Balanced Incomplete Latin Square Designs

Mingyao Ai[1,*], Kang Li[1], Senmao Liu[2], Dennis K. J. Lin[3]

[1]*LMAM, School of Mathematical Sciences and Center for Statistical Science, Peking University, Beijing 100871, China*

[2]*Department of Statistics, Texas A&M University, College Station, Texas 77843, US*

[3]*Department of Statistics, Penn State University, University Park, PA 16802, US*

Abstract. Latin squares have been widely used to design an experiment where the blocking factors and treatment factors are of the same levels. For some experiments, the size of blocks may be less than the number of treatments. Since not all of the treatments can be compared within each block, a new class of designs called balanced incomplete Latin squares (BILS) is proposed to deal with such experiments. A general method for constructing BILS is proposed by an intelligent selection of certain cells from a complete Latin square via orthogonal Latin squares. The optimality of the proposed BILS designs is investigated. It is shown that the proposed transversal BILS designs are asymptotically optimal for all the row, column and treatment effects. The relative efficiencies of a delete-one-transversal BILS design with respect to the optimal designs for both cases are also derived; it is shown to be close to 100%, as the order becomes large.

MSC: Primary 62K15; Secondary 62K10

Keywords: Balanced, incomplete Latin square, information matrix, optimality, orthogonal Latin square.

1 Introduction

A Latin square of order k, denoted by $LS(k)$, is a $k \times k$ square matrix of k symbols, say $1, 2, ..., k$, such that each symbol appears only once in each row and each column. Two Latin squares of the same order are said to be orthogonal, if these two squares when superimposed have the property that each pair of symbols appears exactly once. For detailed constructions of Latin squares and orthogonal Latin squares (OLS) refer to Dénes and Keedwell (1974, 1991).

Latin squares of order k have been widely applied to design an experiment in which three factors each at k levels are investigated by randomly assigning the k levels of the three factors to the rows, columns and the symbols of the squares, respectively. When both row and column factors are treated as two blocking factors, then one treatment factor corresponding

*Corresponding author. E-mail address: myai@math.pku.edu.cn (Ai).

to the symbols of the square can effectively be studied by removing the inter-row and inter-column variations. For detailed discussion refer to, for example, Wu and Hamada (2003). It should be noted that such a design supposes both blocks size is exactly equal to the number of treatments, i.e., a complete block design is adopted for each blocking factor.

For some experiments, however, the size of blocks may be less than the number of treatments. Since not all of the treatments can be compared within each block, a new class of incomplete Latin square (ILS) has to be adopted. An incomplete Latin square of order k and block size r $(r < k)$, denoted by $ILS(k, r)$, is an incomplete Latin square of order k in which each row and each column has r non-empty cells. If an $ILS(k, r)$ satisfies the condition that each symbol appears exactly r times in whole square, then the $ILS(k, r)$ is called a balanced incomplete Latin square, denoted by $BILS(k, r)$. For example, Table 1 presents an example of Latin square of order six, $LS(6)$. If the six cells in boldface are removed, then the rest cells form a $BILS(6, 5)$.

Table 1: $LS(6)$ and $BILS(6, 5)$

1	**2**	3	4	5	6
2	3	**6**	1	4	5
3	6	2	**5**	1	4
4	5	1	2	6	3
5	1	4	6	**3**	2
6	4	5	3	2	**1**

$LS(6)$

$\Longrightarrow$

1		3	4	5	6
2	3		1	4	5
3	6	2		1	4
	5	1	2	6	3
5	1	4	6		2
6	4	5	3	2	

$BILS(6, 5)$

The rest of the paper is unfolded as follows. Section 2 introduces a general method for constructing all kinds of BILS by an intelligent selection of certain cells from a complete Latin square via orthogonal Latin squares. Section 3 gives the application of a BILS design on a practical experiment, which works as nearly equally well as the complete Latin square design. Section 4 reviews the optimality criteria based on the information matrices for the effects of interest in a linear model, and then investigates the optimality of a $BILS(k, k-1)$ design among all designs corresponding to all kinds of discrete distributions on any complete Latin square. It is shown that for a given $LS(k)$, the uniform design on the k^2 cells is optimal for all the row, column and treatment effects. The relative efficiencies of a $BILS(k, k-1)$ design with respect to the foregoing optimal design for both cases are derived to be close to 100% as the order k becomes large. Section 5 concludes this paper with some remarks.

2 Construction of BILS

A natural way for constructing a BILS is to select certain cells from a complete Latin square such that the remaining cells satisfy the condition of balanced occurrance of symbols. It can be done by removing one or more "transversal." For a given $LS(k)$, a *transversal* is a set of k cells such that only one cell is allowed in each row and in each column, and furthermore, each symbol can appear in each cell once. It is known that for two orthogonal Latin squares of the same order k, any k cells of one square corresponding to the same symbol of the other square form a transversal. Bose, Shrikhande and Parker (1960) showed that there always exist at least two orthogonal Latin squares for any order $k \geq 4$ except for $k = 6$. Thus the following conclusion can be obtained.

Construction Method. *For any order $k \geq 4$ (except for $k = 6$), a $BILS(k, r)$ can be constructed by removing $k - r$ disjoint transversals from a $LS(k)$ via a pair of orthogonal Latin squares for any $3 \leq r \leq k - 1$.*

Note that if $r < 3$, the $BILS(k, r)$ design does not offer enough degrees of freedom for data analysis, so we will focus on the cases $r \geq 3$.

