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In this paper we propose a closed-form asymptotic expansion approach to pricing discretely monitored Asian options in general one-dimensional diffusion models. Our expansion is a small-time expansion because the expansion parameter is selected to be the square root of the length of monitoring interval. This expansion method is distinguished from many other pricing-oriented expansion algorithms in the literature because of two appealing features. First, we illustrate that it is possible to explicitly calculate not only the first several expansion terms but also any general expansion term in a systematic way. Second, the convergence of the expansion is proved rigorously under some regularity conditions. Numerical experiments suggest that the closed-form expansion formula with only a few terms (e.g., four terms up to the third order) is accurate, fast, and easy to implement for a broad range of diffusion models, even including those violating the regularity conditions.

1. Introduction. Asian options, whose payoffs depend on the arithmetic average of the underlying asset prices over a prespecified time period, are among the most popular exotic options traded actively in the financial markets. Asian options and Asian-type derivatives have a wide application in equity, currency, interest rate, energy, and commodity, due primarily to the following two attractive features. First, they can help reduce the risk of potential market manipulations by large market participants. Second, they can readily serve as appropriate hedging instruments for the firms with significant revenues collected periodically and associated with certain financial assets.

Consequently, the valuation of Asian options has attracted much attention from both researchers and practitioners. Research on the pricing of continuously monitored Asian options under various models has made significant progress. For example, under the Black-Scholes model (BSM), Linetsky [46] derived a spectral expansion for continuously monitored Asian option prices. Večeř [63] provided a numerically stable one-dimensional partial differential equation (PDE) for the Asian option price. An analytical single Laplace transform for the Asian option price under the BSM was obtained by Geman and Yor [28], and closed-form double transforms were provided by Fu et al. [22] and Fusai [23]. As a generalization of Geman and Yor [28] and Fusai [23], Cai and Kou [12] derived a closed-form double Laplace transform of the Asian option price under the hyper-exponential jump diffusion model. Fouque and Han [21] employed singular perturbation to obtain an asymptotic expansion of the continuously monitored Asian option price under the fast mean-reverting stochastic volatility model. These are only a small portion of a large volume of literature on continuously monitored Asian option pricing. For an extensive literature review, we refer to, e.g., Cai and Kou [12].

It is worth noting that most Asian options traded in the real marketplace are monitored discretely rather than continuously. Nonetheless, this "discretely monitoring" structure poses a great challenge to the associated pricing problem. Pioneering works under the BSM consist of the fast Fourier transform (FFT)-based recursive method by Carverhill and Clewlow [14] and the PDE approach via the change of numeraire by Andreasen [4]. Ju [34] proposed an accurate Taylor expansion approach (around zero volatility) to the pricing of discretely monitored Asian options and basket options under the BSM. Howison [31] developed approximations to various options with discrete structures under the BSM via the multiple timescales method. Recently a variety of sophisticated recursive algorithms have also been developed to evaluate discretely monitored Asian options under general exponential Lévy models. For instance, Fusai and Meucci [24] proposed a new numerical pricing

method combining a recursive numerical quadrature with the FFT algorithm. Fusai et al. [25] presented a numerical scheme based on maturity randomization. Černý and Kyriakou [15] suggested an improved FFT pricing algorithm that complemented the existing literature. Within the nonexponential-Lévy framework, Fusai et al. [26] derived an elegant analytical pricing formula in the Cox-Ingersoll-Ross (CIR) model and applied the results to commodity markets.

This paper aims at pricing discretely monitored Asian options under general one-dimensional diffusion models, all of which are nonexponential-Lévy models except the BSM. Specifically, we manage to derive a closed-form asymptotic expansion for the Asian option price based on the celebrated theory by Watanabe [64]. This theory developed Malliavin calculus for the so-called (generalized) "Wiener functionals," which can be roughly thought of as (generalized) functions of random variables related to Wiener processes or Brownian motions, and applied it to investigate the large-deviations-based asymptotic expansion for heat kernels. Accordingly, it can be applied naturally to evaluate discretely monitored Asian options under the diffusions because the Asian option payoffs are essentially generalized Wiener functionals. Despite the sophisticated theory, the computation for the asymptotic expansion is very much similar to the Taylor expansion of a common function and thereby can be obtained in a systematical manner by differentiating a standardized payoff function and then explicitly calculating some conditional expectations relating to normal distributions.

Note that one important issue associated with asymptotic expansion approaches is which small variable to be selected as the expansion parameter. A small variable naturally involved in discretely monitored Asian options is the length of the monitoring interval. For theoretical convenience we choose its square root as the expansion parameter. Therefore, our expansion can be viewed as a "small-time" expansion. Small-time expansions via different techniques have been widely applied in finance; see, e.g., Broadie et al. [10, 11], Hagan et al. [30], Kou [38], Andersen and Brotherton-Ratcliffe [3], and Takahashi and Yamada [61]. In the real financial markets, the monitoring interval length of discretely monitored Asian options is typically equal to 1/12 (monthly), 1/52 (weekly), and 1/250 (daily), which turns out to be small enough to make our asymptotic expansion converge quite fast. As a result, the closed-form expansion formulas up to the third order have achieved a high accuracy; see numerical results in §5.

Indeed, the applications of Malliavin-calculus-based approximations in option pricing have led to many elegant results; see, e.g., Yoshida [65], Takahashi [58, 59], Benhamou et al. [6, 7, 8], Gobet and Miri [29], Kunitomo and Takahashi [41, 42], Uchida and Yoshida [62], Shiraya and Takahashi [55], and Takahashi et al. [60]. In particular, to approximate the law of the very general average (including both continuous and discrete averages) of the marginal of diffusion processes, Gobet and Miri [29] proposed an efficient approximation with nonasymptotic error bounds and higher accuracy in the cases of small time or small volatility; Kunitomo and Takahashi [40] presented a "small-diffusion" expansion for pricing continuously monitored Asian options under the Black-Scholes model; Shiraya and Takahashi [55] derived a small-diffusion expansion formula (up to the third order) for pricing discretely monitored Asian options with either uniform or nonuniform time steps under the Heston and the λ-SABR models. Fundamentally different from the aforementioned expansions with the expansion parameters selected as auxiliary ones, our small-time expansion is based on a different parameterization, where the expansion parameter comes naturally from the option contract. In general diffusion models, our small-time expansion can lead to closed-form expansion formulas in terms of only the probability density function (pdf) and the cumulative distribution function (cdf) of the standard normal distribution, whereas the small-diffusion expansion formulas may involve some integrals. Although in many practical cases these integrals can be evaluated explicitly, they may need to be calculated numerically in some sophisticated nonlinear cases studied by, e.g., Aït-Sahalia [1] and Bakshi et al. [5]. Another key difference between our work and the literature is that we develop a systematic method to explicitly express general correction terms rather than first several ones. This is made possible because we apply the Itô-Stratonovich calculus, which offers significant computational convenience compared with the Itô calculus employed in many other approaches in the literature. Moreover, we propose a novel method for calculating a new type of conditional expectations involving iterated stochastic integrals, which is potentially useful in a wide range of studies in applied probability and stochastic modeling.

The contribution of our paper is twofold.

First, we propose a closed-form small-time expansion approach to pricing discretely monitored Asian options in general one-dimensional diffusion models, where the expansion parameter is naturally selected to be the square root of the monitoring interval length. Under some regularity conditions, we provide a rigorous proof for the convergence of the expansion formula, which, however, seems unavailable for many existing expansion methods for option pricing.

Second, we develop a systematic method for the explicit calculation of the general correction terms of the asymptotic expansion pricing formula up to an arbitrary order in general one-dimensional diffusion models. Moreover, this systematic method can be implemented via any symbolic computation package such as

TABLE 1.   Some popular asset pricing models nested in the class of one-dimensional diffusion models.

	$\mu(x)$	$\sigma(x)$
The Black-Scholes-Merton (BSM) model	$(r-q)x \; (r>0, q \geq 0)$	$\sigma x \; (\sigma > 0)$
The Brennan-Schwartz model	$\kappa(\theta - x) \; (\kappa > 0, \theta \in \mathbb{R})$	$\sigma x \; (\sigma > 0)$
The Cox-Ingersoll-Ross (CIR) model	$\kappa(\theta - x) \; (\kappa > 0, \theta \in \mathbb{R})$	$\delta \sqrt{x} \; (\delta > 0)$
The constant elasticity of volatility (CEV) model	$rx \; (r > 0)$	$\delta x^{\beta+1} \; (\delta > 0, \beta \in \mathbb{R})$

Mathematica. (It seems that most expansion pricing methods in the literature usually discuss only the explicit computations of the first several terms rather than general terms because this is always challenging, if not impossible.) In particular, we explicitly provide the closed-form expressions for the first four terms. Numerical experiments suggest that the corresponding expansion formula up to the third order performs very well for a broad range of diffusion models, including not only those satisfying these regularity conditions such as the BSM and the Brennan-Schwartz process, but also those violating them, e.g., the CIR model (see Cox and Ross [17]) and the general CEV models (see, e.g., Cox [16], Linetsky [45], Linetsky and Mendoza [47]).

Furthermore, numerical results demonstrate several appealing features of our expansion formula up to the third order:

(I) It is highly accurate and performs consistently well for a wide range of model parameters and contract parameters.

(II) It usually takes less than 0.5 seconds to generate one numerical result. This is mainly because the expansion has a closed-form expression only in terms of the standard normal pdf and cdf.

(III) The closed-form expansion formula is simple to implement. Despite the seemingly complicated expression, the expansion formula consists only of the standard normal pdf and cdf. No other complex numerical procedures such as Fourier or Laplace transform inversions and numerical integrations are involved in the implementation.

(IV) Our pricing method can also be applied to accurately evaluate hedging parameters such as delta and gamma. Indeed, closed-form approximations to these Greeks can be obtained simply by differentiating the closed-form expansion pricing formula in a straightforward way.

The remainder of the paper is organized as follows. Section 2 presents our main theoretical results about the small-time asymptotic expansion of discretely monitored Asian option price. Section 3 exemplifies the explicit calculation of closed-form expansion formulas using the first four terms under the BSM model, and §4 provides a systematic method to compute explicitly the expansion formulas up to an arbitrary order in general diffusions. Numerical results are given in §5. Most proofs are deferred to the appendices.

2. The main result.

2.1. The model and discretely monitored Asian options.   Consider an asset pricing model $\{S(t): t \geq 0\}$, which follows a one-dimensional diffusion governed by the following stochastic differential equation (SDE) under a risk-neutral measure P:

$$dS(t) = \mu(S(t)) \, dt + \sigma(S(t)) \, dW(t), \quad \text{with } S(0) = s_0 > 0, \tag{1}$$

where the functions $\sigma(x) > 0$ and $\mu(x)$ are continuous for $x \in (0, +\infty)$ and $\{W(t): t \geq 0\}$ is a standard Brownian motion. The class of one-dimensional diffusion models nests a variety of popular asset pricing models; see Table 1. It is worth pointing out that if $\{S(t)\}$ is the price process of a traded asset, then the drift $\mu(S(t))$ under the risk-neutral measure P must be of the form $rS(t)$ or $(r-q)S(t)$, where r is the risk free interest rate and d the dividend yield; see, e.g., the BSM model and the CEV model in Table 1. Nonetheless, if $\{S(t)\}$ corresponds to an asset that cannot be directly traded, the drift $\mu(S(t))$ under the risk-neutral measure P is not necessary to take the form of $rS(t)$ or $(r-q)S(t)$ because the discounted price process does not have to be a martingale under the risk-neutral measure. For instance, the Brennan-Schwartz model and the CIR model in Table 1 can be used to model the spot price of a commodity, which is usually not traded directly, and therefore, their drifts can be of the mean-reverting type under the risk-neutral measure. Note that the commodity futures price, however, is a martingale under the risk-neutral measure because the futures contract is assumed to be a traded asset. For more discussions on the spot prices and future prices of the commodities, we refer to, e.g., Li and Linetsky [44], Schwartz [54], and Geman [27].

The price of a discretely monitored Asian call option at time 0 is given by

$$C(\Delta) = E\left[e^{-rT} \left(\frac{1}{m+1} \sum_{j=0}^{m} S(j\Delta) - K \right)^{+} \right], \tag{2}$$

where r is the risk-free interest rate, T the maturity, K the strike price, and the $m+1$ monitoring dates are equally spaced: $0, \Delta, 2\Delta, \dots, m\Delta \equiv T$, with Δ denoting the length of monitoring interval.

2.2. The selection of expansion parameter. The main objective of this paper is to obtain a closed-form asymptotic expansion for the discretely monitored Asian call option price (the Asian put options can be dealt with similarly) under the general diffusion model (1). The first important issue is the selection of an expansion parameter ϵ. For pricing European type options, various effective applications of the so-called small-time expansions have been proposed by choosing the option maturity as a parameter to expand; see, e.g., Andersen and Brotherton-Ratcliffe [3] and Hagan et al. [30].

In this paper, we select the expansion parameter to be the square root of the monitoring interval length (i.e., $\epsilon = \sqrt{\Delta}$) rather than the option maturity. Note that such a choice naturally comes from the contract parameter of the discretely monitored Asian option instead of auxiliary assignments used in the small-diffusion expansions. Furthermore, the typical values of $\sqrt{\Delta}$ in the marketplace such as $\sqrt{1/12}$ (monthly monitored), $\sqrt{1/52}$ (weekly monitored), and $\sqrt{1/250}$ (daily monitored) turn out to be sufficiently small to guarantee a fast convergence of the associated asymptotic expansions. As a result, the expansion pricing formula with only a few terms, e.g., four terms up to the third order, can achieve a high accuracy; see the numerical examples in §5.

2.3. The main result. Before presenting our main result, Theorem 2.1, and proving its validity, we first provide a heuristic approach to the main result to motivate readers by the intuition behind the scenes. Define

$$X(t, \epsilon) := S(\epsilon^2 t), \quad \text{with } \epsilon \equiv \sqrt{\Delta}.$$

Then by (1) we obtain

$$X(t, \epsilon) = s_0 + \epsilon^2 \int_0^t \mu(X(u, \epsilon))\, du + \epsilon \int_0^t \sigma(X(u, \epsilon))\, dW(u, \epsilon), \tag{3}$$

where $W(t, \epsilon) := (1/\epsilon)W(\epsilon^2 t)$ is a standard Brownian motion adapted to the filtration generated by $\{W(\epsilon^2 t): t \geq 0\}$. Without any confusion we write $W(t, \epsilon)$ as $W(t)$ in what follows. For ease of exposition, define

$$B_i := \frac{1}{\sqrt{i}} W(i), \quad \text{for } i = 1, 2, \dots, m. \tag{4}$$

Then $(B_1, B_2, \dots, B_m)$ has a multivariate normal distribution, and a direct calculation yields

$$\rho_{ij} := \mathrm{Corr}(B_i, B_j) = \sqrt{\min(i, j)/\max(i, j)}, \quad \text{for } i, j = 1, \dots, m. \tag{5}$$

Enlightened by the asymptotic expansion procedure proposed in Watanabe [64], we standardize $X(k, \epsilon)$ in the following way:

$$Y_k(\epsilon) := \frac{X(k, \epsilon) - s_0}{\epsilon \sigma(s_0)\gamma}, \quad \text{for } k = 1, 2, \dots, m, \tag{6}$$

where the constant $\gamma > 0$ will be specified later for ease of computation. Note that the goal of such a standardization is to guarantee the convergence of our expansion. Then the Asian call option price $C(\Delta)$ in (2) can be expressed in terms of $\{Y_k(\epsilon): k = 1, \dots, m\}$:

$$C(\Delta) = \frac{e^{-rT}\sqrt{\Delta}\sigma(s_0)\gamma}{m+1} E\left[\left(\sum_{k=1}^{m} Y_k(\epsilon) - z \right)^{+} \right], \quad \text{with } z := \frac{(m+1)(K - s_0)}{\sqrt{\Delta}\sigma(s_0)\gamma}. \tag{7}$$

To obtain an asymptotic expansion for the Asian call option price $C(\Delta)$ in (7), we take the following four steps heuristically.

- *Step* 1. Derive a power-series expansion for $X(k, \epsilon)$ around $\epsilon = 0$. For any $J = 0, 1, 2, \dots$,

$$X(k, \epsilon) = \sum_{j=0}^{J} F_{k, j} \epsilon^j + O(\epsilon^{J+1}), \quad \text{with } F_{k, 0} \equiv s_0 \text{ for } k = 1, \dots, m. \tag{8}$$

This can be done intuitively by regarding $X(k, \epsilon)$ as a function of ϵ and then taking a Taylor expansion.

- *Step* 2. It follows from (6) and (8) that a power-series expansion for $Y_k(\epsilon)$ around $\epsilon = 0$ is

$$Y_k(\epsilon) = \sum_{j=0}^{J} Y_{k,j}\epsilon^j + O(\epsilon^{J+1}), \quad \text{with } Y_{k,j} := \frac{F_{k,j+1}}{\sigma(s_0)\gamma} \text{ for } k = 1, \ldots, m. \tag{9}$$

- *Step* 3. Derive a power-series expansion for $G(\epsilon) := (\sum_{k=1}^{m} Y_k(\epsilon) - z)^+$ around $\epsilon = 0$ intuitively by the chain rule and Taylor expansion.

$$G(\epsilon) = \sum_{j=0}^{J} \Phi_j(z)\epsilon^j + O(\epsilon^{J+1}), \tag{10}$$

where the first four coefficients are explicitly calculated as

$$\begin{cases} \Phi_0(z) = (Z_0 - z)^+, \qquad \Phi_1(z) = \sum_{i=1}^{m} Y_{i,1} 1_{\{Z_0 \geq z\}}, \\ \Phi_2(z) = \sum_{i=1}^{m} Y_{i,2} 1_{\{Z_0 \geq z\}} + \frac{1}{2!} \sum_{i,j=1}^{m} Y_{i,1} Y_{j,1} \delta(Z_0 - z), \\ \Phi_3(z) = \sum_{i=1}^{m} Y_{i,3} 1_{\{Z_0 \geq z\}} + \frac{2}{2!} \sum_{i,j=1}^{m} Y_{i,1} Y_{j,2} \delta(Z_0 - z) + \frac{1}{3!} \sum_{i,j,k=1}^{m} Y_{i,1} Y_{j,1} Y_{k,1} \delta'(Z_0 - z). \end{cases} \tag{11}$$

Here $\delta(\cdot)$ denotes the Dirac delta function and

$$Z_0 := \sum_{k=1}^{m} Y_{k,0}. \tag{12}$$

- *Step* 4. Taking expectations on both sides of (10) and plugging the result into (7) yields

$$C(\Delta) = \frac{e^{-rT}\gamma\sqrt{\Delta}\sigma(s_0)}{m+1} \left(\sum_{j=0}^{J} \Omega_j(z)\epsilon^j + O(\epsilon^{J+1}) \right), \quad \text{with } \Omega_j(z) := E\Phi_j(z) \text{ for } j = 0, \ldots, J.$$

REMARK 2.1. It is worth mentioning that although the above asymptotic expansion of $C(\Delta)$ and the intermediate results (8)–(10) were derived heuristically, their validity can all be justified rigorously in the sense of expansions for random variables via the theory of Watanabe [64]; see the following Theorem 2.1 and its proof given in Appendix A via the Malliavin calculus for generalized random variables proposed in Watanabe [64] and Yoshida [65] as well as the related theory of asymptotic expansion for option pricing developed in Kunitomo and Takahashi [42].

THEOREM 2.1. *Assume that $\sigma(s_0) \neq 0$ and the two functions $\mu(\cdot)$ and $\sigma(\cdot)$ have bounded derivatives of all orders. For any $J = 0, 1, 2, \ldots$, the discretely monitored Asian option price (7) (or (2)) admits the following asymptotic expansion in the sense of classical calculus*

$$C(\Delta) = \frac{e^{-rT}\gamma\sqrt{\Delta}\sigma(s_0)}{m+1} \left(\sum_{j=0}^{J} \Omega_j(z)\Delta^{j/2} + O(\Delta^{(J+1)/2}) \right), \tag{13}$$

where m is the number of monitoring intervals, $\Delta = T/m$ is the length of monitoring intervals, and

$$\Omega_j(z) = E\Phi_j(z), \quad \text{for } j = 0, \ldots, J, \tag{14}$$

with $\Phi_j(z)$, $j = 0, 1, 2, \ldots$, given by (8)–(10).

PROOF. See Appendix A.

REMARK 2.2. The coefficients $\Omega_j(z)$ in (13) can be calculated through (8)–(10) and (14). Note that most existing expansion methods for option pricing focus on the derivation of the first several expansion terms rather than discuss the explicit calculation of general terms. This is because the calculation of higher order terms becomes much more complicated, and it is usually very challenging, if not impossible, to provide a systematic approach to the calculation of general terms even for European type options. In contrast, for our expansion method we manage to illustrate how to explicitly calculate all the coefficients (i.e., the general expansion terms) in a systematic way via only basic mathematical operations (without recursions and numerical integrations involved) under the general one-dimensional diffusions; see §4.3. This is made possible mainly because of the

convenience offered by our particular parameterization (3) and the computational advantage rendered by the application of the Itô-Stratonovich calculus that resembles the classical calculus. However, for other expansion methods, the associated parameterizations and computational methods may not lead to explicit results. For example, the expansion in Takahashi et al. [60] uses a different parameterization (i.e., the small diffusion) and applies the Itô calculus in the computation. As a result, their calculation relies on recursion-based algorithms and (multidimensional) numerical integrations.

