A PROOF OF THE KAZHDAN-LUSZTIG PURITY THEOREM VIA THE DECOMPOSITION THEOREM OF BBD

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1. Introduction

The purpose of this note is to present the "modern" proof of the purity result of Kazhdan-Lusztig [KL2] . It relies on two main ingredients:

- The Decomposition Theorem of Beilinson-Bernstein-Deligne-Gabber ([BBD], Thm.5.4.5, 6.2.5);
- The fibers of the Demazure resolution of a Schubert variety are paved by affine spaces (see below).

The first ingredient is quite deep, and we will not say anything much about its proof, using it simply as a "black box". The second ingredient is elementary, and we take this opportunity to present a simple and self-contained proof which works equally well for both finite and affine flag varieties. Other proofs can be found in [Gau] and [Haer].

The original proof in [KL2] is more elementary than the present one, in that it does not require the Decomposition theorem. However, it is easier to remember the present proof and to understand "how it works". More importantly, the present proof applies equally well to both the finite flag variety and to the affine flag variety, thereby giving a uniform treatment valid for both situations where this kind of result holds.

2. Statement of the theorem

We will discuss the case of the finite flag variety. As stated above, everything goes over, *mutatis mutandis*, for affine flag varieties.

Let $X = G/B$, a projective $\overline{\mathbb{F}}_q$-variety which is defined over $\mathbb{F}_q$. Let us denote the obvious base point by e_0. For any $x \in W$, let $Y(w) = Bwe_0$, a locally closed subvariety of X isomorphic to $\mathbb{A}^{\ell(w)}$, called a **Schubert cell**. Let us define a **Schubert variety** to be $X(w) = \overline{Y(w)}$, the closure of $Y(w)$ in X. Clearly $X(w)$ is an irreducible variety defined over $\mathbb{F}_q$, and in general it is singular. Recall that the closure relations are given by the Bruhat order: for $v, w \in W$, we have $v \leq w$ if and only if $Y(v) \subset X(w)$. Therefore $X(w)$ is a stratified space whose strata are the various $Y(v)$ with $v \leq w$.

Let Fr_q denote the **absolute Frobenius morphism** for the scheme X relative to its field of definition $\mathbb{F}_q$: on points it is the identity map, and on the structure sheaf $\mathcal{O}_X$, it is the map $f \mapsto f^q$. In projective coordinates $[x_0 : x_1 : \cdots : x_n]$, this induces the map $x_i \mapsto x_i^q$. It induces an endomorphism on the étale cohomology groups with compact support $H^i_c(X, \overline{\mathbb{Q}}_\ell)$ (which are the étale cohomology groups $H^i(X, \overline{\mathbb{Q}}_\ell)$, since X is a complete variety). In a similar way, Fr_q induces an endomorphism on the stalk $\mathcal{F}_{\overline{x}}$ at a closed point $x \in X(\mathbb{F}_q)$, where $\mathcal{F}$ is a $\overline{\mathbb{Q}}_\ell$-sheaf on X which is defined over $\mathbb{F}_q$. More generally, we get a Frobenius endomorphism Fr_q on any stalk $\mathcal{F}_{\overline{x}}$, for any object $\mathcal{F}$ in the derived category $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ of "bounded complexes of constructible $\overline{\mathbb{Q}}_\ell$-sheaves on X", as long as $\mathcal{F}$ is "defined over $\mathbb{F}_q$" in a suitable sense. Given such an object $\mathcal{F}$, let $\mathcal{H}^i\mathcal{F}$ be the i-th cohomology sheaf; for $x \in X(\mathbb{F}_q)$, we get the Frobenius endomorphism Fr_q on the stalk $\mathcal{H}^i\mathcal{F}_{\overline{x}}$. This a finite dimensional $\overline{\mathbb{Q}}_\ell$-space, and so the trace

$$\mathrm{Tr}(\mathrm{Fr}_q; \mathcal{H}^i\mathcal{F}_{\overline{x}})$$

is a well-defined element of $\overline{\mathbb{Q}}_\ell$. It does not depend on the choice of geometric point $\overline{x}$ over x, and so in the sequel we will suppress the notation $\overline{x}$ and write simply x when discussing stalks of constructible $\overline{\mathbb{Q}}_\ell$-sheaves.