Example 1. *For $k = 4$, the two orthogonal $LS(4)$ are given in Table 2 (a), denoted by L_1 and L_2, respectively. There are four disjoint transversals in L_1 corresponding to symbols 1, 2, 3, and 4 in L_2, respectively. If we remove the transversal corresponding to 1, i.e., the cells with symbols in boldface, a $BILS(4, 3)$ is obtained, as displayed in Table 2 (b).*

Table 2: Two orthogonal $LS(4)$ and a $BILS(4, 3)$

3	4	2	1
1	**2**	4	3
4	3	**1**	2
2	1	3	**4**

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

(a) Two orthogonal $LS(4)$

$\Longrightarrow$

	4	2	1
1		4	3
4	3		2
2	1	3	

(b) $BILS(4, 3)$

For the $BILS(4, 3)$ in Example 1, each pair of symbols occurs two times in the same rows or the same columns. Actually, the following result can be verified.

Proposition 1. *For every $BILS(k, k - 1)$, the number of times each pair of symbols occur in the same rows (or the same columns) is $k - 2$.*

3 Example

Consider the wear experiment (Wu and Hamada, 2003, p.70) for testing the abrasion resistance of rubber-covered fabric in a Martindale wear tester. The original design is the complete Latin square L_1 in Table 2 (a), where symbols 1, 2, 3, 4 represent the four types of material A, B, C and D, respectively. The response is the loss in weight in 0.1 milligrams (mgm) over a standard period of time. Two blocking variables "application" and "position" are assigned to the rows and columns, respectively. The weight loss data is given in Table 3. Now we consider the $BILS(4, 3)$ design obtained in Example 1, i.e., the data along the diagonal in Table 3 (a) were removed, as shown in Table 3 (b).

Table 3: Weight loss data for $LS(4)$ and $BILS(4,3)$

235	236	218	268
251	241	227	229
234	273	274	226
195	270	230	225

(a) Data of $LS(4)$

	236	218	268
251		227	229
234	273		226
195	270	230	

(b) Data of $BILS(4,3)$

The underlying linear model for a $BILS(k,r)$ design is

$$y_{ijl} = \mu + \alpha_i + \beta_j + \tau_l + \epsilon_{ijl}, \tag{1}$$

where i, j take values in $\{1, 2, ..., k\}$ and l is the symbol in the (i,j)-th cell of the $BILS(k,r)$, μ is the overall mean, α_i is the ith row effect, β_j is the jth column effect, τ_l is the effect of the lth treatment, and the errors ϵ_{ijl} are independent $N(0,\sigma^2)$. Note that the triplet (i,j,l) takes on only the kr values dictated by the particular $BILS(k,r)$ chosen for the experiment. For the estimability of all effects, three zero-sum constraints are as usual imposed on the row, column and treatment effects, i.e., $\boldsymbol{\alpha}'\mathbf{1}_k = 0$, $\boldsymbol{\beta}'\mathbf{1}_k = 0$, $\boldsymbol{\tau}'\mathbf{1}_k = 0$, where $\mathbf{1}_k$ is the k-dimensional vector of ones, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)'$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_k)'$ and $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_k)'$.

Test first the null hypothesis of no treatment effect difference, i.e., $H_0 : \tau_1 = \cdots = \tau_k$. The linear model under the null hypothesis H_0 is reduced to

$$y_{ijl} = \mu + \alpha_i + \beta_j + \epsilon_{ijl}. \tag{2}$$

By using the extra sum of squares principle, the ANOVA table for the $BILS(4,3)$ wear experiment can be obtained, as shown in Table 4. There, we conclude that at the $\alpha = 5\%$ level the treatment factor (material) has the most significance as indicated by a p-value of 0.02179, which is consistent with the result of the complete $LS(4)$ design in Section 2.6 of Wu and Hamada (2003).

Table 4: ANOVA table for the $BILS(4,3)$ wear experiment

Source	Degrees of Freedom	Sum of Squares	Mean Squares	F value	P value(>F)
application	3	278.2	92.75	3.66	0.22192
position	3	2243.5	747.83	29.52	0.03294
material	3	3424.5	1141.50	45.06	0.02179
residual	2	50.7	25.33		

When such an H_0 is rejected, multiple comparisons of the k treatments should be performed. When $r = k-1$, suppose the triplets removed from the complete $LS(k)$ are (i_l, j_l, l), $l = 1, 2, ..., k$. Under model (1), it can be shown that the least squares estimate $\hat{\tau}_l$ is

$$\hat{\tau}_l = [k(k-3)]^{-1}[(k-2)y_{\cdot\cdot l} + y_{i_l \cdot\cdot} + y_{\cdot j_l \cdot} - y_{\cdot\cdot\cdot}],$$

where $y_{\cdot\cdot l}$ is the sum of the y-values for the lth treatment, $y_{i_l\cdot\cdot}$ is the sum of the y-values in the i_lth row, $y_{\cdot j_l\cdot}$ is the sum of the y-values in the j_lth column and $y_{\cdots}$ is the sum of all the y-values for the $BILS(k, k-1)$ experiment. It can also be shown that $Var(\hat{\tau}_j - \hat{\tau}_i) = 2(k-2)[k(k-3)]^{-1}\sigma^2$. Thus the t statistics for testing $\tau_i = \tau_j$, $i, j = 1, ..., k$, has the form

$$t_{ij} = \frac{\hat{\tau}_j - \hat{\tau}_i}{\hat{\sigma}\sqrt{2(k-2)[k(k-3)]^{-1}}},$$

where $\hat{\sigma}^2$ is the residual mean square. Under $H_0 : \tau_1 = \cdots = \tau_k$, each t_{ij} has a t distribution with $k^2 - 4k + 2$ degrees of freedom. The Tukey multiple comparison method identifies treatments i and j as different if $|t_{ij}| > q_{k,k^2-4k+2,\alpha}/\sqrt{2}$, where $q_{k,k^2-4k+2,\alpha}$ is the upper α quantile of the Studentized range distribution with parameters k and $k^2 - 4k + 2$. The simultaneous confidence intervals for $\tau_j - \tau_i$ are given by $\hat{\tau}_j - \hat{\tau}_i \pm q_{k,k^2-4k+2,\alpha}\ \hat{\sigma}\sqrt{(k-2)[k(k-3)]^{-1}}$ for all (i, j) pairs.