REMARK 2.3. Another appealing feature of our expansion method is that, under some regularity conditions, we provide a rigorous proof for its validity, which is unavailable for many existing expansion methods for option pricing. It is worth pointing out that these technical regularity conditions on the functions $\mu(\cdot)$ and $\sigma(\cdot)$ in Theorem 2.1 are conventionally proposed for the study of diffusion models; see, e.g., the monographs by Ikeda and Watanabe [32] and Nualart [50]. It is well known that relaxation on these conditions would pose a great technical challenge for theoretical verification of the validity. However, these conditions are sufficient but not necessary. As we shall see in §5, numerical examples suggest that our expansion method is not limited to these sufficient conditions but applicable in a wide range of models.

3. An illustrative example: The Black-Scholes model (BSM).

Before discussing the explicit computations of the general expansion pricing formula up to an arbitrary order in the general diffusion model in §4, we first use a simple model—the BSM—as an example to illustrate how to explicitly compute the first four coefficients $\Omega_j(z)$ for $j = 0, 1, \ldots, 4$ in (14), or equivalently the expansion pricing formula up to the third order, in an easy way.

As we shall see, the first four coefficients, and in fact, all $\Omega_j(z)$'s, only involve the standard normal pdf and cdf, and so does the expansion formula. Numerical experiments demonstrate that the resulting closed-form expansion pricing formula up to the third order is highly accurate; see §5.1.

Under the BSM specified in Table 1,

$$X(k, \epsilon) = s_0 \exp(\sigma \epsilon W(k) + \epsilon^2 bk), \tag{15}$$

where $b := r - q - \frac{1}{2}\sigma^2$. From (11) and (14) we know that the first four coefficients are given by

$$
\begin{cases}
\Omega_0(z) = E(Z_0 - z) \\
\Omega_1(z) = \sum_{i=1}^{m} E[Y_{i,1} 1_{\{Z_0 \geq z\}}] \\
\Omega_2(z) = \sum_{i=1}^{m} E[Y_{i,2} 1_{\{Z_0 \geq z\}}] + \frac{1}{2!} \sum_{i,j=1}^{m} E[Y_{i,1} Y_{j,1} \delta(Z_0 - z)] \\
\Omega_3(z) = \sum_{i=1}^{m} E[Y_{i,3} 1_{\{Z_0 \geq z\}}] + \frac{2}{2!} \sum_{i,j=1}^{m} E[Y_{i,1} Y_{j,2} \delta(Z_0 - z)] + \frac{1}{3!} \sum_{i,j,k=1}^{m} E[Y_{i,1} Y_{j,1} Y_{k,1} \delta'(Z_0 - z)],
\end{cases} \tag{16}
$$

where $Z_0 := \sum_{k=1}^{m} Y_{k,0}$ and

$$
\begin{cases}
Y_{k,0} = \frac{1}{\gamma} W(k) = \frac{\sqrt{k}}{\gamma} B_k \\
Y_{k,1} = \frac{1}{2\gamma}\left(\sigma W^2(k) + \frac{2bk}{\sigma}\right) = \frac{k}{2\gamma}\left(\sigma B_k^2 + \frac{2b}{\sigma}\right) \\
Y_{k,2} = \frac{1}{6\gamma}(\sigma^2 W^3(k) + 6bk W(k)) = \frac{k\sqrt{k}}{6\gamma}(\sigma^2 B_k^3 + 6b B_k) \\
Y_{k,3} = \frac{1}{24\gamma}\left(\sigma^3 W^4(k) + 12b\sigma k W(k)^2 + \frac{12b^2 k^2}{\sigma}\right) = \frac{k^2}{24\gamma}\left(\sigma^3 B_k^4 + 12b\sigma B_k^2 + \frac{12b^2}{\sigma}\right),
\end{cases} \tag{17}
$$

which are calculated by first differentiating $X(t, \epsilon)$ with respect to (w.r.t.) ϵ to obtain $F_{k,j}$ for $j = 1, \ldots, 4$ and then substituting them into (9). For computational convenience, we choose

$$\gamma = \sqrt{\text{Var}\left(\sum_{k=1}^{m} W(k)\right)} = \sqrt{\frac{m(m+1)(2m+1)}{6}}, \tag{18}$$

such that $Z_0 = \sum_{k=1}^{m} Y_{k,0} = \sum_{k=1}^{m} W(k)/\gamma$ has a standard normal distribution.

First, $\Omega_0(z)$ is easy to compute as follows:

$$\Omega_0(z) = E(Z_0 - z)^+ = \int_{-\infty}^{\infty} (u - z)^+ \phi(u)\, du = zN(z) + \phi(z) - z, \tag{19}$$

where $\phi(\cdot)$ and $N(\cdot)$ denote the standard normal pdf and cdf, respectively.

Second, we can see from (16) that the calculation of $\Omega_1(z)$, $\Omega_2(z)$, and $\Omega_3(z)$ reduces to the computation of the following three types of expectations.

Type 1: $E[Y_{i,1} 1_{\{Z_0 \geq z\}}]$, $E[Y_{i,2} 1_{\{Z_0 \geq z\}}]$, $E[Y_{i,3} 1_{\{Z_0 \geq z\}}]$;

Type 2: $E[Y_{i,1} Y_{j,1} \delta(Z_0 - z)]$, $E[Y_{i,1} Y_{j,2} \delta(Z_0 - z)]$;

Type 3: $E[Y_{i,1} Y_{j,1} Y_{k,1} \delta'(Z_0 - z)]$.

Conditional on Z_0, the above three types of expectations can be evaluated as follows:

$$\text{Type 1:} \quad E[Y_{i,j} 1_{\{Z_0 \geq z\}}] = E[1_{\{Z_0 \geq z\}} E(Y_{i,j} \mid Z_0)]$$
$$= \int_z^{\infty} E[Y_{i,j} \mid Z_0 = z] \phi(z)\, dz, \quad \text{for } j = 1, 2, 3; \tag{20}$$

$$\text{Type 2:} \quad E[Y_{i,1} Y_{j,l} \delta(Z_0 - z)] = \int_{-\infty}^{\infty} \delta(u - z) E[Y_{i,1} Y_{j,l} \mid Z_0 = u] \phi(u)\, du$$
$$= E[Y_{i,1} Y_{j,l} \mid Z_0 = z] \phi(z), \quad \text{for } l = 1, 2; \tag{21}$$

$$\text{Type 3:} \quad E[Y_{i,1} Y_{j,1} Y_{k,1} \delta'(Z_0 - z)] = \int_{-\infty}^{\infty} \delta'(u - z) E[Y_{i,1} Y_{j,1} Y_{k,1} \mid Z_0 = u] \phi(u)\, du$$
$$= -\int_{-\infty}^{\infty} \delta(u - z) \frac{\partial}{\partial u} \{E[Y_{i,1} Y_{j,1} Y_{k,1} \mid Z_0 = u] \phi(u)\}\, du$$
$$= -\frac{\partial}{\partial z} \{E[Y_{i,1} Y_{j,1} Y_{k,1} \mid Z_0 = z] \phi(z)\}, \tag{22}$$

where the derivations of (21) and (22) are based on the properties of the Dirac delta function (see Kanwal [35]).

Since $Y_{i,j}$ is a polynomial in B_i for any $i = 1, \ldots, m$ and $j = 1, \ldots, 4$ (see (17)), we conclude from (20)–(22) that the calculation of $\Omega_j(z)$ for $j = 1, 2, 3$ is reduced to explicit computations of the conditional cross-moment

$$M_{(i_1, i_2, \ldots, i_s)}^{(p_1, p_2, \ldots, p_s)}(z) := E\left(B_{i_1}^{p_1} B_{i_2}^{p_2}, \ldots, B_{i_s}^{p_s} \mid Z_0 = z\right), \tag{23}$$

its integration

$$R_{(i_1, i_2, \ldots, i_s)}^{(p_1, p_2, \ldots, p_s)}(z) := \int_z^{\infty} M_{(i_1, i_2, \ldots, i_s)}^{(p_1, p_2, \ldots, p_s)}(u) \phi(u)\, du, \tag{24}$$

and its differentiation

$$Q_{(i_1, i_2, \ldots, i_s)}^{(p_1, p_2, \ldots, p_s)}(z) := -\frac{\partial}{\partial z}\left[M_{(i_1, i_2, \ldots, i_s)}^{(p_1, p_2, \ldots, p_s)}(z) \phi(z)\right], \tag{25}$$

where $1 \leq i_1, i_2, \ldots, i_s \leq m$ and $s, p_1, p_2, \ldots, p_s \geq 1$ are all integers. Indeed, we point out that this is true not only for the first four coefficients but also for general ones.

Note that the random vector $(B_1, B_2, \ldots, B_m, Z_0)$ has a multivariate normal distribution. Thus the conditional distribution of $(B_1, B_2, \ldots, B_m)$ given $Z_0 = z$ is also normal, and the corresponding conditional moment generating function has a closed-form expression as follows:

$$\varphi(\vartheta_1, \ldots, \vartheta_m; z) := E\left(e^{\sum_{k=1}^m \vartheta_k B_k} \mid Z_0 = z\right) = \exp\left(\sum_{k=1}^m \vartheta_k \rho_k z + \frac{1}{2} \sum_{i,j=1}^m \vartheta_i \vartheta_j (\rho_{ij} - \rho_i \rho_j)\right),$$

where ρ_{ij}'s are defined in (5) and

$$\rho_k := \text{Corr}(Z_0, B_k) = \frac{1}{2\gamma}(2m - k + 1)\sqrt{k}, \quad \text{for } k = 1, \ldots, m.$$

Therefore, we can derive $M_{(i_1, i_2, \ldots, i_s)}^{(p_1, p_2, \ldots, p_s)}(z)$ explicitly by differentiating $\varphi(\vartheta_1, \vartheta_2, \ldots, \vartheta_m; z)$ at $(\vartheta_1, \vartheta_2, \ldots, \vartheta_m) = (0, 0, \ldots, 0)$. As a by-product, it can be seen that $M_{(i_1, i_2, \ldots, i_s)}^{(p_1, p_2, \ldots, p_s)}(z)$ is a polynomial in z with order $p := p_1 + p_2 + \cdots + p_s$. We summarize these results in the following lemma.

LEMMA 3.1. *The conditional cross-moment* $M_{(i_1, i_2, \ldots, i_s)}^{(p_1, p_2, \ldots, p_s)}(z)$ *defined in* (23) *is a polynomial in z with order* $p := p_1 + p_2 + \cdots + p_s$:

$$M_{(i_1, i_2, \ldots, i_s)}^{(p_1, p_2, \ldots, p_s)}(z) = \sum_{n=0}^{p} a_n z^n \equiv \left. \frac{\partial^p [\varphi(\vartheta_1, \vartheta_2, \ldots, \vartheta_m; z)]}{\partial \vartheta_{i_1}^{p_1} \partial \vartheta_{i_2}^{p_2}, \ldots, \partial \vartheta_{i_s}^{p_s}} \right|_{\vartheta_1 = \vartheta_2 = \cdots = \vartheta_m = 0}. \tag{26}$$

REMARK 3.1. To explicitly compute the coefficients $\{a_n\}$ in (26), we need to differentiate $\varphi(\vartheta_1, \vartheta_2, \ldots, \vartheta_m; z)$ at $(\vartheta_1, \vartheta_2, \ldots, \vartheta_m) = (0, 0, \ldots, 0)$. Indeed, this can be done in a simple manner via any symbolic computation package such as *Mathematica*.

REMARK 3.2. To calculate $\Omega_1(z)$, $\Omega_2(z)$, and $\Omega_3(z)$, we provide explicit expressions of all related $M_{(i_1, i_2, \ldots, i_s)}^{(p_1, p_2, \ldots, p_s)}(z)$ as follows:

$$\begin{cases}
M_{(i)}^{(1)}(z) = E(B_i \mid Z_0 = z) = \rho_i z, \\
M_{(i)}^{(2)}(z) = E(B_i^2 \mid Z_0 = z) = \rho_i^2 z^2 + \bar{\rho}_i^2, \\
M_{(i)}^{(3)}(z) = E(B_i^3 \mid Z_0 = z) = \rho_i^3 z^3 + 3\rho_i \bar{\rho}_i^2 z, \\
M_{(i, j)}^{(1, 2)}(z) = E(B_i B_j^2 \mid Z_0 = z) = \rho_i \rho_j^2 z^3 + \rho_i \bar{\rho}_j^2 z + 2\rho_j r_{ij} z, \\
M_{(i, j)}^{(2, 2)}(z) = E(B_i^2 B_j^2 \mid Z_0 = z) = 2r_{ij}^2 + (\rho_i^2 z^2 + \bar{\rho}_i^2)(\rho_j^2 z^2 + \bar{\rho}_j^2) + 4r_{ij}\rho_i \rho_j z^2, \\
M_{(i, j)}^{(3, 2)}(z) = E(B_i^3 B_j^2 \mid Z_0 = z) = \rho_i^3 \rho_j^2 z^5 + (\bar{\rho}_i^2 \rho_j^3 + 6\rho_i^2 \rho_j r_{ij} + 3\rho_i \bar{\rho}_i^2 \rho_j^2) z^3 \\
\qquad\qquad\qquad + (3\rho_i \bar{\rho}_i^2 \bar{\rho}_j^2 + 6\rho_i r_{ij}^2 + 6\rho_j \bar{\rho}_i^2 r_{ij}) z, \\
M_{(i, j, k)}^{(2, 2, 2)}(z) = E(B_i^2 B_j^2 B_k^2 \mid Z_0 = z) \\
\qquad = \rho_i^2 \rho_j^2 \rho_k^2 z^6 + (\rho_i^2 \rho_j^2 \bar{\rho}_k^2 + \rho_i^2 \bar{\rho}_j^2 \rho_k^2 + \bar{\rho}_i^2 \rho_j^2 \rho_k^2 + 4\rho_i \rho_j \rho_k (\rho_i r_{jk} + \rho_j r_{ik} + \rho_k r_{ij})) z^4 \\
\qquad\quad + (\rho_i^2 \bar{\rho}_j^2 \bar{\rho}_k^2 + \bar{\rho}_i^2 \rho_j^2 \bar{\rho}_k^2 + \bar{\rho}_i^2 \bar{\rho}_j^2 \rho_k^2 + 2\rho_i^2 r_{jk}^2 + 2\rho_j^2 r_{ik}^2 + 2\rho_k^2 r_{ij}^2 + 8\rho_i \rho_j r_{ik} r_{jk} \\
\qquad\quad + 8\rho_j \rho_k r_{ij} r_{jk} + 8\rho_i \rho_k r_{ij} r_{ik} + 4\rho_i \rho_j \bar{\rho}_k^2 r_{ij} + 4\rho_i \bar{\rho}_j^2 \rho_k r_{ik} + 4\bar{\rho}_i^2 \rho_j \rho_k r_{jk}) z^2 \\
\qquad\quad + (\bar{\rho}_i^2 \bar{\rho}_j^2 \bar{\rho}_k^2 + 2\bar{\rho}_i^2 r_{jk}^2 + 2\bar{\rho}_j^2 r_{ik}^2 + 2\bar{\rho}_k^2 r_{ij}^2 + 8r_{ij} r_{ik} r_{jk}),
\end{cases} \tag{27}$$

where

$$r_{ij} := \rho_{ij} - \rho_i \rho_j \qquad \text{and} \qquad \bar{\rho}_i := \sqrt{1 - \rho_i^2}. \tag{28}$$

Now let us turn to $R_{(i_1, i_2, \ldots, i_s)}^{(p_1, p_2, \ldots, p_s)}(z)$ and $Q_{(i_1, i_2, \ldots, i_s)}^{(p_1, p_2, \ldots, p_s)}(z)$ defined in (24) and (25), respectively. Before addressing how to compute them, we first present a lemma.

LEMMA 3.2. *Define* $P_n(x) := \int_x^\infty u^n \phi(u) \, du$. *Then* $\{P_n(x): n \geq 0\}$ *can be computed recursively*:

$$\begin{aligned}
P_0(x) &= 1 - N(x), \\
P_1(x) &= \phi(x), \\
P_n(x) &= x^{n-1} \phi(x) + (n-1) P_{n-2}(x), \quad \text{for } n = 2, 3, \ldots.
\end{aligned} \tag{29}$$

PROOF. $P_0(x)$ and $P_1(x)$ can be derived via a straightforward calculation. If $n \geq 2$, the recursion (29) can be obtained simply by integration by parts. □

Based on Lemmas 3.1 and 3.2, we have the following result.

LEMMA 3.3. $R_{(i_1, i_2, \ldots, i_s)}^{(p_1, p_2, \ldots, p_s)}(z)$ *and* $Q_{(i_1, i_2, \ldots, i_s)}^{(p_1, p_2, \ldots, p_s)}(z)$ *defined in* (24) *and* (25) *are given by*

$$R_{(i_1, i_2, \ldots, i_s)}^{(p_1, p_2, \ldots, p_s)}(z) = \sum_{n=0}^{p} a_n P_n(z), \tag{30}$$

$$Q_{(i_1, i_2, \ldots, i_s)}^{(p_1, p_2, \ldots, p_s)}(z) = \left[a_0 z + \sum_{n=1}^{p} a_n (z^{n+1} - n z^{n-1}) \right] \phi(z), \tag{31}$$

where a_n's *are the same as in Lemma* 3.1 *and* $P_n(\cdot)$ *are defined in* (29).

PROOF. Substituting (26) into (24) and (25) and applying Lemma 3.2 yields the results immediately. □

Now we are ready to present the closed-from expansion pricing formula up to the third order.

THEOREM 3.1. *In the BSM, the expansion* (13) *for the discretely monitored Asian option price holds. In particular, the asymptotic expansion pricing formula up to the third order is given by*

$$C(\Delta) = \frac{e^{-rT}\sqrt{\Delta}\sigma(s_0)\gamma}{m+1}\left(\Omega_0(z) + \Omega_1(z)\Delta^{1/2} + \Omega_2(z)\Delta + \Omega_3(z)\Delta^{3/2} + O(\Delta^2)\right), \quad as \ \Delta \to 0, \quad (32)$$

where the coefficients $\Omega_i(z)$ *for* $i = 0, 1, 2, 3$ *are explicitly calculated as follows*

$$\Omega_0(z) = zN(z) + \phi(z) - z,$$

$$\Omega_1(z) = \frac{\sigma}{2\gamma}\sum_{i=1}^{m} iR_{(i)}^{(2)}(z) + \frac{bm(m+1)}{2\sigma\gamma}(1 - N(z)),$$

$$\Omega_2(z) = \sum_{i=1}^{m} IIa_i + \sum_{i,j=1}^{m} IIb_{ij},$$

$$\Omega_3(z) = \sum_{i=1}^{m} IIIa_i + \sum_{i,j=1}^{m} IIIb_{ij} + \sum_{i,j,k=1}^{m} IIIc_{ijk}.$$

Here

$$IIa_i = \frac{i^{3/2}}{\gamma}\left(\frac{1}{6}\sigma^2 R_{(i)}^{(3)}(z) + bR_{(i)}^{(1)}(z)\right),$$

$$IIb_{ij} = \frac{ij}{2\gamma^2}\phi(z)\left(\frac{\sigma^2}{4}M_{(i,j)}^{(2,2)}(z) + \frac{b}{2}M_{(i)}^{(2)}(z) + \frac{b}{2}M_{(j)}^{(2)}(z) + \frac{b^2}{\sigma^2}\right);$$

$$IIIa_i = \frac{i^2}{\gamma}\left(\frac{\sigma^3}{24}R_{(i)}^{(4)}(z) + \frac{b\sigma}{2}R_{(i)}^{(2)}(z) + \frac{b^2}{2\sigma}R_{(i)}^{(0)}(z)\right),$$

$$IIIb_{ij} = \frac{i^{3/2}j}{\gamma^2}\phi(z)\left(\frac{\sigma^3}{12}M_{(i,j)}^{(3,2)}(z) + \frac{b\sigma}{2}M_{(i,j)}^{(1,2)}(z) + \frac{b\sigma}{6}M_{(i)}^{(3)}(z) + \frac{b^2}{\sigma}M_{(i)}^{(1)}(z)\right),$$

$$IIIc_{ijk} = \frac{ijk}{6\gamma^3}\phi(z)\left(\frac{\sigma^3}{8}Q_{(i,j,k)}^{(2,2,2)}(z) + \frac{b\sigma}{4}(Q_{(i,j)}^{(2,2)}(z) + Q_{(j,k)}^{(2,2)}(z) + Q_{(i,k)}^{(2,2)}(z))\right.$$

$$\left. + \frac{b^2}{2\sigma}(Q_{(i)}^{(2)}(z) + Q_{(j)}^{(2)}(z) + Q_{(k)}^{(2)}(z)) + \frac{b^3 z}{\sigma^3}\right),$$

and all the involved $M_{(i_1,i_2,\ldots,i_s)}^{(p_1,p_2,\ldots,p_s)}(z)$ *are explicitly given by* (27). *Moreover, all the involved* $R_{(i_1,i_2,\ldots,i_s)}^{(p_1,p_2,\ldots,p_s)}(z)$ *and* $Q_{(i_1,i_2,\ldots,i_s)}^{(p_1,p_2,\ldots,p_s)}(z)$ *depend on those* $M_{(i_1,i_2,\ldots,i_s)}^{(p_1,p_2,\ldots,p_s)}(z)$ *in* (27) *through* (30) *and* (31).

