# Weighted Estimation of the Dependence Function for an Extreme-Value Distribution

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**Abstract**

Bivariate extreme-value distributions have been employed in modeling extremes in environmental sciences and risk management. An important issue is to estimate the dependence function such as Pickands dependence function. Some estimators for the Pickands dependence function have been studied by assuming the marginals are known. Recently, Genest and Segers (2009) [Ann. Statist. 37, 2990–3022] derived the asymptotic distributions of those proposed estimators with marginal distributions replaced by the empirical distributions. In this paper, we propose a class of weighted estimators including those in Genest and Segers (2009) as special cases. Furthermore, a jackknife empirical likelihood method is proposed for constructing confidence intervals for the Pickands dependence function, which avoids estimating the complicated asymptotic variance. A simulation study demonstrates the effectiveness of the proposed jackknife empirical likelihood method.

**Key-words**: Bivariate extreme; dependence function; jackknife empirical likelihood method

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## 1 Introduction

Let $(X_{11}, X_{12}), \cdots, (X_{n1}, X_{n2})$ be independent random pairs with common distribution function $F$ and continuous marginal distributions $F_1(x) = F(x, \infty)$ and $F_2(y) = F(\infty, y)$. Then the copula of $F$ is defined as

$$C(x, y) = P(F_1(X_{11}) \leq x, F_2(X_{12}) \leq y).$$

When $C^t(u^{1/t}, v^{1/t}) = C(u, v)$ holds for all $u, v \in [0, 1]$ and $t > 0$, $C$ is called an extreme value copula and is determined by the so-called Pickands dependence function $A$ through the equation

$$C(u, v) = \exp\{\log(uv) A(\frac{\log(v)}{\log(uv)})\} \tag{1.1}$$

for all $(u, v) \in (0, 1]^2 \setminus \{(1, 1)\}$, where $A$ is a convex function and satisfies $\max(t, 1-t) \leq A(t) \leq 1$ for all $0 \leq t \leq 1$; see Pickands (1981) and Falk and Reiss (2005).

Put $Y_{ij} = -\log\{F_j(X_{ij})\}$ for $i = 1, \cdots, n$, $j = 1, 2$ and

$$H_n(z) = \frac{1}{n} \sum_{i=1}^{n} I(\frac{Y_{i1}}{Y_{i1} + Y_{i2}} \leq z).$$

We denote $u \wedge v = \min(u, v)$ and $u \vee v = \max(u, v)$ throughout. When the marginal distributions $F_j, j = 1, 2$ are known, estimators for the Pickands dependence function $A(t)$ have been proposed by Pickands (1981), Deheuvels (1991), Hall and Tajvidi (2000) and Capéraà, Fougères and Genest (1997), which are defined as

$$A^P(t) = \frac{n}{\sum_{i=1}^n \{Y_{i1}/t\} \wedge \{Y_{i2}/(1-t)\}},$$

$$A^D(t) = \frac{n}{\sum_{i=1}^n \{Y_{i1}/t\} \wedge \{Y_{i2}/(1-t)\} - t\sum_{i=1}^n Y_{i1} - (1-t)\sum_{i=1}^n Y_{i2} + n},$$

$$A^{HT}(t) = \frac{n}{\sum_{i=1}^n \{\frac{nY_{i1}}{t\sum_{j=1}^n Y_{j1}}\} \wedge \{\frac{nY_{i2}}{(1-t)\sum_{j=1}^n Y_{j2}}\}},$$

$$A^{CFG}(t) = \exp\{\lambda(t) \int_0^t \frac{H_n(z) - z}{z(1-z)}\, dz - (1 - \lambda(t)) \int_t^1 \frac{H_n(z) - z}{z(1-z)}\, dz\}$$

respectively, where $\lambda(t) \in [0, 1]$ is a weight function, and $A^P(t)$ and $A^D(t)$ are defined as corresponding limits when $t = 0$ or 1. When the marginal distributions are unknown, similar non-parametric estimators can be obtained by replacing the marginal distribution $F_j$ by the corresponding empirical distribution $F_{nj}(x) = \frac{1}{n}\sum_{i=1}^n I(X_{ij} \le x)$ or $\hat{F}_{nj}(x) = \frac{1}{n+1}\sum_{i=1}^n I(X_{ij} \le x)$. Let us denote these estimators as $\tilde{A}^P(t), \tilde{A}^D(t), \tilde{A}^{HT}(t)$ and $\tilde{A}^{CFG}(t)$. Recently, Genest and Segers (2009) showed that $\tilde{A}^P(t), \tilde{A}^D(t)$ and $\tilde{A}^{HT}(t)$ have the same asymptotic distribution as

$$\hat{A}^P(t) = \frac{n}{\sum_{i=1}^n \{Z_{i1}/(1-t)\} \wedge \{Z_{i2}/t\}}$$

and $\tilde{A}^{CFG}(t)$ with $\lambda(t) = t$ has the same asymptotic distribution as

$$\hat{A}^{CFG}(t) = \exp\{-\gamma - \frac{1}{n}\sum_{i=1}^n (Z_{i1}/(1-t)) \wedge (Z_{i2}/t)\},$$

where $\gamma = -\int_0^\infty \log(x)e^{-x}\,dx$ is the Euler constant and

$$Z_{ij} = -\log\{\hat{F}_{nj}(X_{ij})\} \quad \text{for} \quad i = 1, \cdots, n, j = 1, 2.$$

Moreover, Genest and Segers (2009) derived the asymptotic distributions of $\hat{A}^P(t)$ and $\hat{A}^{CFG}(t)$ by noticing the following important relationship

$$\hat{A}^P(t) = \{\int_0^1 u^{-1}\hat{C}_n(u^{1-t}, u^t)\,du\}^{-1}$$

and

$$\hat{A}^{CFG}(t) = \exp\left(-\gamma + \int_0^1 \{\hat{C}_n(u^{1-t}, u^t) - I(u > e^{-1})\}\{u\log(u)\}^{-1}\,du\right),$$

where

$$\hat{C}_n(u, v) = \frac{1}{n}\sum_{i=1}^n I(\hat{F}_{n1}(X_{i1}) \le u, \hat{F}_{n2}(X_{i2}) \le v).$$

In this paper, we propose a class of weighted estimators including $\hat{A}^P(t)$ and $\hat{A}^{CFG}(t)$ as special cases, see Section 2 for details. Further, in Section 3 we propose a jackknife empirical likelihood method to construct confidence intervals for the Pickands dependence function. Unlike the normal approximation method, this new method has no need to estimate any additional quantities such as asymptotic variance. A simulation study is conducted in Section 4 to examine the finite sample behavior of the proposed jackknife empirical likelihood method. All proofs are put in Section 5.

## 2 Weighted Estimation

It follows from (1.1) that

$$C(u^{1-t}, u^t) = u^{A(t)} \quad \text{for all} \quad u \in [0,1] \quad \text{and all} \quad t \in [0,1], \tag{2.1}$$

which motivates to estimate $A(t)$ by minimizing the following weighted distance with respect to $\alpha \geq 0$:

$$\int_0^1 \{\hat{C}_n(u^{1-t}, u^t) - u^\alpha\}^2 \bar{\lambda}(u,t)\, du,$$

where $\bar{\lambda}(u,t) \geq 0$ is a weight function. Under some regularity conditions, the above estimator is the solution of $\alpha$ to the equation

$$\int_0^1 \{\hat{C}_n(u^{1-t}, u^t) - u^\alpha\} u^\alpha \{-\log(u)\} \bar{\lambda}(u,t)\, du = 0$$

for $\alpha > 0$. This is a special case of the proposed M-estimators and Z-estimators in Bücher, Dette and Volgushev (2011). By noting that $u^\alpha(-\log u)\bar{\lambda}(u,t) = C(u^{1-t}, u^t)(-\log u)\bar{\lambda}(u,t)$ and $\bar{\lambda}(u,t)$ is any weight function, we propose to treat $C(u^{1-t}, u^t)(-\log u)\bar{\lambda}(u,t)$ as a new weight function. This leads us to estimate $A(t)$ by solving the following equation with respect to $\alpha \geq 0$:

$$\int_0^1 \{\hat{C}_n(u^{1-t}, u^t) - u^\alpha\} \lambda(u,t)\, du = 0, \tag{2.2}$$

where $\lambda(u,t) \geq 0$ is a new weight function. Denote this new estimator by $\hat{A}_n^w(t;\lambda)$. When $\lambda(u,t)$ is taken as $u^{-1}$ or $\{-u\log(u)\}^{-1}$, $\hat{A}_n^w(t;\lambda)$ becomes $\hat{A}^P(t)$ or $\hat{A}^{CFG}(t)$. Thus the above class of estimators includes the known estimators in the literature as special cases.

Put $g(\alpha) = \int_0^1 \{\hat{C}_n(u^{1-t}, u^t) - u^\alpha\}\lambda(u,t)\, du$. Since $u^\alpha$ is a decreasing function of $\alpha$ for each fixed $u \in [0,1]$, $g(\alpha)$ is an increasing function of $\alpha$ for each fixed $t$. Also $g(0) < 0$ and $g(\infty) > 0$ when $n$ is large enough. Hence (2.2) has a unique solution $\hat{A}_n^w(t;\lambda)$ for each large $n$ and $t \in [0,1]$. Note that this unique solution may not satisfy that $\max(t, 1-t) \leq \hat{A}_n^w(t;\lambda) \leq 1$ and $\hat{A}_n^w(0;\lambda) = \hat{A}_n^w(1;\lambda) = 1$.

Let $W(u,v)$ denote a tight Gaussian process with mean zero, covariance

$$E\{W(u_1,v_1)W(u_2,v_2)\} = C(u_1 \wedge u_2, v_1 \wedge v_2) - C(u_1,v_1)C(u_2,v_2)$$

and $W(u,0) = W(0,v) = W(1,1) = 0$ for all $u, v \in [0,1]$. The asymptotic distribution for the proposed estimator $\hat{A}_n^w(t;\lambda)$ is given in the following theorem.

**Theorem 2.1.** *Suppose $\frac{\partial^2}{\partial u^2}C(u,v)$, $\frac{\partial^2}{\partial v^2}C(u,v)$ and $\frac{\partial^2}{\partial u \partial v}C(u,v)$ are defined and continuous on the sets $\mathcal{F}_1 = \{(u,v) : 0 < u < 1 \quad \text{and} \quad 0 \le v \le 1\}$, $\mathcal{F}_2 = \{(u,v) : 0 \le u \le 1, 0 < v < 1\}$ and $\mathcal{F}_3 = \{(u,v) : 0 < u < 1, 0 < v < 1\}$, respectively, and for each fixed $t \in [0,1]$ the function $\lambda(u,t) \ge 0$ is continuous and is not identical to zero as a function of $u \in (0,1)$. Further we assume that*

$$\begin{cases} |\frac{\partial^2}{\partial u^2}C(u,v)| \le \frac{M}{u(1-u)} & \text{for} \quad (u,v) \in \mathcal{F}_1, \\ |\frac{\partial^2}{\partial v^2}C(u,v)| \le \frac{M}{v(1-v)} & \text{for} \quad (u,v) \in \mathcal{F}_2, \\ |\frac{\partial^2}{\partial u \partial v}C(u,v)| \le \frac{M}{u(1-u)} \wedge \frac{M}{v(1-v)} & \text{for} \quad (u,v) \in \mathcal{F}_3, \end{cases}$$