Table 5: Multiple comparison t statistics for the $BILS(4, 3)$ wear experiment

A vs. B	A vs. C	A vs. D	B vs. C	B vs. D	C vs. D
-11.03	-5.96	-8.64	5.07	2.38	-2.68

Returning to our experiment, the regression analysis leads to the estimates

$$\hat{\tau}_1 = 32.25, \qquad \hat{\tau}_2 = -23.25, \qquad \hat{\tau}_3 = 2.25, \qquad \hat{\tau}_4 = -11.25,$$

and $\hat{\sigma}^2 = 25.33$. The corresponding multiple comparison t statistics are given in Table 5. By comparing with the 0.05 critical value $q_{4,2,0.05}/\sqrt{2} = 6.93$ for the Tukey method, we conclude that at the 0.05 level material A wears more than B and D. If comparing with the 0.1 critical value $q_{4,2,0.1}/\sqrt{2} = 4.79$, we can identify that material A wears more than B, C and D, and C wears more than B, which is fully consistent with the result of the complete $LS(4)$ design in Section 2.6 of Wu and Hamada (2003), even though only 12 out of 16 experiments were conducted.

4 Optimality of $BILS(k, k-1)$ designs

Consider the linear model (1) for a given complete Latin square $LS(k)$, where the triplet (i, j, l) takes on the k^2 values dictated by the $LS(k)$ in this case. Let $\boldsymbol{X} = (\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_{k^2})'$ be the model matrix of order $k^2 \times (3k+1)$.

An experimental design D with the weight matrix $\boldsymbol{W} = (w_{ij})_{k\times k}$ is a discrete distribution of the numbers of experimental replications on the k^2 cells of L, where w_{ij} is the design weight on the (i, j)-th cell of L, $0 \le w_{ij} \le 1$ and $\sum_{i,j=1}^{k} w_{ij} = 1$. Denote by Ω the space of all such designs. The moment matrix of a design D is defined as follows (Pukelsheim, 1993):

$$\boldsymbol{M}(D) = \sum_{i,j=1}^{k} w_{ij} \boldsymbol{x}_{(i-1)k+j} \boldsymbol{x}'_{(i-1)k+j} = \begin{bmatrix} 1 & \boldsymbol{r}' & \boldsymbol{s}' & \boldsymbol{t}' \\ \boldsymbol{r} & \boldsymbol{\Delta_r} & \boldsymbol{W} & \boldsymbol{W}_1 \\ \boldsymbol{s} & \boldsymbol{W}' & \boldsymbol{\Delta_s} & \boldsymbol{W}_2 \\ \boldsymbol{t} & \boldsymbol{W}'_1 & \boldsymbol{W}'_2 & \boldsymbol{\Delta_t} \end{bmatrix}, \tag{3}$$

where $\boldsymbol{W}_1$ is a $k \times k$ matrix whose (i, j)-th entry is the weight on the cell of L which lies in the ith row and contains symbol j, $\boldsymbol{W}_2$ is a $k \times k$ matrix whose (i, j)-th entry is the weight on the cell of L which lies in the ith column and contains symbol j, $\boldsymbol{r} = \boldsymbol{W}\boldsymbol{1}_k = \boldsymbol{W}_1\boldsymbol{1}_k$, $\boldsymbol{s} = \boldsymbol{W}'\boldsymbol{1}_k = \boldsymbol{W}_2\boldsymbol{1}_k$, $\boldsymbol{t} = \boldsymbol{W}'_1\boldsymbol{1}_k = \boldsymbol{W}'_2\boldsymbol{1}_k$, and $\boldsymbol{\Delta_r}$, $\boldsymbol{\Delta_s}$ and $\boldsymbol{\Delta_t}$ are three diagonal matrices with the elements of the three vectors $\boldsymbol{r}$, $\boldsymbol{s}$ and $\boldsymbol{t}$, respectively, as the diagonal entries. Here, our interest is in the optimal estimation of the treatment effects $\boldsymbol{\tau}$ and that of all row, column and treatment effects $\boldsymbol{\theta} = (\boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{\tau}')'$, respectively.

Let $\boldsymbol{C}(D)$ be the information matrix of a design D under model (1) and $\phi(\boldsymbol{C}(D))$ be a real-valued function of $\boldsymbol{C}(D)$. A design D_1 is said to be ϕ-optimal in a design space $\mathcal{D}$ if $\phi(\boldsymbol{C}(D_1)) = \max_{D \in \mathcal{D}} \phi(\boldsymbol{C}(D))$. Let $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ be two information matrices corresponding to any two designs. Throughout we only consider the optimality functions $\phi(\cdot)$ which satisfy the following four conditions:

(i) isotonic to the Loewner ordering: if $\boldsymbol{C}_1 \geq \boldsymbol{C}_2$, then $\phi(\boldsymbol{C}_1) \geq \phi(\boldsymbol{C}_2)$;

(ii) concavity: $\phi((1-\gamma)\boldsymbol{C}_1 + \gamma\boldsymbol{C}_2) \geq (1-\gamma)\phi(\boldsymbol{C}_1) + \gamma\phi(\boldsymbol{C}_2)$ for any scalar $\gamma \in (0, 1)$;

(iii) positive homogeneity: $\phi(\delta\boldsymbol{C}_1) = \delta\phi(\boldsymbol{C}_1)$ for any scalar $\delta \geq 0$;

(iv) permutation invariant: $\phi(\boldsymbol{P}'\boldsymbol{C}_1\boldsymbol{P}) = \phi(\boldsymbol{C}_1)$ for any permutation matrix $\boldsymbol{P}$.