PROOF. We note that the drift and volatility functions under the BSM have bounded derivatives of all orders. Therefore, Theorem 2.1 guarantees the convergence of the expansion pricing formula (13) and in particular (32). The closed-form expression of $\Omega_0(z)$ has been derived in (19). As regards $\Omega_1(z)$, $\Omega_2(z)$, and $\Omega_3(z)$, substituting (17) into (20)–(22) and then plugging the results into (16) yields immediately their closed-form expansions in terms of $M_{(i_1,i_2,\ldots,i_s)}^{(p_1,p_2,\ldots,p_s)}(z)$, $R_{(i_1,i_2,\ldots,i_s)}^{(p_1,p_2,\ldots,p_s)}(z)$ and $Q_{(i_1,i_2,\ldots,i_s)}^{(p_1,p_2,\ldots,p_s)}(z)$. □

4. Closed-form asymptotic expansions of discretely monitored Asian option prices in general one-dimensional diffusions. In this section, we shall illustrate that following the road map in §2, it is possible to derive the closed-form expansion pricing formulas (13) up to any order $J \in \mathbb{N}$ under the general one-dimensional diffusion models by explicitly computing related general coefficients $\Omega_j(z)$ for $j = 0, 1, \ldots, J$ defined in (14). The computation procedure can be regarded as a generalization of that under the BSM as discussed in §3.

By (10) and (14) we obtain

$$\Omega_0(z) = E\Phi_0(z), \quad with \ \Phi_0(z) = (Z_0 - z)^+$$

and

$$\Omega_k(z) = E\Phi_k(z), \quad with \ \Phi_k(z) = \sum_{(l,(i_1,i_2,\ldots,i_l),(j_1,j_2,\ldots,j_l))\in\mathscr{S}_k} \frac{1}{l!}\frac{\partial^{(l)}T(Z_0)}{\partial x^l}Y_{i_1,j_1}Y_{i_2,j_2},\ldots,Y_{i_l,j_l}, \quad for \ k \geq 1, \quad (33)$$

where $\Phi_k(z)$ for $k \geq 1$ are derived via the chain rule of differentiation, $T(\cdot)$ is a function defined as

$$T(x) := (x - z)^+,$$

and the index set $\mathscr{S}_k$ is specified as

$$\mathscr{S}_k = \big\{ (l, \mathbf{i}, \mathbf{j}) \mid l \in \mathbb{N}, \mathbf{i} = (i_1, i_2, \ldots, i_l) \in \{1, 2, \ldots, m\}^l, \mathbf{j} = (j_1, j_2, \ldots, j_l)$$
$$\text{with } j_i \geq 1 \text{ for } i = 1, \ldots, l \text{ and } j_1 + j_2 + \cdots + j_l = k \big\}. \tag{34}$$

To calculate $Y_{k,j}$, we need to differentiate $X(k, \epsilon) \equiv S(\epsilon^2 i)$ w.r.t. ϵ to obtain $F_{k,j}$ for $j \in \mathbb{N}$ and $k = 1, \ldots, m$ and then substitute the results into (9).

Unlike the BSM, the SDE (1) for $S(t)$ in the general diffusion may not have an analytical solution. Therefore, $X(k, \epsilon)$ in (3) may not have an analytical expression that can be differentiated directly w.r.t. ϵ. Instead, motivated by Watanabe [64], we can write (3) as an equivalent Stratonovich form

$$dX(t, \epsilon) = \epsilon^2 b(X(t, \epsilon)) \, dt + \epsilon \sigma(X(t, \epsilon)) \circ dW(t),$$

where "∘" represents the Stratonovich integration. Then applying the Itô-Stratonovich formula repeatedly (see, e.g., Kloeden and Platen [37]) yields the expressions of general $F_{k,j}$ for $j \in \mathbb{N}$ and $k = 1, \ldots, m$.

LEMMA 4.1. *The coefficients $F_{k,j}$ in (8) can be expressed as a linear combination of iterated Stratonovich integrals*:

$$F_{k,j} = \sum_{\|\mathbf{i}\| = j} C_{\mathbf{i}}(s_0) J_{\mathbf{i}}(k), \quad \text{for } j \in \mathbb{N} \text{ and } k = 1, \ldots, m, \tag{35}$$

where, for any index $\mathbf{i} = (i_1, \ldots, i_n) \in \{0, 1\}^n$, $J_{\mathbf{i}}(t)$ denotes an iterated Stratonovich integral

$$J_{\mathbf{i}}(t) := \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \circ \, dW_{i_n}(t_n) \cdots \circ \, dW_{i_2}(t_2) \circ \, dW_{i_1}(t_1),$$

with $W_1(t) := W(t)$ and $W_0(t) := t$. The coefficient $C_{\mathbf{i}}(s_0)$ is given by

$$C_{\mathbf{i}}(s_0) = \mathscr{A}_{i_n}(\cdots(\mathscr{A}_{i_2}(\sigma_{i_1})))(s_0), \tag{36}$$

with $\sigma_1(x) \equiv \sigma(x)$ and $\sigma_0(x) \equiv b(x) := \mu(x) - \frac{1}{2}\sigma(x)\sigma'(x)$. Here two differential operators

$$\mathscr{A}_0 := b(s_0)\frac{\partial}{\partial x} \qquad \text{and} \qquad \mathscr{A}_1 := \sigma(s_0)\frac{\partial}{\partial x} \tag{37}$$

map real valued functions to real valued functions, and $\|\mathbf{i}\|$ is a norm of the index $\mathbf{i}$ defined as

$$\|\mathbf{i}\| = n + \#\{\nu \in \{1, 2, \ldots, n\}: i_\nu = 0\}. \tag{38}$$

PROOF. This is an immediate result of Theorem 3.3 in Watanabe [64]. □

REMARK 4.1. Given the explicit forms of $\mu(\cdot)$ and $\sigma(\cdot)$, all the coefficients $C_{\mathbf{i}}(s_0)$ can be explicitly calculated via any symbolic computation package such as *Mathematica*.

In particular, we have

$$F_{k,1} = \sigma(s_0)J_{(1)}(k) \qquad \text{and} \qquad J_{(1)}(k) = \int_0^k \circ \, dW(t_1) = W(k), \quad \text{for } k = 1, \ldots, m.$$

Thus by (9) and (12),

$$Z_0 = \sum_{k=1}^m Y_{k,0} = \sum_{k=1}^m \frac{F_{k,1}}{\sigma(s_0)\gamma} = \sum_{k=1}^m \frac{W(k)}{\gamma},$$

which is the same as in the BSM case. Therefore, γ can also be selected as (18) such as Z_0 has a standard normal distribution. It follows that under any general one-dimensional diffusion model, $\Omega_0(z)$ is the same as in the BSM case and given by (19).

As for $\Omega_k(z)$ for $k \geq 1$, a similar idea of conditioning on Z_0 as under the BSM implies that the calculation can be reduced to the computation of the following conditional expectation

$$P_{(\mathbf{i}_1, \mathbf{i}_2, \ldots, \mathbf{i}_l)}^{(i_1, i_2, \ldots, i_l)}(z) := E\left(\prod_{r=1}^l J_{\mathbf{i}_r}(i_r) \,\Big|\, Z_0 = z\right) \equiv E\left(\prod_{r=1}^l J_{\mathbf{i}_r}(i_r) \,\Big|\, \sum_{k=1}^m W(k) = \gamma z\right), \tag{39}$$

for integers $l, i_1, i_2, \ldots, i_l \geq 1$ and indices $i_1, i_2, \ldots, i_l$, its integration

$$\mathcal{I}(P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)})(z) := \int_z^\infty P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)}(u)\phi(u)\,du, \tag{40}$$

and its differentiations

$$\mathcal{D}^i(P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)})(z) := \frac{\partial^i}{\partial z^i}\big[P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)}(z)\phi(z)\big], \quad \text{for } i = 0, 1, \ldots. \tag{41}$$

In fact, using the operators $\mathcal{I}$ and $\mathcal{D}$, $R_{(i_1,i_2,\ldots,i_s)}^{(p_1,p_2,\ldots,p_s)}(z)$ and $Q_{(i_1,i_2,\ldots,i_s)}^{(p_1,p_2,\ldots,p_s)}(z)$ defined in (24) and (25) under the BSM can be expressed as

$$R_{(i_1,i_2,\ldots,i_s)}^{(p_1,p_2,\ldots,p_s)}(z) \equiv \mathcal{I}(M_{(i_1,i_2,\ldots,i_s)}^{(p_1,p_2,\ldots,p_s)})(z) \qquad \text{and} \qquad Q_{(i_1,i_2,\ldots,i_s)}^{(p_1,p_2,\ldots,p_s)}(z) = -\mathcal{D}^{(1)}(M_{(i_1,i_2,\ldots,i_s)}^{(p_1,p_2,\ldots,p_s)})(z).$$

In the following theorem, we explicitly derive $\Omega_k(z)$ for $k \in \mathbb{N}$ in terms of $P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)}(z)$, $\mathcal{I}(P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)})(z)$, and $\mathcal{D}^i(P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)})(z)$, which immediately leads to the general expansion formula of the discretely monitored Asian option price under the general diffusion model.

THEOREM 4.1. *Assuming the existence of bounded derivatives of all orders for the drift and volatility functions $\mu(\cdot)$ and $\sigma(\cdot)$ as well as $\sigma(s_0) \neq 0$, then the asymptotic expansion* (13) *holds as $\Delta \to 0$. The coefficients $\Omega_k(z)$ can be expressed explicitly as*

$$\Omega_0(z) = zN(z) + \phi(z) - z,$$

$$\Omega_k(z) = \frac{1}{\sigma(s_0)\gamma} \sum_{i=1}^m \sum_{\|\mathbf{i}\|=k+1} \big[C_{\mathbf{i}}(s_0)\mathcal{I}(P_{(\mathbf{i})}^{(i)})(z)\big]$$

$$+ \sum_{\substack{l \geq 2,\, (l,\,(i_1,i_2,\ldots,i_l)), \\ (j_1,j_2,\ldots,j_l)) \in \mathcal{S}_k}} \left\{ \frac{(-1)^{l-2}}{l!}\left(\frac{1}{\sigma(s_0)\gamma}\right)^l \sum_{\|\mathbf{i}_1\|=j_1+1,\,\ldots,\,\|\mathbf{i}_l\|=j_l+1} \left[\left(\prod_{r=1}^l C_{\mathbf{i}_r}(s_0)\right)\mathcal{D}^{l-2}(P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)})(z)\right] \right\},$$

$$\text{for any } k \geq 1, \quad (42)$$

where $\mathcal{S}_k$, $C_{\mathbf{i}}(s_0)$, $\|\cdot\|$, $P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)}$, $\mathcal{I}(\cdot)$, and $\mathcal{D}^i(\cdot)$ are defined in (34), (36), (38), (39), (40), *and* (41), *respectively.*

PROOF. See Appendix B.

To explicitly calculate the expansion pricing formula (13) or equivalently $\Omega_k(z)$ in (42) for $k \in \mathbb{N}$, the crucial step is to explicitly compute $P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)}(z)$, $\mathcal{I}(P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)})(z)$ and $\mathcal{D}^i(P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)})(z)$ defined in (39), (40) and (41), respectively. In §§4.1–4.3, we shall provide a systematic method to achieve this objective.

4.1. Computing the conditional expectation $P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)}(z)$ in (39). Note that (39) is a novel type of iterated-integral-related conditional expectation because on the one hand, the involved iterated Stratonovich integrals have different upper limits, and on the other hand, the condition is path dependent. Iterated stochastic integrals have been playing an important role in (both theoretical and applied) probability and stochastic modeling; see, e.g., Kloeden and Platen [37], Kunitomo and Takahashi [41], Nualart [50], Peccati and Taqqu [51], and Yoshida [65]. However, most existing computational methods for iterated-integral-related conditional expectations are usually devoted to the simpler case where iterated integrals have the same upper limits and are conditional only on the value of the underlying Brownian motion at the time of the upper limit.

One theoretical contribution of our paper is to develop a systematic method to explicitly calculate this novel type of iterated-integral-related conditional expectation (39). To begin with, we apply the law of iterated conditioning to obtain that

$$E\left(\prod_{r=1}^l J_{\mathbf{i}_r}(i_r) \,\bigg|\, \sum_{k=1}^m W(k) = \gamma z\right) = E\left[E\left(\prod_{r=1}^l J_{\mathbf{i}_r}(i_r) \,\bigg|\, W(1), W(2), \ldots, W(m)\right) \,\bigg|\, \sum_{k=1}^m W(k) = \gamma z\right]. \tag{43}$$

If we can show that the inside conditional expectation is a multivariate polynomial in $W(1), W(2), \ldots, W(m)$, then the conditional expectation (39) can be simply represented as a linear combination of conditional cross-moments $M_{(i_1,i_2,\ldots,i_s)}^{(p_1,p_2,\ldots,p_s)}(z)$ defined in (23). Indeed, suppose it can be shown that

$$E\left(\prod_{r=1}^l J_{\mathbf{i}_r}(i_r) \,\bigg|\, W(1), \ldots, W(m)\right) = \sum_{n_1,n_2,\ldots,n_m} c(n_1, n_2, \ldots, n_m)W(1)^{n_1}W(2)^{n_2}, \ldots, W(m)^{n_m}, \tag{44}$$

with coefficients $c(n_1, n_2, \ldots, n_m)$ for nonnegative integers $n_1, n_2, \ldots, n_m$. It follows that

$$
\begin{aligned}
E\left(\prod_{r=1}^{l} J_{\mathbf{i}_r}(i_r) \,\middle|\, \sum_{k=1}^{m} W(k) = \gamma z\right) \\
= \sum_{n_1, n_2, \ldots, n_m} c(n_1, n_2, \ldots, n_m) E\left[W(1)^{n_1} W(2)^{n_2}, \ldots, W(m)^{n_m} \,\middle|\, \sum_{k=1}^{m} W(k) = \gamma z\right] \\
= \sum_{n_1, n_2, \ldots, n_m} c(n_1, n_2, \ldots, n_m) E(B_1^{n_1} (\sqrt{2} B_2)^{n_2}, \ldots, (\sqrt{m} B_m)^{n_m} \mid Z_0 = z) \\
= \sum_{n_1, n_2, \ldots, n_m} \left[c(n_1, n_2, \ldots, n_m) \left(\prod_{i=1}^{m} i^{n_i/2}\right) M_{(1,2\ldots,m)}^{(n_1, n_2, \ldots, n_m)}(z) \right],
\end{aligned}
\tag{45}
$$

where the second equality holds because of the definition of B_i in (4).

REMARK 4.2. Interestingly, (45) implies that $M_{(i_1, i_2, \ldots, i_s)}^{(p_1, p_2, \ldots, p_s)}(z)$ is a fundamental building block of the computations of the expansion pricing formulas not only under the BSM but also under the general diffusions. Moreover, (45) also reflects how the computations under the general diffusions generalize those under the BSM.

As discussed in Lemma 3.1 and Remark 3.1, $M_{(1,2\ldots,m)}^{(n_1, n_2, \ldots, n_m)}(z)$ can be explicitly calculated via any symbolic computation package such as *Mathematica*. Therefore, what is left is to
  (I) show (44) really holds, i.e., the LHS of (44) is a multivariate polynomial of $W(1), \ldots, W(m)$; and
  (II) study how to compute the coefficients $c(n_1, n_2, \ldots, n_m)$ in (44) explicitly.
First, applying the general recursion algorithm in (2.34) of Kloeden and Platen [37], we can convert each iterated Stratonovich integral $J_{\mathbf{i}_r}(i_r)$ on the LHS of (44) to a linear combination of iterated Itô integrals. Thus, by such a conversion algorithm and interchanging the order of multiplication and summation, we have that

$$
\prod_{r=1}^{l} J_{\mathbf{i}_r}(i_r) = \prod_{r=1}^{l} \sum_{\mathbf{j}_r} I_{\mathbf{j}_r}(i_r) = \sum_{(\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_l)} \prod_{r=1}^{l} I_{\mathbf{k}_r}(i_r),
\tag{46}
$$

where the summations are taken over all patterns of the indices resulted from the related conversions, and $I_{\mathbf{i}}(t)$ denotes the iterated Itô integral

$$
I_{\mathbf{i}}(t) := \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} dW_{i_n}(t_n), \ldots, dW_{i_2}(t_2) \, dW_{i_1}(t_1).
$$

We demonstrate this conversion procedure (46) through a representative example in Appendix C. In general cases, the conversion can be done via *Mathematica* systematically; see Kloeden and Platen [37] for more details.

Accordingly, to evaluate the LHS of (44) it suffices to explicitly compute

$$
E\left(\prod_{r=1}^{l} I_{\mathbf{k}_r}(i_r) \,\middle|\, W(1) = w_1, W(2) = w_2, \ldots, W(m) = w_m\right).
$$

To this end, we intend to remove the condition by constructing a multiply pinned Brownian motion $\{\mathscr{W}(t), 0 \le t \le m\}$ such that for any $w_1, w_2, \ldots, w_m \in \mathbb{R}$,

$$
\{\mathscr{W}(t), 0 \le t \le m\} \stackrel{\text{in law}}{=} \{W(t) \mid W(1) = w_1, \ldots, W(m) = w_m, 0 \le t \le m\}.
$$

Indeed, such a multiply pinned Brownian motion can be obtained by generalizing the construction of Brownian bridge (see, e.g., Shreve [56]). Specifically, let $\{B(t): 0 \le t \le m\}$ be a standard Brownian motion, and define $w_0 := 0$. Then $\{\mathscr{W}(t), 0 \le t \le m\}$ can be constructed as

$$
\mathscr{W}(t) = \sum_{i=0}^{m-1} 1_{(i, i+1)}(t) \{w_i(1 - t + i) + w_{i+1}(t - i) + [\mathscr{B}(t) - \mathscr{B}(i) - (t-i)(\mathscr{B}(i+1) - \mathscr{B}(i))]\}.
\tag{47}
$$

It follows that

$$
E\left(\prod_{r=1}^{l} I_{\mathbf{k}_r}(i_r) \,\middle|\, W(1) = w_1, \ldots, W(m) = w_m\right) = E\left(\prod_{r=1}^{l} \mathscr{I}_{\mathbf{k}_r}(i_r)\right),
\tag{48}
$$

where

$$\mathcal{I}_{\mathbf{i}}(t) := \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} d\mathcal{W}_{i_n}(t_n), \ldots, d\mathcal{W}_{i_2}(t_2)\, d\mathcal{W}_{i_1}(t_1), \qquad (49)$$

with $\mathcal{W}_1(t) \equiv \mathcal{W}(t)$ and $\mathcal{W}_0(t) \equiv t$.

Plugging (47) into (49) and using the fundamental properties of Itô calculus, we can calculate $E(\prod_{r=1}^l \mathcal{I}_{\mathbf{k}_r}(i_r))$ explicitly, which leads to a multivariate polynomial in $w_1, w_2, \ldots, w_m$. Then the LHS of (44) must be a multivariate polynomial in $W(1), W(2), \ldots, W(m)$, and its explicit expression follows immediately from explicit expressions of $E(\prod_{r=1}^l \mathcal{I}_{\mathbf{k}_r}(i_r))$ and the conversion procedure (46). By (45) we obtain the explicit expression of $P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)}(z)$ in (39). Without loss of generality, we illustrate the above road map through a representative example in Appendix C. In general cases, such a calculation procedure can be implemented symbolically.

4.2. Computing $\mathcal{I}(P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)})(z)$ in (40) and $\mathcal{D}^i(P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)})(z)$ in (41). Assume that applying the method in §4.1 has yielded an explicit expression (45) for $P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)}(z)$. By (45) we conclude that the calculation of $\mathcal{I}(P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)})(z)$ and $\mathcal{D}^i(P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)})(z)$ is reduced to the computation of $\mathcal{I}(M_{(1,2\ldots,m)}^{(n_1,n_2,\ldots,n_m)})(z)$ and $\mathcal{D}^i(M_{(1,2\ldots,m)}^{(n_1,n_2,\ldots,n_m)})(z)$. We know from Lemma 3.1 that $\mathcal{I}(M_{(1,2\ldots,m)}^{(n_1,n_2,\ldots,n_m)})(z)$ is a polynomial in z. Assume

$$M_{(1,2,\ldots,m)}^{(n_1,n_2,\ldots,n_m)}(z) = \sum_{j=0}^n a_j z^j,$$

where $n := n_1 + \cdots + n_m$ and, as discussed in §2, the coefficients a_j can be obtained explicitly. Then, similarly to Lemma 3.3 under the BSM, we can obtain by Lemma 3.2 that

$$\mathcal{I}(M_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)})(z) = \sum_{j=0}^n a_j P_j(z) \qquad (50)$$

$$\mathcal{D}^i(M_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)})(z) = \sum_{j=0}^n a_j \frac{d^i}{dz^i}[\phi(z)z^j] = \sum_{j=0}^n a_j \sum_{l=\max\{0,i-j\}}^i C_i^l \frac{j!}{(j-i+l)!}\phi^{(l)}(z)z^{j-i+l}, \qquad (51)$$

where $C_i^l = i!/(l!(i-l)!)$ and $\phi^{(l)}(z)$ is the lth derivative of $\phi(z)$ given by

$$\phi^{(l)}(z) = (-1)^l H_l(z)\phi(z)$$

with $H_l(z)$ denoting the Hermite polynomial of order l. Since the Hermite polynomial has a closed-form expression $H_l(x) = l! \sum_{r=0}^{\lfloor l/2 \rfloor}((-1)^r/(r!(l-2r)!))(2x)^{l-2r}$ with $\lfloor x \rfloor$ defined as the largest integer no greater than x, we can obtain an explicit expression for $\mathcal{D}^i(M_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)})(z)$.

4.3. A systematic method for computing the expansion pricing formula up to an arbitrary order under general diffusions. Summarizing the analysis above leads to the following systematic algorithm for computing the expansion pricing formula up to an arbitrary order, say the Jth-order ($J \geq 1$), under general diffusions. This systematic algorithm can be implemented via any symbolic computation package such as *Mathematica*.

Step 1. Computing $M_{(1,\ldots,m)}^{(n_1,\ldots,n_m)}(z)$ involved in (45) by Lemma 3.1.

