Some new classes of orthogonal Latin hypercube designs

Mingyao Ai, Yuanzhen He, Senmao Liu

LMAM, School of Mathematical Sciences, Peking University,

Beijing 100871, P. R. China

Email: myai@math.pku.edu.cn (Ai)

heyuanzhen@gmail.com (He)

liusenmaofh@gmail.com (Liu)

Abstract

Orthogonal Latin hypercube (OLH) is a good choice for computer experiments because it ensures independent estimation of linear effects when a first-order model is fitted. However, when second-order effects are present, a second-order model must be adopted. In such cases, apart from the above two-dimensional orthogonality an OLH also needs to satisfy the property that each column is orthogonal to the elementwise square of all columns and orthogonal to the elementwise product of every pair of columns. Such class of OLHs is called OLHs of order two while the former class just possessing two-dimensional orthogonality is called OLHs of order one. In this paper we present a general method for constructing OLHs of orders one and two for $n = s^m$ runs, where s and m may be any positive integers greater than one, by rotating a grouped orthogonal array with a column-orthogonal rotation matrix. The Kronecker product and the stacking methods are revisited and combined to construct some new classes of OLHs of orders one and two with other flexible numbers of runs. Some useful OLHs of order one or two with moderate runs are tabulated and discussed.

MSC: Primary 62K15; Secondary 62K10

Keywords: Column-orthogonal rotation matrix, full factorial design, Galois field, orthogonal array, orthogonal Latin hypercube

1 Introduction

Latin hypercube was first proposed by Makay et al. (1979) in their pioneering paper on computer experiments. Ever since then, Latin hypercubes have become increasingly popular in the design of computer experiments because of their uniform coverage of each individual factor (Fang, Li and Sudjianto (2005), Santner, Williams and Notz (2003)). A Latin hypercube is typically an $n \times q$ matrix in which each column is a permutation of n uniformly spaced levels, say $\{-1, -(n-3)/(n-1), \ldots, (n-3)/(n-1), 1\}$. However, there is no guarantee that Latin hypercubes will have good multivariate properties. If the levels of columns are simply random permutations, there will usually be some pairs of columns with high correlations. This will make it more complicated to identify the active factors. An orthogonal Latin hypercube, denoted by $OLH(n, q)$, is an $n \times q$ Latin hypercube in which any pair of columns have zero correlation. Thus, an OLH not only retains one-dimensional uniformity but also possesses two-dimensional orthogonality, and ensures independent estimation of linear effects when a first-order model is fitted. However, when second-order effects exist, a second-order model must be adopted. In such cases, apart from the above two-dimensional orthogonality an OLH also needs to satisfy the property that each column is orthogonal to the elementwise square of all columns and orthogonal to the elementwise product of every pair of columns. For the usefulness of such class of OLHs, refer to Ye's (1998) detailed discussion. For ease of later distinguishing, the former class of OLHs is called of order one and the latter class of OLHs with this property is called of order two.

Systematic construction of OLHs has attracted great attentions of many researchers. By using permutation group theory, Ye (1998) constructed two classes of OLHs of order two with the forms of $OLH(2^m, 2m-2)$ and $OLH(2^m+1, 2m-2)$. Subsequently, for any integer $s \geq 2$, Beattie and Lin (2005) successfully obtained a class of OLHs of order two of the form $OLH(s^m, m)$ by rotating an s^m-run full factorial design with a rotation matrix of order m, where m must be a power of two. Subject to the same restriction of m, Steinberg and Lin (2006) rotated grouped two-level regular fractional factorial designs and constructed OLHs of order one with 2^m runs, but the number of columns can reach as many as $\lfloor (2^m-1)/m \rfloor m$, where $\lfloor x \rfloor$ is the integer part of x. Pang et al. (2009) extended this method to obtain $OLH(p^m, (p^m-1)/(p-1))$ of order one, where p is any prime and m must be a power of two. For $n = s^2$ runs, Lin et al. (2009) generated a bigger OLH of order one which has a high factor-to-run ratio by assembling an orthogonal array (OA) with a smaller OLH. Through an inductive construction method, Sun et al. (2009) obtained two classes of OLHs of order two with the forms of $OLH(2^{c+1}, 2^c)$ and $OLH(2^{c+1}+1, 2^c)$ for any positive integer c. Meanwhile, Georgiou (2009) obtained some OLHs of small runs by transferring orthogonal designs. A detailed discussion of OLHs with at most nine runs can be found in Prescott (2009). Recently, Bingham et al. (2009) introduced a new method for constructing a series of larger designs based on a small design with the tool of Kronecker product. Lin et al. (2010) further extended their method to construct OLHs and also proved that an OLH of order one does not exist for $n = 4c+2$ runs.

In this paper we are ready to give a general method for constructing OLHs of orders one and two. The rest of the paper is arranged as follows. Section 2 introduces a unified construction method for OLHs of order one and those of order two for $n = s^m$ runs, where s and m may be any positive integers greater than one, by rotating a grouped OA with a column-orthogonal rotation (COR) matrix. Section 3 presents a general partition scheme to divide any prime power s-level regular saturated factorial design of strength two into some groups of full factorial designs by using the subfield theory. To obtain OLHs of orders one and two with other flexible numbers of runs, the Kronecker product and the stacking methods proposed by Lin et al. (2010) are revisited in Sections 4 and 5, respectively, and some new classes of OLHs are constructed. For illustration, some new OLHs of order one or order two with small prime power numbers of runs are tabulated in Section 6. Section 7 concludes with some remarks.

2 A general method for constructing OLHs

Some notations and definitions are introduced here. An orthogonal array of size n, q constraints, s levels, and strength $t \geq 2$, denoted by $OA(n, s^q, t)$, is an $n \times q$ matrix with entries from a set of s levels such that for every $n \times t$ submatrix, the s^t level combinations occur equally often (Hedayat et al. (1999)). In the article we adopt the s levels as $\{-1, -(s-3)/(s-1), \ldots, (s-3)/(s-1), 1\}$. For an $n \times m$ matrix $D = (d_{ij}) = (d_1, \ldots, d_m)$, the J-characteristic of its any k columns $d_{j_1}, \ldots, d_{j_k}$ is defined to be $J(d_{j_1}, \ldots, d_{j_k}) = \sum_{i=1}^{n} d_{ij_1} \cdots d_{ij_k}$, which measures the non-orthogonality among these k columns. For any $n_1 \times m_1$ matrix $A = (a_{ij})$ and $n_2 \times m_2$ matrix B, their Kronecker product $A \otimes B$ is the $n_1 n_2 \times m_1 m_2$ blocked matrix $(a_{ij}B)_{1 \leq i \leq n_1, 1 \leq j \leq m_1}$, where the block $a_{ij}B$ itself is the $n_2 \times m_2$ matrix with entries $a_{ij} b_{rv}, r = 1, \ldots, n_2, v = 1, \ldots, m_2$. A vector is said to be equally-spaced if all of its entries are equally-spaced.