A design D_1 is said to be universally optimal in a design space $\mathcal{D}$ if it is ϕ-optimal in the space $\mathcal{D}$ for all functions $\phi(\cdot)$ which satisfy the above four conditions (Kiefer, 1975).

4.1 Optimality of $BILS(k, k-1)$ designs for the effects $\boldsymbol{\tau}$

Consider the optimality of a $BILS(k, k-1)$ design in Ω for the estimation of the effects $\boldsymbol{\tau}$. Following Bailey and Druilhet (2004) and Ai et al. (2009), for any design $D \in \Omega$, the information matrix for $\boldsymbol{\tau}$ can be derived from the moment matrix (3) as follows:

$$\begin{aligned} \boldsymbol{C_\tau}(D) &= \boldsymbol{\Delta_t} - \left(\boldsymbol{t},\ \boldsymbol{W}'_1,\ \boldsymbol{W}'_2\right) \begin{pmatrix} 1 & \boldsymbol{r}' & \boldsymbol{s}' \\ \boldsymbol{r} & \boldsymbol{\Delta_r} & \boldsymbol{W} \\ \boldsymbol{s} & \boldsymbol{W}' & \boldsymbol{\Delta_s} \end{pmatrix}^{-} \begin{pmatrix} \boldsymbol{t}' \\ \boldsymbol{W}_1 \\ \boldsymbol{W}_2 \end{pmatrix} \\ &= \boldsymbol{\Delta_t} - \boldsymbol{W}'_1\boldsymbol{\Delta_r^-}\boldsymbol{W}_1 - (\boldsymbol{W}'_2 - \boldsymbol{W}'_1\boldsymbol{\Delta_r^-}\boldsymbol{W})\boldsymbol{Q}^-(\boldsymbol{W}_2 - \boldsymbol{W}'\boldsymbol{\Delta_r^-}\boldsymbol{W}_1), \end{aligned} \tag{4}$$

where $\boldsymbol{Q} = \boldsymbol{\Delta_s} - \boldsymbol{W}'\boldsymbol{\Delta_r^-}\boldsymbol{W}$ and the notation $\boldsymbol{A}^-$ denotes a generalized inverse of a matrix $\boldsymbol{A}$ such that $\boldsymbol{A}\boldsymbol{A}^-\boldsymbol{A} = \boldsymbol{A}$. It is known from Pukelsheim (1993) that $\boldsymbol{C_\tau}(D)$ in (4) doesn't depend on the choice of the generalized inverses.

For a given $LS(k)$, denote by D^* the design with the weight matrix $\boldsymbol{W} = k^{-2}\mathbf{1}_k\mathbf{1}_k'$. Note that for the design D^*, $\boldsymbol{W} = \boldsymbol{W}_1 = \boldsymbol{W}_2$ and $\boldsymbol{\Delta_r} = \boldsymbol{\Delta_s} = \boldsymbol{\Delta_t} = k^{-1}\boldsymbol{I}_k$, where $\boldsymbol{I}_k$ is the identity matrix of order k. It can easily be verified that $\boldsymbol{C_\tau}(D^*) = k^{-1}\boldsymbol{H}_k$, where $\boldsymbol{H}_k = \boldsymbol{I}_k - k^{-1}\mathbf{1}_k\mathbf{1}_k'$. It should be mentioned that the information matrix of design D^* is independent of the choice of the original $LS(k)$. Thus the following result can be obtained, whose proof is given in Appendix.

Theorem 1. *For any design D based on a given $LS(k)$, $\phi(\boldsymbol{C_\tau}(D)) \leq k^{-1}\phi(\boldsymbol{H}_k)$.*

Theorem 1 shows that design D^* is universally optimal in Ω for the effects $\boldsymbol{\tau}$. Note that a $BILS(k, k-1)$ design based on a given $LS(k)$ is typically a design on the $LS(k)$ with the weight 0 on each of the k deleted cells and the weight $[k(k-1)]^{-1}$ on each of the remaining $k(k-1)$ cells. The following lemma gives the information matrix of a $BILS(k, k-1)$ design and its proof is postponed in Appendix.

Lemma 1. *For any $ILS(k, k-1)$ design D based on a given $LS(k)$, the entries of $\boldsymbol{C_\tau}(D)$ have the following forms*

$$\boldsymbol{C_\tau}(D)(i,j) = \begin{cases} \frac{k}{k-2}t_it_j - \frac{2[(k-1)t_i+(k-1)t_j-1]}{(k-1)(k-2)}, & \textit{for } i \neq j, \\ \frac{k}{k-2}t_i^2 + \frac{k-4}{k-2}t_i, & \textit{otherwise,} \end{cases} \tag{5}$$

where t_i is the ith element of $\boldsymbol{t}$. Especially, the information matrix for a $BILS(k, k-1)$ deign has the form $(k-3)/[(k-1)(k-2)]\boldsymbol{H}_k$.

The asymptotic optimality of a $BILS(k, k-1)$ design can be revealed by its relative efficiency with respect to the optimal design D^* under the optimality function $\phi(\cdot)$,

$$\text{Eff}_{\boldsymbol{\tau}}(D, \phi) = \frac{\phi(\boldsymbol{C_\tau}(D))}{\phi(\boldsymbol{C_\tau}(D^*))}. \tag{6}$$

Based on Lemma 1, the following result can be obtained.