Let $IC_w = j_{!*}\overline{\mathbb{Q}}_\ell$, the Goresky-MacPherson middle extension of the constant sheaf $\overline{\mathbb{Q}}_\ell$ along the open embedding $j : Y(w) \hookrightarrow X(w)$. The complex IC_w is an object of the derived category $D^b_c(X(w), \overline{\mathbb{Q}}_\ell)$, and is defined over $\mathbb{F}_q$ in a suitable sense that we may speak of the endomorphism Fr_q acting on its cohomology stalks, as above. We will call IC_w the **intersection complex** for the variety $X(w)$. Our normalization is such that IC_w is not perverse, but its cohomological shift $IC_w[\ell(w)]$ by $\ell(w)$ degrees to the left, is perverse. In fact $IC_w[\ell(w)]$ is the unique Verdier self-dual[1] object in $D^b_c(X(w), \overline{\mathbb{Q}}_\ell)$ with the properties

(1) $IC_w[\ell(w)]|_{Y(w)} = \overline{\mathbb{Q}}_\ell[\ell(w)]$; and
(2) for every stratum $Y(v) \neq Y(w)$ in $X(w)$, we have $\mathcal{H}^i(IC_w)[\ell(w)]|_{Y(v)} = 0$ for all $i \geq -\dim Y(v)$.

In what follows, we will denote by $\mathcal{H}^i(X(w))$ the i-th cohomology sheaf of IC_w. Now we can state the purity theorem.

Theorem 2.0.1 (Kazhdan-Lusztig [KL2])**.** *For every $v \leq w$, and every closed point $y \in Y(v)(\mathbb{F}_q)$, we have $\mathcal{H}^i(X(w))_y = 0$ for odd i, and for even i all the eigenvalues of* Fr_q *on $\mathcal{H}^i(X(w))_y$ are equal to $q^{i/2}$.*

3. The fibers of the Demazure resolution are paved by affine spaces

Let $w \in W$ and choose a reduced expression $w = s_1 \cdots s_r$. Let $\widetilde{X}(w)$ be the subscheme of $(G/B)^r$ consisting of r-tuples $(g_1B, g_2B, \dots, g_rB)$ such that for all i, we have $g_{i-1}^{-1}g_i \in \overline{Bs_iB}$, the closure of Bs_iB in G (by convention, $g_0 = 1$). Projection onto the r-th factor gives a projective birational morphism

$$\pi_{s_\bullet} : \widetilde{X}(w) \to X(w),$$

called the **Demazure resolution**. Since $\widetilde{X}(w)$ is non-singular (being a succession of $\mathbb{P}^1$-bundles), $\pi_{s_\bullet}$ is a resolution of the singularities of $X(w)$.

We recall the notion of **paving by affine spaces**. We say a scheme Z is *paved by affine spaces* provided that there is a finite filtration by closed subspaces $Z = Z_N \supset Z_{N-1} \cdots \supset Z_1 \supset Z_0 = \emptyset$, such that each difference $Z_j - Z_{j-1}$ is a (topological) disjoint union of certain affine spaces $\mathbb{A}^{n_{ji}}$.

It is convenient to think of X as the space of all Borel subgroups in G. We identify the coset $gB \in X$ with the Borel subgroup ${}^gB := gBg^{-1}$. Further, we will write ${}^{g_1}B \xrightarrow{w} {}^{g_2}B$ when $g_1^{-1}g_2 \in BwB$ and ${}^{g_1}B \xrightarrow{\leq w} {}^{g_2}B$ when $g_1^{-1}g_2 \in \overline{BwB}$.

[1]In the category $D^{b,\mathrm{Weil}}_c(X, \overline{\mathbb{Q}}_\ell)$, the self-dual intersection complex is the Tate-twist $IC_w[\ell(w)](\frac{\ell(w)}{2})$.

Proposition 3.0.2. *For each $v \leq w$ and any point ${}^yB \in Y(v)$, the fiber $\pi_{s_\bullet}^{-1}({}^yB)$ is paved by affine spaces.*

Proof. We argue by induction on $r = \ell(w)$. Since $\pi_{s_\bullet}$ is B-equivariant (with respect to the obvious left-actions of B), we may as well assume ${}^yB = {}^vB$.

Consider an element $(B_1, \cdots, B_{r-1}, {}^vB)$ in $\pi_{s_\bullet}^{-1}({}^vB)$. We have

$$B \xrightarrow{v} {}^vB \xrightarrow{\leq s_r} B_{r-1}.$$

It follows that $B_{r-1} \in Y(v) \cup Y(vs_r)$. We consider the map

$$p : \pi_{s_\bullet}^{-1}({}^vB) \to Y(v) \cup Y(vs_r)$$

given by $(B_1, B_2, \ldots, B_{r-1}, {}^vB) \mapsto B_{r-1}$.