*for some constant $M > 0$, $A'(t)$ is continuous on $[0,1]$, and there exist $\delta_1 > 0$ and $\delta_2 \in [0, 1/2)$ such that*

$$\begin{cases} \sup_{0\le t\le 1} \sqrt{n} \int_0^{(n+1)^{-1/((1-t)\vee t)}} u^{1/2}\lambda(u,t)\, du \to 0 \\ \sup_{0\le t\le 1} \sqrt{n} \int_{(\frac{n}{n+1})^{1/((1-t)\vee t)}}^1 (1-u)\lambda(u,t)\, du \to 0 \\ \sup_{0\le t\le 1} n^{-1/4+\delta_1} \int_{(n+1)^{-1/((1-t)\vee t)}}^{(\frac{n}{n+1})^{1/((1-t)\vee t)}} \lambda(u,t)\, du \to 0 \\ \sup_{0\le t\le 1} \int_0^1 \{u^{(1-t)\vee t}(1-u^{(1-t)\vee t})\}^{\delta_2}\lambda(u,t)\, du < \infty \\ \sup_{0\le t\le 1} \int_0^1 u^{(1-t)\vee t-(1-t)}u^{(1-t)\delta_2}(1-u^{1-t})^{\delta_2}\lambda(u,t)\, du < \infty \\ \sup_{0\le t\le 1} \int_0^1 u^{(1-t)\vee t-t}u^{t\delta_2}(1-u^{t})^{\delta_2}\lambda(u,t)\, du < \infty \\ \sup_{0\le t\le 1} \int_{1/2}^1 (-\log u)\lambda(u,t)\, dt < \infty. \end{cases} \tag{2.3}$$

*Then as $n \to \infty$, $\sup_{0\le t\le 1} |\hat{A}_n^w(t;\lambda) - A(t)| = o_p(1)$. Moreover, suppose that $\lambda(u,t)$ is continuous in $(0,1)\times[0,1]$, and*

$$|\lambda(u,t_1) - \lambda(u,t_2)| \le |t_1 - t_2|^{\delta_0}\lambda_0(u), t_1, t_2 \in [0,1], u \in (0,1)$$

*for some constant $\delta_0 > 0$ and function $\lambda_0(u), u \in (0,1)$, where $\lambda_0(u)$ satisfies that*

$$\int_0^{1/2} u^\alpha \lambda_0(u)du < \infty, \int_{1/2}^1 (1-u^\alpha)\lambda_0(u)du < \infty$$

*for all $\alpha > 0$. Then as $n \to \infty$, $\sqrt{n}\{\hat{A}_n^w(t;\lambda) - A(t)\}$ converges to $B(t)$ in $C([0,1])$, where*

$$\begin{aligned} B(t) &= \{\int_0^1 C(u^{1-t},u^t)\lambda(u,t)\log(u)\, du\}^{-1} \int_0^1 \{W(u^{1-t},u^t) \\ &\quad -C_1(u^{1-t},u^t)W(u^{1-t},1) - C_2(u^{1-t},u^t)W(1,u^t)\}\lambda(u,t)\, du, \end{aligned}$$

$C_1(u,v) = \frac{\partial}{\partial u} C(u,v)$ *and* $C_2(u,v) = \frac{\partial}{\partial v} C(u,v)$.

**Remark 2.1.** *Theorem 2.1 still holds when condition (2.3) is replaced by*

$$\begin{cases} \sup_{0\le t\le 1} \sqrt{n} \int_0^{(n+1)^{-2}} u^{1/2} \lambda(u,t)\, du \to 0 \\ \sup_{0\le t\le 1} \sqrt{n} \int_{(\frac{n}{n+1})^2}^1 (1-u)\lambda(u,t)\, du \to 0 \\ \sup_{0\le t\le 1} n^{-1/4+\delta_1} \int_{(n+1)^{-2}}^{(\frac{n}{n+1})^2} \lambda(u,t)\, du \to 0 \\ \sup_{0\le t\le 1} \int_0^1 u^{\delta_2/2}(1-u)^{\delta_2} \lambda(u,t)\, du < \infty \end{cases}$$

*for some* $\delta_1 > 0$ *and* $\delta_2 \in [0, 1/2)$. *This follows from the proof of Theorem 2.1 by replacing* $\int_0^{(n+1)^{-1/((1-t)\vee t)}}$, $\int_{(n+1)^{-1/((1-t)\vee t)}}^{(\frac{n}{n+1})^{1/((1-t)\vee t)}}$ *and* $\int_{(\frac{n}{n+1})^{1/((1-t)\vee t)}}^1$ *in (5.5) by* $\int_0^{(n+1)^{-2}}$, $\int_{(n+1)^{-2}}^{(\frac{n}{n+1})^2}$ *and* $\int_{(\frac{n}{n+1})^2}^1$, *respectively.*

**Remark 2.2.** *To choose* $\lambda(u,t)$, *a common way is to minimize the asymptotic variance of* $\hat{A}_n^w(t;\lambda)$. *This is hard to obtain analytically. Instead, one can consider linear combinations of some known estimators. For example, suppose the weight functions* $\lambda_1(u), \cdots, \lambda_q(u)$ *give the corresponding estimators* $\hat{A}_{n,1}^w(t), \cdots, \hat{A}_{n,q}^w(t)$. *Define the class of new weight functions as*

$$\mathcal{F}_0 = \{\lambda(u,t) : \lambda(u,t) = \sum_{i=1}^q a_i(t)\lambda_i(u), a_1(t) \ge 0, \cdots, a_q(t) \ge 0, \sum_{i=1}^q a_i(t) = 1\}.$$

*Then one can choose* $a_i's$ *to minimize the asymptotic variance of* $\hat{A}_n^w(t;\lambda)$ *in this class* $\mathcal{F}_0$, *which results in explicit formulas for* $a_i's$.

**An example.** Consider $\lambda(u,t) = u^{-1}(-\log u)^{-q(t)}$ for some $q(t) \in [0,1]$. Then $\hat{A}^P(t)$ and $\hat{A}^{CFG}(t)$ correspond to $q(t) = 0$ and $q(t) = 1$, respectively. When $q(t) < 1$, we can write

$$\begin{aligned} & \int_0^1 \{\hat{C}_n(u^{1-t}, u^t) - u^\theta\}\lambda(u,t)\, du \\ = \; & -\tfrac{1}{1-q(t)} \int_0^1 \{\hat{C}_n(u^{1-t}, u^t) - u^\theta\}\, d(-\log u)^{1-q(t)} \\ = \; & \tfrac{1}{1-q(t)} \int_0^1 (-\log u)^{1-q(t)}\, d(\hat{C}_n(u^{1-t}, u^t) - u^\theta) \\ = \; & \tfrac{1}{1-q(t)} \int_0^1 (-\log u)^{1-q(t)}\, d\hat{C}_n(u^{1-t}, u^t) - \tfrac{\theta^{q(t)-1}}{1-q(t)} \int_0^\infty u^{1-q} e^{-u} du \\ = \; & \tfrac{1}{1-q(t)} \{\tfrac{1}{n} \textstyle\sum_{i=1}^n \{\tfrac{Z_{i1}}{1-t} \wedge \tfrac{Z_{i2}}{t}\}^{1-q(t)} - \theta^{q(t)-1}\Gamma(2-q(t))\}, \end{aligned}$$

where $Z_{ij}'s$ are defined in Section 1. Hence

$$\hat{A}_n^w(t;\lambda) = \exp\{-(\log(\frac{1}{n}\sum_{i=1}^n (\frac{Z_{i1}}{1-t} \wedge \frac{Z_{i2}}{t})^{1-q(t)}) - \log\Gamma(2-q(t)))/(1-q(t))\}$$

for $0 \le q(t) \le 1$. Note that, when $q(t) = 1$, the above expression is defined as the limit, which becomes the same as $\hat{A}^{CFG}(t)$. In particular, we propose to choose $q(t) = \min\{\hat{A}^{CFG}(t), 1\}$ and denote the resulting estimator by $\hat{A}_n^w(t)$. To compare this new estimator with $\hat{A}^{CFG}(t)$, we draw $1,000$ random samples with size $n = 100, 1000, 5000$ from Gumbel copula with $A(t) = \{t^\theta + (1-t)^\theta\}^{1/\theta}$, Hüsler-Reiss copula with $A(t) = (1-t)\Phi(\theta + \frac{1}{2\theta}\log\frac{1-t}{t}) + t\Phi(\theta + \frac{1}{2\theta}\log\frac{t}{1-t})$ and Tawn copula with $A(t) = 1 - \theta t + \theta t^2$, where $\Phi(x)$ denotes the distribution function of $N(0,1)$. In Figure 5.1, we plot the ratios of the mean squared error of $\hat{A}_n^w(t)$ to that of $\hat{A}^{CFG}(t)$ for $t = 0.1, 0.2, \cdots, 0.9$, which shows that the new estimator has a smaller mean squared error than $\hat{A}^{CFG}(t)$ in all considered cases.

## 3 Jackknife Empirical Likelihood Method

In this section, we consider interval estimation for the Pickands dependence function $A(t)$, which plays an important role in risk management since one may concern with an interval estimation for $C(u, v)$ at some particular values of $u$ and $v$. Note that an interval for $A(t)$ can be easily transformed to an interval for a monotone function of $A(t)$. Moreover, these two intervals have the same coverage probability, but different interval length. Since upper tail dependence coefficient can be written as a monotone function of $A(1/2)$, an interval can be constructed via an interval for $A(1/2)$.

An obvious approach to construct an interval for $A(t)$ is to employ the normal approximation method based on any one of the estimators for $A(t)$. Since the asymptotic distribution of any one of the estimators for $A(t)$ depends on its derivative $A'(t)$, the normal approximation method requires to estimate $A'(t)$ first. As an alternative way of constructing confidence intervals, empirical likelihood method has been extended and applied in various fields since Owen (1988, 1990) introduced it for constructing a confidence interval/region for a mean. We refer to Owen (2001) for an overview. An important feature of the empirical likelihood method is the property of self-studentization, which avoids estimating the asymptotic variance explicitly. A general method to formulate the empirical likelihood function is based on estimating equations as in Qin and Lawless

(1994).

Since the proposed weighted estimator is defined as the solution to equation (2.2), one may apply the method in Qin and Lawless (1994) directly by defining the empirical likelihood function as

$$\sup\big\{\Pi_{i=1}^n (np_i) : p_1 \ge 0, \cdots, p_n \ge 0, \sum_{i=1}^n p_i = 1,$$
$$\sum_{i=1}^n p_i \int_0^1 \{I(\hat F_{n1}(X_{i1}) \le u^{1-t}, \hat F_{n2}(X_{i2}) \le u^t) - u^\theta\}\lambda(u,t)\, du = 0\big\}.$$

However, this method can not catch the variation introduced by the marginal empirical distributions. In other words, the limit is no longer a chi-square distribution. In general, one has to linearize the nonlinear functional by introducing some link variables before using the profile empirical likelihood method. See Chen, Peng and Zhao (2009) for applying the profile empirical likelihood method to copulas. Unfortunately, this linearization idea is not applicable to the estimation of $A(t)$. Recently, Jing, Yuan and Zhou (2009) proposed a so-called jackknife empirical likelihood method to construct confidence intervals for U-statistics. More specifically, Jing, Yuan and Zhou (2009) proposed to employ the empirical likelihood method to jackknife samples, which may result in a chi-square limit. Motivated by the study of using smoothed jackknife empirical likelihood method to construct a confidence interval for a receiver operating characteristic curve in Gong, Peng and Qi (2010), one needs to work with a smoothed version of the left hand side of (2.2). The reason to smooth is to separate marginals from the copula estimator in expanding the jackknife empirical likelihood ratio. Here, we employ the smoothed empirical copula in Fermanian, Radulović and Wegkamp (2004), which is defined as

$$\hat C_n^s(u^{1-t}, u^t) = \frac{1}{n}\sum_{i=1}^n K(\frac{u - \hat F_{n1}^{1/(1-t)}(X_{i1})}{h}) K(\frac{u - \hat F_{n2}^{1/t}(X_{i2})}{h}),$$

where $K(x) = \int_{-\infty}^x k(s)\, ds$, $k$ is a symmetric density function with support $[-1, 1]$, and $h = h(n) > 0$ is a bandwidth. Based on this smoothed estimation, a jackknife empirical likelihood function can be constructed as follows.