Beattie and Lin (2005) established a fundamental lemma about the equally-spaced property, which plays a pivotal role in our construction and is rewritten as follows.

Lemma 2.1 *Let A_1 be a full factorial design for m factors each at s levels and u be an m-dimensional vector. Then the vector $A_1 u$ is equally-spaced if and only if u is a permutation of $\{\pm\lambda, \pm\lambda s, \pm\lambda s^2, \ldots, \pm\lambda s^{m-1}\}$ up to sign specification, where λ is any constant.*

Let $A_2, \ldots, A_f$ be $f - 1$ different matrices obtained by permutating some rows of A_1. Clearly, $A_1, \ldots, A_f$ are all s^m full factorial designs for m factors each at s levels. Let U be a column-orthogonal rotation (COR) matrix of order $m \times d$ in which the d columns are d permutations of $\{\pm 1, \pm s, \pm s^2, \ldots, \pm s^{m-1}\}$ and any pair of columns have zero correlation. Let $A = (A_1, \ldots, A_f)$ and $R = I_f \otimes U$. Construct an $s^m \times fd$ matrix

$$M = (s-1)/(s^m - 1)AR. \tag{1}$$

Then we can obtain the following result, whose proof is given in the Appendix.

Theorem 2.1 *(i) If A is an OA of strength two, then the M constructed in (1) is an $OLH(s^m, fd)$ of order one;*

(ii) If A is an OA of strength three, then the M constructed in (1) is an $OLH(s^m, fd)$ of order two.

Let B be an $OLH(s, l)$. For $t = 1, \ldots, l$, obtain a matrix $A^{(t)}$ by replacing the s levels of A with $b_{1t}, \ldots, b_{st}$, respectively. Construct an $s^m \times fdl$ matrix

$$M = (s-1)/(s^m - 1)(A^{(1)}R, A^{(2)}R, \ldots, A^{(l)}R). \qquad (2)$$

Then we can further obtain the following result, whose proof is also postponed to the Appendix.

Theorem 2.2 *(i) If A is an OA of strength two and B is an OLH of order one, then the M constructed in (2) is an $OLH(s^m, fdl)$ of order one;*

(ii) If A is an OA of strength three and B is an OLH of order two, then the M constructed in (2) is an $OLH(s^m, fdl)$ of order two.

For applying the general method to construct new OLHs, we need two components, a grouped OA and a COR matrix. It can be shown that for any numbers $s, m \geq 2$, the m-row COR matrix U has at least one column $(1, s, \ldots, s^{m-2}, s^{m-1})'$, and for any even number $m \geq 2$, the m-row COR matrix U has at least one more column $(s, -1, \ldots, s^{m-1}, -s^{m-2})'$. Furthermore, for $m = 2^c$ with any positive integer c, the m-row COR matrix U has the following recursive square form

$$V_c = \begin{pmatrix} 1 & s^{2^{c-1}} \\ s^{2^{c-1}} & -1 \end{pmatrix} \otimes V_{c-1}, \text{ where } V_0 = 1. \qquad (3)$$

Example 2.1 *Consider the construction of an OLH of* 81 *runs. Let A be a* 3*-level regular saturated factorial design of strength two and* 81 *runs, whose generator matrix can be partitioned into* 10 *groups as follows*

$$\begin{pmatrix} 1000 & 1002 & 1011 & 1200 & 2201 & 0221 & 1010 & 1212 & 2120 & 1222 \\ 0100 & 2101 & 1120 & 0220 & 1022 & 1101 & 0121 & 2212 & 0122 & 2200 \\ 0010 & 0210 & 1112 & 0022 & 0102 & 2110 & 1012 & 1221 & 2012 & 2220 \\ 0001 & 0021 & 0111 & 2002 & 2010 & 2211 & 0101 & 2122 & 1201 & 2222 \end{pmatrix},$$

where the four columns in each group are linearly independent and form the generator matrix of a 3^4 full factorial design. Replace the three symbols $0, 1, 2$ in A with $-1, 0, 1$, respectively. Let $U = V_2$ in (3) with $s = 3$ and $R = I_{10} \otimes U$. Then by item (i) of Theorem 2.1, the matrix M in (1) gives an $OLH(81, 40)$ of order one.

On the other hand, let A be a 9*-level regular saturated factorial design of strength two and* 81 *runs. Note that among the* 10 *columns of A, any two columns form a 9^2 full* *factorial design. Then A can be partitioned into 5 groups each containing two columns. Replace the nine symbols of A with $-1, -3/4, \ldots, 3/4, 1$, respectively. Let $U = V_1$ in (3) with $s = 9$ and $R = I_5 \otimes U$. Then by item (i) of Theorem 2.1, the matrix M in (1) gives an $OLH(81, 10)$ of order one. Furthermore, let B be the $OLH(9, 5)$ of order one given in Table 3. By item (i) of Theorem 2.2, the matrix M in (2) gives an $OLH(81, 50)$ of order one.*

Example 2.1 shows that for a same prime power n, if it can be expressed in two different forms, $n = s_1^{m_1} = s_2^{m_2}$ with $m_1 > m_2$, and if an OLH of s_2 runs exists with most columns, then the OLH constructed by applying the s_2-level OA tends to have more columns than the one by applying the s_1-level OA. Since regular fractional factorial designs as discussed in Wu and Hamada (2000) are the most familiar examples of OAs, their grouping schemes will be discussed in detail in the next section.

3 Construction of OLHs via the general method

Pang et al. (2009) presented a partition scheme to divide a regular fractional factorial design of prime levels into some groups of full factorial designs. Here we generalize this method to suit for any regular saturated factorial design with any prime power levels and then construct OLHs with more columns for the same number of runs. The generalization is not straightforward and is based on the following subfield theory.

For a prime p and an integer $\nu \geq 1$, let $GF(p^\nu)$ denote a Galois field of order p^ν. For $\nu = 1$, the set of residues $\{0, 1, \ldots, p-1\}$ modulo p forms a $GF(p)$ of order p under addition and multiplication modulo p. For $\nu > 1$, the set of all polynomials of degree $\nu - 1$ or lower $\{a_0 + a_1 x + \cdots + a_{\nu-1} x^{\nu-1} | a_j \in GF(p)\}$ is a $GF(p^\nu)$ of order p^ν under addition and multiplication of polynomials modulo $g(x)$, where $g(x) = b_0 + b_1 x + \cdots + b_\nu x^\nu$ with $b_j \in GF(p)$ and $b_\nu = 1$ is an irreducible polynomial of degree ν. Both the additive group $GF(p^\nu)$ and the multiplicative group $GF(p^\nu) \setminus \{0\}$ are cyclic.

Let $\mathcal{F}$ denote $GF(p^{cm})$ with a primitive irreducible polynomial $g(x)$ of degree cm. Then x is a primitive element of $\mathcal{F}$ and every non-zero element of $\mathcal{F}$ can also be expressed as a power of x. Note that $x^{p^{cm}-1} = 1$. Define $\alpha = x^{(p^{cm}-1)/(p^c-1)}$. Let $\mathcal{G}$ be the set $\{0, \alpha, \alpha^2, \ldots, \alpha^{p^c-1}\}$ with $\alpha^{p^c-1} = 1$. Obviously, $\mathcal{G}$ is a subfield of $\mathcal{F}$ with the same addition and multiplication as those in $\mathcal{F}$. Furthermore, we have the following result about the expression of the elements of $\mathcal{F}$, whose proof is given in the Appendix.