We will examine the subsets $\mathrm{Im}(p) \cap Y(v)$ and $\mathrm{Im}(p) \cap Y(vs_r)$, and we will show that

(a) each of these is an affine space (either empty, a point, or $\mathbb{A}^1$); one of them (denoted $\mathbb{A}_1$) is closed (and possibly empty) and the other (denoted $\mathbb{A}_2$) is open and dense in $\mathrm{Im}(p)$;

(b) let $\pi_{s'_\bullet}$ denote the Demazure resolution associated to the reduced word $s'_\bullet = s_1 \cdots s_{r-1}$; if $\mathbb{A}_1 \neq \emptyset$, then $\mathbb{A}_1$ belongs to the image of $\pi_{s'_\bullet}$; furthermore, we have $p^{-1}(\mathbb{A}_1) = \pi_{s'_\bullet}^{-1}(\mathbb{A}_1)$, and $p : p^{-1}(\mathbb{A}_1) \to \mathbb{A}_1$ is simply the morphism $\pi_{s'_\bullet} : \pi_{s'_\bullet}^{-1}(\mathbb{A}_1) \to \mathbb{A}_1$ (similar remarks apply to $\mathbb{A}_2$); and

(c) the morphism $\pi_{s'_\bullet} : \pi_{s'_\bullet}^{-1}(\mathbb{A}_1) \to \mathbb{A}_1$ is *trivial* (similarly for $\mathbb{A}_2$).

This will be enough to prove the proposition. Indeed, applying p^{-1} to the decomposition

$$\mathrm{Im}(p) = \mathbb{A}_1 \cup \mathbb{A}_2$$

gives us a decomposition

$$\pi_{s_\bullet}^{-1}({}^vB) = p^{-1}(\mathbb{A}_1) \cup p^{-1}(\mathbb{A}_2)$$

where the first subset is closed (possibly empty) and the second is open. Furthermore, the triviality statement in (c) together with our induction hypothesis applied to $\pi_{s'_\bullet}$ show that $p^{-1}(\mathbb{A}_1)$ and $p^{-1}(\mathbb{A}_2)$ are each paved by affine spaces. Putting all this together, we see that $\pi_{s_\bullet}^{-1}({}^vB)$ is indeed paved by affine spaces.

To verify (a-c), we need to consider various cases. The cases break up according to whether $v < vs_r$ or $vs_r < v$ in the Bruhat order. We will break these cases up further, using the following general fact about the Bruhat order: $v \leq s_1 \cdots s_r$ implies that either $v \leq s_1 \cdots s_{r-1}$ or $vs_r \leq s_1 \cdots s_{r-1}$ (or both). Thus we get the following four cases:

Case 1 : $v < vs_r$.
Case 1a : $v < vs_r \leq s_1 \cdots s_{r-1}$;
Case 1b : $v \leq s_1 \cdots s_{r-1}$ and $vs_r \nleq s_1 \cdots s_{r-1}$.
Case 2 : $vs_r < v$.
Case 2a : $vs_r < v \leq s_1 \cdots s_{r-1}$;
Case 2b : $vs_r \leq s_1 \cdots s_{r-1}$ and $v \nleq s_1 \cdots s_{r-1}$.

Before we analyze the various cases, let us make a few preliminary remarks. Let ξ denote one of the elements v or vs_r. It is immediate that $\mathrm{Im}(p) \subset \mathrm{Im}(\pi_{s'_\bullet})$, and that the latter is simply the union of all $Y(y)$ with $y \leq s_1 \cdots s_{r-1}$. It follows easily that $\mathrm{Im}(p) \cap Y(\xi)$ is empty unless $\xi \leq s_1 \cdots s_{r-1}$, in which case it is the variety of those Borel subgroups B' which satisfy

$$B \xrightarrow{\xi} B' \xrightarrow{\leq s_r} {}^vB.$$

To condense notation somewhat, we make the following definitions for any $w \in W$, any simple reflection $s \in W$, and any Borel subgroup B_0:

$$\mathbb{P}^1(B_0, s) := \{B' \mid B_0 \xrightarrow{\leq s} B'\}$$
$$\mathbb{A}(B_0, w) := \{B' \mid B_0 \xrightarrow{w} B'\}.$$

(The former (resp. latter) is clearly isomorphic to a copy of $\mathbb{P}^1$ (resp. $\mathbb{A}^{\ell(w)}$), thus the notation.)