Theorem 2. *For any $BILS(k, k-1)$ design D based on a given $LS(k)$ and for any optimality function $\phi(\cdot)$, we have*

$$\textit{Eff}_{\boldsymbol{\tau}}(D, \phi) = \frac{k(k-3)}{(k-1)(k-2)}. \tag{7}$$

Theorem 2 shows that for any $BILS(k, k-1)$ design D, its relative efficiency $\text{Eff}_{\boldsymbol{\tau}}(D, \phi)$ quickly approaches 100% as k becomes large. Thus, a $BILS(k, k-1)$ design is asymptotically universally optimal for the estimation of the effects $\boldsymbol{\tau}$ in the space Ω of all such designs.

4.2 Optimality of $BILS(k, k-1)$ designs for the effects $\boldsymbol{\theta}$

We next consider the optimality of a $BILS(k, k-1)$ design in Ω for the estimation of all the row, column and treatment effects $\boldsymbol{\theta}$. For any design D based on a given $LS(k)$, the information matrix for $\boldsymbol{\theta}$ under model (1) can be similarly derived as

$$\boldsymbol{C}_{\boldsymbol{\theta}}(D) = \begin{pmatrix} \boldsymbol{\Delta}_{\boldsymbol{r}} & \boldsymbol{W} & \boldsymbol{W}_1 \\ \boldsymbol{W}' & \boldsymbol{\Delta}_{\boldsymbol{s}} & \boldsymbol{W}_2 \\ \boldsymbol{W}_1' & \boldsymbol{W}_2' & \boldsymbol{\Delta}_{\boldsymbol{t}} \end{pmatrix} - \begin{pmatrix} \boldsymbol{r} \\ \boldsymbol{s} \\ \boldsymbol{t} \end{pmatrix} (\boldsymbol{r}', \boldsymbol{s}', \boldsymbol{t}'). \tag{8}$$

For the discrete uniform design D^* based on a given $LS(k)$ introduced in the former subsection, we can easily obtain that $\boldsymbol{C}_{\boldsymbol{\theta}}(D^*) = k^{-1}\boldsymbol{I}_3 \otimes \boldsymbol{H}_k$, where $\otimes$ denotes the Kronecker product. Similar to Theorem 1, the following result can also be obtained and its proof is given in Appendix.

Theorem 3. *For any design D based on a given $LS(k)$, $\phi(\boldsymbol{C}_{\boldsymbol{\theta}}(D)) \leq k^{-1}\phi(\boldsymbol{I}_3 \otimes \boldsymbol{H}_k)$.*

For any $BILS(k, k-1)$ design D based on a given $LS(k)$, since the function $\phi(\boldsymbol{C}_{\boldsymbol{\theta}}(D))$ is invariant under any permutation operation of $\boldsymbol{C}_{\boldsymbol{\theta}}(D)$; without loss of generality, we can assume that the (i, i)-th cell of the $LS(k)$ contains symbol i for $i = 1, \ldots, k$, and the deleted k cells of the $BILS(k, k-1)$ design D are exactly the k main diagonal cells. Then $\boldsymbol{C}_{\boldsymbol{\theta}}(D)$ has the form

$$\boldsymbol{C}_{\boldsymbol{\theta}}(D) = (k-1)^{-1}\boldsymbol{I}_3 \otimes \boldsymbol{H}_k - [k(k-1)]^{-1}\boldsymbol{1}_3\boldsymbol{1}_3' \otimes \boldsymbol{H}_k. \tag{9}$$

Note that the relative efficiency of a $BILS(k, k-1)$ design D with respect to the optimal design D^* is depend on the choice of optimality functions $\phi(\cdot)$ unlike the case of optimality for the effects $\boldsymbol{\tau}$. Some specific classes of optimality functions $\phi(\cdot)$ have to be defined in order to calculate the relative efficiency of a design D, i.e., $\text{Eff}_{\boldsymbol{\theta}}(D, \phi)$ in (6).

For any design D based on a given $LS(k)$, it is easy to see that $\boldsymbol{C}_{\boldsymbol{\theta}}(D)(\boldsymbol{1}_k', \boldsymbol{0}_k', \boldsymbol{0}_k')' = \boldsymbol{0}$, $\boldsymbol{C}_{\boldsymbol{\theta}}(D)(\boldsymbol{0}_k', \boldsymbol{1}_k', \boldsymbol{0}_k')' = \boldsymbol{0}$, and $\boldsymbol{C}_{\boldsymbol{\theta}}(D)(\boldsymbol{0}_k', \boldsymbol{0}_k', \boldsymbol{1}_k')' = \boldsymbol{0}$, where $\boldsymbol{0}_k$ is the k-dimensional vector of zeroes. Namely, the information matrix $\boldsymbol{C}_{\boldsymbol{\theta}}(D)$ in (8) of design D has the first three eigenvalues being zero. Denote $\lambda_4 \leq \cdots \leq \lambda_{3k}$ the other eigenvalues of $\boldsymbol{C}_{\boldsymbol{\theta}}(D)$ except for the first three zero eigenvalues. The function $\phi_p(\cdot)$ on the rank deficient matrix $\boldsymbol{C}_{\boldsymbol{\theta}}(D)$ can be defined as follow (see, Pukelsheim, 1993):

$$\phi_p(\boldsymbol{C}_{\boldsymbol{\theta}}(D)) = \begin{cases} \max_{4\leq j\leq 3k} \lambda_j, & \text{for } p = \infty, \\ \min_{4\leq j\leq 3k} \lambda_j, & \text{for } p = -\infty, \\ (\prod_{4\leq j\leq 3k} \lambda_j)^{1/(3k-3)}, & \text{for } p = 0, \\ \left[(3k-3)^{-1} \sum_{4\leq j\leq 3k} \lambda_j^p\right]^{1/p}, & \text{otherwise}. \end{cases} \tag{10}$$