Step 2. Substituting $M_{(1,\ldots,m)}^{(n_1,\ldots,n_m)}(z)$ obtained in Step 1 into (45), we can obtain $P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)}(z)$ involved in (42), where the coefficients $c(n_1,\ldots,n_m)$ are derived by (46), (48), and (49).

Step 3. Plugging $P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)}(z)$ obtained in Step 2 into (50) and (51) yields $\mathcal{I}(P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)})(z)$ and $\mathcal{D}^i(P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)})(z)$ involved in (42).

Step 4. Computing $C_{\mathbf{i}}(s_0)$ involved in (42).

Step 5. Substituting $P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)}(z)$, $\mathcal{I}(P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)})(z)$, $\mathcal{D}^i(P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)})(z)$ and $C_{\mathbf{i}}(s_0)$ obtained in Steps 2–4 into (42) gives the general coefficient $\Omega_k(z)$ for $k = 0, 1, \ldots, J$.

Step 6. Given $\Omega_k(z)$ for $k = 0, 1, \ldots, J$, the expansion pricing formula up to the Jth-order follows immediately from (13).

4.4. Explicit expansion pricing formulas up to the third order. From the general algorithm presented in §4.3, we can see that the key step is to derive $P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)}(z)$ involved in (42). For illustration, in this subsection we provide the explicit expressions of $P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)}(z)$ that are required for the computations of $\Omega_1(z)$, $\Omega_2(z)$, and $\Omega_3(z)$, leading to a closed-form expansion pricing formula up to the third order. (For these first three correction terms, the explicit computation can be done either simply by hand or by a *Mathematica* program following the general algorithm in §4.3.) More precisely, from (42) in Theorem 4.1 we obtain

$$\Omega_1(z) = \sum_{i=1}^m \frac{1}{\sigma(s_0)\gamma}\big[C_{(0)}(s_0)\mathcal{I}(P_{(0)}^{(i)})(z) + C_{(1,1)}(s_0)\mathcal{I}(P_{(1,1)}^{(i)})(z)\big], \tag{52}$$

$$\Omega_2(z) = \sum_{i=1}^m \frac{1}{\sigma(s_0)\gamma}\big[C_{(0,1)}(s_0)\mathcal{I}(P_{(0,1)}^{(i)})(z) + C_{(1,0)}(s_0)\mathcal{I}(P_{(1,0)}^{(i)})(z) + C_{(1,1,1)}(s_0)\mathcal{I}(P_{(1,1,1)}^{(i)})(z)\big]$$
$$+ \frac{1}{2}\sum_{i_1,i_2=1}^m \left(\frac{1}{\sigma(s_0)\gamma}\right)^2\big[C_{(0)}(s_0)C_{(0)}(s_0)P_{((0),(0))}^{(i_1,i_2)}(z) + 2C_{(0)}(s_0)C_{(1,1)}(s_0)P_{((0),(1,1))}^{(i_1,i_2)}(z)$$
$$+ C_{(1,1)}(s_0)C_{(1,1)}(s_0)P_{((1,1),(1,1))}^{(i_1,i_2)}(z)\big]\phi(z), \tag{53}$$

$$\Omega_3(z) = \sum_{i=1}^m \frac{1}{\sigma(s_0)\gamma}\big[C_{(0,0)}(s_0)\mathcal{I}(P_{(0,0)}^{(i)})(z) + C_{(0,1,1)}(s_0)\mathcal{I}(P_{(0,1,1)}^{(i)})(z) + C_{(1,0,1)}(s_0)\mathcal{I}(P_{(1,0,1)}^{(i)})(z)$$
$$+ C_{(1,1,0)}(s_0)\mathcal{I}(P_{(1,1,0)}^{(i)})(z) + C_{(1,1,1,1)}(s_0)\mathcal{I}(P_{(1,1,1,1)}^{(i)})(z)\big] - \frac{1}{6}\left(\frac{1}{\sigma(s_0)\gamma}\right)^3$$
$$\times \sum_{i_1,i_2,i_3=1}^m \big[C_{(0)}(s_0)^3\mathcal{D}(P_{((0),(0),(0))}^{(i_1,i_2,i_3)})(z) + 3C_{(0)}(s_0)^2 C_{(1,1)}(s_0)\mathcal{D}(P_{((0),(0),(1,1))}^{(i_1,i_2,i_3)})(z)$$
$$+ 3C_{(0)}(s_0)C_{(1,1)}(s_0)^2\mathcal{D}(P_{((0),(1,1),(1,1))}^{(i_1,i_2,i_3)})(z) + C_{(1,1)}(s_0)^3\mathcal{D}(P_{((1,1),(1,1),(1,1))}^{(i_1,i_2,i_3)})(z)\big]$$
$$+ \left(\frac{1}{\sigma(s_0)\gamma}\right)^2 \sum_{i_1,i_2=1}^m \big[C_{(0)}(s_0)C_{(0,1)}(s_0)P_{((0),(0,1))}^{(i_1,i_2)}(z) + C_{(0)}(s_0)C_{(1,0)}(s_0)P_{((0),(1,0))}^{(i_1,i_2)}(z)$$
$$+ C_{(0)}(s_0)C_{(1,1,1)}(s_0)P_{((0),(1,1,1))}^{(i_1,i_2)}(z) + C_{(1,1)}(s_0)C_{(0,1)}(s_0)P_{((1,1),(0,1))}^{(i_1,i_2)}(z)$$
$$+ C_{(1,1)}(s_0)C_{(1,0)}(s_0)P_{((1,1),(1,0))}^{(i_1,i_2)}(z) + C_{(1,1)}(s_0)C_{(1,1,1)}(s_0)P_{((1,1),(1,1,1))}^{(i_1,i_2)}(z)\big]. \tag{54}$$

The following lemma provides explicit expressions of $P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)}(z)$ involved in (52), (53), and (54).

LEMMA 4.2. *The involved conditional expectations* $P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)}(z)$ *in* (52), (53), *and* (54) *are explicitly given as follows in terms of conditional cross-moments* $M_{(i_1,i_2,\ldots,i_s)}^{(p_1,p_2,\ldots,p_s)}(z)$ *defined in* (23):
Those involved in $\Omega_1(z)$:

$$P_{(0)}^{(i)}(z) = i, \qquad P_{(1,1)}^{(i)}(z) = \frac{i}{2}M_{(i)}^{(2)}(z);$$

Those involved in $\Omega_2(z)$:

$$P_{(1,1,1)}^{(i)}(z) = \frac{i\sqrt{i}}{6}M_{(i)}^{(3)}(z), \qquad P_{((0),(0))}^{(i_1,i_2)}(z) = i_1 i_2,$$

$$P_{((0),(1,1))}^{(i_1,i_2)}(z) = \frac{i_1 i_2}{2}M_{(i_2)}^{(2)}(z), \qquad P_{((1,1),(1,1))}^{(i_1,i_2)}(z) = \frac{i_1 i_2}{4}M_{(i_1,i_2)}^{(2,2)}(z),$$

$$P_{(0,1)}^{(i)}(z) = \sum_{k=1}^{i-1}\sqrt{k}M_{(k)}^{(1)}(z) + \frac{1}{2}\sqrt{i}M_{(i)}^{(1)}(z), \qquad P_{(1,0)}^{(i)}(z) = -\sum_{k=1}^{i-1}\sqrt{k}M_{(k)}^{(1)}(z) + \left(i - \frac{1}{2}\right)\sqrt{i}M_{(i)}^{(1)}(z);$$

Those involved in $\Omega_3(z)$

$$P_{(0,0)}^{(i)}(z) = \frac{i^2}{2}, \qquad P_{(1,1,1,1)}^{(i)}(z) = \frac{i^2}{24}M_{(i)}^{(3)}(z),$$

$$P_{((0),(0),(0))}^{(i_1,i_2,i_3)}(z) = i_1 i_2 i_3, \qquad P_{((0),(0),(1,1))}^{(i_1,i_2,i_3)}(z) = \frac{i_1 i_2 i_3}{2}M_{(i_3)}^{(2)}(z),$$

$$P_{((0),(1,1),(1,1))}^{(i_1,i_2,i_3)}(z) = \frac{i_1 i_2 i_3}{4}M_{(i_2,i_3)}^{(2,2)}(z), \qquad P_{((1,1),(1,1),(1,1))}^{(i_1,i_2,i_3)}(z) = \frac{i_1 i_2 i_3}{8}M_{(i_1,i_2,i_3)}^{(2,2,2)}(z),$$

$$P_{((0),(1,1,1))}^{(i_1,i_2)}(z) = \frac{i_1 i_2 \sqrt{i_2}}{6} M_{(i_2)}^{(3)}(z), \qquad P_{((1,1),(1,1,1))}^{(i_1,i_2)}(z) = \frac{i_1 i_2 \sqrt{i_2}}{12} M_{(i_2,i_1)}^{(3,2)}(z),$$

$$P_{((0),(1,0))}^{(i_1,i_2)}(z) = i_1 P_{(1,0)}^{(i_2)}(z), \qquad P_{((0),(0,1))}^{(i_1,i_2)}(z) = i_1 P_{(0,1)}^{(i_2)}(z),$$

$$P_{((1,1),(0,1))}^{(i_1,i_2)}(z) = \frac{i_1}{2} \sum_{k=1}^{i_2-1} \sqrt{k} M_{(k,i_1)}^{(1,2)}(z) + \frac{i_1 \sqrt{i_2}}{4} M_{(i_2,i_1)}^{(1,2)}(z),$$

$$P_{((1,1),(1,0))}^{(i_1,i_2)}(z) = -\frac{i_1}{2} \sum_{k=1}^{i_2-1} \sqrt{k} M_{(k,i_1)}^{(1,2)}(z) + i_1 \sqrt{i_2}\left(i_2 - \frac{1}{2}\right) M_{(i_2,i_1)}^{(1,2)}(z),$$

$$P_{(0,1,1)}^{(i)}(z) = \frac{i}{12} + \frac{1}{3} \sum_{k=1}^{i-1} k M_{(k)}^{(2)}(z) + \frac{1}{6} i M_{(i)}^{(2)}(z) + \frac{1}{6} \sum_{k=1}^{i-1} \sqrt{k(k+1)} M_{(k,k+1)}^{(1,1)}(z),$$

$$P_{(1,0,1)}^{(i)}(z) = -\frac{i}{6} - \frac{2}{3} \sum_{k=1}^{i-1} k M_{(k)}^{(2)}(z) + \frac{1}{6} i M_{(i)}^{(2)}(z) - \frac{1}{3} \sum_{k=1}^{i-1} \sqrt{k(k+1)} M_{(k,k+1)}^{(1,1)}(z) + \sum_{k=1}^{i-1} \sqrt{k}i M_{(k,i)}^{(1,1)}(z),$$

$$P_{(1,1,0)}^{(i)}(z) = \frac{i^2}{2} M_{(i)}^{(2)}(z) - P_{(0,1,1)}^{(i)}(z) - P_{(1,0,1)}^{(i)}(z).$$

PROOF. See Appendix C.

5. Numerical examples. This section illustrates the numerical performance of our closed-form expansion formulas up to the third order under several one-dimensional diffusion models, including the BSM, the CIR model, the general CEV model, and the Brennan-Schwartz process. Note that these tested models encompass not only diffusion models that satisfy the regularity conditions, e.g., the BSM and the Brennan-Schwartz process, but also those that violate the conditions such as the CIR model and the general CEV model. However, we can see that in comparison with the benchmarks existing in the literature or obtained via Monte Carlo simulation, our numerical method is consistently accurate, efficient, and robust in all of these models.

All the computations in the numerical parts of this paper are conducted on a desktop computer with 2.85 GB of RAM and an Intel Core i5-2500 (3.3 GHz) processor.

5.1. The BSM. Table 2 gives numerical results of prices, deltas, and gammas (denoted by "AE") of discretely monitored Asian options under the BSM via our asymptotic expansion formula up to the third order. We let the strike K vary from 80 to 120 with increment 5 and consider two different monitoring frequencies, monthly ($m = 12$) and daily ($m = 250$). It can be seen that all the AE results stay within the 95% confidence intervals of the Monte Carlo simulation results (denoted by "MC"). The average of (absolute values of) absolute errors for prices, deltas, and gammas are 0.00132, 0.00003, and 0.00028, respectively, when $m = 250$, and 0.00358, 0.00022, and 0.00032, Respectively, when $m = 12$. This implies that our method is accurate and robust in that it performs well for a wide range of strikes and even for seemingly long monitoring intervals such as $\Delta t = 1/12$, i.e., the monthly monitored case. Besides, to generate one AE result of price, delta, and gamma, it takes only approximately 0.002, 0.05, and 0.07 seconds, respectively, when $m = 12$, and 0.13, 0.30, and 0.45 seconds, respectively, when $m = 250$. It is worth mentioning that the CPU times reported in this paper correspond to the numerical calculations given that the coefficients of the expansion have been precomputed. Otherwise, it would be computationally expensive to recompute the expansion formula.

Figure 1 demonstrates how the expansion formulas converge across the strikes as the number of correction terms increases in all six cases of Table 2, namely, for prices, deltas, and gammas with $m = 12$ and $m = 250$, respectively. It suggests that our expansion formulas converge quite fast so that the results up to the third order have achieved a high accuracy.

We also compare our asymptotic expansion numerical results with those obtained through other existing methods in the literature, including the recursive integration method by Fusai and Meucci [24], the maturity-randomization-based recursive method by Fusai et al. [25], and an improved convolution pricing algorithm by Cerny and Kyriakou [15]; see Tables 3, 4, and 5, respectively. We find that all the absolute errors between our AE results and the 30 benchmarks are no greater than 0.003 (the average error is 0.00151). It is worth pointing out that these benchmarks consist of various parameter settings, including different monitoring frequencies such as daily, weekly, and monthly, different model parameters such as interest rate and volatility, and different contract parameters such as the strike. Therefore, Tables 3–5 also imply that our pricing method is accurate and robust.

TABLE 2. Prices, deltas, and gammas of discretely monitored Asian options under the BSM. Parameters are $r = 0.05$, $\sigma = 0.3$, $S_0 = 100$, and $T = 1$. The columns "AE" denote our asymptotic expansion results up to the third order, and the columns "MC" and "Std. err." denote Monte Carlo simulation estimates and associated standard errors obtained by simulating 1,000,000 sample paths. "Abs. err." is the absolute error between "AE" and "MC." We can see all the AE results lie in the 95% confidence intervals of the associated MC estimates even in the case of quite small $m = 12$. Besides, to generate one AE result of price, delta, and gamma via our method, it takes only approximately 0.002, 0.05, and 0.07 seconds respectively when $m = 12$, and 0.13, 0.30, and 0.45 seconds, respectively, when $m = 250$.

	$m = 250$				$m = 12$			
K	AE	MC	Std. err.	Abs. err.	AE	MC	Std. err.	Abs. err.
	Prices of discretely monitored Asian call options under the BSM							
80	21.90348	21.90509	0.00180	−0.00161	21.85858	21.86143	0.00169	−0.00285
85	17.72261	17.72419	0.00268	−0.00158	17.65195	17.65537	0.00256	−0.00342
90	13.94619	13.94775	0.00359	−0.00156	13.85170	13.85599	0.00348	−0.00429
95	10.66810	10.66958	0.00438	−0.00148	10.55718	10.56157	0.00428	−0.00439
100	7.93672	7.93805	0.00493	−0.00133	7.81970	7.82374	0.00484	−0.00404
105	5.74995	5.75069	0.00519	−0.00074	5.63691	5.64012	0.00511	−0.00321
110	4.06364	4.06453	0.00517	−0.00089	3.96220	3.96543	0.00509	−0.00323
115	2.80720	2.80835	0.00490	−0.00115	2.72161	2.72514	0.00483	−0.00353
120	1.89964	1.90117	0.00447	−0.00153	1.83112	1.83440	0.00440	−0.00328
	Deltas of discretely monitored Asian call options under the BSM							
80	0.94898	0.94891	0.00012	0.00007	0.91759	0.91758	0.00017	0.00001
85	0.85677	0.85679	0.00022	−0.00002	0.86045	0.86079	0.00022	−0.00034
90	0.77658	0.77661	0.00026	−0.00003	0.77939	0.77966	0.00026	−0.00027
95	0.67830	0.67832	0.00028	−0.00002	0.67937	0.67936	0.00028	0.00001
100	0.57032	0.57034	0.00029	−0.00002	0.56931	0.56895	0.00029	0.00036
105	0.46205	0.46207	0.00030	−0.00002	0.45917	0.45867	0.00030	0.00050
110	0.36145	0.36147	0.00031	−0.00002	0.35729	0.35742	0.00031	−0.00013
115	0.27379	0.27381	0.00031	−0.00002	0.26905	0.26922	0.00031	−0.00017
120	0.20140	0.20145	0.00031	−0.00005	0.19667	0.19688	0.00031	−0.00021
	Gammas of discretely monitored Asian call options under the BSM							
80	0.00730	0.00764	0.00024	−0.00034	0.00719	0.00719	0.00023	0.00000
85	0.01175	0.01176	0.00031	−0.00001	0.01181	0.01206	0.00031	−0.00025
90	0.01628	0.01621	0.00037	0.00007	0.01652	0.01687	0.00038	−0.00035
95	0.01989	0.02030	0.00043	−0.00041	0.02028	0.01988	0.00042	0.00040
100	0.02193	0.02246	0.00046	−0.00053	0.02234	0.02204	0.00046	0.00030
105	0.02217	0.02217	0.00047	0.00000	0.02251	0.02174	0.00047	0.00077
110	0.02085	0.02056	0.00046	0.00029	0.02105	0.02145	0.00047	−0.00040
115	0.01844	0.01790	0.00044	0.00054	0.01850	0.01848	0.00045	0.00002
120	0.01551	0.01582	0.00042	−0.00031	0.01544	0.01585	0.00042	−0.00041

TABLE 3. Comparison with numerical results obtained by Fusai and Meucci [24]. Other parameters are $S_0 = 100$, $r = 0.0367$, $\sigma = 0.17801$, and $T = 1$. The column "AE" represents our asymptotic expansion results up to the third order, and the column "Fusai and Meucci" is taken from Table 5 in Fusai and Meucci [24].

m	K	AE	Fusai and Meucci	Abs. err.
		Comparison with numerical results in Fusai and Meucci [24]		
12	90	11.90363	11.90497	−0.00134
	100	4.88072	4.88210	−0.00138
	110	1.36173	1.36314	−0.00141
50	90	11.93171	11.93301	−0.00130
	100	4.93602	4.93736	−0.00134
	110	1.40127	1.40264	−0.00137
250	90	11.93935	11.94068	−0.00133
	100	4.95098	4.95233	−0.00135
	110	1.41214	1.41351	−0.00137

Note that Fusai and Meucci [24], Fusai et al. [25], and Cerny and Kyriakou [15] are all focused on general exponential Lévy models, whereas our paper deals with one-dimensional diffusion models, which are nonexponential-Lévy except the BSM. In §§5.2 and 5.3, we shall provide numerical examples to demonstrate our method's performance in the nonexponential-Lévy case.
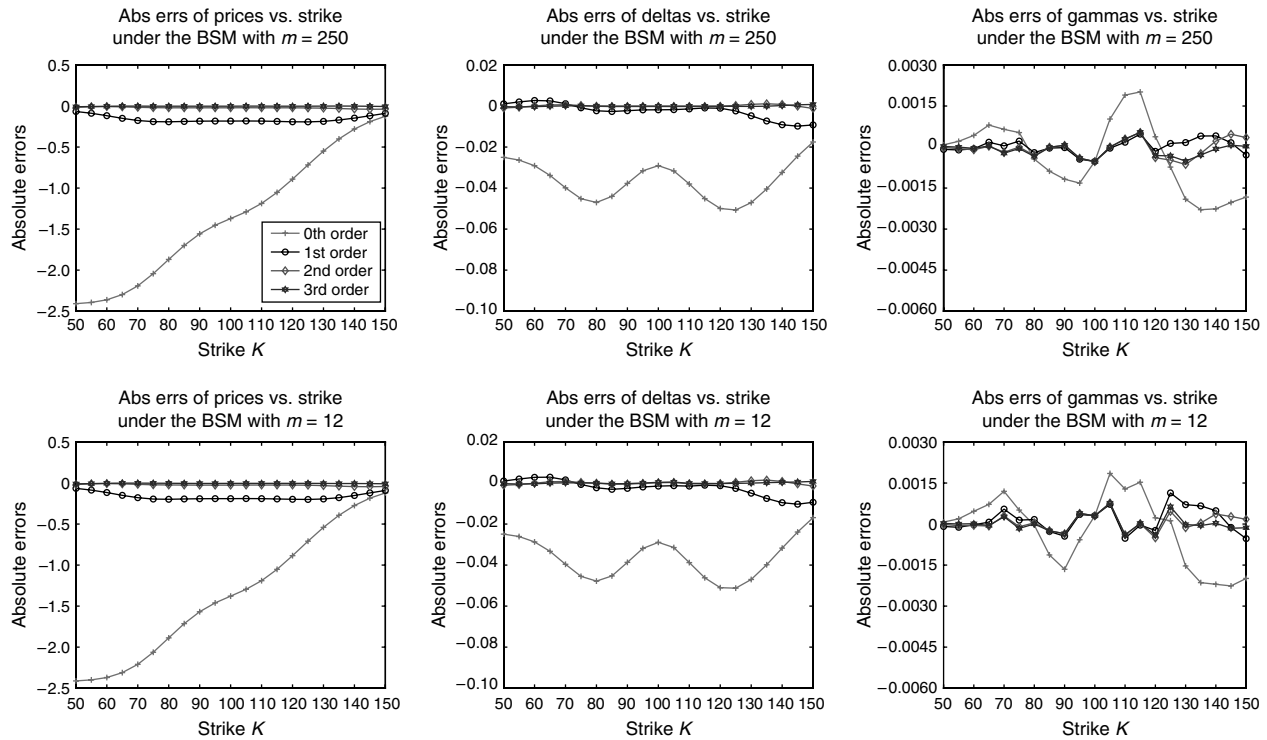
FIGURE 1. Convergence of asymptotic expansions for prices, deltas, and gammas of discretely monitored Asian options under the BSM. Parameters are $r = 0.05$, $\sigma = 0.3$, $S_0 = 100$, and $T = 1$. The upper three plots correspond to $m = 250$, whereas the lower three $m = 12$. The reference true values are Monte Carlo simulation estimates obtained by simulating 1,000,000 sample paths.

