1,0}(t-\sigma)^{-\frac{1}{2}} + M_{1,1}(t-\sigma)^{-1}|\vec{\xi}-\vec{y}|\right) \\
&\quad \times \sum_{i=0}^{n-1} M_{n-1,i}(\tau-\sigma)^{\frac{n-3-i}{2}}|\vec{y}-\vec{\eta}|^i Z(\vec{\xi},t;\vec{y},\sigma)Z(\vec{y},\sigma;\vec{\eta},\tau)d\vec{y}d\sigma.
\end{aligned}$$

Applying equations (B.2) and (B.3) and the definition of the series $\{M_{n,i} : n \geq 2, 0 \leq i \leq n\}$ to the above equation, we can get the conclusion. Thus we complete the proof.

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