It is known that $\phi_p(\cdot)$ cover the commonly used optimality functions as special cases. For example, ϕ_0-, ϕ_{-1}-, $\phi_{-\infty}$- and ϕ_1-optimality are simply the D-, A-, E- and T-optimality. The universal optimality in Kiefer's (1975) sense must be ϕ_p-optimality for all $p \leq 0$, but may not for $p > 0$ (Ai and Hickernell, 2009). Because the function $\phi_p(\cdot)$ can be used as an optimality function only when $p \leq 1$; in the following we need only to consider the case of $p \leq 1$. As for the relative efficiency of a $BILS(k, k-1)$ design based on a given $LS(k)$ under the above optimality functions $\phi_p(\cdot)$, we can obtain the following conclusion. The proof of

Table 6: $BILS(6,3)$ and $BILS(6,4)$

5			2		6
2		4		1	
	6			4	3
3		6	1		
	2	3		5	
	5		4		1

$BILS(6,3)$

2		3		4	5
	4	6	3	5	
6	1		2		3
3		2		1	4
5	2		1		6
	5	1	4	6	

$BILS(6,4)$

Theorem 4 is also postponed in Appendix.

Theorem 4. *For any $BILS(k, k-1)$ design D based on a given $LS(k)$, its relative efficiency in (6) under the optimality functions $\phi_p(\cdot)$ $(p \leq 1)$ has the following forms*

$$\mathit{Eff}_{\boldsymbol{\theta}}(D, \phi_p) = \begin{cases} \frac{k-3}{k-1}, & \text{for } p = -\infty, \\ (\frac{k-3}{k-1})^{1/3}(\frac{k}{k-1})^{2/3}, & \text{for } p = 0, \\ \frac{k}{k-1}[\frac{2}{3} + \frac{1}{3}(\frac{k-3}{k})^p]^{1/p}, & \text{otherwise.} \end{cases}$$

Theorem 4 shows that for any $BILS(k, k-1)$ design D based on a given $LS(k)$, its relative efficiency $\mathrm{Eff}_{\boldsymbol{\theta}}(D, \phi_p)$ quickly goes to 100% as k becomes large. Thus a $BILS(k, k-1)$ design is asymptotically ϕ_p-optimal for the estimation of the effects $\boldsymbol{\theta}$ in the space Ω of all such designs.

5 Concluding remarks

In this paper we introduce a new class of designs, called balanced incomplete Latin square (BILS) designs, to deal with the experiments with two blocking and treatment variables where the size of both blocks may be less than the number of treatments. A general construction method of BILS designs is proposed via orthogonal Latin squares. An application shows that BILS designs works well on practical experiments. Furthermore, the asymptotic optimality of these BILS designs of block size $k-1$ is derived. The optimality issue of the BILS designs with other block sizes becomes much more complicated and is under investigation.

Note that when $k = 6$, where there do not exist two orthogonal Latin squares, the foregoing construction method can not be used. Table 1 presents a Latin square with one transversal consisting of the six symbols in boldface and a $BILS(6,5)$ can be obtained by removing the transversal from the complete Latin square. For the block size $r = 3$ and 4, computer searching gives $BILS(6,3)$ and $BILS(6,4)$ with better balance property, shown in Table 6.

It should be mentioned that the concept of "balance" in the BILS simply requires equal times of occurance of each treatment. But the "balance" in a balanced incomplete block design (BIBD) further demands the balance condition that each pair of treatments is compared in the same number of blocks. The $BILS(k, k-1)$ designs constructed in this paper satisfy all the balance conditions such that the designs reduce to a BIBD when one of the two blocking factors is only considered. If we redefine all BILS designs in this strict sense, the construction of this new kind of BILS designs becomes an issue of great sparsity, since in this strict sense of balance, BILS designs do not exist at all for most parameters except for block size $k-1$.

Acknowledgements. The work is supported by NNSF of China Grant 11271032.

References

Ai, M. and Hickernell F. J. (2009). Universal optimality of digital nets and lattice designs. *Science China Ser. A: Math.* **52**, 2309C2320.

Ai, M., Yu, Y., and He, S. (2009). Optimality of circular neighbor-balanced designs for total effects with autoregressive correlated observations. *J. Statist. Plann. Inference* **139**, 2293-2304.

Bailey, R. A. and Druilhet, P. (2004). Optimality of neighbor-balanced designs for total effects. *Ann. Statist.* **32**, 1650-1661.

Bose, R. C., Shrikhande, S. S., and Parker, E.T. (1960). Further results on the construction of mutually orthogonal Latin squares and the falsity of Euler's conjecture. *Canad. J. Math.* **12**, 189-203.

Denés, J. and Keedwell, A. D. (1974). *Latin squares and their applications.* Akadémiai Kiadó, Budapest.

Denés, J. and Keedwell, A. D. (1991). *Latin squares: new developments in the theory and applications.* Elsvier, Amsterdam.

Kiefer, J.C. (1975). Construction and optimality of generalized Youden designs. In *A Survey of Statistical Design and Linear Models* (Ed. J.N. Srivastava). North-Holland, Amsterdam, 333-353.

Pukelsheim, F. (1993). *Optimal design of experiments.* Wiley, New York.

Wu, C. F. J. and Hamada, M. (2003). *Experiments: planning, analysis, and parameter design optimization.* Wiley, New York.