Proposition 3.1 *Let $\mathcal{F}$ be a $GF(p^{cm})$ with a primitive irreducible polynomial $g(x)$ and $\mathcal{G}$ is a subfield $GF(p^c)$ of $\mathcal{F}$ with $\alpha = x^{(p^{cm}-1)/(p^c-1)}$ as a primitive element. Then every element of $\mathcal{F}$ can be uniquely represented by*

$$\lambda_0 + \lambda_1 x + \cdots + \lambda_{m-1} x^{m-1},\ \lambda_i \in \mathcal{G},\ i = 0, 1, \ldots, m-1. \tag{4}$$

Let $s = p^c$. For an s-level regular fractional factorial design A of s^m runs, let $1, x, \dots, x^{m-1}$ label its m independent columns. By Proposition 3.1, for $i = 1, 2, \dots, p^{cm} - 1$, the element x^i of $\mathcal{F}$ can be uniquely represented by $\lambda_{i0} + \lambda_{i1}x + \cdots + \lambda_{i,m-1}x^{m-1}$, $\lambda_{ij} \in \mathcal{G}$, $j = 0, 1, \dots, m-1$. Then the column of A labeled by x^i can be uniquely generated by the linear combination of the m columns labeled by $1, x, \dots, x^{m-1}$ with the m coefficients λ_{ij}'s in the expression of x^i. Note that for a regular saturated design A of m factors each at s levels, apart from the m independent columns, columns $x^m, \dots, x^{(s^m-1)/(s-1)-1}$ exactly form the remaining additional columns of A. Furthermore, for any positive integer k, columns $x^k, x^{k+1}, \dots, x^{k+m-1}$ are linear independent over $\mathcal{G}$ if and only if columns $1, x, \dots, x^{m-1}$ are linearly independent over $\mathcal{G}$. Thus we can partition a regular saturated design A of s^m runs into at most $\lfloor (s^m-1)/(m(s-1)) \rfloor$ groups each consisting of m columns labeled by m consecutive powers of x from zero power on and forming an s^m full factorial design. So the grouped A can be used as a grouped OA of strength two in Theorems 2.1 and 2.2.

An example is presented to illustrate the details of the construction of OLHs of order one.

Example 3.1 *Consider the construction of an OLH of order one in* 9^4 *runs. Let* $\mathcal{F}$ *be the* $GF(3^8)$ *with a primitive polynomial* $g(x) = x^8 + x + 2$. x *is a primitive element of* $GF(3^8)$. *Let* $\mathcal{G} = GF(3^2)$ *be a subfield of* $\mathcal{F}$, *then* $\alpha = x^{820}$ *is a primitive element of* $\mathcal{G}$, *whose* 9 *elements can be expressed as* $\{0, 1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6, \alpha^7\}$. *Let* A *be a* 9*-level saturated factorial design of* 9^4 *runs in which the* 820 *factors are labeled by* x^i,$i = 0, \dots, 819$. *Let columns* $1, x, x^2$ *and* x^3 *be the four independent columns and the entries of column* x^i *are generated by the linear combination of columns* $1, x, x^2, x^3$ *with the four coefficients* λ_{ij}*'s in the expression of* x^i. *Then for* $k = 1, \dots, 205$, *the four columns of the* k*-th group labeled by* $x^{4(k-1)+j}$, $j = 0, 1, 2, 3$ *form a* 9*-level full factorial design for* 4 *factors. Replacing the nine symbols of* A *with* -1, $-3/4$, $\dots, 3/4, 1$, *respectively, we obtain a scaled* $OA(9^4, 9^{820}, 2)$ A. *Let* $U = V_2$ *in (3) with* $s = 9$ *and* $B = OLH(9, 5)$ *given in Table 3, then the matrix* M *constructed in (2) gives an* $OLH(9^4, 4100)$ *of order one according to item* (i) *of Theorem 2.2.*

For $m = 3$, any s-level OA of strength three can be directly partitioned into exclusive groups of three columns since any three columns form a full factorial design. For $m \geq 4$, it is a hard work to partition an OA of strength three into groups of s^m full factorial designs for m factors. Here we carry out this grouping process by computer to construct some useful OLHs of order two.

Example 3.2 *Consider the construction of an OLH of order two with* 9^4 *runs. Along with the notations used in Example 3.1, we search out an* $OA(9^4, 9^{36}, 3)$, *denoted by* A, *and present its grouped generator matrix as follows.*

1	x	x^2	x^3	x^5	x^{13}	x^{255}	x^{86}	x^{107}	x^{694}	x^{756}	x^{60}
1	0	0	0	α^3	α^2	α^6	α^5	α^2	α^6	α^6	0
0	1	0	0	α	α	1	α^3	α^6	α^3	α^7	α
0	0	1	0	0	0	0	1	0	0	0	α
0	0	0	1	α^3	α^5	α^3	0	α^4	α^2	α^4	1

x^{784}	x^{222}	x^{58}	x^{385}	x^{136}	x^{103}	x^{247}	x^{626}	x^{90}	x^{787}	x^{512}	x^{501}
α^3	1	α^2	1	α^3	α^3	α^6	α^4	1	0	α^7	0
α^3	α^5	α^2	α^4	1	1	α^3	α^6	α^7	α^7	α^2	α^7
0	α^3	α^6	α^4	α	α^4	α^2	1	α^4	α^5	1	1
α^4	α^6	α^4	α^4	α^5	α^6	α^6	0	α^7	α^3	α^5	α^5

x^{322}	x^{470}	x^{19}	x^{610}	x^{35}	x^{620}	x^{358}	x^{517}	x^{200}	x^{376}	x^{39}	x^{47}
0	1	α^7	α^6	α^3	α^2	1	α^7	α^7	α^7	α	α^6
α^4	α^2	0	α^5	α^5	α^5	1	α^3	0	α	α^4	α
α	α^2	α^2	α^3	α^6	α^4	α	α^4	α	α^4	1	α^4
α^5	α^4	α^4	0	α^6	α^4	1	α^6	α^7	α	α^4	α^3

By replacing the nine symbols of A with -1, $-3/4$, $\ldots, 3/4, 1$, respectively, and letting $U = V_2$ in (3) with $s = 9$ and $B = OLH(9, 4)$ of order two given in Table 3, we can construct the matrix M in (2), which gives an $OLH(9^4, 144)$ of order two.

4 Construction of OLHs via a Kronecker product

Lin et al. (2010) proposed a Kronecker product approach to the construction of OLHs of order one. In this section this approach is revisited to construct a new class of OLHs of order one and also is extended to suit for the construction of OLHs of order two.