Put $\hat F_{nj,-i}(x) = \frac{1}{n}\sum_{l=1, l\ne i}^n I(X_{lj} \le x)$ for $j = 1, 2$ and $i = 1, \cdots, n$,

$$\hat C_{n,-i}^s(u^{1-t}, u^t) = \frac{1}{n-1}\sum_{j=1, j\ne i}^n K(\frac{u - \hat F_{n1,-i}^{1/(1-t)}(X_{j1})}{h}) K(\frac{u - \hat F_{n2,-i}^{1/t}(X_{j2})}{h})$$

for $i = 1, \cdots, n$, and define the jackknife sample as

$$\hat{V}_i(u,t) = n\hat{C}_n^s(u^{1-t}, u^t) - (n-1)\hat{C}_{n,-i}^s(u^{1-t}, u^t)$$

for $i = 1, \cdots, n$. Next we apply the empirical likelihood method based on estimating equations in Qin and Lawless (1994) to the above jackknife sample, which gives the jackknife empirical likelihood function for $\theta = A(t)$ as

$$\begin{aligned} L(\theta) \; &= \sup\{\Pi_{i=1}^n (np_i) : p_1 \geq 0, \cdots, p_n \geq 0, \textstyle\sum_{i=1}^n p_i = 1, \\ &\textstyle\sum_{i=1}^n p_i \int_{a_n}^{1-b_n} \{\hat{V}_i(u,t) - u^\theta\}\lambda(u,t)\, du = 0\}, \end{aligned}$$

where $a_n > 0$ and $b_n > 0$. Note that we use $\int_{a_n}^{1-b_n}$ instead of $\int_0^1$ in defining the above jackknife empirical likelihood function. The reason is to control the bias term and to allow the possibility of $\lambda(0,t) = \infty$ and $\lambda(1,t) = \infty$.

By the standard Lagrange multiplier technique, we obtain the log jackknife empirical likelihood ratio as

$$l(\theta) = -2\log L(\theta) = 2\sum_{i=1}^n \log\{1 + \beta Q_i(\theta)\},$$

where

$$Q_i(\theta) = \int_{a_n}^{1-b_n} \{\hat{V}_i(u,t) - u^\theta\}\lambda(u,t)\, du$$

and $\beta = \beta(\theta)$ satisfies

$$\frac{1}{n}\sum_{i=1}^n \frac{Q_i(\theta)}{1 + \beta Q_i(\theta)} = 0. \tag{3.1}$$

**Theorem 3.1.** *Suppose* $\frac{\partial^2}{\partial u^2}C(u,v)$, $\frac{\partial^2}{\partial v^2}C(u,v)$ *and* $\frac{\partial^2}{\partial u \partial v}C(u,v)$ *are defined and continuous on the set* $\mathcal{F}_3 = \{(u,v) : 0 < u < 1 \quad \text{and} \quad 0 < v < 1\}$, *and*

$$|\frac{\partial^2}{\partial u^2}C(u,v)| \leq \frac{M}{u(1-u)}, \quad |\frac{\partial^2}{\partial v^2}C(u,v)| \leq \frac{M}{v(1-v)}, \quad |\frac{\partial^2}{\partial u \partial v}C(u,v)| \leq \frac{M}{u(1-u)} \wedge \frac{M}{v(1-v)}$$

*for* $(u,v) \in \mathcal{F}_3$ *and some constant* $M > 0$. *Let* $t$ *denote a fixed point in* $(0,1)$. *Assume that the function* $\lambda(u,t) \geq 0$ *is continuous and is not identical to zero as a function of*

*$u \in (0,1)$, $A'(s)$ is continuous at $s=t$, and*

$$\begin{cases} h = h(n) \to 0, \quad nh \to \infty \\ a_n \to 0, \quad b_n \to 0, \quad h/a_n \to 0, \quad h/b_n \to 0 \\ n^{-1/4+\delta_1} \int_{a_n}^{1-b_n} \lambda(u,t)\, du \to 0 \quad \text{for some} \quad \delta_1 > 0 \\ \int_0^1 u^{\delta_2}(1-u^{\delta_2})\lambda(u,t)\, du < \infty \quad \text{for some} \quad \delta_2 \in [0, 1/2) \\ \sqrt{n}h^2 \int_{a_n}^{1-b_n} u^{-3/2}\lambda(u,t)\, du \to 0 \\ \sqrt{n}h^2 \int_{a_n}^{1-b_n} \{\log u\}^{-1} u^{-3/2}\lambda(u,t)\, du \to 0 \\ \frac{1}{\sqrt{nh}} \int_{a_n}^{1-b_n} u^{-1}\lambda(u,t)\, du \to 0 \\ n^{-3/2} \int_{a_n}^{1-b_n} u^{-2}\lambda(u,t)\, du \to 0 \end{cases} \tag{3.2}$$

*as $n \to \infty$. Then $l(A_0(t)) \xrightarrow{d} \chi^2(1)$ as $n \to \infty$, where $A_0(t)$ denotes the true value of $A(t)$.*

For any fixed $t \in (0,1)$, based on the above theorem, a jackknife empirical likelihood confidence interval for $A_0(t)$ with level $\gamma_0$ can be constructed as

$$I_{\gamma_0}(t) = \{\theta : l(\theta) \le \chi^2_{\gamma_0}\},$$

where $\chi^2_{\gamma_0}$ is the $\gamma_0$ quantile of $\chi^2(1)$.

**Remark 3.1.** *i) When $\lambda(u,t) = \{-u \log u\}^{-1}$, we have $\sup_{0 \le u \le 1} \lambda(u,t) = \infty$. One can choose*

$$a_n = d_1 n^{-a}, \quad b_n = d_2 n^{-b}, \quad h = d_3 n^{-1/3}$$

*for some $d_1, d_2, d_3 > 0$, $0 < a < 1/9$ and $0 < b < 1/6$;*

*ii) When $\sup_{0 \le u \le 1} \lambda(u,t) < \infty$, we can choose*

$$a_n = d_1 n^{-a}, \quad b_n = d_2 n^{-b}, \quad h = d_3 n^{-1/3}$$

*for some $d_1, d_2, d_3 > 0, b > 0$ and $0 < a < 1/3$. Here, we fix the rate for $h$ since the optimal rate for the bandwidth in smoothing distribution estimation is $n^{-1/3}$.*

*iii) Theorem 3.1 still holds when $a_n \to a \in (0, 1/2)$ and $b_n \to b \in (0, 1/2]$ as $n \to \infty$.*

## 4 Simulation Study

In this section we examine the finite sample behavior of the proposed jackknife empirical likelihood method based on $\lambda(u,t) = u^{-1}(-\log u)^{-\min\{\hat{A}^{CFG}(t),1\}}$ in terms of coverage probability and compare it with the method based on the asymptotic distribution of $\hat{A}^{CFG}(t)$.

For computing the coverage probability of the proposed jackknife empirical likelihood method, we choose $k(x) = \frac{15}{16}(1-x^2)^2 I(|x| \le 1)$, $h = 0.5n^{-1/3}$, $a_n = b_n = 0.1$, $\lambda(u,t) = u^{-1}(-\log u)^{-\min\{\hat{A}^{CFG}(t),1\}}$ and employ the package 'emplik' in R (see Zhou). For computing the confidence interval based on the asymptotic distribution of $\hat{A}^{CFG}(t)$, we employ the multiplier method proposed by Kojadinovic and Yan (2010). More specifically, we use equation (7) in Kojadinovic and Yan (2010) with $N = 500$ and $\{Z_i^{(k)} : i = 1, \cdots, n, k = 1, \cdots, N\}$ being independent random variables from $N(0,1)$ to calculate the critical points of the asymptotic distribution of $\sqrt{n}\{\hat{A}^{CFG}(t) - A(t)\}$. We do not use a larger $N$ since this multiplier method is computationally intensive. A comparison study on bootstrap approximations can be found in Bücher and Dette (2010).

We draw $1,000$ random samples with size $n = 100, 1000$ from Gumbel copula, Hüsler-Reiss copula and Tawn copula with Pickands dependence functions given in the end of Section 2. In Table 5.1, we report the coverage probabilities at levels 0.9 and 0.95 for $t = 0.1, 0.5, 0.8$, which show that i) the proposed jackknife empirical likelihood method gives much more accurate coverage probabilities than the multiplier method based on the asymptotic distribution of $\hat{A}^{CFG}(t)$; ii) the proposed jackknife empirical likelihood method performs badly for the boundary case $t = 0.1$ when $n = 100$, but its performance improves as $n$ becomes large.

## 5 Proofs

*Proof of Theorem 2.1.* Define

$$\alpha_n(u,v) = \sqrt{n}\{\frac{1}{n}\sum_{i=1}^{n} I(F_1(X_{i1}) \le u, F_2(X_{i2}) \le v) - C(u,v)\}.$$

Then it follows Proposition 4.2 of Segers (2011) and Theorem G.1 of Genest and Segers (2009) that

$$\begin{aligned}&\sup_{0\le u,v\le 1}|\sqrt{n}\{\hat{C}_n(u,v)-C(u,v)\}-\alpha_n(u,v)+C_1(u,v)\alpha_n(u,1)\\&+C_2(u,v)\alpha_n(1,v)|=O(n^{-1/4}(\log n)^{1/2}(\log\log n))\quad\text{a.s.}\end{aligned}$$

and

$$\frac{\alpha_n(u,v)}{(u\wedge v)^{\delta}(1-u\wedge v)^{\delta}}\overset{D}{\to}\frac{W(u,v)}{(u\wedge v)^{\delta}(1-u\wedge v)^{\delta}}$$

in the space $l^{\infty}([0,1]^2)$ of bounded, real-valued functions on $[0,1]^2$ for any $\delta\in[0,1/2)$, where $W$ is defined before Theorem 2.1. By the Skorohod construction, there exists a probability space carrying $\hat{C}_n^*,\alpha_n^*,W^*$ such that

$$(\hat{C}_n^*,\alpha_n^*)\overset{d}{=}(\hat{C}_n,\alpha_n),\quad W^*\overset{d}{=}W, \tag{5.1}$$

$$\begin{aligned}&\sup_{0\le u,v\le 1}|\sqrt{n}\{\hat{C}_n^*(u,v)-C(u,v)\}-\alpha_n^*(u,v)+C_1(u,v)\alpha_n^*(u,1)\\&+C_2(u,v)\alpha_n^*(1,v)|=O(n^{-1/4}(\log n)^{1/2}(\log\log n))\quad\text{a.s.}\end{aligned} \tag{5.2}$$

and

$$\sup_{0\le u,v\le 1}\left|\frac{\alpha_n^*(u,v)}{(u\wedge v)^{\delta}(1-u\wedge v)^{\delta}}-\frac{W^*(u,v)}{(u\wedge v)^{\delta}(1-u\wedge v)^{\delta}}\right|=o_p(1) \tag{5.3}$$

Let $\hat{A}_n^{w*}(t;\lambda)$ denote the solution to

$$\int_0^1\{\hat{C}_n^*(u^{1-t},u^t)-u^{\alpha}\}\lambda(u,t)\,du=0.$$

Then (5.1) implies that

$$\{\hat{A}_n^{w*}(t;\lambda):0\le t\le 1\}\overset{d}{=}\{\hat{A}_n^{w}(t;\lambda):0\le t\le 1\}. \tag{5.4}$$