TABLE 4. Comparison with three numerical results obtained by Fusai et al. [25]. Other parameters are $S_0 = K = 100$ and $T = 1$. The column "AE" represents our asymptotic expansion results up to the third order, and the column "Fusai et al." is taken from Tables 2, 3 and 6 in Fusai et al. [25].

m	r	σ	AE	Fusai et al.	Abs. err.
			Comparison with numerical results in Fusai et al. [25]		
50	0.04	0.3	7.69712	7.69859	−0.00147
50	0.04	0.1	4.48589	4.48842	−0.00253
250	0.0367	0.17801	4.95098	4.95212	−0.00214

TABLE 5. Comparison with numerical results in Cerny and Kyriakou [15]. Other parameters are $S_0 = 100$, $r = 0.0367$, $\sigma = 0.17801$, and $T = 1$ for the left panel, and $S_0 = 100$, $r = 0.04$, $m = 50$, and $T = 1$ for the right panel. The columns "AE" represent our asymptotic expansion results up to the third order, and the columns "C&K" are taken from Tables 4 and 7 in Černỳ and Kyriakou [15].

		Prices when m varies					Prices when σ varies		
m	K	AE	C&K	Abs. err.	σ	K	AE	C&K	Abs. err.
				Comparison with numerical results in Černỳ and Kyriakou [15]					
12	90	11.90363	11.90492	−0.00129	0.1	90	11.57841	11.58113	−0.00272
	100	4.88072	4.88196	−0.00124		100	3.33766	3.33861	−0.00095
	110	1.36173	1.36304	−0.00141		110	0.27085	0.27375	0.00290
50	90	11.93171	11.93294	−0.00123	0.2	90	13.66835	13.66981	−0.00146
	100	4.93602	4.93720	−0.00118		100	7.69712	7.69859	−0.00147
	110	1.40127	1.40252	−0.00125		110	3.89489	3.89639	−0.00150
250	90	11.93935	11.94056	−0.00121	0.3	90	17.19090	17.19239	−0.00149
	100	4.95098	4.95216	−0.00118		100	12.09000	12.09153	−0.00153
	110	1.41214	1.41337	−0.00123		110	8.31281	8.31441	−0.00160

TABLE 6. Comparison with numerical results in Ju [34]. The parameters are the same as in Table 4 in Ju [34], i.e., $S_0 = 100$, $r = 0.09$ and $T = 3$. The column "Ju" obtained by Ju's Taylor expansion method up to the sixth order is taken from Table 4 in Ju [34], and the column "AE" denotes our asymptotic expansion results up to the sixth order. The columns "MC" (Monte Carlo simulation estimates) and "Std. err." (standard errors) are also taken from Table 4 in Ju [34]. "Abs. err. of AE" (respectively, "Abs. err. of Ju") is the absolute error between "AE" (resp., "Ju") and "MC." Following Ju [34], we also use the root of mean-squared error (RMSE) to measure the overall accuracy for a whole set of options, and the maximum absolute error (MAE) to indicate the worst case. We can see that our method produces more accurate results than Ju's because our RMSE and MAE are both smaller than Ju's. Besides, it takes 0.1 seconds to generate one AE result.

| (σ, K)	AE	Ju	MC	Std. err.	Abs. err. of AE	Abs. err. of Ju
			Comparison with numerical results in Ju [34]			
(0.05, 95)	15.0951	15.1197	15.1199	0.0002	−0.0248	−0.0002
(0.05, 100)	11.3119	11.3069	11.3071	0.0002	0.0048	−0.0002
(0.05, 105)	7.5417	7.5562	7.5563	0.0002	−0.0146	−0.0001
(0.1, 95)	15.2137	15.2165	15.2171	0.0007	−0.0034	−0.0006
(0.1, 100)	11.6349	11.6394	11.6399	0.0007	−0.0050	−0.0005
(0.1, 105)	8.3923	8.3913	8.3919	0.0007	0.0004	−0.0006
(0.2, 95)	16.6366	16.6365	16.6366	0.0026	0.0000	−0.0001
(0.2, 100)	13.7691	13.7634	13.7654	0.0026	0.0037	−0.0020
(0.2, 105)	11.2176	11.2135	11.2174	0.0026	0.0002	−0.0039
(0.3, 95)	19.0160	19.0179	19.0194	0.0009	−0.0034	−0.0015
(0.3, 100)	16.5882	16.5755	16.5812	0.0009	0.0070	−0.0057
(0.3, 105)	14.3789	14.3774	14.3874	0.0009	−0.0085	−0.0100
(0.4, 95)	21.7338	21.7307	21.7331	0.0018	0.0007	−0.0024
(0.4, 100)	19.5788	19.5690	19.5798	0.0018	−0.0010	−0.0108
(0.4, 105)	17.6168	17.5978	17.6164	0.0018	0.0004	−0.0186
(0.5, 95)	24.5577	24.5583	24.5585	0.0031	−0.0008	−0.0002
(0.5, 100)	22.6183	22.6032	22.6167	0.0031	0.0016	−0.0135
(0.5, 105)	20.8352	20.8023	20.8289	0.0031	0.0063	−0.0266
RMSE	0.0077	0.0092				
MAE	0.0248	0.0266				
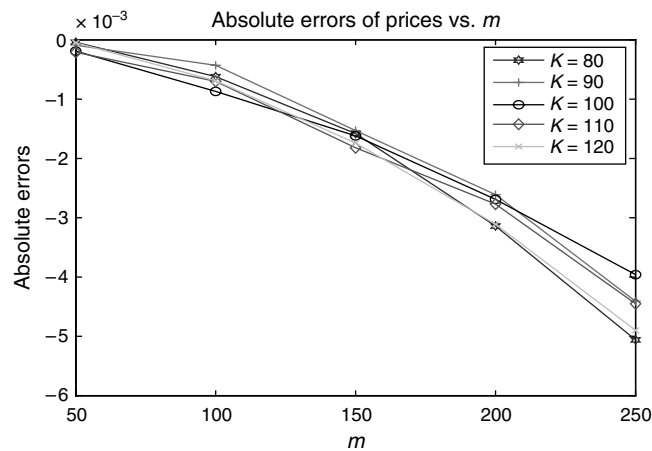
FIGURE 2. How the absolute errors (between the prices obtained via our expansion formula up to the third order and the benchmark computed through Monte Carlo simulation) change as m varies in the cases of in the money ($K = 80$ and 90), at the money ($K = 100$), and out of the money ($K = 110$ and 120). Parameters are $r = 0.06$, $\sigma = 0.2$, $S_0 = 100$, and $\Delta = 1/250$. We can see that the absolute errors tend to increase as m rises.

Besides, we compare our numerical results with those obtained via the Taylor expansion method (up to the sixth order) under the BSM by Ju [34]. See Table 6. Following Ju [34], we also use the root of mean-squared error (RMSE) to measure the overall accuracy for a whole set of options, and use the maximum absolute error (MAE) to indicate the worst case. It can be seen that our expansion formula up to the sixth order can produce more accurate numerical results than Ju's expansion method up to the sixth order because our RMSE and MAE are both smaller than Ju's. It is worth pointing out that the parameters used in Ju [34] are not that usual. For example, the risk-free interest rate is 9%; the maturity is three years; the volatility can be as small as 0.05.

TABLE 7. Prices and deltas of discretely monitored Asian options under the Brennan-Schwartz process. Parameters are $r = 0.05$, $\kappa = 0.1$, $\mu = 120$, $\sigma = 0.25$, $S_0 = 100$, and $T = 1$. The columns "AE" represent our asymptotic expansion results up to the third order, and the columns "MC" and "Std. err." denote Monte Carlo simulation estimates and associated standard errors obtained by simulating 200,000 sample paths. "Abs. err." is the absolute error between "AE" and "MC." We can see that all the AE results lie in the 95% confidence intervals of the associated MC estimates. In addition, to generate one AE result of price and delta, it takes approximately 0.001 and 0.002 seconds respectively when $m = 12$, and 0.1 and 0.2 seconds, respectively, when $m = 250$.

	$m = 250$				$m = 12$			
K	AE	MC	Std. err.	Abs. err.	AE	MC	Std. err.	Abs. err.
	Prices of discretely monitored Asian call options under the Brennan-Schwartz process							
80	20.13363	20.13789	0.00242	−0.00426	20.10687	20.10805	0.00224	−0.00118
85	15.75659	15.76048	0.00411	−0.00389	15.70458	15.70572	0.00389	−0.00114
90	11.79768	11.79996	0.00604	−0.00228	11.71851	11.71667	0.00583	0.00184
95	8.42197	8.41832	0.00775	0.00365	8.32298	8.32048	0.00756	0.00250
100	5.72567	5.72168	0.00881	0.00399	5.62035	5.62267	0.00862	−0.00232
105	3.71040	3.70959	0.00903	0.00081	3.61241	3.61994	0.00884	−0.00753
110	2.29709	2.29621	0.00850	0.00089	2.21557	2.22243	0.00830	−0.00686
115	1.36297	1.36258	0.00745	0.00039	1.30124	1.30592	0.00725	−0.00468
120	0.77865	0.77857	0.00616	0.00008	0.73563	0.74014	0.00596	−0.00451
	Deltas of discretely monitored Asian call options under the Brennan-Schwartz process							
80	0.87182	0.87230	0.00052	−0.00049	0.87482	0.87515	0.00050	−0.00033
85	0.82117	0.82232	0.00071	−0.00115	0.82508	0.82616	0.00069	−0.00107
90	0.73757	0.73855	0.00091	−0.00099	0.74098	0.74227	0.00090	−0.00128
95	0.62515	0.62634	0.00106	−0.00119	0.62657	0.62694	0.00106	−0.00037
100	0.49754	0.49694	0.00114	0.00060	0.49627	0.49486	0.00113	0.00142
105	0.37177	0.37112	0.00112	0.00065	0.36822	0.36790	0.00112	0.00032
110	0.26156	0.26110	0.00104	0.00046	0.25678	0.25733	0.00103	−0.00055
115	0.17391	0.17305	0.00091	0.00086	0.16898	0.16859	0.00090	0.00039
120	0.10957	0.10961	0.00076	−0.00004	0.10521	0.10593	0.00075	−0.00072

In fact, for more usual parameters, our expansion formula up to the third order usually has achieved a high accuracy.

To illustrate the effect of the number of monitoring intervals m on the accuracy of our asymptotic expansion method, we fix $\Delta = 1/250$ and let m vary from 50 to 250 with increment 50. Figure 2 demonstrates how the absolute errors (between the prices obtained via our expansion formula up to the third order and the benchmark computed through Monte Carlo simulation) change as m varies in the cases of in the money ($K = 80$ and 90), at the money ($K = 100$), and out of the money ($K = 110$ and 120). We can see that the absolute errors tend to increase as m rises.

5.2. The Brennan-Schwartz process. Table 7 provides numerical results of prices and deltas (denoted by AE) of discretely monitored Asian options under the Brennan-Schwartz process (specified in Table 1) via our asymptotic expansion formula up to the third order; see, e.g., Pilipovic [52] for the applications of the Brennan-Schwatz process in financial modeling of the commodity market. We can see that all the AE results stay within the 95% confidence intervals of the Monte Carlo simulation results (denoted by MC). The average of (absolute values of) absolute errors for prices and deltas are 0.00225 and 0.00071, respectively, when $m = 250$, and 0.00362 and 0.00072, respectively, when $m = 12$. In addition, to generate one AE result of price and delta, it takes approximately 0.001 and 0.002 seconds, respectively, when $m = 12$, and 0.1 and 0.2 seconds, respectively, when $m = 250$. Similarly to Figure 1, Figure 3 indicates that our expansion formulas converge quite fast as the number of correction terms increases in all four cases of Table 7.

5.3. The general CEV model. The CEV model (specified in Table 1) is a very important asset pricing model. On the one hand, it includes several well-known models as special cases, e.g., the BSM ($\beta = 0$) and the CIR model ($\beta = -1/2$). On the other hand, the flexibility of the selection of β makes it capable of modeling the volatility smile effect in the equity index option market (see Jackwerth and Rubinstein [33]). Furthermore, some novel financial models derived from the CEV model have also won great popularity in the financial industry. For example, the jump to default extended CEV model proposed by Carr and Linetsky [13] unifies the valuation of credit derivatives and equity derivatives (see also Mendoza-Arriaga and Linetsky [49]), and the stochastic alpha-beta-rho (SABR) model proposed by Hagan et al. [30] is able to provide good fits to various types of implied volatility curves observed in the marketplace.
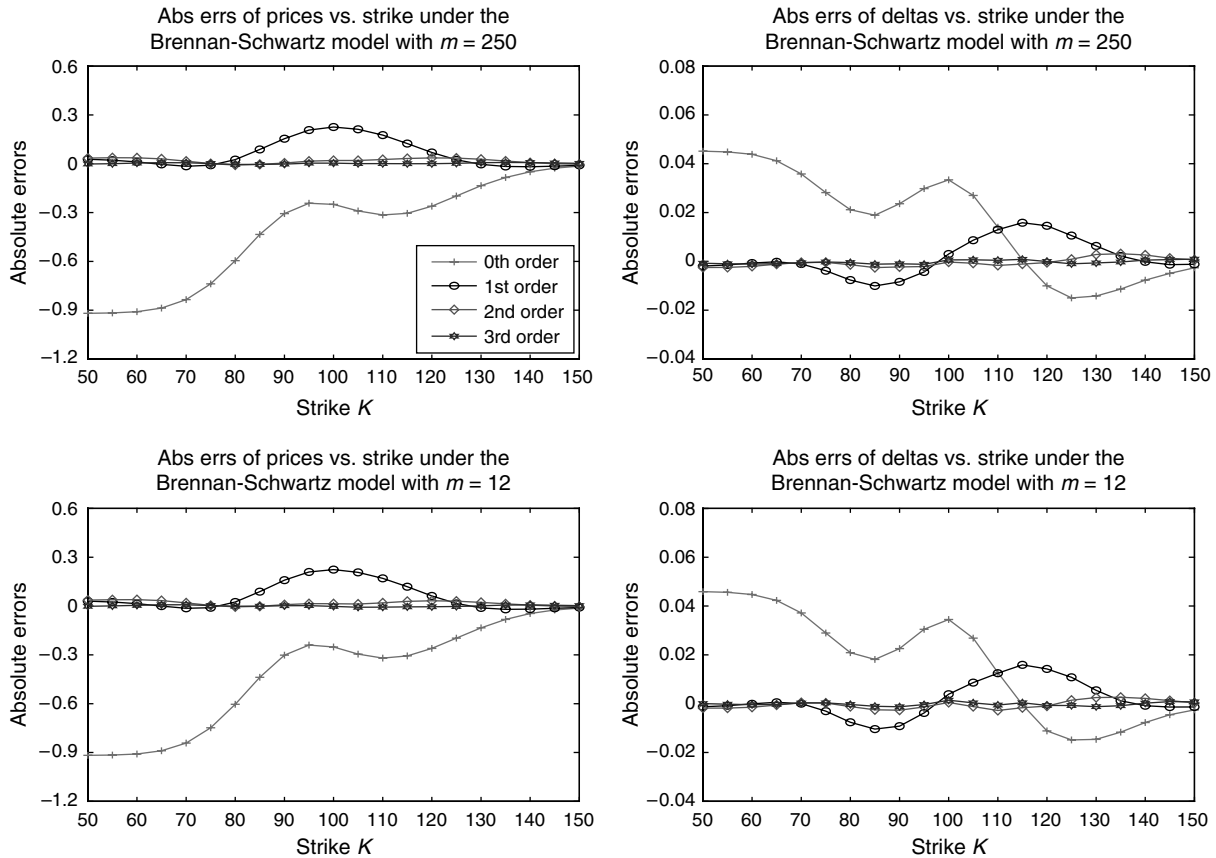
FIGURE 3. Convergence of asymptotic expansions for prices and deltas of discretely monitored Asian options under the Brennan-Schwartz process. Parameters are $r = 0.05$, $\kappa = 0.1$, $\mu = 120$, $\sigma = 0.25$, $S_0 = 100$, and $T = 1$. The reference true values are Monte Carlo simulation estimates obtained by simulating 200,000 sample paths.

It is worth pointing out that option pricing under the CEV model needs to be dealt with carefully. Specifically, when $\beta > 0$, the discounted CEV process is a strict local martingale, and when $\beta < 0$, the CEV process has a killing boundary at zero (zero is either an exit boundary when $\beta \in [-1/2, 0)$, or is a regular boundary when $\beta < -1/2$ and is then specified as a killing boundary by adjoining a killing boundary condition) and the transition density is norm defective. To deal with these two issues, one can regularize the CEV process to achieve bounded volatility by "freezing" the volatility for the stock prices above certain high level and below certain low level, respectively. For more details, we refer to, e.g., Emanuel and MacBeth [20], Andersen and Andreasen [2], Davydov and Linetsky [18], Carr and Linetsky [13], and Lewis [43]. It turns out that in either case, our asymptotic expansion produces approximations to the Asian option prices for the aforementioned "regularized" CEV process with bounded volatility. See Appendix E for more detailed discussions.

Since the BSM, a special case of CEV with $\beta = 0$, has been discussed extensively in §5.1, we concentrate on three other cases of CEV with β equal to 1/4, −1/4, and −1/2 (CIR), respectively. It turns out that for all these three cases, our asymptotic expansions up to the third order are accurate and efficient for both prices and Greeks such as deltas. Indeed, Table 8 indicates that all the prices and deltas lie in the 95% confidence intervals of associated Monte Carlo simulation estimates, and the average errors of prices and deltas are 0.00931 and 0.00074, respectively. Similarly to the case of the BSM, our method remains efficient and it takes approximately 0.2 seconds (0.4 seconds, respectively) to produce one numerical result of the price (the delta, respectively). Moreover, our method is quite robust because it performs consistently well across various strikes K and elasticities β. Similarly as in §§5.1 and 5.2, Figure 4 demonstrates that our expansion formulas under general CEV models converge quite fast.

Note that our expansion method can also deal with European options that correspond to the special case $m = 1$ of the discretely monitored Asian options. Table 9 reports the European option prices and deltas obtained via our expansion method as well as the analytical formula (see, e.g., (32) in Davydov and Linetsky [18]) under the CEV model. We can see that our method is accurate with the average errors of prices and deltas being

TABLE 8.   Prices and deltas of discretely monitored Asian options under the CEV model with $\beta = 1/4$, $-1/4$, and $-1/2$ (CIR model). Parameters are $r = 0.05$, $S_0 = 100$, $\delta S_0^\beta = 0.25$, $T = 1$, and $m = 250$. The columns "AE" represent our asymptotic expansion results up to the third order, and the columns "MC" and "Std. err." denote Monte Carlo simulation estimates and associated standard errors obtained by simulating 2,000,000 sample paths for prices and 200,000 sample paths for deltas. "Abs. err." is the absolute error between "AE" and "MC." We can see all the AE results lie in the 95% confidence intervals of the associated MC estimates. Besides, it takes around 0.2 seconds (0.4 seconds, respectively) to produce one numerical result of the price (the delta, respectively).

	Prices				Deltas			
K	AE	MC	Std. err.	Abs. err.	AE	MC	Std. err.	Abs. err.
Discretely monitored Asian call options under the CEV model with $\beta = 1/4$								
80	21.60167	21.61365	0.01000	−0.01198	0.94653	0.94600	0.00059	0.00053
85	17.18468	17.19687	0.00967	−0.01219	0.89976	0.89929	0.00078	0.00047
90	13.15550	13.16780	0.00909	−0.01230	0.81995	0.81994	0.00098	0.00001
95	9.67509	9.68666	0.00826	−0.01157	0.71007	0.71048	0.00116	−0.00041
100	6.84034	6.85112	0.00726	−0.01078	0.58252	0.58365	0.00126	−0.00113
105	4.66083	4.66997	0.00617	−0.00914	0.45335	0.45362	0.00127	−0.00027
110	3.07180	3.07792	0.00510	−0.00612	0.33614	0.33723	0.00122	−0.00109
115	1.96653	1.96998	0.00413	−0.00345	0.23873	0.23727	0.00111	0.00146
120	1.22841	1.23003	0.00327	−0.00162	0.16335	0.16206	0.00097	0.00129
Discretely monitored Asian call options under the CEV model with $\beta = -1/4$								
80	21.67122	21.68465	0.00971	−0.01343	−0.93506	0.93460	0.00054	0.00046
85	17.28990	17.30278	0.00933	−0.01288	0.88533	0.88540	0.00073	−0.00007
90	13.26903	13.28168	0.00872	−0.01265	0.80568	0.80598	0.00093	−0.00030
95	9.75397	9.76576	0.00787	−0.01179	0.69779	0.69856	0.00109	−0.00077
100	6.84853	6.85961	0.00684	−0.01108	0.57141	0.57236	0.00118	−0.00095
105	4.58662	4.59619	0.00573	−0.00957	0.44099	0.44145	0.00119	−0.00046
110	2.92962	2.93639	0.00462	−0.00677	0.32057	0.32173	0.00112	−0.00116
115	1.78606	1.79026	0.00360	−0.00420	0.21979	0.21879	0.00100	0.00100
120	1.04072	1.04369	0.00273	−0.00297	0.14247	0.14132	0.00085	0.00115
Discretely monitored Asian call options under the CIR model, i.e., the CEV model with $\beta = -1/2$								
80	21.71428	21.72781	0.00958	−0.01353	0.92843	0.92810	0.00053	0.00033
85	17.34831	17.36129	0.00918	−0.01298	0.87747	0.87767	0.00071	−0.00020
90	13.32877	13.34152	0.00855	−0.01275	0.79806	0.79859	0.00090	−0.00053
95	9.79478	9.80661	0.00770	−0.01183	0.69126	0.69242	0.00106	−0.00116
100	6.85365	6.86478	0.00666	−0.01113	0.56562	0.56685	0.00114	−0.00123
105	4.55096	4.56065	0.00553	−0.00969	0.43478	0.43547	0.00115	−0.00069
110	2.86119	2.86827	0.00441	−0.00708	0.31297	0.31387	0.00108	−0.00090
115	1.70087	1.70538	0.00338	−0.00451	0.21070	0.20992	0.00095	0.00078
120	0.95542	0.95870	0.00250	−0.00328	0.13268	0.13161	0.00079	0.00107

0.00932 and 0.00072, respectively. Besides, it takes approximately 0.002 seconds (0.003 seconds, respectively) to generate one numerical result of the price (the delta, respectively).