Appendix. Proofs of All Theorems and A Lemma

Proof of Theorem 1

For any design D based on a given $LS(k)$, it can easily be verified that $\boldsymbol{W}'\Delta_{\boldsymbol{r}}^{-}\boldsymbol{r} = \boldsymbol{W}'\boldsymbol{1}_k = \boldsymbol{s}$ and $\boldsymbol{W}_1'\Delta_{\boldsymbol{r}}^{-}\boldsymbol{r} = \boldsymbol{W}_1'\boldsymbol{1}_k = \boldsymbol{t}$, whether the $\boldsymbol{r}$ has elements zero or not. Then it can be derived that $\boldsymbol{C}_{\boldsymbol{\tau}}(D)\boldsymbol{1}_k = \boldsymbol{0}$.

Denote by $\mathcal{P}_k$ the set of all possible $k \times k$ permutation matrices. Let

$$\overline{\boldsymbol{C}_{\boldsymbol{\tau}}(D)} = (k!)^{-1} \sum_{\boldsymbol{P}\in\mathcal{P}_k} \boldsymbol{P}'\boldsymbol{C}_{\boldsymbol{\tau}}(D)\boldsymbol{P}.$$

Because $\boldsymbol{P}_1'\overline{\boldsymbol{C}_{\boldsymbol{\tau}}(D)}\boldsymbol{P}_1 = (k!)^{-1}\sum_{\boldsymbol{P}\in\mathcal{P}_k}\boldsymbol{P}_1'\boldsymbol{P}'\boldsymbol{C}_{\boldsymbol{\tau}}(D)\boldsymbol{P}\boldsymbol{P}_1 = \overline{\boldsymbol{C}_{\boldsymbol{\tau}}(D)}$ for any permutation matrix $\boldsymbol{P}_1$, $\overline{\boldsymbol{C}_{\boldsymbol{\tau}}(D)}$ is completely symmetric, i.e., $\overline{\boldsymbol{C}_{\boldsymbol{\tau}}(D)} = a\boldsymbol{I}_k + b\boldsymbol{1}_k\boldsymbol{1}_k'$, where a and b are two scalars. Furthermore, since $\boldsymbol{1}_k'\overline{\boldsymbol{C}_{\boldsymbol{\tau}}(D)}\boldsymbol{1}_k = \boldsymbol{1}_k'\boldsymbol{C}_{\boldsymbol{\tau}}(D)\boldsymbol{1}_k = 0$ and $\mathrm{tr}\overline{\boldsymbol{C}_{\boldsymbol{\tau}}(D)} = \mathrm{tr}\boldsymbol{C}_{\boldsymbol{\tau}}(D)$, we can obtain that $a = (k-1)^{-1}\mathrm{tr}\boldsymbol{C}_{\boldsymbol{\tau}}(D)$, $b = -[k(k-1)]^{-1}\mathrm{tr}\boldsymbol{C}_{\boldsymbol{\tau}}(D)$, and so $\overline{\boldsymbol{C}_{\boldsymbol{\tau}}(D)} = [(k-1)^{-1}\mathrm{tr}\boldsymbol{C}_{\boldsymbol{\tau}}(D)]\boldsymbol{H}_k$.

Now we are ready to prove that $\mathrm{tr}\boldsymbol{C}_{\boldsymbol{\tau}}(D) \leq 1-k^{-1}$. By Lemma 3.12 of Pukelsheim (1993), it is known that both $\boldsymbol{Q} = \boldsymbol{\Delta}_{\boldsymbol{s}} - \boldsymbol{W}'\boldsymbol{\Delta}_{\boldsymbol{r}}^{-}\boldsymbol{W}$ and $(\boldsymbol{W}_2' - \boldsymbol{W}_1'\boldsymbol{\Delta}_{\boldsymbol{r}}^{-}\boldsymbol{W})\boldsymbol{Q}^{-}(\boldsymbol{W}_2 - \boldsymbol{W}'\boldsymbol{\Delta}_{\boldsymbol{r}}^{-}\boldsymbol{W}_1)$ are nonnegative definite and hence their traces are not less than zero. Then we can obtain that $\mathrm{tr}\boldsymbol{C}_{\boldsymbol{\tau}}(D) \leq 1 - \mathrm{tr}\boldsymbol{W}_1'\boldsymbol{\Delta}_{\boldsymbol{r}}^{-}\boldsymbol{W}_1 = 1 - \sum_{i=1,r_i\neq 0}^{k} r_i^{-1}\sum_{j=1}^{k} w_{ij}^2 \leq 1 - k^{-1}\sum_{i=1}^{k} r_i = 1-k^{-1}$, where r_i is the ith element of $\boldsymbol{r}$.

Thus, by applying the properties of the function $\phi(\cdot)$, we further have that $k^{-1}\phi(\boldsymbol{H}_k) \geq \phi(\overline{\boldsymbol{C}_{\boldsymbol{\tau}}(D)}) \geq (k!)^{-1}\sum_{\boldsymbol{P}\in\mathcal{P}_k}\phi(\boldsymbol{P}'\boldsymbol{C}_{\boldsymbol{\tau}}(D)\boldsymbol{P}) = \phi(\boldsymbol{C}_{\boldsymbol{\tau}}(D))$. The proof of Theorem 1 is complete.