Write

$$\begin{aligned}&\int_0^1\{\hat{C}_n^*(u^{1-t},u^t)-u^{A(t)}\}\lambda(u,t)\,du\\=\;&\int_0^{(n+1)^{-1/((1-t)\vee t)}}\{-u^{A(t)}\}\lambda(u,t)\,du\\&+\int_{(n+1)^{-1/((1-t)\vee t)}}^{(\frac{n}{n+1})^{1/((1-t)\vee t)}}\{\hat{C}_n^*(u^{1-t},u^t)-u^{A(t)}\}\lambda(u,t)\,du\\&+\int_{(\frac{n}{n+1})^{1/((1-t)\vee t)}}^{1}\{1-u^{A(t)}\}\lambda(u,t)\,du\\=:\;&I_1(t)+I_2(t)+I_3(t).\end{aligned} \tag{5.5}$$

Since $1 \geq A(t) \geq (1-t) \vee t \geq 1/2$, (2.3) implies that $I_1(t)$ and $I_3(t)$ are finite and

$$\begin{cases} \sup_{0\leq t\leq 1} \sqrt{n}|I_1(t)| \leq \sup_{0\leq t\leq 1} \sqrt{n} \int_0^{(n+1)^{-1/((1-t)\vee t)}} u^{1/2}\lambda(u,t)\,du = o(1) \\ \sup_{0\leq t\leq 1} \sqrt{n}|I_3(t)| \leq \sup_{0\leq t\leq 1} \sqrt{n} \int_{(\frac{n}{n+1})^{1/((1-t)\vee t)}}^1 (1-u)\lambda(u,t)\,du = o(1). \end{cases} \tag{5.6}$$

It follows from the condition

$$\sup_{0\leq t\leq 1} \int_0^1 \{u^{(1-t)\vee t}(1-u^{(1-t)\vee t})\}^{\delta_2}\lambda(u,t)\,du < \infty$$

in (2.3) and (5.3) that

$$\sup_{0\leq t\leq 1} \left|\int_0^1 \left(\alpha_n^*(u^{1-t},u^t) - W^*(u^{1-t},u^t)\right)\lambda(u,t)\,du\right| \xrightarrow{p} 0. \tag{5.7}$$

By (1.1), we have

$$\begin{aligned} 0 \leq C_1(u^{1-t},u^t) &= u^{A(t)-(1-t)}\{A(t) - tA'(t)\} \\ &\leq u^{(1-t)\vee t-(1-t)}\{A(t) - tA'(t)\} \end{aligned}$$

and

$$\begin{aligned} 0 \leq C_2(u^{1-t},u^t) &= u^{A(t)-t}\{A(t) + (1-t)A'(t)\} \\ &\leq u^{(1-t)\vee t-t}\{A(t) + (1-t)A'(t)\}. \end{aligned}$$

Since $A(t)$ and $A'(t)$ are bounded on [0,1], it follows from the conditions

$$\sup_{0\leq t\leq 1} \int_0^1 u^{(1-t)\vee t-(1-t)}u^{(1-t)\delta_2}(1-u^{1-t})^{\delta_2}\lambda(u,t)\,du < \infty$$

and

$$\sup_{0\leq t\leq 1} \int_0^1 u^{(1-t)\vee t-t}u^{t\delta_2}(1-u^t)^{\delta_2}\lambda(u,t)\,du < \infty$$

in (2.3) that

$$\begin{cases} \sup_{0\leq t\leq 1} \left|\int_0^1 \alpha_n^*(u^{1-t},1)C_1(u^{1-t},u^t)\lambda(u,t)\,du \right. \\ \qquad \left. - \int_0^1 W^*(u^{1-t},1)C_1(u^{1-t},u^t)\lambda(u,t)\,du\right| \xrightarrow{p} 0, \\ \sup_{0\leq t\leq 1} \left|\int_0^1 \alpha_n^*(1,u^t)C_2(u^{1-t},u^t)\lambda(u,t)\,du \right. \\ \qquad \left. - \int_0^1 W^*(1,u^t)C_2(u^{1-t},u^t)\lambda(u,t)\,du\right| \xrightarrow{p} 0. \end{cases} \tag{5.8}$$

By the condition

$$\sup_{0\leq t\leq 1} n^{-1/4+\delta_1} \int_{(n+1)^{-1/((1-t)\vee t)}}^{(\frac{n}{n+1})^{1/((1-t)\vee t)}} \lambda(u,t)\,du \to 0$$

in (2.3), (5.2), (5.7) and (5.8), we have

$$\begin{aligned}
&\sup_{0\le t\le 1} |\sqrt{n} I_2(t) - \textstyle\int_0^1 \{W^*(u^{1-t},u^t) - W^*(u^{1-t},1)C_1(u^{1-t},u^t) \\
&\qquad -W^*(1,u^t)C_2(u^{1-t},u^t)\}\lambda(u,t)\,du| \\
=\ & O_p\left(n^{-1/4}(\log n)^{1/2}(\log\log n)^{1/4}\sup_{0\le t\le 1}\textstyle\int_{(n+1)^{-1/((1-t)\vee t)}}^{(\frac{n}{n+1})^{1/((1-t)\vee t)}}\lambda(u,t)\,du\right) + o_p(1) \\
=\ & o_p(1).
\end{aligned} \tag{5.9}$$

By (5.6) and (5.9), we have

$$\begin{aligned}
&\sup_{0\le t\le 1} | \textstyle\int_0^1 \sqrt{n}\{\hat{C}_n^*(u^{1-t},u^t) - u^{A(t)}\}\lambda(u,t)\,du \\
&\quad - \textstyle\int_0^1 \{W^*(u^{1-t},u^t) - W^*(u^{1-t},1)C_1(u^{1-t},u^t) \\
&\quad -W^*(1,u^t)C_2(u^{1-t},u^t)\}\lambda(u,t)\,du| = o_p(1),
\end{aligned}$$

which is equivalent to that

$$\begin{aligned}
&\sup_{0\le t\le 1} | \textstyle\int_0^1 \sqrt{n}\{u^{\hat{A}_n^{w*}(t;\lambda)} - u^{A(t)}\}\lambda(u,t)\,du \\
&\quad - \textstyle\int_0^1 \{W^*(u^{1-t},u^t) - W^*(u^{1-t},1)C_1(u^{1-t},u^t) \\
&\quad -W^*(1,u^t)C_2(u^{1-t},u^t)\}\lambda(u,t)\,du| = o_p(1).
\end{aligned} \tag{5.10}$$

The above equation shows that as $n \to \infty$,

$$\sup_{0\le t\le 1} |\int_0^1 \{u^{\hat{A}_n^{w*}(t;\lambda)} - u^{A(t)}\}\lambda(u,t)\,du| = o_p(1), \tag{5.11}$$

which implies that

$$\begin{aligned}
&P\left(\hat{A}_n^{w*}(t;\lambda) > 4/3 \quad \text{for some} \quad t\in[0,1]\right) \\
&\le P\left(\sup_{0\le t\le 1} |\int_0^1 \{u^{\hat{A}_n^{w*}(t;\lambda)} - u^{A(t)}\}\lambda(u,t)\,du| \right. \\
&\qquad \left. \ge \inf_{0\le t\le 1}\int_0^1 (u^{A(t)} - u^{4/3})\lambda(u,t)\,du\right) \\
&\to 0
\end{aligned}$$

since $1/2 \le A(t) \le 1$ for all $0 \le t \le 1$. Similarly,

$$
\begin{aligned}
& P\left(\hat{A}_n^{w*}(t;\lambda) < 1/3 \quad \text{for some} \quad t \in [0,1]\right) \\
& \le P\left(\sup_{0\le t\le 1} \Big| \int_0^1 \{u^{\hat{A}_n^{w*}(t;\lambda)} - u^{A(t)}\}\lambda(u,t)\,du\Big| \right. \\
& \qquad \left. \ge \inf_{0\le t\le 1} \int_0^1 (u^{1/3} - u^{A(t)})\lambda(u,t)\,du\right) \\
& \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

Thus

$$
P\left(1/3 \le \hat{A}_n^{w*}(t;\lambda) \le 4/3 \text{ for all } t \in [0,1]\right) \to 1. \tag{5.12}
$$

By the mean-value theorem,

$$
\begin{aligned}
& \textstyle\int_0^1 \{u^{\hat{A}_n^{w*}(t;\lambda)} - u^{A(t)}\}\lambda(u,t)\,du \\
& = \textstyle\int_0^1 u^{a(u,t)A(t)+(1-a(u,t))\hat{A}_n^{w*}(t;\lambda)}(\log u)\lambda(u,t)\,du \\
& \qquad \times (A(t) - \hat{A}_n^{w*}(t;\lambda))
\end{aligned} \tag{5.13}
$$

for some $a(u,t) \in [0,1]$. Since $1/2 \le A(t) \le 1$, we have $0 < a(u,t)A(t) + (1 - a(u,t))\hat{A}_n^{w*}(t;\lambda) \le 7/3$ when $0 < \hat{A}_n^{w*}(t;\lambda) \le 4/3$. Hence, it follows from (5.12) that

$$
\begin{aligned}
& P\Big(\textstyle\inf_{0\le t\le 1}\int_0^1 u^{a(u,t)A(t)+(1-a(u,t))\hat{A}_n^{w*}(t;\lambda)}(-\log u)\lambda(u,t)\,du \\
& \qquad \ge \textstyle\sup_{0\le t\le 1}\int_0^1 u^{7/3}(-\log u)\lambda(u,t)\,du\Big) \to 1
\end{aligned}
$$

as $n \to \infty$, which, combining with (5.11), (5.13) and (5.1) , implies that

$$
\sup_{0\le t\le 1} |\hat{A}_n^{w*}(t;\lambda) - A(t)| = o_p(1). \tag{5.14}
$$

Then $\sup_{0\le t\le 1} |\hat{A}_n^{w}(t;\lambda) - A(t)| = o_p(1)$ follows from (5.14) and (5.4).

Next we prove that $\hat{A}_n^w(t;\lambda)$ is continuous for $t \in [0,1]$. For $t_m, t \in [0,1]$ and $t_m \to t \in [0,1]$ as $m \to \infty$, we have

$$
\begin{aligned}
& \int_0^{1/2} u^{\hat{A}_n^w(t_m;\lambda)}\lambda(u,t_m)du + \int_{1/2}^1 (u^{\hat{A}_n^w(t_m;\lambda)} - 1)\lambda(u,t_m)du \\
& = \int_0^{1/2} \hat{C}_n(u^{1-t_m}, u^{t_m})\lambda(u,t_m)du + \int_{1/2}^1 (\hat{C}_n(u^{1-t_m}, u^{t_m}) - 1)\lambda(u,t_m)du.
\end{aligned}
$$

Note that the function

$$\int_0^{1/2} \hat{C}_n(u^{1-t}, u^t)\lambda(u,t)du + \int_{1/2}^1 (\hat{C}_n(u^{1-t}, u^t) - 1)\lambda(u,t)du$$

is continuous in $t \in [0,1]$, thus we have

$$\begin{aligned}
&\lim_{m\to\infty} \Big( \int_0^{1/2} u^{\hat{A}_n^w(t_m;\lambda)}\lambda(u,t_m)du + \int_{1/2}^1 (u^{\hat{A}_n^w(t_m;\lambda)} - 1)\lambda(u,t_m)du\Big) \\
&= \int_0^{1/2} \hat{C}_n(u^{1-t}, u^t)\lambda(u,t)du + \int_{1/2}^1 (\hat{C}_n(u^{1-t}, u^t) - 1)\lambda(u,t)du.
\end{aligned}$$

Since

$$\int_0^{1/2} u^{\alpha}\lambda(u,t)du + \int_{1/2}^1 (u^{\alpha} - 1)\lambda(u,t)du$$

is continuous in $t \in [0,1]$, and is monotone in $\alpha$ for each $t \in [0,1]$, then we conclude that $\hat{A}_n^w(t_m;\lambda) \to \hat{A}_n^w(t;\lambda)$ as $m \to \infty$. Thus $\hat{A}_n^w(t;\lambda)$ is continuous in $[0,1]$.