Let G and V be $n_1 \times q_1$ and $n_2 \times q_2$, respectively, orthogonal designs with entries ± 1, and W and H be $n_1 \times q_1$ and $n_2 \times q_2$, respectively, OLHs of order one. Define

$$M = \pi_1 G \otimes H + \pi_2 W \otimes V, \tag{5}$$

where $\pi_1 = (n_2 - 1)/(n_1 n_2 - 1)$ and $\pi_2 = n_2(n_1 - 1)/(n_1 n_2 - 1)$. Note that the scale numbers π_1 and π_2 are different from those used in Lin et al. (2010) because the two different regularization systems of the levels of an OLH have been adopted. Then Theorem 1 of Lin et al. (2010) can be rewritten in the following Lemma 4.1.

Lemma 4.1 *The matrix M in (5) is an $OLH(n_1 n_2,\ q_1 q_2)$ of order one if:*

(i) $G'W = 0$ or $H'V = 0$.

(ii) At least one of the following two conditions is true: (a) for any $1 \le i \le q_1$, if $w_{pi} = -w_{p'i}$, then $g_{pi} = g_{p'i}$; (b) for any $1 \le j \le q_2$, if $h_{pj} = -h_{p'j}$ then $v_{pj} = v_{p'j}$.

Now similar to Sun et al. (2009), we present an actual construction of G and W used in (5). Let

$$S_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad T_1 = \begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix}.$$

For any positive integer $c \geq 2$, let

$$S_c = \begin{pmatrix} S_{c-1} & -S^*_{c-1} \\ S_{c-1} & S^*_{c-1} \end{pmatrix}, \; T_c = \begin{pmatrix} T_{c-1} & -(T^*_{c-1} + 2^c S^*_{c-1}) \\ T_{c-1} + 2^c S_{c-1} & T^*_{c-1} \end{pmatrix}, \tag{6}$$

where the * operator works on any matrix of an even number of rows by multiplying the entries in the top half of the matrix by -1 and leaving those in the bottom unchanged.

Let $G = (S'_{c-1}, S'_{c-1})'$ and $W = 1/(2^c - 1)(T'_{c-1}, -T'_{c-1})'$. Similar to Theorem 2 of Sun et al. (2009), it can be shown that W forms an $OLH(2^c, 2^{c-1})$. Furthermore, it can be verified that $G'W = 0$ and the matrices G and W satisfy the condition (a) in Lemma 4.1. Suppose that an $OLH(r, k)$ is available where $r \geq k$ and r is a multiple of 4 such that a Hadamard matrix of order r exists. Let H be the $OLH(r, k)$ and V be the submatrix consisting of k columns of the associated Hadamard matrix of order r. Then the matrix M in (5) gives an $OLH(r2^c, k2^{c-1})$ of order one by Lemma 4.1.

Proposition 4.1 *Suppose that an $OLH(r, k)$ is available where $r \geq k$ and r is a multiple of 4 such that a Hadamard matrix of order r exists. Then for any positive integer c, an $OLH(r2^c, k2^{c-1})$ of order one can be constructed.*

A conclusion for constructing OLHs of order two by applying the Kronecker product method is presented as follows and its proof is delegated to the Appendix.

Lemma 4.2 *The matrix M constructed in (5) is an $OLH(n_1n_2, q_1q_2)$ of order two if the conditions (i) and (ii) in Lemma 4.1 hold, and at least one of the following two conditions is true:*

(a) the J-characteristic of any three columns of the matrix (G, W) is zero, or

(b) the J-characteristic of any three columns of the matrix (H, V) is zero.

Note that if an OLH can be expressed as the full foldover form of $(H', -H')'$, then its J-characteristic of any odd number of columns must be zero. Based on this property, a new construction of OLHs of order two is presented in the following.

Let

$$M_1 = \lambda_1 \begin{pmatrix} G \\ -G \end{pmatrix} \otimes \begin{pmatrix} -H_0 \\ H_0 \end{pmatrix} + \lambda_2 \begin{pmatrix} W_0 \\ -W_0 \end{pmatrix} \otimes \begin{pmatrix} V \\ V \end{pmatrix}, \tag{7}$$

$$M_2 = \lambda_1 \begin{pmatrix} G \\ -G \end{pmatrix} \otimes \begin{pmatrix} H_0 \\ H_0 \end{pmatrix} + \lambda_2 \begin{pmatrix} W_0 \\ -W_0 \end{pmatrix} \otimes \begin{pmatrix} V \\ -V \end{pmatrix}, \tag{8}$$

and

$$M = [M_1, M_2], \tag{9}$$

where $\lambda_1 = (2n_2-1)/(4n_1n_2-1)$ and $\lambda_2 = 2n_2(2n_1-1)/(4n_1n_2-1)$, G and V are $n_1 \times q_1$ and $n_2 \times q_2$, respectively, orthogonal designs with entries ± 1, and W_0 and H_0 are $n_1 \times q_1$ and $n_2 \times q_2$, respectively, orthogonal matrices such that both $(H_0', -H_0')'$ and $(W_0', -W_0')'$ are OLHs of order two.

By Lemma 4.2 we can easily verify that M_1 is an OLH of order two. Note that M_2 can be represented as a row juxtaposition of four blocks each with the form $\pm\lambda_1 G \otimes H_0 \pm \lambda_2 W_0 \otimes V$. By exchanging the second and the third blocks, it can be changed to

$$\lambda_1 \begin{pmatrix} G \\ G \end{pmatrix} \otimes \begin{pmatrix} H_0 \\ -H_0 \end{pmatrix} + \lambda_2 \begin{pmatrix} W_0 \\ -W_0 \end{pmatrix} \otimes \begin{pmatrix} V \\ -V \end{pmatrix}.$$

Thus M_2 is also an OLH of order two by Lemma 4.2. Noticing that $M_1' M_2 = 4\lambda_1\lambda_2[(W_0'G)\otimes(V'H_0) - (G'W_0)\otimes(H_0'V)]$, we can obtain the following conclusion.

Proposition 4.2 *When $(W_0', -W_0')'$ and $(H_0', -H_0')'$ are OLHs of order two, the matrix M constructed in (9) is an $OLH(4n_1n_2, 2q_1q_2)$ of order two if and only if $(G'W_0)\otimes(H_0'V)$ is symmetric.*

5 Construction of OLHs via a stacking method

Suppose that D_a and D_b are two orthogonal designs with sizes of $n_a \times q$ and $n_b \times q$, respectively. Let $n = n_a + n_b$. Construct an $n \times q$ matrix M by stacking D_a and D_b as

$$M = (D_a', D_b')'. \tag{10}$$

Then Lin et al. (2010) gave the following fact.

Lemma 5.1 *The constructed M in (10) is an $OLH(n, q)$ of order one if*

(i) each column of D_a is a permutation of $\{-(n_a-1)/(n-1), -(n_a-3)/(n-1), \ldots, (n_a-3)/(n-1), (n_a-1)/(n-1)\}$, and

(ii) each column of D_b is a permutation of $\{-1, \ldots, -(n_a+1)/(n-1), (n_a+1)/(n-1), \ldots, 1\}$.