Proof of Lemma 1

For an $ILS(k, k-1)$ deign D based on a given $LS(k)$, the weight on each of the remaining $k(k-1)$ cells is $[k(k-1)]^{-1}$. It can be derived that $\boldsymbol{\Delta}_{\boldsymbol{r}} = \boldsymbol{\Delta}_{\boldsymbol{s}} = k^{-1}\boldsymbol{I}_k$, $\boldsymbol{W}'\boldsymbol{\Delta}_{\boldsymbol{r}}^{-}\boldsymbol{W} = [k(k-1)^2]^{-1}[\boldsymbol{I}_k + (k-2)\boldsymbol{1}_k\boldsymbol{1}_k']$, and so $\boldsymbol{Q} = \frac{k-2}{(k-1)^2}\boldsymbol{H}_k$. Then the information matrix $\boldsymbol{C}_{\boldsymbol{\tau}}(D)$ in (4) is given by

$$\begin{aligned}\boldsymbol{C}_{\boldsymbol{\tau}}(D) \;=\; & \boldsymbol{\Delta}_{\boldsymbol{t}} - k\boldsymbol{W}_1'\boldsymbol{W}_1 - (k-1)^2(k-2)^{-1}\big(\boldsymbol{W}_2'\boldsymbol{W}_2 + \\ & k^2\boldsymbol{W}_1'\boldsymbol{W}\boldsymbol{W}'\boldsymbol{W}_1 - k\boldsymbol{W}_1'\boldsymbol{W}\boldsymbol{W}_2 - k\boldsymbol{W}_2'\boldsymbol{W}'\boldsymbol{W}_1\big).\end{aligned} \tag{11}$$

It can be verified that the entries of the matrices in (11) have the following forms:

$$\begin{aligned}
\boldsymbol{W}_1'\boldsymbol{W}_1(i,j) &= k^{-1}(k-1)^{-2}\big[(k-1)t_i + (k-1)t_jI_{\{i\neq j\}} - I_{\{i\neq j\}}\big],\\
\boldsymbol{W}_2'\boldsymbol{W}_2(i,j) &= k^{-1}(k-1)^{-2}\big[(k-1)t_i + (k-1)t_jI_{\{i\neq j\}} - I_{\{i\neq j\}}\big],\\
\boldsymbol{W}_1'\boldsymbol{W}\boldsymbol{W}_2(i,j) &= k^{-2}(k-1)^{-3}\big[k(k-1)^2t_it_j - (k-1)t_i - (k-1)t_jI_{\{i\neq j\}} + I_{\{i\neq j\}}\big],\\
\boldsymbol{W}_1'\boldsymbol{W}\boldsymbol{W}'\boldsymbol{W}_1(i,j) &= k^{-3}(k-1)^{-4}\big[k(k-1)^2(k-2)t_it_j + (k-1)t_i + \\
&\quad (k-1)t_jI_{\{i\neq j\}} - I_{\{i\neq j\}}\big],
\end{aligned}$$

where $I_{\{\cdot\}}$ is the indicator function. Thus, applying the above expressions, the formula (5) can easily be obtained. For a $BILS(k, k-1)$ design, the conclusion is followed just by letting $t_1 = \cdots = t_k = k^{-1}$ in (5). So the proof of Lemma 1 is complete.

Proof of Theorem 3

Denote by $\mathcal{P}_k^3$ the set of all permutation matrices of the form $\boldsymbol{P} = \text{diag}(\boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P}_3)$, where $\boldsymbol{P}_1$, $\boldsymbol{P}_2$ and $\boldsymbol{P}_3$ are any three $k \times k$ permutation matrices. Let

$$\overline{\boldsymbol{C_\theta}(D)} = (k!)^{-3} \sum_{\boldsymbol{P} \in \mathcal{P}_k^3} \boldsymbol{P}' \boldsymbol{C_\theta}(D) \boldsymbol{P}.$$

For any design D based on a given $LS(k)$, following the proof of Theorem 1, we can obtain

$$\overline{\boldsymbol{C_\theta}(D)} = (k-1)^{-1} \text{diag}(1 - \boldsymbol{r}'\boldsymbol{r}, 1 - \boldsymbol{s}'\boldsymbol{s}, 1 - \boldsymbol{t}'\boldsymbol{t}) \otimes \boldsymbol{H}_k.$$

Because $\boldsymbol{r}'\boldsymbol{r}$, $\boldsymbol{s}'\boldsymbol{s}$ and $\boldsymbol{t}'\boldsymbol{t}$ are all not less than k^{-1}, we have that $\overline{\boldsymbol{C_\theta}(D)} \leq k^{-1} \boldsymbol{I}_3 \otimes \boldsymbol{H}_k$. Then it holds that $k^{-1}\phi(\boldsymbol{I}_3 \otimes \boldsymbol{H}_k) \geq \phi(\overline{\boldsymbol{C_\theta}(D)}) \geq \phi(\boldsymbol{C_\theta}(D))$. The proof of Theorem 3 is complete.

Proof of Theorem 4

Since $\boldsymbol{H}_k$ has $(k-1)$ eigenvalues being 1 and one being 0, it can be obtained that $\boldsymbol{I}_3 \otimes \boldsymbol{H}_k$ has $(3k-3)$ positive eigenvalues being 1 and $\mathbf{1}_3\mathbf{1}_3' \otimes \boldsymbol{H}_k$ has $(k-1)$ positive eigenvalues being 3. Note that $(\boldsymbol{I}_3 \otimes \boldsymbol{H}_k)(\mathbf{1}_{3\times 3} \otimes \boldsymbol{H}_k) = (\mathbf{1}_{3\times 3} \otimes \boldsymbol{H}_k)(\boldsymbol{I}_3 \otimes \boldsymbol{H}_k)$. Since those two matrices can be diagonalized simultaneously, it can easily be verified that the information matrix $\boldsymbol{C_\theta}(D)$ in (9) has three eigenvalues being zero, $(k-1)$ eigenvalues being $(k-3)[k(k-1)]^{-1}$ and $(2k-2)$ eigenvalues being $(k-1)^{-1}$. Then the expression of $\text{Eff}_{\boldsymbol{\theta}}(D, \phi_p)$ follows directly under the optimality function $\phi_p(\cdot)$ for different values $p \leq 1$. The proof of Theorem 4 is complete.