Note that

$$\sup_{0\le t\le 1} \int_0^1 \{u^{(1-t)\vee t}(1 - u^{(1-t)\vee t})\}^{\delta_2}\lambda(u,t)\,du < \infty$$

for some $\delta_2 \in [0,1/2)$ in (2.3) implies that $\int_0^{1/2} u^{\delta_2}\lambda(u,t)\,du < \infty$. Thus using

$$\begin{aligned}
&u^{a(u,t)A(t)+(1-a(u,t))\hat{A}_n^{w*}(u;\lambda)}(-\log u)\lambda(u,t) \\
&= u^{A(t)}(-\log u)\lambda(u,t)u^{(1-a(u,t))(\hat{A}_n^{w*}(u;\lambda)-A(t))} \\
&\le u^{A(t)}(-\log u)\lambda(u,t)u^{-(1-a(u,t))\sup_{0\le t\le 1}|\hat{A}_n^{w*}(t;\lambda)-A(t)|},
\end{aligned}$$

$A(t) \ge 1/2$ for all $t \in [0,1]$, (5.14) and

$$0 \le u^{-s_1} - 1 \le \frac{s_1}{s_2}u^{-s_2} \quad \text{for all} \quad u \in [0,1] \quad \text{and any fixed} \quad 0 < s_1 < s_2 < 1,$$

we get that

$$\begin{aligned}
&\sup_{0\le t\le 1} | \textstyle\int_0^1 u^{a(u,t)A(t)+(1-a(u,t))\hat{A}_n^{w*}(t;\lambda)}(\log u)\lambda(u,t)\,du \\
&\qquad - \textstyle\int_0^1 u^{A(t)}(\log u)\lambda(u,t)\,du| \\
&\le \sup_{0\le t\le 1} | \textstyle\int_0^1 u^{A(t)}(-\log u)\lambda(u,t)(u^{-(1-a(u,t))\sup_{0\le s\le 1}|\hat{A}_n^{w*}(s;\lambda)-A(s)|} - 1)\,du| \\
&\le \sup_{0\le t\le 1} \Big( \frac{(1-a(u,t))\sup_{0\le s\le 1}|\hat{A}_n^{w*}(s;\lambda)-A(s)|}{(1-a)\sup_{0\le s\le 1}|\hat{A}_n^{w*}(s;\lambda)-A(s)|+(A(t)-\delta_2)/2} \\
&\qquad \textstyle\int_0^1 u^{-(1-a(u,t))\sup_{0\le s\le 1}|\hat{A}_n^{w*}(s;\lambda)-A(s)|+(A(t)+\delta_2)/2}(-\log u)\lambda(u,t)\,du\Big) \\
&= o_p(1)O_p\Big(\sup_{0\le t\le 1} \textstyle\int_0^1 u^{-(1-a(u,t))\sup_{0\le s\le 1}|\hat{A}_n^{w*}(s;\lambda)-A(s)|+(A(t)+\delta_2)/2}(-\log u)\lambda(u,t)\,du\Big) \\
&= o_p(1)O_p\Big(\sup_{0\le t\le 1} \textstyle\int_0^1 u^{\delta_2}(1 - u^{\delta_2})\lambda(u,t)\,du\Big) = o_p(1).
\end{aligned} \tag{5.15}$$

Note that the two processes $\hat{A}_n^w(t;\lambda), B(t)$ are continuous for $t \in [0,1]$. Thus from (5.1), (5.10), (5.13), (5.15) and (5.4), we conclude that $\sqrt{n}\{\hat{A}_n^w(t;\lambda) - A(t)\}$ converges to $B(t)$ in $C([0,1])$. □

Before proving Theorem 3.1, we show some lemmas. Throughout, we assume that $t$ is a given point in $(0,1)$ and we use $\theta_0$ to denote $A_0(t)$ .

**Lemma 5.1.** *Under conditions of Theorem 3.1, as $n \to \infty$ we have*

$$\begin{aligned} \tfrac{1}{\sqrt{n}} \textstyle\sum_{i=1}^n Q_i(\theta_0) \quad &\xrightarrow{d} \textstyle\int_0^1 \big\{ W(u^{1-t}, u^t) - W(u^{1-t}, 1) C_1(u^{1-t}, u^t) \\ &\qquad - W(1, u^t) C_2(u^{1-t}, u^t) \big\} \lambda(u,t)\, du. \end{aligned}$$

*Proof.* Write

$$\begin{aligned} \hat{V}_i(u,t) \quad &= K\big(\tfrac{u - \hat{F}_{n1,-i}^{1/(1-t)}(X_{i1})}{h}\big) K\big(\tfrac{u - \hat{F}_{n2,-i}^{1/t}(X_{i2})}{h}\big) \\ &\quad + \textstyle\sum_{j=1}^n \{ K\big(\tfrac{u - \hat{F}_{n1}^{1/(1-t)}(X_{j1})}{h}\big) K\big(\tfrac{u - \hat{F}_{n2}^{1/t}(X_{j2})}{h}\big) - \\ &\qquad K\big(\tfrac{u - F_{n1,-i}^{1/(1-t)}(X_{j1})}{h}\big) K\big(\tfrac{u - F_{n2,-i}^{1/t}(X_{j2})}{h}\big) \} \\ &=: \hat{V}_{i1}(u,t) + \hat{V}_{i2}(u,t) \end{aligned} \tag{5.16}$$

and

$$\begin{aligned} & \tfrac{1}{n} \textstyle\sum_{i=1}^n Q_i(\theta_0) \\ = \quad & n^{-1} \textstyle\int_{a_n}^{1-b_n} \sum_{i=1}^n \{ \hat{V}_{i1}(u,t) - u^{\theta} \} \lambda(u,t)\, du \\ & + n^{-1} \textstyle\int_{a_n}^{1-b_n} \sum_{i=1}^n \hat{V}_{i2}(u,t) \lambda(u,t)\, du \\ = \quad & n^{-1} \textstyle\int_{a_n}^{1-b_n} \big\{ \sum_{i=1}^n K\big(\tfrac{u - \hat{F}_{n1,-i}^{1/(1-t)}(X_{i1})}{h}\big) K\big(\tfrac{u - \hat{F}_{n2,-i}^{1/t}(X_{i2})}{h}\big) - u^{\theta_0} \big\} \lambda(u,t)\, du \\ & + n^{-1} \textstyle\int_{a_n}^{1-b_n} \sum_{i=1}^n \sum_{j=1}^n \big\{ K\big(\tfrac{u - \hat{F}_{n1}^{1/(1-t)}(X_{j1})}{h}\big) - K\big(\tfrac{u - \hat{F}_{n1,-i}^{1/(1-t)}(X_{j1})}{h}\big) \big\} \times \\ & K\big(\tfrac{u - \hat{F}_{n2}^{1/t}(X_{j2})}{h}\big) \lambda(u,t)\, du \\ & + n^{-1} \textstyle\int_{a_n}^{1-b_n} \sum_{i=1}^n \sum_{j=1}^n K\big(\tfrac{u - \hat{F}_{n1}^{1/(1-t)}(X_{j1})}{h}\big) \times \\ & \big\{ K\big(\tfrac{u - \hat{F}_{n2}^{1/t}(X_{j2})}{h}\big) - K\big(\tfrac{u - \hat{F}_{n2,-i}^{1/t}(X_{j2})}{h}\big) \big\} \lambda(u,t)\, du \\ & + n^{-1} \textstyle\int_{a_n}^{1-b_n} \sum_{i=1}^n \sum_{j=1}^n \big\{ K\big(\tfrac{u - \hat{F}_{n1,-i}^{1/(1-t)}(X_{j1})}{h}\big) - K\big(\tfrac{u - \hat{F}_{n1}^{1/(1-t)}(X_{j1})}{h}\big) \big\} \times \\ & \big\{ K\big(\tfrac{u - F_{n2}^{1/t}(X_{j2})}{h}\big) - K\big(\tfrac{u - \hat{F}_{n2,-i}^{1/t}(X_{j2})}{h}\big) \big\} \lambda(u,t)\, du \\ =: \quad & I_1 + \cdots + I_4. \end{aligned} \tag{5.17}$$

Further, the first term $I_1$ can be expressed as

$$
\begin{aligned}
& I_1 \\
= \quad & \int_{a_n}^{1-b_n} \lambda(u,t)\big\{ \int_0^1 \int_0^1 \tfrac{1}{n} \sum_{i=1}^n I(\hat{F}_{n1,-i}(X_{i1}) \le s_1, \hat{F}_{n2,-i}(X_{i2}) \le s_2) h^{-2} \\
& \times k(\tfrac{u - s_1^{1/(1-t)}}{h}) k(\tfrac{u - s_2^{1/t}}{h}) d s_1^{1/(1-t)} d s_2^{1/t} - u^{\theta_0} \big\}\, du \\
= \quad & \int_{a_n}^{1-b_n} \lambda(u,t)\big\{ \int_0^1 \int_0^1 \tfrac{1}{n} \sum_{i=1}^n I\left(\hat{F}_{n1}(X_{i1}) \le \tfrac{n}{n+1}(s_1 + \tfrac{1}{n}),\right. \\
& \left.\hat{F}_{n2}(X_{i2}) \le \tfrac{n}{n+1}(s_2 + \tfrac{1}{n})\right) \\
& \times h^{-2} k(\tfrac{u - s_1^{1/(1-t)}}{h}) k(\tfrac{u - s_2^{1/t}}{h}) d s_1^{1/(1-t)} d s_2^{1/t} - u^{\theta_0} \big\} du \qquad (5.18) \\
= \quad & \int_{a_n}^{1-b_n} \int_{-1}^1 \int_{-1}^1 \lambda(u,t)\big\{ \hat{C}_n(\tfrac{n}{n+1}(u - s_1 h)^{1-t} + \tfrac{1}{n+1}, \tfrac{n}{n+1}(u - s_2 h)^t + \tfrac{1}{n+1}) - \\
& C(\tfrac{n}{n+1}(u - s_1 h)^{1-t} + \tfrac{1}{n+1}, \tfrac{n}{n+1}(u - s_2 h)^t + \tfrac{1}{n+1})\big\} k(s_1) k(s_2)\, ds_1 ds_2 du \\
& + \int_{a_n}^{1-b_n} \int_{-1}^1 \int_{-1}^1 \lambda(u,t)\big\{ C(\tfrac{n}{n+1}(u - s_1 h)^{1-t} + \tfrac{1}{n+1}, \tfrac{n}{n+1}(u - s_2 h)^t + \tfrac{1}{n+1}) - \\
& C(u^{1-t}, u^t)\big\} k(s_1) k(s_2)\, ds_1 ds_2 du \\
=: \quad & II_1 + II_2.
\end{aligned}
$$