Thus, in the case $\xi \leq s_1 \cdots s_{r-1}$, we have to show

$$\mathbb{A}(B, \xi) \cap \mathbb{P}^1({}^vB, s_r)$$

is always an affine space, and moreover we need to give an explicit description of it (needed to verify (c)).

To do so, we use the following way to think about the relative position of Borel subgroups in terms of alcoves for the spherical building associated to G [2]. For any $w \in W$, $\mathbb{A}(B_0, w)$ is an affine space whose points correspond to those alcoves in relative position w from the alcove fixed by B_0 (more precisely we fix a reduced expression $w = s_{j_1} \cdots s_{j_p}$; then $\mathbb{A}(B_0, w)$ consists of the terminal alcoves in the galleries of length $\ell(w)$ which start at that alcove, where the wall-crossings of the gallery are of type $(s_{j_1}, s_{j_2}, \ldots, s_{j_p})$, in that order). The description of $\mathbb{P}^1(B_0, s) = \mathbb{A}(B_0, s) \coprod \mathbb{A}(B_0, 1)$ is similar. Using this description, or simply by using BN-pair relations, one can easily verify the results in the table below.

Case	$\mathrm{Im}(p) \cap Y(v)$	$\mathrm{Im}(p) \cap Y(vs_r)$
1a	$\mathbb{A}({}^vB, 1)$	$\mathbb{A}({}^vB, s_r)$
1b	$\mathbb{A}({}^vB, 1)$	$\emptyset$
2a	$\mathbb{A}({}^{vs_r}B, s_r)$	$\mathbb{A}({}^{vs_r}B, 1)$
2b	$\emptyset$	$\mathbb{A}({}^{vs_r}B, 1)$

In each case it is clear which piece should be labelled $\mathbb{A}_1$ or $\mathbb{A}_2$.

At this stage we have verified (a-b) in every case. It remains to check that (c) holds in the two non-trivial cases 1a and 2a. For each of those cases, we need to show that $\pi_{s'_\bullet} : \pi_{s'_\bullet}^{-1}(\mathbb{A}_2) \to \mathbb{A}_2$ is trivial. Let us consider the case 1a, where we have $\mathbb{A}_2 = \{B' \mid B' \xrightarrow{s_r} {}^vB\}$. Any such element B' can be written in the form

$$B' = {}^{vus_r}B$$

for some unique element $u \in U \cap {}^{s_r}\overline{U}$, where $\overline{U}$ is the unipotent radical opposite to U (to see this, note that conjugation by v^{-1} reduces us to the special case $v = 1$, where the statement is clear). We can then define an isomorphism

$$\pi_{s'_\bullet}^{-1}(\mathbb{A}_2) \xrightarrow{\sim} \pi_{s'_\bullet}^{-1}({}^{vs_r}B) \times \mathbb{A}_2$$

by sending $(B_1, \cdots, B_{r-2}, {}^{vus_r}B)$ to $({}^{vu^{-1}v^{-1}}B_1, \ldots, {}^{vu^{-1}v^{-1}}B_{r-2}, {}^{vs_r}B) \times {}^{vus_r}B$. In order to check that the first factor belongs to $\pi_{s'_\bullet}^{-1}({}^{vs_r}B)$, we need to check that

[2]For affine flag varieties, "Borel" would be replaced with "Iwahori", and we would think instead about the Bruhat-Tits building for G.

$vuv^{-1} \in U$. But if α_r denotes the simple positive root corresponding to s_r, with associated root homomorphism $x_{\alpha_r} : \mathbb{G}_a \to U$, then we may write $u = x_{\alpha_r}(\lambda)$ for some $\lambda \in \overline{\mathbb{F}}_q$. Then vuv^{-1} belongs to the image of the homomorphism $x_{v\alpha_r}$, which still takes values in U because our assumption $v < vs_r$ implies that $v\alpha_r$ is a positive root.

Finally, we note that the case 2a is handled in the same way, but with the roles of v and vs_r interchanged. (Indeed, in that case we have $\mathbb{A}_2 = \{B' \mid B' \xrightarrow{s_r} {}^{vs_r}B\}$, and any such B' can be written as ${}^{vs_r u s_r}B$ for a unique $u \in U \cap {}^{s_r}\overline{U}$; now proceed as before.) We have now verified (c) in every case, and this completes the proof of the proposition. □

Remark 3.0.3. *The above proof is parallel to the proof of Theorem 3.1 in* [Ha]*, which concerns convolution morphisms related to the affine Grassmannian for G.*