Under the CIR model, Fusai et al. [26] derived a recursion-based analytical expression for the moment generating function of the joint distribution of the spot price's terminal value at maturity and its discretely monitored average. Then applying the Fourier inversion algorithm, they can price discretely monitored Asian options numerically in a fast way. Table 10 presents a comparison between our asymptotic expansion results with those obtained via Fourier inversion in Fusai et al. [26]. It can be seen that our results are very accurate because the absolute errors are no greater than 0.00002.

6. Concluding remarks.   In this paper, a closed-form asymptotic expansion approach is proposed to price discretely monitored Asian options in one-dimensional diffusion models. We show the convergence of the expansion rigorously under some regularity conditions and moreover, develop a systematic method for the purpose of calculating general expansion terms. Numerical experiments suggest that our expansion method with only a few terms (e.g., four terms up to the third order) is accurate, fast, and easy to implement for a wide range of diffusion models. Potential future research topics include the extensions of our expansion method to time-dependent one-dimensional diffusions, multidimensional diffusions, or nonuniform discrete structures. One may first generalize our approach for time-independent one-dimensional diffusions to the time-independent multidimensional case. Then the time-dependent one-dimensional diffusion might be dealt with as a special time-independent two-dimensional diffusion. As regards the nonuniform discrete structure, one potential idea is

TABLE 9. Prices and deltas of European call options under the CEV model with $\beta = 1/4$, $-1/4$, and $-1/2$ (CIR model). The parameters are the same as in Table 8. The columns "AE" represent our asymptotic expansion results up to the third order, and the columns "True value" are obtained from the analytical formula (see, e.g., Davydov and Linetsky [18]). "Abs. err." denotes the absolute error between "AE" and "True value." We can see our expansion method is quite accurate. Besides, it takes around 0.002 seconds (0.003 seconds, respectively) to produce one numerical result of the price (the delta, respectively).

	Prices			Deltas		
K	AE	True value	Abs. err.	AE	True value	Abs. err.
	European call options under the CEV model with $\beta = 1/4$					
80	25.27467	25.28380	−0.00913	0.89921	0.89897	0.00024
85	21.47303	21.48223	−0.00920	0.84662	0.84633	0.00029
90	18.03386	18.04303	−0.00917	0.78389	0.78348	0.00041
95	14.98378	14.99294	−0.00916	0.71381	0.71333	0.00048
100	12.32843	12.33758	−0.00915	0.63972	0.63921	0.00051
105	10.05528	10.06444	−0.00916	0.56486	0.56439	0.00047
110	8.13829	8.14747	−0.00918	0.49205	0.49168	0.00037
115	6.54279	6.55200	−0.00921	0.42343	0.42322	0.00021
120	5.22998	5.23919	−0.00921	0.36041	0.36039	0.00002
	European call options under the CEV model with $\beta = -1/4$					
80	25.53970	25.54908	−0.00938	0.87517	0.87681	−0.00164
85	21.72499	21.73426	−0.00927	0.82223	0.82350	−0.00127
90	18.23275	18.24296	−0.00921	0.75981	0.76065	−0.00084
95	15.09495	15.10412	−0.00917	0.69000	0.69041	−0.00041
100	12.32843	12.33758	−0.00915	0.61552	0.61555	−0.00003
105	9.93458	9.94375	−0.00917	0.53931	0.53908	0.00023
110	7.90087	7.91008	−0.00921	0.46420	0.46386	0.00034
115	6.20347	6.21276	−0.00929	0.39267	0.39235	0.00032
120	4.81063	4.82003	−0.00940	0.32661	0.32642	0.00019
	European call options under the CEV model with $\beta = -1/2$					
80	25.68429	25.69403	−0.00974	0.86140	0.86439	−0.00299
85	21.85874	21.86830	−0.00956	0.80844	0.81106	−0.00262
90	18.33753	18.34695	−0.00942	0.74639	0.74849	−0.00210
95	15.15501	15.16433	−0.00932	0.67700	0.67850	−0.00150
100	12.33306	12.34234	−0.00928	0.60269	0.60359	−0.00090
105	9.87961	9.88895	−0.00934	0.52623	0.52663	−0.00040
110	7.78877	7.79824	−0.00947	0.45043	0.45049	−0.00006
115	6.04236	6.05203	−0.00967	0.37788	0.37778	0.00010
120	4.61249	4.62239	−0.00990	0.31071	0.30159	0.00012

to select the smallest time step as the expansion parameter by expressing other larger time steps as (fractional) multiples of the smallest one.

Appendix A. Proof of Theorem 2.1.

PROOF.   Our proof is an application of the Malliavin calculus for generalized random variables proposed in Watanabe [64] and Yoshida [65] as well as the related theory of asymptotic expansion for option pricing established in Kunitomo and Takahashi [42]. We employ standard notations of the Malliavin calculus (see, e.g., Ikeda and Watanabe [32], Nualart [50]). Let $Z(\epsilon) := \sum_{k=1}^{m} Y_k(\epsilon)$, where $Y_k(\epsilon)$ is defined in (6). For any integer $n \in \mathbb{N}$, we have

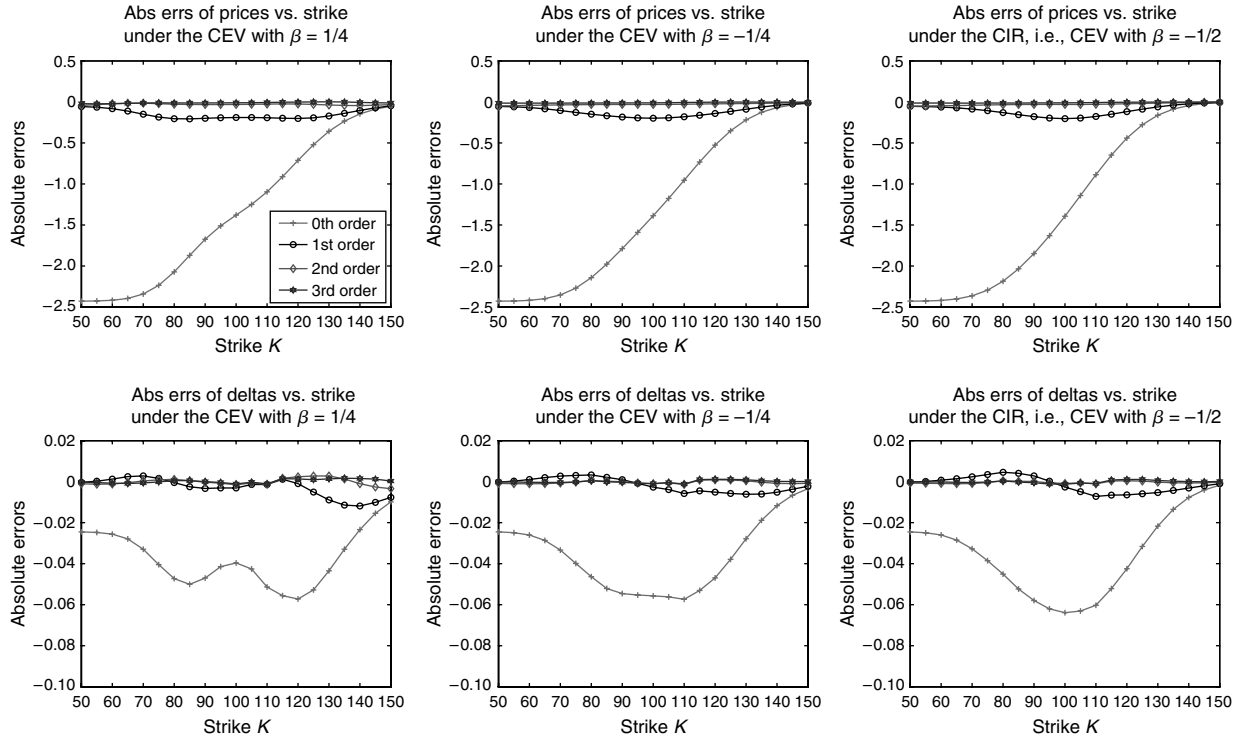$$Z(\epsilon) := \sum_{k=1}^{m} Y_k(\epsilon) = \sum_{j=0}^{n} Z_j \epsilon^j + O(\epsilon^{n+1}), \tag{A1}$$

FIGURE 4. Convergence of the asymptotic expansions for prices and deltas of discretely monitored Asian options under the CEV models with $\beta = 1/4$, $-1/4$, and $-1/2$, respectively. Parameters are $r = 0.05$, $S_0 = 100$, $\delta S_0^{\beta} = 0.25$, $T = 1$ and $m = 250$. The reference true values for prices and deltas are Monte Carlo simulation estimates obtained by using 10,000 time steps and by simulating 2,000,000 and 200,000 sample paths, respectively.

TABLE 10. Comparison with numerical results in Fusai et al. [26] under the CIR model. Other parameters are $S_0 = 1$, $r = 0.04$, $\sigma = 0.7$, and $T = 1$. The rows "AE" represent our asymptotic expansion results up to the third order, and the rows "Fusai et al." are taken from Table 4 in Fusai et al. [26]. The rows "Abs. err." denote the absolute errors between "AE" and "Fusai et al." It can be seen that our results are very accurate with absolute errors no greater than 0.00002.

		Comparison with numerical results in Fusai et al. [26]			
K	0.90	0.95	1.00	1.05	1.10
			$m = 12$		
AE	0.21280	0.18660	0.16283	0.14141	0.12224
Fusai et al.	0.21279	0.18659	0.16282	0.14140	0.12223
Abs. err.	0.00001	0.00001	0.00001	0.00001	0.00001
			$m = 25$		
AE	0.21430	0.18811	0.16433	0.14288	0.12366
Fusai et al.	0.21428	0.18810	0.16432	0.14287	0.12365
Abs. err.	0.00002	0.00001	0.00001	0.00001	0.00001
			$m = 50$		
AE	0.21502	0.18884	0.16506	0.14359	0.12435
Fusai et al.	0.21501	0.18883	0.16505	0.14359	0.12434
Abs. err.	0.00001	0.00001	0.00001	0.00000	0.00001
			$m = 100$		
AE	0.21539	0.18921	0.16543	0.14396	0.12471
Fusai et al.	0.21538	0.18920	0.16542	0.14395	0.12470
Abs. err.	0.00001	0.00001	0.00001	0.00001	0.00001
			$m = 250$		
AE	0.21562	0.18944	0.16566	0.14419	0.12493
Fusai et al.	0.21560	0.18943	0.16565	0.14418	0.12492
Abs. err.	0.00002	0.00001	0.00001	0.00001	0.00001

where $Z_j = \sum_{k=1}^m Y_{k,j}$ with $Y_{k,j}$ constructed from (9) and (35). Standard arguments as employed in the proof of Theorem 7.1 in Malliavin and Thalmaier [48] yield that the expansion (A1) can be interpreted in the following sense

$$\left\| Z(\epsilon) - \sum_{j=0}^n Z_j \epsilon^j \right\|_{D_p^s} = O(\epsilon^{n+1}),$$

for any $p > 1$ and $s > 0$, under the conditions that $\mu(\cdot)$ and $\sigma(\cdot)$ have bounded derivatives of all orders and $\sigma(s_0) \neq 0$. Here D_p^s denotes the space of Malliavin differentiable variables equipped with the norm

$$\|F\|_{D_p^s} = \left[E\|F\|^p + \sum_{j=1}^s E\|D^{(j)}F\|_{H^{\otimes j}}^p \right]^{1/p} \quad \text{for any } F \in D_p^s,$$

where $D^{(j)}F$ is the jth order Malliavin derivative of F. For simplicity of notations, we use D to represent the first order Malliavin differentiation operator $D^{(1)}$.

To prove that $E[(Z(\epsilon) - z)^+]$ admits the asymptotic expansion

$$E[(Z(\epsilon) - z)^+] = \sum_{j=0}^n \Omega_j(z)\epsilon^j + O(\epsilon^{n+1}), \tag{A2}$$

our immediate task is to verify that there exists a random sequence $\{\eta^\epsilon\}$ such that the Malliavin covariance matrix $\Sigma(\epsilon) := \langle DZ(\epsilon), DZ(\epsilon) \rangle_{L^2([0,T])}$ satisfies the following two conditions:
- Uniform nondegeneracy under truncation:

$$\sup_{\epsilon \in [0,1]} E\left[1_{\{\eta^\epsilon \leq 1\}} (\det(\Sigma(\epsilon)))^{-p} \right] < +\infty, \tag{A3}$$

- Negligible probability of truncation:

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^n} P\left(|\eta^\epsilon| > \frac{1}{2} \right) = 0, \quad n \in \mathbb{N}. \tag{A4}$$

We will employ the method developed in Kunitomo and Takahashi [41, 42] to complete the verification. Let us verify (A3) first. Since $Z(0) = \sum_{k=1}^m W(k)/\gamma$, its Malliavin covariance can be easily obtained

$$\Sigma(0) := \langle DZ(0), DZ(0) \rangle_H = \int_0^m |D_s Z(0)|^2 \, ds = \int_0^m \left| \sum_{k=1}^m \frac{1}{\gamma} 1_{\{s \leq k\}} \right|^2 ds = \frac{1}{\gamma^2} \sum_{k=1}^m (m+1-k)^2 = 1.$$

Define

$$\eta^\epsilon := c \int_0^m \left| D_s Z(\epsilon) - D_s Z(0) \right|^2 ds,$$

where $c > 0$ is an arbitrary constant. Then we claim that there exists a constant $c_0 > 0$ such that for any $c > c_0$ and any $p > 1$, the condition (A3) holds. Indeed, it follows from the triangle inequality that

$$\left| (D_s Z(\epsilon))^2 - (D_s Z(0))^2 \right| \leq \left| D_s Z(\epsilon) - D_s Z(0) \right|^2 + 2|D_s Z(0)| \left| D_s Z(\epsilon) - D_s Z(0) \right|.$$

On the set $\{\eta^\epsilon \leq 1\}$ we have

$$
\begin{aligned}
|\Sigma(\epsilon) - \Sigma(0)| &= \left| \int_0^m (D_s Z(\epsilon))^2 \, ds - \int_0^m (D_s Z(0))^2 \, ds \right| \leq \int_0^m \left| (D_s Z(\epsilon))^2 - (D_s Z(0))^2 \right| ds \\
&\leq \int_0^m \left| D_s Z(\epsilon) - D_s Z(0) \right|^2 ds + \int_0^m 2|D_s Z(0)| \left| D_s Z(\epsilon) - D_s Z(0) \right| ds \\
&\leq \int_0^m \left| D_s Z(\epsilon) - D_s Z(0) \right|^2 ds + 2\left(\int_0^m |D_s Z(0)|^2 \, ds \right)^{1/2} \left(\int_0^m |D_s Z(\epsilon) - D_s Z(0)|^2 \, ds \right)^{1/2} \\
&\leq \frac{1}{c} + 2\sqrt{\frac{\Sigma(0)}{c}}.
\end{aligned}
$$

Hence, there exists c_0 such that, for any $c > c_0 > 0$,

$$|\Sigma(\epsilon)| \geq \Sigma(0) - |\Sigma(\epsilon) - \Sigma(0)| > \Sigma(0) - \left(\frac{1}{c_0} + 2\sqrt{\frac{\Sigma(0)}{c_0}} \right).$$

Thus we obtain (A3) immediately.

Next, we justify the condition (A4). The Cauchy inequality implies

$$|D_s Z(\epsilon) - D_s Z(0)|^2 = \left| \sum_{k=1}^{m} \left(D_s \frac{X(k,\epsilon) - s_0}{\epsilon \sigma(s_0)\gamma} - \frac{1}{\gamma} 1_{\{s \leq k\}} \right) \right|^2 \leq m \sum_{k=1}^{m} \left(D_s \frac{X(k,\epsilon) - s_0}{\epsilon \sigma(s_0)\gamma} - \frac{1}{\gamma} 1_{\{s \leq k\}} \right)^2,$$

where we use the fact that $Z(\epsilon) = \sum_{k=1}^{m} (X(k,\epsilon) - s_0)/(\epsilon \sigma(s_0)\gamma)$. Therefore,

$$\int_0^m |D_s Z(\epsilon) - D_s Z(0)|^2 \, ds \leq m \sum_{k=1}^{m} \int_0^m \left(D_s \frac{X(k,\epsilon) - s_0}{\epsilon \sigma(s_0)\gamma} - \frac{1}{\gamma} 1_{\{s \leq k\}} \right)^2 ds.$$

It follows that

$$P\left(|\eta^\epsilon| \geq \frac{1}{2} \right) \leq \sum_{k=1}^{m} P\left(\int_0^m \left(D_s \frac{X(k,\epsilon) - s_0}{\epsilon \sigma(s_0)\gamma} - \frac{1}{\gamma} 1_{\{s \leq k\}} \right)^2 ds \geq \frac{1}{2cm^2} \right)$$

$$= \sum_{k=1}^{m} P\left(\int_0^m (D_u Y_k(\epsilon) - D_u Y_k(0))^2 \, du \geq \frac{1}{2cm^2} \right).$$

Following Lemma 7.2 in Kunitomo and Takahashi [42], we conclude that for any $k = 1, 2, \ldots, m$ and any $n \in \mathbb{N}$,

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^n} P\left(\int_0^m (D_u Y_k(\epsilon) - D_u Y_k(0))^2 \, du \geq \frac{\Delta}{2cm^2} \right) = 0,$$

which leads to (A4) immediately. The proof is completed.

Appendix B. Proof of Theorem 4.1.

PROOF. Since Z_0 also has a standard normal distribution under the general diffusions, $\Omega_0(z)$ is the same as in the BSM. Here we focus on the derivation of $\Omega_k(z)$ for $k \geq 1$. Observing that

$$\frac{\partial^{(l)} T(x)}{\partial x^l} = \begin{cases} 1_{\{x \geq z\}}, & \text{if } l = 1; \\ \delta^{(l-2)}(x - z), & \text{if } l \geq 2, \end{cases}$$

we deduce from (33) that

$$\Omega_k(z) = \sum_{i=1}^{m} E\left(1_{\{Z_0 \geq z\}} \frac{F_{i,k+1}}{\sigma(s_0)\gamma} \right)$$

$$+ \sum_{l \geq 2, (l,(i_1,i_2,\ldots,i_l),(j_1,j_2,\ldots,j_l)) \in \mathscr{S}_k} \frac{1}{l!} E\left(\delta^{(l-2)}(Z_0 - z) \frac{F_{i_1,j_1+1}}{\sigma(s_0)\gamma} \frac{F_{i_2,j_2+1}}{\sigma(s_0)\gamma} \cdots \frac{F_{i_l,j_l+1}}{\sigma(s_0)\gamma} \right).$$

The involved expectations can be further expressed as

$$E\left(1_{\{Z_0 \geq z\}} \frac{F_{i,k+1}}{\sigma(s_0)\gamma} \right) = \frac{1}{\sigma(s_0)\gamma} \int_z^\infty E(F_{i,k+1} \mid Z_0 = z)\phi(z) \, dz$$

and

$$E\left(\delta^{(l-2)}(Z_0 - z) \frac{F_{i_1,j_1+1}}{\sigma(s_0)\gamma} \frac{F_{i_2,j_2+1}}{\sigma(s_0)\gamma} \cdots \frac{F_{i_l,j_l+1}}{\sigma(s_0)\gamma} \right)$$

$$= \left(\frac{1}{\sigma(s_0)\gamma} \right)^l \int_{\mathbb{R}} \delta^{(l-2)}(u - z) E(F_{i_1,j_1+1} F_{i_2,j_2+1} ldots, F_{i_l,j_l+1} \mid Z_0 = u)\phi(u) \, du$$

$$= \left(\frac{1}{\sigma(s_0)\gamma} \right)^l (-1)^{l-2} \int_{\mathbf{R}} \delta(u - z) \frac{\partial^{(l-2)}}{\partial z^{l-2}} \left[E(F_{i_1,j_1+1} F_{i_2,j_2+1}, \ldots, F_{i_l,j_l+1} \mid Z_0 = u)\phi(u) \right] du$$

$$= (-1)^{l-2} \left(\frac{1}{\sigma(s_0)\gamma} \right)^l \frac{\partial^{(l-2)}}{\partial z^{l-2}} \left[E(F_{i_1,j_1+1} F_{i_2,j_2+1}, \ldots, F_{i_l,j_l+1} \mid Z_0 = z)\phi(z) \right],$$

where the second equality holds because of the integration-by-parts formula for the Dirac-delta function. Indeed, the above calculations can be rigorously justified through Watanabe [64], which untangled the puzzle on the calculation of expectations involving Dirac-delta functions.