Since $\sup_{a_n \le u \le 1-b_n} (h/u) \le h/a_n \to 0$ and

$$\inf_{a_n \le u \le 1-b_n} \min\{(n+1)u^t, (n+1)u^{1-t}\} \ge (n+1)a_n \to \infty$$

as $n \to \infty$, we have

$$
\begin{aligned}
& \sup_{a_n \le u \le 1-b_n, -1 \le s \le 1} |\tfrac{\log(u - sh)}{\log u} - 1| \\
\le \quad & \sup_{a_n \le u \le 1-b_n} \tfrac{2h/u}{-\log u} \le \tfrac{2h}{-a_n \log(a_n)} + \tfrac{2h}{-(1-b_n)\log(1-b_n)} \to 0,
\end{aligned}
\qquad (5.19)
$$

$$\sup_{a_n \le u \le 1-b_n, -1 \le s \le 1} |u^{t-1}\{\frac{n}{n+1}(u - sh)^{1-t} + \frac{1}{n+1}\} - 1| \to 0 \qquad (5.20)$$

and

$$\sup_{a_n \le u \le 1-b_n, -1 \le s \le 1} |u^{-t}\{\frac{n}{n+1}(u - sh)^t + \frac{1}{n+1}\} - 1| \to 0, \qquad (5.21)$$

which, in together with (1.1), imply that

$$
\left\{
\begin{array}{l}
\sup_{a_n\le u\le 1-b_n,-1\le s_1,s_2\le 1}\left|\frac{\log\{\frac{n}{n+1}(u-s_1h)^{1-t}+\frac{1}{n+1}\}+\log\{\frac{n}{n+1}(u-s_2h)^{t}+\frac{1}{n+1}\}}{\log u}\right. \\
\qquad \left.-1\right| \to 0, \\
\sup_{a_n\le u\le 1-b_n,-1\le s_1,s_2\le 1}\left|A\left(\frac{\log\{\frac{n}{n+1}(u-s_2h)^{t}+\frac{1}{n+1}\}}{\log\{\frac{n}{n+1}(u-s_1h)^{1-t}+\frac{1}{n+1}\}+\log\{\frac{n}{n+1}(u-s_2h)^{t}+\frac{1}{n+1}\}}\right)\right. \\
\qquad \left.-A(t)\right| \to 0, \\
\sup_{a_n\le u\le 1-b_n,-1\le s_1,s_2\le 1}\left|C(\tfrac{n}{n+1}(u-s_1h)^{1-t}+\tfrac{1}{n+1},\tfrac{n}{n+1}(u-s_2h)^{t}+\tfrac{1}{n+1})\right. \\
\qquad \left.-C(u^{1-t},u^t)\right| \to 0, \\
\sup_{a_n\le u\le 1-b_n,-1\le s_1,s_2\le 1}\left|C_1(\tfrac{n}{n+1}(u-s_1h)^{1-t}+\tfrac{1}{n+1},\tfrac{n}{n+1}(u-s_2h)^{t}+\tfrac{1}{n+1})\right. \\
\qquad \left.-C_1(u^{1-t},u^t)\right| \to 0, \\
\sup_{a_n\le u\le 1-b_n,-1\le s_1,s_2\le 1}\left|C_2(\tfrac{n}{n+1}(u-s_1h)^{1-t}+\tfrac{1}{n+1},\tfrac{n}{n+1}(u-s_2h)^{t}+\tfrac{1}{n+1})\right. \\
\qquad \left.-C_2(u^{1-t},u^t)\right| \to 0, \\
\sup_{a_n\le u\le 1-b_n,-1\le s_1,s_2\le 1}\left|C_{11}(\tfrac{n}{n+1}(u-s_1h)^{1-t}+\tfrac{1}{n+1},\tfrac{n}{n+1}(u-s_2h)^{t}+\tfrac{1}{n+1})\right. \\
\qquad \left.-C_{11}(u^{1-t},u^t)\right| \to 0, \\
\sup_{a_n\le u\le 1-b_n,-1\le s_1,s_2\le 1}\left|C_{12}(\tfrac{n}{n+1}(u-s_1h)^{1-t}+\tfrac{1}{n+1},\tfrac{n}{n+1}(u-s_2h)^{t}+\tfrac{1}{n+1})\right. \\
\qquad \left.-C_{12}(u^{1-t},u^t)\right| \to 0, \\
\sup_{a_n\le u\le 1-b_n,-1\le s_1,s_2\le 1}\left|C_{22}(\tfrac{n}{n+1}(u-s_1h)^{1-t}+\tfrac{1}{n+1},\tfrac{n}{n+1}(u-s_2h)^{t}+\tfrac{1}{n+1})\right. \\
\qquad \left.-C_{22}(u^{1-t},u^t)\right| \to 0.
\end{array}
\right.
\tag{5.22}
$$

Hence, by (3.2), (5.22) and similar arguments used in the proof of Theorem 2.1, we can show that

$$
\begin{array}{rl}
\sqrt{n}II_1 & \xrightarrow{d} \int_0^1\{W(u^{1-t},u^t)-W(u^{1-t},1)C_1(u^{1-t},u^t) \\
& \quad -W(1,u^t)C_2(u^{1-t},u^t)\}\lambda(u,t)\,du.
\end{array}
\tag{5.23}
$$

It is straightforward to verify that

$$
\left\{
\begin{array}{l}
|C_1(u^{1-t},u^t)u^{1-t}| = O(u^{A(t)}) = O(u^{1/2}) \\
|C_2(u^{1-t},u^t)u^{t}| = O(u^{A(t)}) = O(u^{1/2}) \\
|C_{11}(u^{1-t},u^t)u^{2-2t}\{1-\log u\}| = O(u^{A(t)}) = O(u^{1/2}) \\
|C_{22}(u^{1-t},u^t)u^{2t}\log u| = O(u^{A(t)}) = O(u^{1/2}) \\
|C_{12}(u^{1-t},u^t)u\{1-\log u\}| = O(u^{A(t)}) = O(u^{1/2})
\end{array}
\right.
\tag{5.24}
$$

uniformly for $u \in [a_n, 1-b_n]$. By Taylor expansion, we have

$$\begin{aligned} II_2 &= \int_{a_n}^{1-b_n}\int_{-1}^{1}\int_{-1}^{1}\big\{C_1(u^{1-t},u^t)u^{1-t}\big(\tfrac{n}{n+1}(1-\tfrac{s_1h}{u})^{1-t}+\tfrac{1}{(n+1)u^{1-t}}-1\big)\\ &+C_2(u^{1-t},u^t)u^t\big(\tfrac{n}{n+1}(1-\tfrac{s_2h}{u})^{t}+\tfrac{1}{(n+1)u^{t}}-1\big)\\ &+\tfrac{1}{2}C_{11}(u^{1-t},u^t)(1+o(1))u^{2-2t}\big(\tfrac{n}{n+1}(1-\tfrac{s_1h}{u})^{1-t}+\tfrac{1}{(n+1)u^{1-t}}-1\big)^2\\ &+\tfrac{1}{2}C_{22}(u^{1-t},u^t)(1+o(1))u^{2t}\big(\tfrac{n}{n+1}(1-\tfrac{s_2h}{u})^{t}+\tfrac{1}{(n+1)u^{t}}-1\big)^2\\ &+C_{12}(u^{1-t},u^t)(1+o(1))u\big(\tfrac{n}{n+1}(1-\tfrac{s_1h}{u})^{1-t}+\tfrac{1}{(n+1)u^{1-t}}-1\big)\times\\ &\big(\tfrac{n}{n+1}(1-\tfrac{s_2h}{u})^{t}+\tfrac{1}{(n+1)u^{t}}-1\big)\big\}k(s_1)k(s_2)\lambda(u,t)\,ds_1ds_2du. \end{aligned} \tag{5.25}$$

Consider the first term in the above expression. By (3.2),(5.22), (5.24) and the symmetry of $k(s)$, we have

$$\begin{aligned} &\int_{a_n}^{1-b_n}\int_{-1}^{1}\int_{-1}^{1}C_1(u^{1-t},u^t)u^{1-t}\big(\tfrac{n}{n+1}(1-\tfrac{s_1h}{u})^{1-t}+\tfrac{1}{(n+1)u^{1-t}}-1\big)\\ &\times k(s_1)k(s_2)\lambda(u,t)\,ds_1ds_2du\\ =\;&\int_{a_n}^{1-b_n}\int_{-1}^{1}C_1(u^{1-t},u^t)u^{1-t}\big(\tfrac{n}{n+1}(1-\tfrac{s_1h}{u})^{1-t}-1\big)k(s_1)\lambda(u,t)\,ds_1du\\ &+\tfrac{1}{n+1}\int_{a_n}^{1-b_n}C_1(u^{1-t},u^t)\lambda(u,t)du\\ =\;&(1+o(1))\int_{a_n}^{1-b_n}\int_{-1}^{1}C_1(u^{1-t},u^t)u^{-1-t}\tfrac{nh^2}{2(n+1)}(1-t)(-t)s_1^2k(s_1)\lambda(u,t)\,ds_1du\\ &+\tfrac{1}{n+1}\int_{a_n}^{1-b_n}C_1(u^{1-t},u^t)(1-u^{1-t})\lambda(u,t)du\\ =\;&O(h^2\int_{a_n}^{1-b_n}u^{-3/2}\lambda(u,t)du)+O(n^{-1}\int_{a_n}^{1-b_n}u^{-1/2}\lambda(u,t)du)=o(1/\sqrt{n}). \end{aligned}$$

Other terms of (5.25) can be handled in the same way, which results in

$$\begin{aligned} II_2 &= o(1/\sqrt{n})+O\left(\int_{a_n}^{1-b_n}|C_2(u^{1-t},u^t)u^t(\tfrac{h^2}{u^2}+\tfrac{1}{(n+1)u})\lambda(u,t)|\,du\right)\\ &+O\left(\int_{a_n}^{1-b_n}|C_{11}(u^{1-t},u^t)u^{2-2t}(\tfrac{h}{u}+\tfrac{1}{(n+1)u})^2\lambda(u,t)|\,du\right)\\ &+O\left(\int_{a_n}^{1-b_n}|C_{22}(u^{1-t},u^t)u^{2t}(\tfrac{h}{u}+\tfrac{1}{(n+1)u})^2\lambda(u,t)|\,du\right)\\ &+O\left(\int_{a_n}^{1-b_n}|C_{12}(u^{1-t},u^t)u(\tfrac{h}{u}+\tfrac{1}{(n+1)u})(\tfrac{h}{u}+\tfrac{1}{(n+1)u})\lambda(u,t)|\,du\right)\\ &= o(1/\sqrt{n})+O\left(h^2\int_{a_n}^{1-b_n}u^{-3/2}\lambda(u,t)\,du\right)\\ &+O\left(h^2\int_{a_n}^{1-b_n}\{\log u\}^{-1}u^{-3/2}\lambda(u,t)\,du\right)\\ &= o(1/\sqrt{n}). \end{aligned} \tag{5.26}$$

For the second term $I_2$ in (5.17), by the mean value theorem we can write

$$\begin{aligned} &I_2\\ =\;&n^{-1}\int_{a_n}^{1-b_n}\sum_{i=1}^{n}\sum_{j=1}^{n}\big\{\tfrac{\hat{F}_{n1,-i}^{1/(1-t)}(X_{j1})-\hat{F}_{n1}^{1/(1-t)}(X_{j1})}{h}k(\tfrac{u-\hat{F}_{n1}^{1/(1-t)}(X_{j1})}{h})\\ &+\tfrac{1}{2}(\tfrac{\hat{F}_{n1,-i}^{1/(1-t)}(X_{j1})-\hat{F}_{n1}^{1/(1-t)}(X_{j1})}{h})^2k'(\tfrac{u-\xi_{n,i,j}^{1/(1-t)}}{h})\big\}K(\tfrac{u-F_{n2}^{1/t}(X_{j2})}{h})\lambda(u,t)\,du, \end{aligned} \tag{5.27}$$

where $\xi_{n,i,j}$ is between $\hat{F}_{n1}(X_{j1})$ and $\hat{F}_{n1,-i}(X_{j1})$. Using the equation

$$F_{n1,-i}(X_{j1}) - \hat{F}_{n1}(X_{j1}) = \frac{1}{n}\hat{F}_{n1}(X_{j1}) - \frac{1}{n}I(X_{i1} \le X_{j1}),$$

we have

$$\begin{cases} \sup_{1\le i,j\le n} |\hat{F}_{n1}(X_{j1}) - \hat{F}_{n1,-i}(X_{j1})| \le n^{-1}, \\ \sup_{1\le i,j\le n} |\hat{F}_{n1}^{1/(1-t)}(X_{j1}) - \hat{F}_{n1,-i}^{1/(1-t)}(X_{j1})| \le \frac{1}{1-t}n^{-1}. \end{cases} \tag{5.28}$$