Let $n_a = 1$. Then D_a reduces to a zero row vector. Suppose that an $OLH(r, k)$ is available where $r \geq k$ and r is a multiple of 4 such that a Hadamard matrix of order r exists. Let H be the $OLH(r, k)$ and V be the submatrix consisting of k columns of the associated Hadamard matrix of order r. Put $G = (S_{c-1}', S_{c-1}')'$ and $W = (rT_{c-1}' + n_a S_{c-1}', -rT_{c-1}' - n_a S_{c-1}')$, where S_c and T_c are defined in (6). Construct

$$D_b = 1/(n-1)((r-1)G \otimes H + W \otimes V). \tag{11}$$

Similar to Proposition 2 of Lin et al. (2010), it can be shown that the $r2^c \times k2^{c-1}$ matrix D_b constructed in (11) is an orthogonal design with each column being a permutation of $\{-1, \ldots, -1/(r2^{c-1}), 1/(r2^{c-1}), \ldots, 1\}$. Then by stacking D_a and D_b, we can construct an $OLH(r2^c + 1, k2^{c-1})$ of order one.

Proposition 5.1 *Suppose that an $OLH(r, k)$ is available where $r \geq k$ and r is a multiple of 4 such that a Hadamard matrix of order r exists. Then for any positive integer c, an $OLH(r2^c + 1, k2^{c-1})$ of order one can be constructed.*

It can be seen that if D_a is an orthogonal design and each column of D_a is a permutation of $\{-(n_a - 1)/(n - 1), -(n_a - 3)/(n - 1), \ldots, (n_a - 3)/(n - 1), (n_a - 1)/(n - 1)\}$, then $H = (n - 1)/(n_a - 1)D_a$ is an $OLH(n_a, q)$ of order one. Conversely, if an $OLH(n_a, q)$ H of order one is available, then $D_a = (n_a - 1)/(n - 1)H$ is a desired orthogonal design in Lemma 5.1. Next, construct a $2^c \times 2^{c-1}$ matrix

$$D_b = 1/(n - 1)(T'_{c-1} + n_a S'_{c-1}, -T'_{c-1} - n_a S'_{c-1})'. \tag{12}$$

It can be shown that the matrix D_b constructed in (12) is an orthogonal design with each column being a permutation of $\{-1, \ldots, -(n_a + 1)/(n - 1), (n_a + 1)/(n - 1), \ldots, 1\}$. Furthermore, it can be verified that the J-characteristic of any three columns of the D_b in (12) is zero. So if an $OLH(n_a, q)$ L of order two is available, Then by stacking D_a and D_b, we can construct an $OLH(n_a + 2^c, \min(q, 2^{c-1}))$ of order two.

Proposition 5.2 *Suppose that an $OLH(n_a, q)$ of order one (or two) is available. Then for any positive integer c, an $OLH(n_a + 2^c, \min(q, 2^{c-1}))$ of order one (or two) can be constructed.*

In particular, for $n_a = 1$, Proposition 5.2 shows that a class of $OLH(k2^c + 1, 2^{c-1})$ of order two can be constructed for any positive integers k and c. This is an extension of Theorem 1 of Sun et al. (2009). Using Proposition 5.2, we can also construct an $OLH(27, 7)$ of order one and an $OLH(25, 4)$ of order two by letting D_a be the orthogonal designs in Lemma 5.1 associated with the $OLH(11, 7)$ and $OLH(9, 4)$, respectively, given in Table 3.

6 Some new OLHs obtained

Some small OLHs of order one and order two with prime power runs are constructed and presented in Tables 1 and 2, respectively. In both tables, the first column displays the new obtained OLHs. The + symbol on the superscript of a design means that the design constructed here is new or has more columns than those constructed by using the previous methods. The three components, a grouped OA (A), a column-orthogonal rotation (COR) matrix (U) and a small OLH (B), used in Theorems 2.1 and 2.2 to generate a bigger OLH are listed in the second, the third and the fourth columns, respectively. The used theorem (or proposition), abbreviated as Th (or Prop), is listed in the last column. Note that if the stacking method is applied, the small OLH column lists the OLHs associated with the orthogonal designs D_a.

The OA's in the "Grouped OA" column of Table 1 are all regular saturated factorial designs of strength two. The subfield grouping scheme displayed in Section 3 is used to partition them into groups of full factorial designs. In Table 2, the grouped OAs of the form $OA(s^3, s^{s+1}, 3)$ can be constructed through the Bush's method (Hedayat et al. (1999)) and be easily partitioned into exclusive groups of three columns. For $m = 4$, the five OA's $OA(3^4, 3^8, 3)$, $OA(5^4, 5^{24}, 3)$, $OA(7^4, 7^{44}, 3)$, $OA(9^4, 9^{36}, 3)$ and $OA(11^4, 11^{116}, 3)$ are regular factorial designs of strength three searched out by computer. The generator matrix of $OA(9^4, 9^{36}, 3)$ has been displayed in Example 3.2. The grouped generator matrices of $OA(3^4, 3^8, 3)$, $OA(5^4, 5^{24}, 3)$, $OA(7^4, 7^{44}, 3)$ are given by

$$\begin{pmatrix} 1000 & 1011 \\ 0100 & 2112 \\ 0010 & 0111 \\ 0001 & 2101 \end{pmatrix}, \quad \begin{pmatrix} 1000 & 0001 & 1111 & 1111 & 1111 & 1111 \\ 0100 & 1110 & 0011 & 1122 & 2244 & 3344 \\ 0010 & 1231 & 2301 & 2401 & 3434 & 3412 \\ 0001 & 2131 & 3420 & 4312 & 3402 & 2141 \end{pmatrix},$$

$$\begin{pmatrix} 1000 & 1111 & 1111 & 1101 & 1101 & 1110 & 1011 & 1111 & 1011 & 1111 & 1111 \\ 0100 & 2655 & 1455 & 5512 & 0211 & 3041 & 1164 & 1042 & 3123 & 3666 & 3604 \\ 0010 & 3454 & 6110 & 2652 & 3410 & 0146 & 2452 & 3665 & 2261 & 6302 & 3153 \\ 0001 & 0006 & 0024 & 3534 & 6153 & 6531 & 1222 & 2315 & 5661 & 2416 & 3315 \end{pmatrix},$$

respectively. The generator matrix of $OA(11^4, 11^{116}, 3)$ is broken into three parts as follows

$$\left(\begin{matrix} 1000 & 1101 & 1111 & 1110 & 0111 & 1111 & 1011 & 1111 & 1111 & 1111 \\ 0100 & 6111 & 1287 & 7061 & 1948 & 21t0 & t138 & 5379 & 6t73 & 6292 \\ 0010 & 0317 & 2647 & 0898 & t43t & 1536 & 8978 & 4t22 & 3230 & 5368 \\ 0001 & t034 & 3817 & 4634 & 7363 & 4914 & t819 & 8915 & 7233 & 4992 \end{matrix}\right.$$