It follows that

$$
\Omega_k(z) = \sum_{i=1}^{m} \frac{1}{\sigma(s_0)\gamma} \int_z^{\infty} E(F_{i,k+1} \mid Z_0 = u)\phi(u)\, du + \sum_{l \geq 2, (l,(i_1,i_2,\ldots,i_l),(j_1,j_2,\ldots,j_l)) \in \mathscr{S}_k} \frac{(-1)^{l-2}}{l!}\left(\frac{1}{\sigma(s_0)\gamma}\right)^l
$$
$$
\times \frac{\partial^{(l-2)}}{\partial z^{l-2}}\big[E(F_{i_1,j_1+1}F_{i_2,j_2+1},\ldots,F_{i_l,j_l+1} \mid Z_0 = z)\phi(z)\big]. \tag{B1}
$$

Applying the general formula for pathwise expansion in Lemma 4.1 yields

$$
E(F_{i,k+1} \mid Z_0 = z) = \sum_{\|\mathbf{i}\|=k+1} C_{\mathbf{i}}(s_0)E(J_{\mathbf{i}}(i) \mid Z_0 = z) = \sum_{\|\mathbf{i}\|=k+1} C_{\mathbf{i}}(s_0)P_{(\mathbf{i})}^{(i)}(z),
$$

$$
E\big(F_{i_1,j_1+1}F_{i_2,j_2+1},\ldots,F_{i_l,j_l+1} \mid Z_0 = z\big) = \sum_{\|\mathbf{i}_1\|=j_1+1,\ldots,\|\mathbf{i}_l\|=j_l+1} \left(\prod_{r=1}^{l} C_{\mathbf{i}_r}(s_0)\right)E\left(\prod_{r=1}^{l} J_{\mathbf{i}_r}(i_r) \mid Z_0 = z\right)
$$

$$
= \sum_{\|\mathbf{i}_1\|=j_1+1,\ldots,\|\mathbf{i}_l\|=j_l+1} \left(\prod_{r=1}^{l} C_{\mathbf{i}_r}(s_0)\right)P_{(\mathbf{i}_1,\mathbf{i}_2,\ldots,\mathbf{i}_l)}^{(i_1,i_2,\ldots,i_l)}(z).
$$

Substituting the above into (B1) results in (42) immediately. $\square$

Appendix C. Proof of Lemma 4.2.

PROOF. Since the calculations of these conditional expectations are quite similar, we only demonstrate how to compute $P_{((1,1),(0,1))}^{(i_1,i_2)}(z)$, and others can be dealt with similarly. Following (43), we deduce that

$$
P_{((1,1),(0,1))}^{(i_1,i_2)}(z) := E\left(J_{(1,1)}(i_1)J_{(0,1)}(i_2) \,\bigg|\, \sum_{k=1}^{m} W(k) = \gamma z\right)
$$
$$
= E\left[E\big(J_{(1,1)}(i_1)J_{(0,1)}(i_2) \mid W(1), W(2), \ldots, W(m)\big) \,\bigg|\, \sum_{k=1}^{m} W(k) = \gamma z\right].
$$

Converting related iterated Stratonovich integrals to iterated Itô integrals (see Kloeden and Platen [37]), we have

$$
J_{(1,1)}(i_1) := \int_0^{i_1}\int_0^{t_1} \circ\, dW(t_2) \circ dW(t_1) = I_{(1,1)}(i_1) + \frac{i_1}{2} \equiv \frac{1}{2}W(i_1)^2,
$$
$$
J_{(0,1)}(i_2) := \int_0^{i_2}\int_0^{t_1} \circ\, dW(t_2) \circ dt_1 = I_{(0,1)}(i_2) \equiv \int_0^{i_2} W(t_1)\, dt_1.
$$

Thus,

$$
P_{((1,1),(0,1))}^{(i_1,i_2)}(z) = E\left[\frac{1}{2}W(i_1)^2 E\left(\int_0^{i_2} W(t_1)\, dt_1 \,\bigg|\, W(1), W(2), \ldots, W(m)\right) \,\bigg|\, \sum_{k=1}^{m} W(k) = \gamma z\right]. \tag{C1}
$$

Applying the construction of the multiply pinned Brownian motion defined in (47) yields

$$
E\left(\int_0^{i_2} W(t_1)\, dt_1 \,\bigg|\, W(1) = w_1, \ldots, W(m) = w_m\right) = E\left(\int_0^{i_2} \mathscr{W}(t)\, dt\right)
$$
$$
= E\sum_{k=0}^{i_2-1}\int_k^{k+1} \big\{w_k(1-t+k) + w_{k+1}(t-k) + \big[\mathscr{B}(t) - \mathscr{B}(k) - (t-k)(\mathscr{B}(k+1) - \mathscr{B}(k))\big]\big\}\, dt
$$
$$
= E\sum_{k=0}^{i_2-1}\left(\frac{1}{2}w_k + \frac{1}{2}w_{k+1} + \int_k^{k+1}\mathscr{B}(t)\, dt - \frac{1}{2}\mathscr{B}(k) - \frac{1}{2}\mathscr{B}(k+1)\right) = \sum_{k=1}^{i_2-1} w_k + \frac{1}{2}w_{i_2}. \tag{C2}
$$

Then plugging (C2) into (C1), we obtain

$$
P_{((1,1),(0,1))}^{(i_1,i_2)}(z) = E\left[\frac{1}{2}W(i_1)^2\left(\sum_{k=1}^{i_2-1} W(k) + \frac{1}{2}W(i_2)\right) \,\bigg|\, \sum_{k=1}^{m} W(k) = \gamma z\right]
$$
$$
= \frac{1}{2}\sum_{k=1}^{i_2-1} i_1\sqrt{k}\, M_{(k,i_1)}^{(1,2)}(z) + \frac{1}{4}i_1\sqrt{i_2}\, M_{(i_2,i_1)}^{(1,2)}(z),
$$

which completes the proof.

Appendix D. Validity of the Greeks approximation. To emphasize the dependence on the parameter s_0, we express the asymptotic expansion proposed in Theorem 2.1 as

$$C(\Delta) = \frac{e^{-rT}\gamma\sqrt{\Delta} \cdot \sigma(s_0)}{m+1}\left(\sum_{j=0}^{J}\Omega_j(z(s_0), s_0)\Delta^{j/2} + O(\Delta^{(J+1)/2})\right) \quad \text{with } z(s_0) = \frac{(m+1)(K-s_0)}{\sqrt{\Delta}\sigma(s_0)\gamma},$$

where $\Omega_j(z, s_0)$, $j = 0, 1, 2, \ldots$, are equal to $\Omega_j(z)$ given by (14).

As exhibited in the computational examples in §5, we approximate Greeks (price sensitivities) by directly differentiating the expansion of option price. For example, the Jth order approximation to delta $\partial C(\Delta)/\partial s_0$ is calculated as

$$\text{Delta}_J = \frac{\partial}{\partial s_0}\left[\frac{e^{-rT}\gamma\sqrt{\Delta}\sigma(s_0)}{m+1}\left(\sum_{j=0}^{J}\Omega_j(z(s_0), s_0)\Delta^{j/2}\right)\right] = \frac{e^{-rT}\gamma\sqrt{\Delta}}{m+1}\sum_{j=0}^{J}D_j(z(s_0), s_0)\Delta^{j/2}, \quad \text{(D1)}$$

where

$$D_j(z, s_0) := \sigma'(s_0)\Omega_j(z, s_0) + \sigma(s_0)\left(\frac{\partial\Omega_j}{\partial z}(z, s_0)z'(s_0) + \frac{\partial\Omega_j}{\partial s_0}(z, s_0)\right). \quad \text{(D2)}$$

The Jth order approximation to gamma $\partial^2 C(\Delta)/\partial s_0^2$ is given by

$$\text{Gamma}_J = \frac{\partial^2}{\partial s_0^2}\left[\frac{e^{-rT}\gamma\sqrt{\Delta}\sigma(s_0)}{m+1}\left(\sum_{j=0}^{J}\Omega_j(z(s_0), s_0)\Delta^{j/2}\right)\right] = \frac{e^{-rT}\gamma\sqrt{\Delta}}{m+1}\sum_{j=0}^{J}G_j(z(s_0), s_0)\Delta^{j/2},$$

where

$$G_j(z, s_0) = \sigma''(s_0)\Omega_j(z, s_0) + 2\sigma'(s_0)\left(\frac{\partial\Omega_j}{\partial z}(z, s_0)z'(s_0) + \frac{\partial\Omega_j}{\partial s_0}(z, s_0)\right)$$

$$+ \sigma(s_0)\left[\left(\frac{\partial^2\Omega_j}{\partial z^2}(z, s_0)z'(s_0) + \frac{\partial^2\Omega_j}{\partial s_0\partial z}(z, s_0)\right)(z'(s_0) + 1) + \frac{\partial\Omega_j}{\partial z}(z, s_0)z''(s_0)\right]. \quad \text{(D3)}$$

We set up the following proposition to clarify the validity of the above approximations.

PROPOSITION D.1. *Assume that $\sigma(s_0) \neq 0$ and the two functions $\mu(\cdot)$ and $\sigma(\cdot)$ have bounded derivatives of all orders. For any $J = 0, 1, 2, \ldots$, the following asymptotic expansions hold in the sense of classical calculus:*

$$Delta = \frac{\partial C(\Delta)}{\partial s_0} = \frac{e^{-rT}\gamma\sqrt{\Delta}}{m+1}\left(\sum_{j=0}^{J}D_j(z(s_0), s_0)\Delta^{j/2} + O(\Delta^{(J+1)/2})\right), \quad \text{(D4)}$$

and

$$Gamma = \frac{\partial^2 C(\Delta)}{\partial s_0^2} = \frac{e^{-rT}\gamma\sqrt{\Delta}}{m+1}\left(\sum_{j=0}^{J}G_j(z(s_0), s_0)\Delta^{j/2} + O(\Delta^{(J+1)/2})\right), \quad \text{(D5)}$$

where $D_j(z, s_0)$ and $G_j(z, s_0)$ are defined in (D2) and (D3), respectively.

PROOF. Without loss of generality, we focus on the proof of (D4). Similar to the proof of Theorem 2.1, our argument is based on the Malliavin calculus for generalized random variables and the related theory of asymptotic expansion established in Watanabe [64], Yoshida [65], and Kunitomo and Takahashi [42].

To emphasize the dependence on s_0, we write $Z(\epsilon)$ in (A1) by $Z(\epsilon, s_0)$. From (7), we recall that the price of the Asian option satisfies

$$C(\Delta) = \frac{e^{-rT}\sqrt{\Delta}\sigma(s_0)\gamma}{m+1}E[(Z(\epsilon, s_0) - z(s_0))^+], \quad \text{with } z(s_0) = \frac{(m+1)(K-s_0)}{\sqrt{\Delta}\sigma(s_0)\gamma}.$$

Thus, differentiation of the above expression yields that

$$\text{Delta} = \frac{\partial C(\Delta)}{\partial s_0} = \frac{e^{-rT}\sqrt{\Delta}\gamma}{m+1}\sigma'(s_0)E[(Z(\epsilon, s_0) - z(s_0))^+] + \frac{e^{-rT}\sqrt{\Delta}\gamma}{m+1}\sigma(s_0)\frac{\partial}{\partial s_0}E(Z(\epsilon, s_0) - z(s_0))^+.$$

In the proof of Theorem 2.1 in Appendix A, we have obtained (A2), i.e.,

$$E[(Z(\epsilon, s_0) - z(s_0))^+] = \sum_{j=0}^{J}\Omega_j(z(s_0))\epsilon^j + O(\epsilon^{J+1}).$$

On the other hand, according to Watanabe [64], we have

$$\frac{\partial}{\partial s_0}\big(E[(Z(\epsilon, s_0) - z(s_0))^+]\big) = E\bigg[1_{\{Z(\epsilon, s_0) - z(s_0) \geq 0\}}\frac{\partial}{\partial s_0}Z(\epsilon, s_0)\bigg] - z'(s_0)E[1_{\{Z(\epsilon, s_0) - z(s_0) \geq 0\}}].$$

In what follows, we will prove that

$$E[1_{\{Z(\epsilon, s_0) - z \geq 0\}}] = -\sum_{j=0}^{J}\frac{\partial\Omega_j}{\partial z}(z, s_0)\epsilon^j + O(\epsilon^{J+1}), \tag{D6}$$

and

$$E\bigg[1_{\{Z(\epsilon, s_0) - z \geq 0\}}\frac{\partial}{\partial s_0}Z(\epsilon, s_0)\bigg] = \sum_{j=0}^{J}\frac{\partial\Omega_j}{\partial s_0}(z, s_0)\epsilon^j + O(\epsilon^{J+1}). \tag{D7}$$

The uniform nondegeneracy of $Z(\epsilon, s_0)$ allows us to apply the Watanabe-Yoshida theory (see Watanabe [64], Yoshida [65]) to obtain an expansion for $E[1_{\{Z(\epsilon, s_0) - z \geq 0\}}]$. Indeed, the following expansion is valid in the sense of $D^{-\infty}$, i.e.,

$$1_{\{Z(\epsilon, s_0) - z \geq 0\}} = \sum_{k=0}^{J}\Upsilon_k(z)\epsilon^k + O(\epsilon^{J+1}).$$

Here, the leading term is given by $\Upsilon_0(z) = 1_{\{Z_0 - z \geq 0\}}$ and the higher order terms are given by

$$\Upsilon_k(z) = \sum_{(l, (i_1, i_2, \ldots, i_l), (j_1, j_2, \ldots, j_l)) \in \mathscr{S}_k}\frac{1}{l!}\frac{\partial^{(l)}H(Z_0)}{\partial x^l}Y_{i_1, j_1}Y_{i_2, j_2}, \ldots, Y_{i_l, j_l}, \quad \text{for } k \geq 1,$$

where $H(x) := 1_{\{x - z \geq 0\}}$ is a Heaviside function and the index set $\mathscr{S}_k$ is specified in (34). Thus, the theory of Watanabe-Yoshida guarantees the validity of the following expansion:

$$E[1_{\{Z(\epsilon, s_0) - z \geq 0\}}] = \sum_{k=0}^{J}\Theta_k(z)\epsilon^k + O(\epsilon^{J+1}),$$

where

$$\Theta_0(z) = E\Upsilon_0(z) = E1_{\{Z_0 - z \geq 0\}} = 1 - N(z),$$

and

$$\Theta_k(z) = E\Upsilon_k(z) = \sum_{(l, (i_1, i_2, \ldots, i_l), (j_1, j_2, \ldots, j_l)) \in \mathscr{S}_k}\frac{1}{l!}E\bigg(\frac{\partial^{(l)}H(Z_0)}{\partial x^l}Y_{i_1, j_1}Y_{i_2, j_2}, \ldots, Y_{i_l, j_l}\bigg), \quad \text{for } k \geq 1.$$

To show (D6), we will verify that $\Theta_k(z) \equiv -\partial\Omega_k/\partial z(z, s_0)$. By (14), it is sufficient to show that $\Upsilon_k(z) \equiv -\partial\Phi_k(z)/\partial z(z, s_0)$. Indeed, it is obvious that

$$\Upsilon_0(z) \equiv 1_{\{Z_0 - z \geq 0\}} \equiv -\frac{\partial\Phi_0(z)}{\partial z}(z, s_0) \equiv -\frac{\partial}{\partial z}(Z_0 - z)^+.$$

For higher order terms, based on (33), we deduce that

$$\frac{\partial\Phi_k(z)}{\partial z}(z, s_0) = \frac{\partial}{\partial z}\Bigg(\sum_{(l, (i_1, i_2, \ldots, i_l), (j_1, j_2, \ldots, j_l)) \in \mathscr{S}_k}\frac{1}{l!}\frac{\partial^{(l)}T(Z_0)}{\partial x^l}Y_{i_1, j_1}Y_{i_2, j_2}, \ldots, Y_{i_l, j_l}\Bigg)$$

$$\equiv -\sum_{(l, (i_1, i_2, \ldots, i_l), (j_1, j_2, \ldots, j_l)) \in \mathscr{S}_k}\frac{1}{l!}\frac{\partial^{(l)}H(Z_0)}{\partial x^l}Y_{i_1, j_1}Y_{i_2, j_2}, \ldots, Y_{i_l, j_l} = -\Upsilon_k(z).$$

Here we have used the following algebraic facts for the Heaviside function and the Dirac delta function:

$$\frac{\partial}{\partial z}\frac{\partial^{(l)}T(x)}{\partial x^l} = \frac{\partial}{\partial z}1_{\{x - z \geq 0\}} = -\delta(x - z) = -\frac{\partial^{(l)}H(x)}{\partial x^l}, \quad \text{for } l = 1;$$

$$\frac{\partial}{\partial z}\frac{\partial^{(l)}T(x)}{\partial x^l} = \frac{\partial}{\partial z}\delta^{(l-2)}(x - z) = -\delta^{(l-1)}(x - z) = -\frac{\partial^{(l)}H(x)}{\partial x^l}, \quad \text{for } l \geq 2.$$

Thus, the above arguments lead to the expansion (D6).

Next, we prove (D7). Based on the theory of stochastic flows (see, e.g., Ikeda and Watanabe [32], Kunita [39]), a standard argument as employed in the proof of Theorem 7.1 in Malliavin and Thalmaier [48] guarantees the following expansion in D^∞:

$$\frac{\partial}{\partial s_0} Z(\epsilon, s_0) = \frac{\partial}{\partial s_0} \sum_{k=1}^{m} Y_k(\epsilon) = \sum_{j=0}^{J} \left(\sum_{k=1}^{m} \frac{\partial}{\partial s_0} Y_{k,j} \right) \epsilon^j + O(\epsilon^{J+1}).$$

Thus, we have that

$$1_{\{Z(\epsilon, s_0) - z \geq 0\}} \frac{\partial}{\partial s_0} Z(\epsilon, s_0) = \left(\sum_{i=0}^{J} \Upsilon_i(z) \epsilon^i + O(\epsilon^{J+1}) \right) \left(\sum_{j=0}^{J} \left(\sum_{k=1}^{m} \frac{\partial}{\partial s_0} Y_{k,j} \right) \epsilon^j + O(\epsilon^{J+1}) \right)$$

$$= \sum_{k=0}^{J} \Xi_k(z) \epsilon^k + O(\epsilon^{J+1}), \tag{D8}$$

where the correction term is given by

$$\Xi_k(z) = \sum_{i=0}^{k} \Upsilon_{k-i}(z) \left(\sum_{n=1}^{m} \frac{\partial}{\partial s_0} Y_{n,i} \right).$$

According to Watanabe [64, Theorem 2.2], the pathwise expansion (D8) is valid in the sense of $D^{-\infty}$. Thus, the theory of Watanabe-Yoshida guarantees the validity of the following expansion:

$$E\left[1_{\{Z(\epsilon, s_0) - z \geq 0\}} \frac{\partial}{\partial s_0} Z(\epsilon, s_0) \right] = \sum_{k=0}^{J} E\Xi_k(z) \epsilon^k + O(\epsilon^{J+1}).$$

Thus, to show (D7), we will prove that $E\Xi_k(z) = \partial \Omega_j / \partial s_0 (z, s_0)$. It is sufficient to show that

$$\Xi_k(z) \equiv \frac{\partial \Phi_k(z)}{\partial s_0}. \tag{D9}$$

Indeed, for $k = 0$, we notice that

$$\Xi_0(z) = \Upsilon_0(z) \sum_{n=1}^{m} \frac{\partial}{\partial s_0} Y_{n,0} = \Upsilon_0(z) \sum_{n=1}^{m} \frac{\partial}{\partial s_0} \frac{F_{n,1}}{\sigma(s_0)\gamma} \equiv 1_{\{Z_0 - z \geq 0\}} \Upsilon_0(z) \sum_{n=1}^{m} \frac{\partial}{\partial s_0} \frac{\sigma(s_0) W(n)}{\sigma(s_0)\gamma} \equiv 0,$$

and, on the other hand,

$$\frac{\partial \Phi_0(z)}{\partial s_0} = \frac{\partial}{\partial s_0} (Z_0 - z)^+ \equiv 0.$$