Then uniformly for $u \in [a_n, 1-b_n]$,

$$\begin{aligned} & \textstyle\sum_{i=1}^n \sum_{j=1}^n I(|\frac{u-\xi_{n,i,j}^{1/(1-t)}}{h}| \le 1) \\ \le\ & \textstyle\sum_{i=1}^n \sum_{j=1}^n P((u-h)^{1-t} - \frac{1}{n} \le \hat{F}_{n1}(X_{j1}) \le (u+h)^{1-t} + \frac{1}{n}) \\ \le\ & n \times \{\frac{(n+1)(u+h)^{1-t}+(n+1)/n}{n} - \frac{(n+1)(u-h)^{1-t}-(n+1)/n-1}{n}\} \\ =\ & O(u^{-1}nh) \end{aligned} \tag{5.29}$$

and

$$\sum_{j=1}^n I(|\frac{u - F_{n1}^{1/(1-t)}(X_{j1})}{h}| \le 1) = O(u^{-1}h). \tag{5.30}$$

Since $k(s)$ is a density function with support on $[-1, 1]$, it follows from (5.27), (5.29) and (5.30) that

$$\begin{aligned} I_2 &= O\left(h^{-1}n^{-2}\int_{a_n}^{1-b_n} \textstyle\sum_{i=1}^n \sum_{j=1}^n I(|\frac{u-F_{n1}^{1/(1-t)}(X_{j1})}{h}| \le 1)\lambda(u,t)\,du\right) \\ &\quad + O\left(h^{-2}n^{-3}\int_{a_n}^{1-b_n} \textstyle\sum_{i=1}^n \sum_{j=1}^n I(|\frac{u-\xi_{n,i,j}^{1/(1-t)}}{h}| \le 1)\lambda(u,t)\,du\right) \\ &= O\left(n^{-1}\int_{a_n}^{1-b_n} u^{-1}\lambda(u,t)\,du\right) = o(1/\sqrt{n}). \end{aligned} \tag{5.31}$$

Similarly we can show that

$$I_3 = o(1/\sqrt{n}) \quad \text{and} \quad I_4 = o(1/\sqrt{n}). \tag{5.32}$$

Hence, the Lemma follows from (5.23), (5.26), (5.31) and (5.32). □

**Lemma 5.2.** *Under conditions of Theorem 3.1, we have*

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^n Q_i^2(\theta_0) \ &\xrightarrow{p} E\left(\int_0^1 \{W(u^{1-t}, u^t) - W(u^{1-t}, 1)C_1(u^{1-t}, u^t)\right. \\ &\qquad \left. - W(1, u^t)C_2(u^{1-t}, u^t)\}\lambda(u,t)\,du\right)^2 \end{aligned}$$

*as* $n \to \infty$.

*Proof.* By (5.16), we can write

$$
\begin{aligned}
& Q_i^2(\theta) \\
= \;& \int_{a_n}^{1-b_n}\int_{a_n}^{1-b_n}\{\hat{V}_{i1}(u_1,t)\hat{V}_{i1}(u_2,t)+\hat{V}_{i1}(u_1,t)\hat{V}_{i2}(u_2,t)-\hat{V}_{i1}(u_1,t)u_2^\theta \\
& +\hat{V}_{i2}(u_1,t)\hat{V}_{i1}(u_2,t)+\hat{V}_{i2}(u_1,t)\hat{V}_{i2}(u_2,t)-\hat{V}_{i2}(u_1,t)u_2^\theta \\
& -u_1^\theta\hat{V}_{i1}(u_2,t)-u_1^\theta\hat{V}_{i2}(u_2,t)+u_1^\theta u_2^\theta\}\lambda(u_1,t)\lambda(u_2,t)\,du_1du_2.
\end{aligned}
$$

Using the similar arguments as in (5.27), we have

$$
\begin{aligned}
& \tfrac{1}{n}\sum_{i=1}^n\int_{a_n}^{1-b_n}\int_{a_n}^{1-b_n}\hat{V}_{i2}(u_1,t)\hat{V}_{i2}(u_2,t)\lambda(u_1,t)\lambda(u_2,t)\,du_1du_2 \\
= \;& \int_{a_n}^{1-b_n}\int_{a_n}^{1-b_n}\Big(\tfrac{1}{n}\sum_{i=1}^n\sum_{j=1}^n\sum_{l=1}^n \\
& \big\{\tfrac{\hat{F}_{n1,-i}(X_{j1})-\hat{F}_{n1}(X_{j1})}{h}\tfrac{1}{1-t}\hat{F}_{n1}^{t/(1-t)}(X_{j1})k(\tfrac{u_1-\hat{F}_{n1}^{1/(1-t)}(X_{j1})}{h})K(\tfrac{u_1-\hat{F}_{n2}^{1/t}(X_{j2})}{h}) \\
& +\tfrac{\hat{F}_{n2,-i}(X_{j2})-\hat{F}_{n2}(X_{j2})}{h}\tfrac{1}{t}\hat{F}_{n2}^{(1-t)/t}(X_{j2})k(\tfrac{u_1-\hat{F}_{n2}^{1/t}(X_{j2})}{h})K(\tfrac{u_1-\hat{F}_{n1}^{1/(1-t)}(X_{j1})}{h})\big\}\times \\
& \big\{\tfrac{\hat{F}_{n1,-i}(X_{l1})-\hat{F}_{n1}(X_{l1})}{h}\tfrac{1}{1-t}\hat{F}_{n1}^{t/(1-t)}(X_{l1})k(\tfrac{u_2-\hat{F}_{n1}^{1/(1-t)}(X_{l1})}{h})K(\tfrac{u_2-\hat{F}_{n2}^{1/t}(X_{l2})}{h}) \\
& +\tfrac{\hat{F}_{n2,-i}(X_{l2})-\hat{F}_{n2}(X_{l2})}{h}\tfrac{1}{t}\hat{F}_{n2}^{(1-t)/t}(X_{l2})k(\tfrac{u_2-\hat{F}_{n2}^{1/t}(X_{l2})}{h})K(\tfrac{u_2-\hat{F}_{n1}^{1/(1-t)}(X_{l1})}{h})\big\}\Big)\times \\
& \lambda(u_1,t)\lambda(u_2,t)\,du_1du_2+o_p(1).
\end{aligned} \tag{5.33}
$$

It is straightforward to check that

$$
\begin{aligned}
& \tfrac{1}{n}\sum_{i=1}^n\{\hat{F}_{n1}(x)-\hat{F}_{n1,-i}(x)\}\{\hat{F}_{n1}(y)-\hat{F}_{n1,-i}(y)\} \\
= \;& \tfrac{n+1}{n^3}\hat{F}_{n1}(x\wedge y)-\tfrac{n+2}{n^3}\hat{F}_{n1}(x)\hat{F}_{n1}(y),
\end{aligned}
$$

$$
\begin{aligned}
& \tfrac{1}{n}\sum_{i=1}^n\{\hat{F}_{n2}(x)-\hat{F}_{n2,-i}(x)\}\{\hat{F}_{n2}(y)-\hat{F}_{n2,-i}(y)\} \\
= \;& \tfrac{n+1}{n^3}\hat{F}_{n2}(x\wedge y)-\tfrac{n+2}{n^3}\hat{F}_{n2}(x)\hat{F}_{n2}(y)
\end{aligned}
$$

and

$$
\begin{aligned}
& \tfrac{1}{n}\sum_{i=1}^n\{\hat{F}_{n1}(x)-\hat{F}_{n1,-i}(x)\}\{\hat{F}_{n2}(y)-\hat{F}_{n2,-i}(y)\} \\
= \;& \tfrac{1}{n^2}\hat{C}_n(\hat{F}_{n1}(x),\hat{F}_{n2}(y))-\tfrac{n+2}{n^3}\hat{F}_{n1}(x)\hat{F}_{n2}(y).
\end{aligned}
$$

Then (5.33) can be written as

$$
\begin{aligned}
& \tfrac{1}{n}\textstyle\sum_{i=1}^n \int_{a_n}^{1-b_n}\int_{a_n}^{1-b_n} \hat{V}_{i2}(u_1,t)\hat{V}_{i2}(u_2,t)\lambda(u_1,t)\lambda(u_2,t)\,du_1du_2 \\
= & \tfrac{1}{h^2}\int_{a_n}^{1-b_n}\int_{a_n}^{1-b_n}\Big(\tfrac{1}{n^2h^2}\textstyle\sum_{j=1}^n\sum_{l=1}^n \\
& \big\{(\hat{F}_{n1}(X_{j1}\wedge X_{l1}) - \hat{F}_{n1}(X_{j1})\hat{F}_{n1}(X_{l1}))\tfrac{1}{(1-t)^2}\times \\
& \hat{F}_{n1}^{t/(1-t)}(X_{j1})\hat{F}_{n1}^{t/(1-t)}(X_{l1})k(\tfrac{u_1-\hat{F}_{n1}^{1/(1-t)}(X_{j1})}{h})\times \\
& K(\tfrac{u_1-\hat{F}_{n2}^{1/t}(X_{j2})}{h})k(\tfrac{u_2-\hat{F}_{n1}^{1/(1-t)}(X_{l1})}{h})K(\tfrac{u_2-\hat{F}_{n2}^{1/t}(X_{l2})}{h}) \\
& +(\hat{F}_{n2}(X_{j2}\wedge X_{l2}) - \hat{F}_{n2}(X_{j2})\hat{F}_{n2}(X_{l2}))\tfrac{1}{t^2}\times \\
& \hat{F}_{n2}^{(1-t)/t}(X_{j2})\hat{F}_{n2}^{(1-t)/t}(X_{l2})k(\tfrac{u_1-\hat{F}_{n2}^{1/t}(X_{j2})}{h})\times \\
& K(\tfrac{u_1-\hat{F}_{n1}^{1/(1-t)}(X_{j1})}{h})k(\tfrac{u_2-\hat{F}_{n2}^{1/t}(X_{l2})}{h})K(\tfrac{u_2-\hat{F}_{n1}^{1/(1-t)}(X_{l1})}{h}) \\
& +(\hat{C}_n(\hat{F}_{n1}(X_{j1}),\hat{F}_{n2}(X_{l2})) - \hat{F}_{n1}(X_{j1})\hat{F}_{n2}(X_{l2}))\tfrac{1}{t(1-t)}\times \\
& \hat{F}_{n1}^{1/(1-t)}(X_{j1})\hat{F}_{n2}^{(1-t)/t}(X_{l2})k(\tfrac{u_1-\hat{F}_{n1}^{1/(1-t)}(X_{j1})}{h})\times \\
& K(\tfrac{u_1-\hat{F}_{n2}^{1/t}(X_{j2})}{h})k(\tfrac{u_2-\hat{F}_{n2}^{1/t}(X_{l2})}{h})K(\tfrac{u_2-\hat{F}_{n1}^{1/(1-t)}(X_{l1})}{h}) \\
& +(\hat{C}_n(\hat{F}_{n1}(X_{l1}),\hat{F}_{n2}(X_{j2})) - \hat{F}_{n1}(X_{l1})\hat{F}_{n2}(X_{j2}))\tfrac{1}{t(1-t)}\times \\
& \hat{F}_{n1}^{1/(1-t)}(X_{l1})\hat{F}_{n2}^{(1-t)/t}(X_{j2})k(\tfrac{u_2-\hat{F}_{n1}^{1/(1-t)}(X_{l1})}{h})\times \\
& K(\tfrac{u_2-\hat{F}_{n2}^{1/t}(X_{l2})}{h})k(\tfrac{u_1-\hat{F}_{n2}^{1/t}(X_{j2})}{h})K(\tfrac{u_1-\hat{F}_{n1}^{1/(1-t)}(X_{j1})}{h})\big\}\Big)\times \\
& \lambda(u_1,t)\lambda(u_2,t)\,du_1du_2 + o_p(1).
\end{aligned}
$$