$$\begin{matrix} 1111 & 1111 & 1111 & 0111 & 1111 & 1011 & 1111 & 1101 & 1111 & 1111 \\ 989t & 7422 & 4350 & 1176 & 3452 & 617t & 428t & t110 & 4109 & 5362 \\ 9537 & 1452 & t691 & 2457 & 1510 & 8665 & 9931 & 0057 & 019t & 34t4 \\ 4tt9 & 9270 & 1701 & 9505 & 2555 & 16t3 & 9150 & 76t3 & 8750 & 408t \end{matrix}$$

$$\left.\begin{matrix} 1110 & 1111 & 1111 & 1111 & 1111 & 1111 & 0111 & 1111 & 1111 \\ t791 & 9630 & 5858 & 3715 & 4350 & 9617 & 1906 & 41t8 & 3052 \\ 6473 & 1625 & t026 & 5998 & 2863 & 018t & 78t2 & 1tt2 & 9457 \\ 5625 & 82t2 & 2270 & 8823 & 4418 & 1685 & 1779 & tt66 & 6966 \end{matrix}\right),$$

where t represents the level 10. For $m = 5$, the $OA(3^5, 3^{20}, 3)$ is obtained from the dual code of Kschischang-Pasupathy cyclic code (Chapter 5.12, Hedayat et al. (1999)) and its grouped generator matrix is displayed as follows

$$\begin{pmatrix} 20211 & 02121 & 01001 & 10000 \\ 02021 & 10212 & 10100 & 11000 \\ 00202 & 11021 & 21010 & 01100 \\ 00020 & 21102 & 12101 & 00110 \\ 00002 & 02110 & 21210 & 10011 \end{pmatrix}.$$

The $OA(5^5, 5^{55}, 3)$ and $OA(7^5, 7^{70}, 3)$ are regular factorial designs of strength three searched out by computer. The generator matrix of $OA(5^5, 5^{55}, 3)$ is given by

$$\begin{pmatrix} 10000 & 11010 & 11111 & 11110 & 11111 & 11001 & 10110 & 11111 & 11011 & 01111 & 11111 \\ 01000 & 33101 & 20241 & 40141 & 43110 & 44113 & 21441 & 13123 & 32113 & 12301 & 13122 \\ 00100 & 42310 & 11344 & 02403 & 10243 & 34414 & 22313 & 01322 & 34331 & 14421 & 00201 \\ 00010 & 20241 & 00112 & 10403 & 04000 & 04000 & 22320 & 03101 & 02100 & 20112 & 12304 \\ 00001 & 10004 & 01334 & 43332 & 23124 & 11123 & 42244 & 42423 & 22134 & 34441 & 20413 \end{pmatrix},$$

and the generator matrix of $OA(7^5, 7^{70}, 3)$ is broken into two parts given by

$$\left(\begin{matrix} 10000 & 11001 & 11111 & 11111 & 01111 & 11100 & 11111 \\ 01000 & 24102 & 54116 & 34446 & 12636 & 46411 & 61053 \\ 00100 & 04410 & 63364 & 56112 & 06165 & 02540 & 61505 \\ 00010 & 00443 & 15443 & 33524 & 63313 & 50452 & 46510 \\ 00001 & 60045 & 62232 & 66631 & 53534 & 36042 & 61251 \end{matrix}\right.$$

$$\left.\begin{matrix} 11011 & 11101 & 11101 & 10011 & 01101 & 10111 & 11111 \\ 05103 & 35616 & 03610 & 51102 & 03610 & 01045 & 23551 \\ 10531 & 32631 & 40161 & 53403 & 14420 & 23401 & 62420 \\ 46054 & 12115 & 14162 & 15132 & 65064 & 52015 & 06155 \\ 00665 & 43141 & 22016 & 32161 & 10304 & 46603 & 50400 \end{matrix}\right)$$

The six small OLHs used as the B matrices in Theorems 2.1 and 2.2 are presented in Table 3.

7 Concluding remarks

A general construction method for OLHs of order one and order two with s^m runs is proposed in this paper, where s and m may be any positive integers greater than one, and is used to construct OLHs of order one and order two with more columns than those provided by the previous methods. The key point is how to partition a saturated OA

Table 1: Some new $OLH(s^m, q)$ of order one

$OLH(s^m, q)$	Grouped OA (A)	COR (U)	Small OLH (B)	Method
$OLH(3^3, 7)$			$OLH(11, 7)$	Prop 5.2
$OLH(9^2, 50)$	$OA(9^2, 9^{10}, 2)$	$U_{2\times 2}$	$OLH(9, 5)$	(i) of Th 2.2
$OLH(3^5, 24)^+$	$OA(3^5, 3^{121}, 2)$	$U_{5\times 1}$		(i) of Th 2.1
$OLH(27^2, 196)^+$	$OA(27^2, 27^{28}, 2)$	$U_{2\times 2}$	$OLH(27, 7)$	(i) of Th 2.2
$OLH(3^7, 156)^+$	$OA(3^7, 3^{1093}, 2)$	$U_{7\times 1}$		(i) of Th 2.1
$OLH(9^4, 4100)^+$	$OA(9^4, 9^{820}, 2)$	$U_{4\times 4}$	$OLH(9, 5)$	(i) of Th 2.2
$OLH(27^3, 1764)^+$	$OA(27^3, 27^{7381}, 2)$	$U_{3\times 1}$	$OLH(27, 7)$	(i) of Th 2.2
$OLH(5^2, 12)$	$OA(5^2, 5^6, 2)$	$U_{2\times 2}$	$OLH(5, 2)$	(i) of Th 2.2
$OLH(5^3, 20)^+$	$OA(5^3, 5^{31}, 2)$	$U_{3\times 1}$	$OLH(5, 2)$	(i) of Th 2.2
$OLH(5^4, 312)$	$OA(5^4, 5^{156}, 2)$	$U_{4\times 4}$	$OLH(5, 2)$	(i) of Th 2.2
$OLH(5^5, 312)^+$	$OA(5^5, 5^{781}, 2)$	$U_{5\times 1}$	$OLH(5, 2)$	(i) of Th 2.2
$OLH(5^6, 2604)$	$OA(5^6, 5^{3906}, 2)$	$U_{6\times 2}$	$OLH(5, 2)$	(i) of Th 2.2
$OLH(7^2, 24)$	$OA(7^2, 7^8, 2)$	$U_{2\times 2}$	$OLH(7, 3)$	(i) of Th 2.2
$OLH(7^3, 57)^+$	$OA(7^3, 7^{57}, 2)$	$U_{3\times 1}$	$OLH(7, 3)$	(i) of Th 2.2
$OLH(7^4, 1200)$	$OA(7^4, 7^{400}, 2)$	$U_{4\times 4}$	$OLH(7, 3)$	(i) of Th 2.2
$OLH(11^2, 84)$	$OA(11^2, 11^{12}, 2)$	$U_{2\times 2}$	$OLH(11, 7)$	(i) of Th 2.2
$OLH(11^3, 308)^+$	$OA(11^3, 11^{133}, 2)$	$U_{3\times 1}$	$OLH(11, 7)$	(i) of Th 2.2
$OLH(11^4, 10248)$	$OA(11^4, 11^{1464}, 2)$	$U_{4\times 4}$	$OLH(11, 7)$	(i) of Th 2.2