For higher order terms, we have

$$\Xi_k(z) \equiv \sum_{i=1}^{k} \left(\sum_{(l, (i_1, i_2, \ldots, i_l), (j_1, j_2, \ldots, j_l)) \in \mathscr{S}_{k-i}} \frac{1}{l!} \frac{\partial^{(l)} H(Z_0)}{\partial x^l} Y_{i_1, j_1} Y_{i_2, j_2}, \ldots, Y_{i_l, j_l} \right) \left(\sum_{n=1}^{m} \frac{\partial}{\partial s_0} Y_{n,i} \right),$$

where we have used the fact that $\partial Y_{n,0}/\partial s_0 = 0$. On the other hand, based on (33), we note that

$$\frac{\partial \Phi_k(z)}{\partial s_0} = \sum_{(l, (i_1, i_2, \ldots, i_l), (j_1, j_2, \ldots, j_l)) \in \mathscr{S}_k} \frac{1}{l!} \frac{\partial^{(l)} T(Z_0)}{\partial x^l} \frac{\partial}{\partial s_0} (Y_{i_1, j_1} Y_{i_2, j_2}, \ldots, Y_{i_l, j_l}),$$

from where we deduce that the right-hand side of the above equation equals to

$$\sum_{(l, (i_1, i_2, \ldots, i_l), (j_1, j_2, \ldots, j_l)) \in \mathscr{S}_k} \frac{1}{l!} \frac{\partial^{(l-1)} H(Z_0)}{\partial x^l} \sum_{r=1}^{l} \frac{\partial Y_{i_r, j_r}}{\partial s_0} Y_{i_1, j_1} Y_{i_2, j_2}, \ldots, Y_{i_{r-1}, j_{r-1}} Y_{i_{r+1}, j_{r+1}}, \ldots, Y_{i_l, j_l}$$

$$= \sum_{(l, (i_1, i_2, \ldots, i_l), (j_1, j_2, \ldots, j_l)) \in \mathscr{S}_k} \frac{1}{l!} \frac{\partial^{(l-1)} H(Z_0)}{\partial x^l} l \frac{\partial Y_{i_l, j_l}}{\partial s_0} Y_{i_1, j_1} Y_{i_2, j_2}, \ldots, Y_{i_{l-1}, j_{l-1}}$$

$$= \sum_{j_l=1}^{k} \sum_{i_l=1}^{m} \left(\sum_{(l-1, (i_1, i_2, \ldots, i_{l-1}), (j_1, j_2, \ldots, j_{l-1})) \in \mathscr{S}_{k-j_l}} \frac{1}{(l-1)!} \frac{\partial^{(l-1)} H(Z_0)}{\partial x^l} Y_{i_1, j_1} Y_{i_2, j_2}, \ldots, Y_{i_{l-1}, j_{l-1}} \right) \frac{\partial Y_{i_l, j_l}}{\partial s_0}$$

$$= \sum_{i=1}^{k} \left(\sum_{(l, (i_1, i_2, \ldots, i_l), (j_1, j_2, \ldots, j_l)) \in \mathscr{S}_{k-i}} \frac{1}{l!} \frac{\partial^{(l)} H(Z_0)}{\partial x^l} Y_{i_1, j_1} Y_{i_2, j_2}, \ldots, Y_{i_l, j_l} \right) \left(\sum_{n=1}^{m} \frac{\partial}{\partial s_0} Y_{n,i} \right).$$

Here, we have used the following simple facts:

$$\frac{\partial^{(l)} H}{\partial x^l} = \frac{\partial^{(l+1)} T}{\partial x^l}, \quad \text{for } l = 0, 1, 2, \ldots,$$

as well as, for any $r, p = 1, 2, 3, \ldots, l$;

$$\sum_{(l,(i_1,i_2,\ldots,i_l),(j_1,j_2,\ldots,j_l)) \in \mathscr{S}_k} \frac{1}{l!} \frac{\partial^{(l-1)} H(Z_0)}{\partial x^l} \frac{\partial Y_{i_r,j_r}}{\partial s_0} Y_{i_1,j_1} Y_{i_2,j_2}, \ldots, Y_{i_{r-1},j_{r-1}} Y_{i_{r+1},j_{r+1}}, \ldots, Y_{i_l,j_l}$$

$$= \sum_{(l,(i_1,i_2,\ldots,i_l),(j_1,j_2,\ldots,j_l)) \in \mathscr{S}_k} \frac{1}{l!} \frac{\partial^{(l-1)} H(Z_0)}{\partial x^l} \frac{\partial Y_{i_p,j_p}}{\partial s_0} Y_{i_1,j_1} Y_{i_2,j_2}, \ldots, Y_{i_{p-1},j_{p-1}} Y_{i_{p+1},j_{p+1}}, \ldots, Y_{i_l,j_l};$$

owing to the definition of the index set $\mathscr{S}_k$ given in (34). Hence, we obtain the expansion (D9).

Finally, we obtain that

$$\text{Delta} = \frac{\partial C(\Delta)}{\partial s_0} = \frac{e^{-rT} \gamma \sqrt{\Delta}}{m+1} \sigma'(s_0) \left(\sum_{j=0}^{J} \Omega_j(z(s_0), s_0) \Delta^{j/2} + O(\Delta^{(J+1)/2}) \right)$$

$$+ \frac{e^{-rT} \gamma \sqrt{\Delta}}{m+1} \sigma(s_0) \left(\sum_{j=0}^{J} \left[\frac{\partial \Omega_j}{\partial z}(z(s_0), s_0) z'(s_0) + \frac{\partial \Omega_j}{\partial s_0}(z(s_0), s_0) \right] \Delta^{j/2} + O(\Delta^{(J+1)/2}) \right),$$

which leads to (D4) for validating the approximation for delta.

Similar argument leads to (D5) for validating the approximation for gamma. Thus, owing to the limited space of the paper, we omit the tedious calculations. $\square$

Appendix E. On the expansions for CEV processes. Option pricing under the CEV process needs to be dealt with carefully because of its interesting and important properties for different β (see, e.g., Emanuel and MacBeth [20], Davydov and Linetsky [18], Carr and Linetsky [13], Andersen and Andreasen [2], Lewis [43]). Following the associate editor's instruction, we interpret our expansions in the following sense.

E.1. The case of $\beta > 0$. When $\beta > 0$, the local volatility function of the CEV process is unbounded at high asset prices. According to the studies in Emanuel and MacBeth [20] (see also Davydove and Linetsky [18, Appendix B] and Lewis [43, Chapter 8]), the discounted CEV price is a strict local martingale and indeed a strict super-martingale on the time interval $[0, T]$ (see Elworthy et al. [19]), which represents a financial bubble in the terminology of Protter [53] and the references therein. As a result, no equivalent martingale measures exist (see Sin [57]). To avoid this problem, one can use the following limited CEV (LCEV hereafter) process proposed by Andersen and Andreasen [2]:

$$dS_U(t) = r S_U(t)\, dt + \delta S_U(t) \min\{U^\beta, S_U(t)^\beta\}\, dW(t), \quad \text{with } S_U(0) = s_0, \tag{E1}$$

where $U > 0$ is a large positive number. We can see that whenever the asset price crosses over the upper "switching level" U, the LCEV process switches to a geometric Brownian motion. In what follows, we will argue that our expansion formula (13) can be interpreted as an approximation to the Asian option price under the LCEV model (E1) with a large enough U.

To guarantee the validity of Theorem 2.1, we consider the following double-sided LCEV (DLCEV hereafter) model $\{S_D(t)\}$ with both the upper switching level $U > 0$ and the lower switching level $L > 0$,

$$dS_D(t) = r S_D(t)\, dt + \varphi(S_D(t))\, dW(t), \quad \text{with } S_D(0) = s_0, \tag{E2}$$

and its "smoothed" version $\{S_\varepsilon(t)\}$,

$$dS_\varepsilon(t) = r S_\varepsilon(t)\, dt + \varphi_\varepsilon(S_\varepsilon(t))\, dW(t), \quad \text{with } S_\varepsilon(0) = s_0. \tag{E3}$$

Here the function $\varphi(\cdot)$ is defined as

$$\varphi(x) := \delta x \left(1_{\{x \geq U\}} U^\beta + 1_{\{L < x < U\}} x^\beta + 1_{\{x \leq L\}} L^\beta \right) \tag{E4}$$

and the function $\varphi_\varepsilon(\cdot)$ is an infinitely smooth modification of $\varphi(\cdot)$ constructed by smoothening the "corners" in $(L - \varepsilon, L + \varepsilon)$ and $(U - \varepsilon, U + \varepsilon)$ for a small enough positive number $\varepsilon(< \min\{L, (U - L)/2\})$,

$$\varphi_\varepsilon(x) := \begin{cases} \delta x U^\beta, & \text{if } x > U + \varepsilon, \\ \delta x[f(x)x^\beta + (1 - f(x))U^\beta], & \text{if } U - \varepsilon < x \le U + \varepsilon, \\ \delta x^{\beta+1}, & \text{if } L + \varepsilon < x \le U - \varepsilon, \\ \delta x[(1 - f(x))x^\beta + f(x)L^\beta], & \text{if } L - \varepsilon < x \le L + \varepsilon, \\ \delta x L^\beta, & \text{if } x \le L - \varepsilon, \end{cases}$$

where the function $f(\cdot)$ is defined as

$$f(x) := \frac{\psi(\varepsilon + \delta - x)}{\psi(x - \varepsilon + \delta) + \psi(\varepsilon + \delta - x)}, \tag{E5}$$

with $\psi(x) := \exp(-1/x)$ for $x > 0$ and $\psi(x) := 0$ for $x \le 0$.

It is straightforward to verify that $\varphi_\varepsilon(\cdot)$ is infinitely smooth with bounded derivatives of all orders. Therefore, the model (E3) satisfies the condition of Theorem 2.1. We shall show that

$$E\left[\left(\frac{1}{m+1}\sum_{j=0}^m S_\varepsilon(j\Delta) - K\right)^+\right] - E\left[\left(\frac{1}{m+1}\sum_{j=0}^m S_U(j\Delta) - K\right)^+\right] \to 0, \quad \text{as } L \to 0 \text{ and } U \to +\infty. \tag{E6}$$

Then for any initial price s_0, our expansion formula (13), which provides an expansion for the Asian option price under the model (E3) with a small enough L and a large enough U, can be interpreted as an approximation to the Asian option price under the LCEV model (E1) with a large enough U.

To prove (E6), we first use a similar argument as in the proof of Theorem 4 in Andersen and Andreasen [2] to show

$$E\left[\left(\frac{1}{m+1}\sum_{j=0}^m S_D(j\Delta) - K\right)^+\right] - E\left[\left(\frac{1}{m+1}\sum_{j=0}^m S_U(j\Delta) - K\right)^+\right] \to 0, \quad \text{as } L \to 0 \text{ and } U \to +\infty. \tag{E7}$$

Applying Jensen's inequality twice yields (the function x^+ is convex)

$$E\left[\left(\frac{1}{m+1}\sum_{j=0}^m S_D(j\Delta) - K\right)^+\right] - E\left[\left(\frac{1}{m+1}\sum_{j=0}^m S_U(j\Delta) - K\right)^+\right]$$

$$\le E\left[\left(\frac{1}{m+1}\sum_{j=0}^m (S_D(j\Delta) - S_U(j\Delta))\right)^+\right] \le \frac{1}{m+1}\sum_{j=0}^m E[(S_D(j\Delta) - S_U(j\Delta))^+].$$

Therefore, it suffices to show that for any $j = 1, \ldots, m$,

$$E[(S_D(j\Delta) - S_U(j\Delta))^+] \to 0, \quad \text{as } L \to 0 \text{ and } U \to +\infty.$$

Denote by $\{\mathcal{F}(t)\}$ the filtration generated by the Brownian motion $\{W(t)\}$, and define the stopping time

$$\tau_L := \inf\{t \ge 0; S_D(t) \le L\} \equiv \inf\{t \ge 0; S_U(t) \le L\}.$$

Note that for any $j = 1, \ldots, m$,

$$E[(S_D(j\Delta) - S_U(j\Delta))^+] = E[(S_D(j\Delta) - S_U(j\Delta))^+ 1_{\{\tau_L < j\Delta\}}] \le E[S_D(j\Delta)1_{\{\tau_L < j\Delta\}}].$$

By the iterated conditioning and the optional sampling theorem, we obtain that for any $j = 1, \ldots, m$,

$$E[(S_D(j\Delta) - S_U(j\Delta))^+] \le E[E[S_D(j\Delta)1_{\{\tau_L < j\Delta\}} \mid \mathcal{F}(\min\{\tau_L, j\Delta\})]]$$

$$= e^{rj\Delta}E[1_{\{\tau_L < j\Delta\}}e^{-r\min\{\tau_L, j\Delta\}}S_D(\min\{\tau_L, j\Delta\})]$$

$$\le e^{rj\Delta}L \to 0, \quad \text{as } L \to 0 \text{ and } U \to +\infty,$$

which concludes the proof of (E7).

On the other hand, it can be shown that

$$E\left[\left(\frac{1}{m+1}\sum_{j=0}^{m}S_\varepsilon(j\Delta)-K\right)^+\right]-E\left[\left(\frac{1}{m+1}\sum_{j=0}^{m}S_D(j\Delta)-K\right)^+\right]\to 0,\quad\text{as }L\to 0\text{ and }U\to+\infty,$$

Indeed, if we define the stopping time

$$\tau_{U,L}=\inf\{t\geq 0;\,S_D(t)\geq U-\varepsilon\text{ or }S_D(t)\leq L+\varepsilon\}$$
$$\equiv\inf\{t\geq 0;\,S_\varepsilon(t)\geq U-\varepsilon\text{ or }S_\varepsilon(t)\leq L+\varepsilon\},$$

then it follows from the put-call parity that

$$\left|E\left[\left(\frac{1}{m+1}\sum_{j=0}^{m}S_\varepsilon(j\Delta)-K\right)^+\right]-E\left[\left(\frac{1}{m+1}\sum_{j=0}^{m}S_D(j\Delta)-K\right)^+\right]\right|$$

$$=\left|E\left\{\left[\left(K-\frac{1}{m+1}\sum_{j=0}^{m}S_\varepsilon(j\Delta)\right)^+-\left(K-\frac{1}{m+1}\sum_{j=0}^{m}S_D(j\Delta)\right)^+\right]1_{\{\tau_{U,L}<T\}}\right\}\right|$$

$$\leq 2KP(\tau_{U,L}<T)\to 0,\quad\text{as }L\to 0\text{ and }U\to+\infty,\tag{E8}$$

where the last inequality follows from the nonexplosiveness at infinity and the nonattainability at zero of the CEV process when $\beta>0$ (note that as $L\to 0$, $\varepsilon\to 0$ because $\varepsilon<L$). Then, combining (E7) with (E8) yields (E6) immediately.

E.2. The case of $\beta<0$. When $\beta<0$, zero is either an exit boundary (for $\beta\in[-1/2,0)$), or is a regular boundary (for $\beta<-1/2$) and is then specified as a killing boundary by adjoining a killing boundary condition (see Borodin and Salminen [9], Karlin and Taylor [36], Davydov and Linetsky [18]). Therefore, in either case the CEV process is stopped once it hits zero (the transition density is norm defective; see Lewis [43, Chapter 8]). Such a feature is employed to model bankruptcy; see, e.g., Carr and Linetsky [13].

To deal with this case, one can use the following LCEV process with a lower switching level $L>0$ (see Andersen and Andreasen [2]):

$$dS_L(t)=rS_L(t)\,dt+\hat{\varphi}(S_L(t))\,dW(t),\quad\text{with }S_L(0)=s_0,\tag{E9}$$

where the function $\hat{\varphi}(\cdot)$ is defined as

$$\hat{\varphi}(x):=\delta x\big(1_{\{x>L\}}x^\beta+1_{\{x\leq L\}}L^\beta\big).$$

In what follows, we shall argue that our expansion formula (13) can be interpreted as an approximation to the Asian option price under the LCEV model with a small enough L.

Similarly to the case of $\beta>0$, we consider the following "smoothed" version of the LCEV model (E9) to guarantee the validity of Theorem 2.1:

$$d\hat{S}_\varepsilon(t)=r\hat{S}_\varepsilon(t)\,dt+\hat{\varphi}_\varepsilon(\hat{S}_\varepsilon(t))\,dW(t),\quad\text{with }\hat{S}_\varepsilon(0)=s_0,\tag{E10}$$

where the function $\hat{\varphi}_\varepsilon(\cdot)$ is an infinitely smooth modification of $\hat{\varphi}(\cdot)$. By smoothening the corners in $(L-\varepsilon,L+\varepsilon)$ for a small enough positive number $\varepsilon(<L)$, we can construct the function $\hat{\varphi}_\varepsilon(\cdot)$ as follows

$$\hat{\varphi}_\varepsilon(x)=\begin{cases}\delta x^{\beta+1}, & \text{if }x>L+\varepsilon,\\ \delta x[(1-f(x))x^\beta+f(x)L^\beta], & \text{if }L-\varepsilon<x\leq L+\varepsilon,\\ \delta xL^\beta, & \text{if }x\leq L-\varepsilon,\end{cases}$$

where the function $f(\cdot)$ is defined in (E5). It is straightforward to verify that $\hat{\varphi}_\varepsilon(\cdot)$ is infinitely smooth with bounded derivatives of all orders. Therefore, the model (E10) satisfies the condition of Theorem 2.1.

Similarly to the argument as in the proof of Theorem 4 in Andersen and Andreasen [2], we can show

$$E\left[\left(\frac{1}{m+1}\sum_{j=0}^{m}\hat{S}_\varepsilon(j\Delta)-K\right)^+\right]-E\left[\left(\frac{1}{m+1}\sum_{j=0}^{m}S_L(j\Delta)-K\right)^+\right]\to 0,\quad\text{as }L\to 0.\tag{E11}$$

Accordingly, for any initial price s_0, our expansion formula (13), which provides an expansion for the Asian option price under the model (E10) with a small enough L, can be interpreted as an approximation to the Asian option price under the LCEV model (E9) with a small enough L.

To prove (E11), we shall first show by induction that for any $n \in \mathbb{N}$, $h_j \in \mathbb{R}^+$, $j = 1, \ldots, n$, and $0 < T_1 < T_2 < \cdots < T_n$,

$$P\big(\hat{S}_\varepsilon(T_j) \le h_j, j = 1, \ldots, n\big) \to P\big(S(T_j) \le h_j, j = 1, \ldots, n\big), \quad \text{as } L \to 0, \tag{E12}$$

where $\{S(t)\}$ is the CEV process with the killing boundary zero. When $n = 1$, (E12) has been proved in Andersen and Andreasen [2, Theorem 4]. Assume (E12) holds for $n = k$. Then when $n = k + 1$,

$$
\begin{aligned}
&\big|P(\hat{S}_\varepsilon(T_j) \le h_j, j = 1, \ldots, k+1) - P(S(T_j) \le h_j, j = 1, \ldots, k+1)\big| \\
&= \big|P(\hat{S}_\varepsilon(T_j) \le h_j, j = 1, \ldots, k) - P(\hat{S}_\varepsilon(T_j) \le h_j, j = 1, \ldots, k, \hat{S}_\varepsilon(T_{k+1}) > h_{k+1}) \\
&\quad - P(S(T_j) \le h_j, j = 1, \ldots, k) + P(S(T_j) \le h_j, j = 1, \ldots, k, S(T_{k+1}) > h_{k+1})\big| \\
&\le \big|P(\hat{S}_\varepsilon(T_j) \le h_j, j = 1, \ldots, k) - P(S(T_j) \le h_j, j = 1, \ldots, k)\big| \\
&\quad + \big|P(\hat{S}_\varepsilon(T_j) \le h_j, j = 1, \ldots, k, \hat{S}_\varepsilon(T_{k+1}) > h_{k+1}) - P(S(T_j) \le h_j, j = 1, \ldots, k, S(T_{k+1}) > h_{k+1})\big|. \tag{E13}
\end{aligned}
$$

The first term of the RHS of (E13) goes to 0 as $L \to 0$ by the induction hypothesis. As for the second term, we define

$$\hat{\tau}_L := \inf\{t \ge 0; \hat{S}_\varepsilon(t) \le L + \varepsilon\} \equiv \inf\{t \ge 0; S(t) \le L + \varepsilon\}.$$

Then we have

$$
\begin{aligned}
&\big|P(\hat{S}_\varepsilon(T_j) \le h_j, j = 1, \ldots, k, \hat{S}_\varepsilon(T_{k+1}) > h_{k+1}) - P(S(T_j) \le h_j, j = 1, \ldots, k, S(T_{k+1}) > h_{k+1})\big| \\
&= \big|P(\hat{S}_\varepsilon(T_j) \le h_j, j = 1, \ldots, k, \hat{S}_\varepsilon(T_{k+1}) > h_{k+1}, \hat{\tau}_L < T_{k+1}) \\
&\quad - P(S(T_j) \le h_j, j = 1, \ldots, k, S(T_{k+1}) > h_{k+1}, \hat{\tau}_L < T_{k+1})\big| \\
&\le P(\hat{S}_\varepsilon(T_{k+1}) > h_{k+1}, \hat{\tau}_L < T_{k+1}) + P(S(T_{k+1}) > h_{k+1}, \hat{\tau}_L < T_{k+1}) \\
&\to 0, \quad \text{as } L \to 0,
\end{aligned}
$$

where the limit holds thanks to (A.4) and (A.5) of the proof for Theorem 4 in Andersen and Andreasen [2]. Therefore, (E12) holds for $n = k + 1$, and hence the proof of (E12) is completed. In particular, this implies that $(\hat{S}_\varepsilon(\Delta), \ldots, \hat{S}_\varepsilon(m\Delta))$ converges weakly to $(S(\Delta), \ldots, S(m\Delta))$ as $L \to 0$. Note that $\pi(y_1, \ldots, y_m) := (K - (s_0 + \sum_{j=1}^m y_j)/(m+1))^+$ is a bounded, continuous function. Then it follows that

$$
\begin{aligned}
&\left|E\left[\left(\frac{1}{m+1}\sum_{j=0}^m \hat{S}_\varepsilon(j\Delta) - K\right)^+\right] - E\left[\left(\frac{1}{m+1}\sum_{j=0}^m S(j\Delta) - K\right)^+\right]\right| \\
&= \left|E\left[\left(K - \frac{1}{m+1}\sum_{j=0}^m \hat{S}_\varepsilon(j\Delta)\right)^+\right] - E\left[\left(K - \frac{1}{m+1}\sum_{j=0}^m S(j\Delta)\right)^+\right]\right| \\
&\to 0, \quad \text{as } L \to 0, \tag{E14}
\end{aligned}
$$

where the first equality holds due to the put-call parity.

Similarly, we can show that

$$\left|E\left[\left(\frac{1}{m+1}\sum_{j=0}^m S_L(j\Delta) - K\right)^+\right] - E\left[\left(\frac{1}{m+1}\sum_{j=0}^m S(j\Delta) - K\right)^+\right]\right| \to 0, \quad \text{as } L \to 0. \tag{E15}$$

Then (E11) follows immediately from (E14) and (E15).

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