Based on the above decomposition, we can show that

$$
\begin{aligned}
& \tfrac{1}{n}\textstyle\sum_{i=1}^n \int_{a_n}^{1-b_n}\int_{a_n}^{1-b_n} \hat{V}_{i2}(u_1,t)\hat{V}_{i2}(u_2,t)\lambda(u_1,t)\lambda(u_2,t)\,du_1du_2 \\
= & \int_0^1\int_0^1 \big(\{u_1^{1-t}\wedge u_2^{1-t} - u_1^{1-t}u_2^{1-t}\}C_1(u_1^{1-t},u_1^t)C_1(u_2^{1-t},u_2^t) \\
& +\{u_1^t\wedge u_2^t - u_1^tu_2^t\}C_2(u_1^{1-t},u_1^t)C_2(u_2^{1-t},u_2^t) \\
& +\{C(u_1^{1-t},u_2^t) - u_1^{1-t}u_2^t\}C_1(u_1^{1-t},u_1^t)C_2(u_2^{1-t},u_2^t) \\
& +\{C(u_2^{1-t},u_1^t) - u_2^{1-t}u_1^t\}C_1(u_2^{1-t},u_2^t)C_2(u_1^{1-t},u_1^t)\big)\times \\
& \lambda(u_1,t)\lambda(u_2,t)\,du_1du_2 + o_p(1).
\end{aligned} \tag{5.34}
$$

Similarly, we have

$$
\begin{aligned}
& \tfrac{1}{n}\textstyle\sum_{i=1}^n \int_{a_n}^{1-b_n}\int_{a_n}^{1-b_n} \hat{V}_{i1}(u_1,t)\hat{V}_{i2}(u_2,t)\,du_1du_2 \\
= & \int_0^1\int_0^1\{C(u_1^{1-t},u_1^t)u_2^{1-t}C_1(u_2^{1-t},u_2^t) - C(u_1^{1-t}\wedge u_2^{1-t},u_1^t)C_1(u_2^{1-t},u_2^t) \\
& +C(u_1^{1-t},u_1^t)u_2^tC_2(u_2^{1-t},u_2^t) - C(u_1^{1-t},u_1^t\wedge u_2^t)C_2(u_2^{1-t},u_2^t)\}\times \\
& \lambda(u_1,t)\lambda(u_2,t)\,du_1du_2 + o_p(1),
\end{aligned} \tag{5.35}
$$

$$\begin{aligned} &\tfrac{1}{n}\textstyle\sum_{i=1}^n \int_{a_n}^{1-b_n}\int_{a_n}^{1-b_n} \hat{V}_{i1}(u_1,t)\hat{V}_{i1}(u_2,t)\lambda(u_1,t)\lambda(u_2,t)\,du_1du_2 \\ = &\textstyle\int_0^1\int_0^1 C(u_1^{1-t}\wedge u_2^{1-t}, u_1^t\wedge u_2^t)\lambda(u_1,t)\lambda(u_2,t)\,du_1du_2 + o_p(1), \end{aligned} \tag{5.36}$$

$$\begin{aligned} &\tfrac{1}{n}\textstyle\sum_{i=1}^n \int_{a_n}^{1-b_n}\int_{a_n}^{1-b_n} \hat{V}_{i1}(u_1,t)u_2^{\theta_0}\lambda(u_1,t)\lambda(u_2,t)\,du_1du_2 \\ = &\textstyle\int_0^1\int_0^1 C(u_1^{1-t},u_1^t)C(u_2^{1-t},u_2^t)\lambda(u_1,t)\lambda(u_2,t)\,du_1du_2 + o_p(1), \end{aligned} \tag{5.37}$$

and

$$\tfrac{1}{n}\textstyle\sum_{i=1}^n \int_{a_n}^{1-b_n}\int_{a_n}^{1-b_n} \hat{V}_{i2}(u_1,t)u_2^{\theta_0}\lambda(u_1,t)\lambda(u_2,t)\,du_1du_2 = o_p(1). \tag{5.38}$$

Hence the lemma follows from (5.34)–(5.38) and the fact that

$$\begin{aligned} &E\Big(\textstyle\int_0^1\{W(u^{1-t},u^t) - W(u^{1-t},1)C_1(u^{1-t},u^t) \\ &-W(1,u^t)C_2(u^{1-t},u^t)\}\lambda(u,t)\,du\Big)^2 \\ = &\textstyle\int_0^1\int_0^1 \{C(u_1^{1-t}\wedge u_2^{1-t}, u_1^t\wedge u_2^t) - C(u_1^{1-t},u_1^t)C(u_2^{1-t},u_2^t) \\ &-(C(u_1^{1-t}\wedge u_2^{1-t},u_1^t) - C(u_1^{1-t},u_1^t)u_2^{1-t})C_1(u_2^{1-t},u_2^t) \\ &-(C(u_1^{1-t},u_1^t\wedge u_2^t) - C(u_1^{1-t},u_1^t)u_2^t)C_2(u_2^{1-t},u_2^t) \\ &-(C(u_1^{1-t}\wedge u_2^{1-t},u_2^{1-t}) - u_1^{1-t}C(u_2^{1-t},u_2^t))C_1(u_1^{1-t},u_1^t) \\ &+(u_1^{1-t}\wedge u_2^{1-t} - u_1^{1-t}u_2^{1-t})C_1(u_1^{1-t},u_1^t)C_1(u_2^{1-t},u_2^t) \\ &+(C(u_1^{1-t},u_2^t) - u_1^{1-t}u_2^t)C_1(u_1^{1-t},u_1^t)C_2(u_2^{1-t},u_2^t) \\ &-(C(u_2^{1-t},u_1^t\wedge u_2^t) - u_1^tC(u_2^{1-t},u_2^t))C_2(u_1^{1-t},u_1^t) \\ &+(C(u_2^{1-t},u_1^t) - u_2^{1-t}u_1^t)C_2(u_1^{1-t},u_1^t)C_1(u_2^{1-t},u_2^t) \\ &+(u_1^t\wedge u_2^t - u_1^tu_2^t)C_2(u_1^{1-t},u_1^t)C_2(u_2^{1-t},u_2^t)\}\lambda(u_1,t)\lambda(u_2,t)\,du_1du_2. \end{aligned}$$

□

*Proof of Theorem 3.1.* Using similar expansions as in the proof of Lemma 5.1, we can show that $\max_{1\le i\le n}|Q_i(\theta_0)| = o_p(n^{1/2})$. Hence, by using Lemmas 5.1–5.2 and standard arguments in expanding the empirical likelihood ratio (see, for example, Owen (1988)), we obtain that as $n\to\infty$,

$$l(\theta_0) = \{\sum_{i=1}^n Q_i(\theta_0)\}^2/\sum_{i=1}^n Q_i^2(\theta_0) + o_p(1) \xrightarrow{d} \chi^2(1).$$

□

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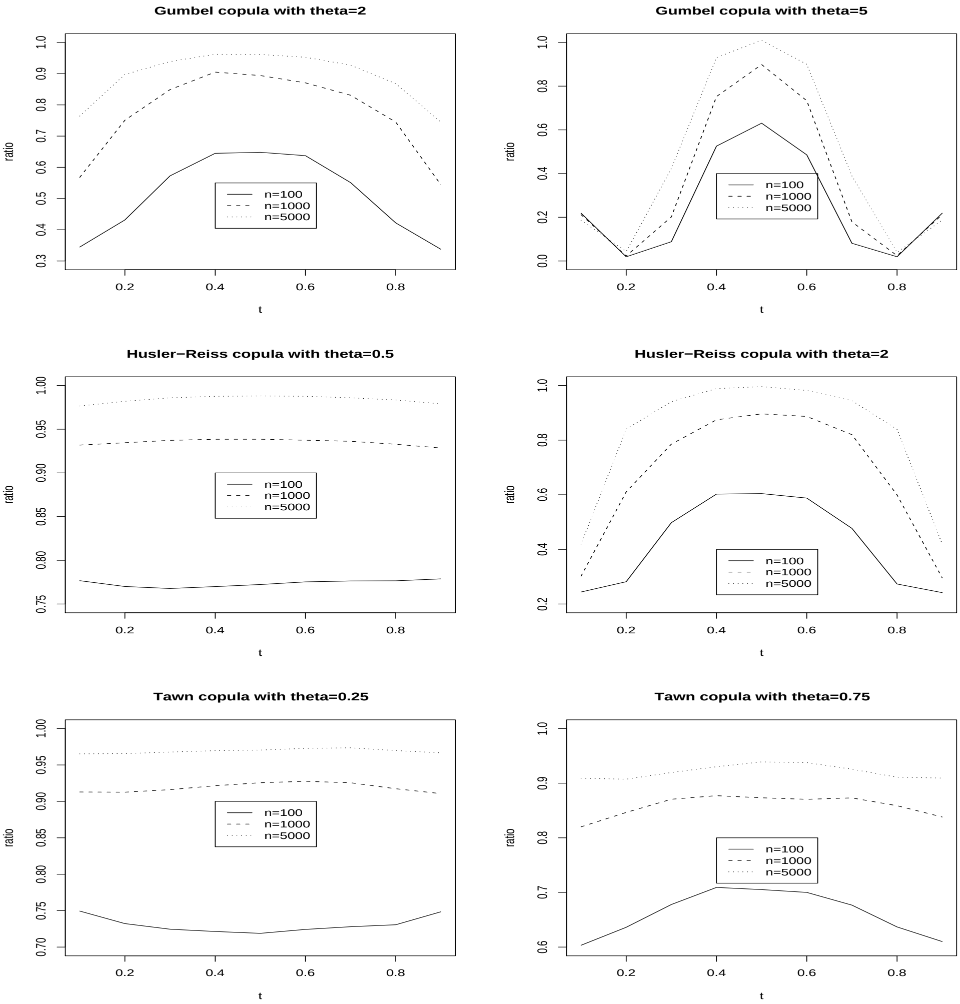


Figure 5.1: Ratios of the mean squared error of the new estimator $\hat{A}_n^w(t)$ to that of $\hat{A}^{CFG}(t)$ for $t = 0.1, 0.2, \cdots, 0.9$.

Table 5.1: Empirical coverage probabilities are reported for the proposed jackknife empirical likelihood confidence interval (JELCI) based on $\lambda(u,t) = u^{-1}(-\log u)^{-\min\{\hat{A}^{CFG}(t),1\}}$, and the confidence interval based on the multiplier method for $\hat{A}^{CFG}(t)$ (MCI) with nominal levels 0.9 and 0.95.

| (n,t,Copula,$\theta$) | level 0.9 JELCI | level 0.9 MCI | level 0.95 JELCI | level 0.95 MCI |
|---|---|---|---|---|
| (100,0.1,Gumbel,2) | 0.604 | 0.276 | 0.639 | 0.366 |
| (100,0.1,Hüsler-Reiss,0.5) | 0.845 | 0.566 | 0.899 | 0.655 |
| (100,0.1,Tawn,0.25) | 0.817 | 0.571 | 0.872 | 0.670 |
| (100,0.5,Gumbel,2) | 0.871 | 0.722 | 0.941 | 0.784 |
| (100,0.5,Hüsler-Reiss,0.5) | 0.888 | 0.715 | 0.941 | 0.802 |
| (100,0.5,Tawn,0.25) | 0.886 | 0.750 | 0.941 | 0.825 |
| (100,0.8,Gumbel,2) | 0.841 | 0.531 | 0.889 | 0.599 |
| (100,0.8,Hüsler-Reiss,0.5) | 0.889 | 0.646 | 0.947 | 0.758 |
| (100,0.8,Tawn,0.25) | 0.884 | 0.677 | 0.938 | 0.758 |
| (1000,0.1,Gumbel,2) | 0.888 | 0.655 | 0.935 | 0.740 |
| (1000,0.1,Hüsler-Reiss,0.5) | 0.892 | 0.813 | 0.942 | 0.883 |
| (1000,0.1,Tawn,0.25) | 0.900 | 0.820 | 0.957 | 0.891 |