Table 2: Some new $OLH(s^m, q)$ of order two

$OLH(s^m, q)$	Grouped OA (A)	COR (U)	Small OLH (B)	Method
$OLH(3^4, 8)^+$	$OA(3^4, 3^8, 3)$	$U_{4\times 4}$		(ii) of Th 2.1
$OLH(3^5, 4)^+$	$OA(3^5, 3^{20}, 3)$	$U_{5\times 1}$		(ii) of Th 2.1
$OLH(9^3, 12)^+$	$OA(9^3, 9^{10}, 3)$	$U_{3\times 1}$	$OLH(9, 4)$	(ii) of Th 2.2
$OLH(9^4, 144)^+$	$OA(9^4, 9^{36}, 3)$	$U_{4\times 4}$	$OLH(9, 4)$	(ii) of Th 2.2
$OLH(5^2, 4)$			$OLH(9, 4)$	Prop 5.2
$OLH(5^3, 4)^+$	$OA(5^3, 5^6, 3)$	$U_{3\times 1}$	$OLH(5, 2)$	(ii) of Th 2.2
$OLH(5^4, 48)^+$	$OA(5^4, 5^{24}, 3)$	$U_{4\times 4}$	$OLH(5, 2)$	(ii) of Th 2.2
$OLH(5^5, 22)^+$	$OA(5^5, 5^{55}, 3)$	$U_{5\times 1}$	$OLH(5, 2)$	(ii) of Th 2.2
$OLH(25^3, 32)^+$	$OA(25^3, 25^{26}, 3)$	$U_{3\times 1}$	$OLH(25, 4)$	(ii) of Th 2.2
$OLH(7^2, 8)$			$OLH(17, 8)$	Prop 5.2
$OLH(7^3, 6)^+$	$OA(7^3, 7^8, 3)$	$U_{3\times 1}$	$OLH(7, 3)$	(ii) of Th 2.2
$OLH(7^4, 132)^+$	$OA(7^4, 7^{44}, 3)$	$U_{4\times 4}$	$OLH(7, 3)$	(ii) of Th 2.2
$OLH(7^5, 42)^+$	$OA(7^5, 7^{70}, 3)$	$U_{5\times 1}$	$OLH(7, 3)$	(ii) of Th 2.2
$OLH(11^3, 8)^+$	$OA(11^3, 11^{12}, 2)$	$U_{3\times 1}$	$OLH(11, 2)$	(i) of Th 2.2
$OLH(11^4, 232)^+$	$OA(11^4, 11^{116}, 2)$	$U_{4\times 4}$	$OLH(11, 2)$	(i) of Th 2.2

Table 3: Six small OLHs

Order Two											Order One											
OLH(5,2)		OLH(7,3)			OLH(9,4)				OLH(11,2)		OLH(9,5)					OLH(11,7)						
-2	-1	-3	1	2	-4	-3	-2	-1	-5	5	-4	2	2	-3	-1	-5	-4	-5	-5	-3	0	0
-1	2	-2	-3	-1	-3	4	1	-2	-4	-2	-3	-1	-1	4	-2	-4	2	-1	3	4	5	4
0	0	-1	2	-3	-2	-1	4	3	-3	-4	-2	3	-4	1	2	-3	-2	4	5	-4	-2	-1
1	-2	0	0	0	-1	2	-3	4	-2	-1	-1	-2	4	-2	3	-2	3	-3	4	1	-4	-2
2	1	1	-2	3	0	0	0	0	-1	-3	0	0	0	0	0	-1	4	2	-4	3	2	-4
		2	3	1	1	-2	3	-4	0	0	1	-4	1	2	-3	0	-5	5	-2	5	-3	2
		3	-1	-2	2	1	-4	-3	1	3	2	-3	-3	-1	4	1	5	3	-3	-5	-1	5
					3	-4	-1	2	2	1	3	1	-2	-4	-4	2	-1	1	1	-2	3	-5
					4	3	2	1	3	4	4	4	3	3	1	3	0	0	-1	0	1	-3
									4	2						4	1	-4	0	2	-5	1
									5	-5						5	-3	-2	2	-1	4	3

Note that the first three OLHs and the fifth OLH come from Prescott (2009) and the last one is from Lin et al. (2009).

into as more groups of full factorial designs as possible. The general partition scheme of a regular prime power-level saturated factorial design of strength two is given. But for a general regular factorial design of strength three, the partition process is currently carried out by computer and is under investigation. In addition, the Kronecker product and the stacking methods proposed by Lin et al. (2010) are revisited and combined to construct some new classes of OLHs of orders one and two with other flexible numbers of runs.

Acknowledgements. The work is supported by NNSF of China Grant 10971004, 973 Program of China Grant 2007CB512605, and SRF for ROCS, SEM.

Appendix

Proof of Theorem 2.1

Since all s levels of A sum to zero and the strength of A is two, any pair of columns of A have zero correlation. Lemma 2.1 ensures the equally-spaced property of any column of M, therefore part (i) is easy to obtain.

Let $A = (a_{ij})$, $R = (r_{ij})$ and β_k be the kth column of AR. For any three columns k_1,

k_2 and k_3 of AR, where k_1, k_2 and k_3 may be equal, we have:

$$\begin{aligned}
J(\beta_{k_1}, \beta_{k_2}, \beta_{k_3}) &= \sum_{i=1}^{s^m} \left(\sum_{j=1}^{fm} a_{ij} r_{jk_1} \right) \left(\sum_{j=1}^{fm} a_{ij} r_{jk_2} \right) \left(\sum_{j=1}^{fm} a_{ij} r_{jk_3} \right) \\
&= \sum_{j_1=1}^{fm} \sum_{j_2=1}^{fm} \sum_{j_3=1}^{fm} r_{j_1k_1} r_{j_2k_2} r_{j_3k_3} \sum_{i=1}^{p^m} a_{ij_1} a_{ij_2} a_{ij_3} \\
&= \sum_{j_1=1}^{fm} \sum_{j_2=1}^{fm} \sum_{j_3=1}^{fm} r_{j_1k_1} r_{j_2k_2} r_{j_3k_3} J(a_{j_1}, a_{j_2}, a_{j_3}) \\
&= 0,
\end{aligned}$$

where the last equality is obtained by noting that all s levels of A sum to zero and its strength is three. Thus Part (ii) follows.

Proof of Theorem 2.2

Following the proof process of Theorem 1 of Lin et al. (2009), part (i) of Theorem 2.2 can be similarly obtained. We now focus on the proof of part (ii). For $t = 1, \ldots, l$, let $A^{(t)} = (a_{ij}^{(t)})$, $R = (r_{ij})$ and $\beta_k^{(t)}$ be the kth column of $A^{(t)}R$. Similar to the proof of part (ii) of Theorem 2.1, we need to prove that $J(\beta_{k_1}^{(t_1)}, \beta_{k_2}^{(t_2)}, \beta_{k_3}^{(t_3)}) = 0$ for any $1 \le t_j \le l$ and $1 \le k_j \le fm$, $j = 1, 2, 3$, where t_1, t_2 and t_3 may be equal, and k_1, k_2 and k_3 may be equal.

Now we consider the following two cases: (1) at least two of k_1, k_2 and k_3 are different, (2) $k_1 = k_2 = k_3$. For case (1), the equation holds by noting that substituting the level symbols in any two different columns doesn't affect their combinatorial orthogonality and the sum of all the s levels in any $a^{(t)}$ is zero. For case (2), the equation can also be ensured because B is an OLH of order two. Thus the proof is complete.

Proof of Proposition 3.1

We just need to prove that $1, x, \ldots, x^{m-1}$ are linearly independent over $\mathcal{G}$. If not, there exist m elements $\alpha_i \in \mathcal{G}$ for $i = 0, 1, \ldots, m-1$, which are not all zero, such that $\sum_{i=0}^{m-1} \alpha_i x^i = 0$. Suppose that among the m elements $\alpha_i \in \mathcal{G}$, $i = 0, 1, \ldots, m-1$, the last nonzero element is α_r $(1 \le r \le m-1)$. Then $x^r = \alpha_r^{-1} \sum_{i=0}^{r-1} \alpha_i x^i$. Thus, x^r, x^{r+1} and so on can be expressed as linear combinations of $1, x, \ldots, x^{r-1}$ with coefficients from $\mathcal{G}$. Note that the total number of such linear combinations of $1, x, \ldots, x^{r-1}$ is at most $(p^c)^r$, a contradiction to the number of elements of $\mathcal{F}$. The proof is complete.

Proof of Lemma 4.2

According to Theorem 1 of Lin et al. (2010), conditions (i) and (ii) in Lemma 4.1 ensure that M is an $OLH(n_1n_2, q_1q_2)$ of order one, so we need only to prove that M is of order two. Let g_k, h_k, w_k and v_k be the kth column of G, H, W and V, respectively. Also let β_k be the kth column of M. For any three columns k_1, k_2 and k_3 of M, where

$k_i = (j_i - 1)q_2 + r_i$ with $0 \le r_i < q_2, 1 \le j_i \le q_1$, $i = 1, 2, 3$. then the expression $M = \pi_1 G \otimes H + \pi_2 W \otimes V$ gives that $\beta_{k_i} = \pi_1 g_{j_i} \otimes h_{r_i} + \pi_2 w_{j_i} \otimes v_{r_i}$. Therefore, $J(\beta_{k_1}, \beta_{k_2}, \beta_{k_3})$ can be expressed as

$$
\begin{aligned}
&\rho_1 J(g_{j_1}, g_{j_2}, g_{j_3}) J(h_{r_1}, h_{r_2}, h_{r_3}) + \rho_2 J(w_{j_1}, g_{j_2}, g_{j_3}) J(v_{r_1}, h_{r_2}, h_{r_3}) \\
+&\rho_2 J(g_{j_1}, w_{j_2}, g_{j_3}) J(h_{r_1}, v_{r_2}, h_{r_3}) + \rho_2 J(g_{j_1}, g_{j_2}, w_{j_3}) J(v_{r_1}, h_{r_2}, v_{r_3}) \\
+&\rho_3 J(w_{j_1}, w_{j_2}, g_{j_3}) J(v_{r_1}, v_{r_2}, h_{r_3}) + \rho_3 J(w_{j_1}, g_{j_2}, w_{j_3}) J(v_{r_1}, h_{r_2}, v_{r_3}) \\
+&\rho_3 J(g_{j_1}, w_{j_2}, w_{j_3}) J(h_{r_1}, v_{r_2}, v_{r_3}) + \rho_4 J(w_{j_1}, w_{j_2}, w_{j_3}) J(v_{r_1}, v_{r_2}, v_{r_3}),
\end{aligned}
$$

where $\rho_1 = \pi_1^3$, $\rho_2 = \pi_1^2 \pi_2$, $\rho_3 = \pi_1 \pi_2^2$ and $\rho_4 = \pi_2^3$. Because the J-characteristic of any three columns of the matrix (G, W) or of the matrix (H, V) is zero, the eight components in the above expression are all zero. So the proof is complete.

References

Beattie, S. D., Lin, D. K. J., 2005. A new class of Latin hypercubes for computer experiments. In Contemporary Multivariate Analysis and Designs of Experiments, in Celebration of Prof. Kai-Tai Fang's 65th Birthday. Ed. J. Fan and G. Li, pp. 205-226. World Scientific, Singapore.

Bingham, D., Sitter, R. D., Tang, B., 2009. Orthogonal and nearly orthogonal designs for computer experiments. Biometrika 96, 51-65.

Bose, R. C., 1947. Mathematical theory of symmetrical factorial designs. Sankhyā 8, 107-166.

Fang, K. T., Li, R. Z., Sudjianto, A., 2005. Design and Modeling for Computer Experiments. Chapman & Hall/CRC Press, New York.

Georgious, S. D., 2009. Orthogonal Latin hypercube designs from generalized orthogonal designs. J. Statist. Plann. Inference 139, 1530-1540.

Geramita, A. V., Seberry, J., 1979. Orthogonal designs: Quadratic forms and Hadamard matrices. Marcel Dekker, New York, Basel.

Hedayat, A. S., Sloane, N. J. A., Stufken, J., 1999. Orthogonal arrays: theory and applications. Springer, New York.

Lin, C. D., Mukerjee, R., Tang, B., 2009. Construction of orthogonal and nearly orthogonal Latin hypercubes. Biometrika 96, 243-247.

Lin, C. D., Bingham, D., Sitter, R. R., Tang, B., 2010. A new and flexible method for constructing designs for computer experiments. Ann. Statist. 38, 1460-1477.

Mckay, M., Beckman, R., Conover, W., 1979. A comparison of three methods for selecting values of input variables in the analysis of output from a computer code. Technometrics 21, 239-246.

Pang, F., Liu, M. Q., Lin, D. K. J., 2009. A construction method for orthogonal Latin hypercube designs with prime power levels. Statist. Sinica 19, 1721-1728.

Prescott, P., 2009. Orthogonal-column Latin hypercube designs with small samples. Computational Statist. Data Analysis 53, 1191-1200.

Santner, T. J., Williams, B. J., Notz, W. I., 2003. The Design and Analysis of Computer Experiments. Springer, New York.

Steinberg, M., Lin, D. K. J., 2006. A construction method for orthogonal Latin hypercube designs. Biometrika 93, 279-288.

Sun, F. S., Liu, M. Q., Lin, D. K. J., 2009. Construction of orthogonal Latin hypercube designs. Biometrika 96, 971-974.

Ye, K. Q., 1998. Orthogonal column Latin hypercubes and their application in computer experiments. J. Amer. Statist. Assoc. 93, 1430-1439.