4. Proof of Theorem 2.0.1

In this section we abbreviate $\pi_{s_\bullet}$ by π. Since $\widetilde{X}(w)$ is non-singular, the constant sheaf $\overline{\mathbb{Q}}_\ell$ is perverse (up to a cohomological shift). The decomposition theorem of BBD states that $R\pi_*(\overline{\mathbb{Q}}_\ell)$ is a direct sum in the category $D^b_c(X(w), \overline{\mathbb{Q}}_\ell)$ of shifts of irreducible perverse sheaves on $X(w)$:

$$R\pi_*(\overline{\mathbb{Q}}_\ell) = \bigoplus_{\substack{\mathcal{L}_Z \\ i \in \mathbb{Z}}} IC(Z, \mathcal{L}_Z)[i]^{\oplus m(i, \mathcal{L}_Z)} \tag{4.0.1}$$

where i ranges over a finite set of integers, Z ranges over closed irreducible subvarieties $Z \subset X(w)$, and $\mathcal{L}_Z$ ranges over a finite set of irreducible locally constant $\overline{\mathbb{Q}}_\ell$-sheaves on nonsingular open dense subsets of the various subvarieties Z. Such a pair $(Z, \mathcal{L}_Z)$ gives rise to an intersection complex $IC(Z, \mathcal{L}_Z)$ supported on Z, and all the irreducible perverse sheaves on $X(w)$ are of this form (up to a shift). The multiplicity $m(i, \mathcal{L}_Z)$ is a non-negative integer.

Since π is an isomorphism over the dense open subset $Y(w) \subset X(w)$, by restricting everything to $Y(w)$ it is easy to see that the intersection complex IC_w must appear in the above decomposition. Thus for each y, we see that $\mathcal{H}^i(X(w))_y$ is a direct summand of the vector space

$$R\pi_*(\overline{\mathbb{Q}}_\ell)_y = H^i_c(\pi^{-1}(y), \overline{\mathbb{Q}}_\ell).$$

Since $\pi^{-1}(y)$ is paved by affine spaces $\mathbb{A}^t$, and

$$H^i_c(\mathbb{A}^t, \overline{\mathbb{Q}}_\ell) = \begin{cases} 0, & \text{if } i \neq 2t \\ \overline{\mathbb{Q}}_\ell(-t), & \text{if } i = 2t, \end{cases}$$

we immediately get the vanishing of $\mathcal{H}^i(X(w))_y$ for odd i. (Recall that $\overline{\mathbb{Q}}_\ell(d)$ denotes the Tate-twist of the constant sheaf: Fr_q acts on $\overline{\mathbb{Q}}_\ell(d)$ by the scalar $q^{-d} \in \overline{\mathbb{Q}}_\ell$.)

The same argument would give us the eigenvalues of Fr_q on $\mathcal{H}^i(X(w))_y$, provided that the decomposition in (4.0.1) were compatible with the Galois structures. However, it is important to note that the decomposition of $R\pi_*(\overline{\mathbb{Q}}_\ell)$ holds in the category $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ *where we have forgotten the Galois structures*. Therefore we need to argue in a more abstract fashion, taking care to keep track of Galois structures, as follows.

Let $D_c^{b,\mathrm{Weil}}(X,\overline{\mathbb{Q}}_\ell)$ denote the category consisting of objects $\mathcal{F}$ in $D_c^b(X,\overline{\mathbb{Q}}_\ell)$ equipped with a **Weil structure**, that is, an isomorphism $\mathrm{Fr}_q^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$. The cohomology stalks of objects of $D_c^{b,\mathrm{Weil}}(X,\overline{\mathbb{Q}}_\ell)$ are equipped with the Frobenius endomorphism Fr_q.

Let $P(X,\overline{\mathbb{Q}}_\ell)$ denote the subcategory of $D_c^b(X,\overline{\mathbb{Q}}_\ell)$ consisting of (middle) perverse sheaves, and let $P^{\mathrm{Weil}}(X,\overline{\mathbb{Q}}_\ell)$ denotes the subcategory of $D_c^{b,\mathrm{Weil}}(X,\overline{\mathbb{Q}}_\ell)$ consisting of perverse sheaves equipped with a Weil structure. The category of perverse sheaves in $P(X,\overline{\mathbb{Q}}_\ell)$ which are "defined over $\mathbb{F}_q$" is a full subcategory of $P^{\mathrm{Weil}}(X,\overline{\mathbb{Q}}_\ell)$, whose essential image is stable under extensions and subquotients ([BBD], Prop. 5.1.2.).

Let ${}^p\pi_* := {}^p\mathrm{H}^0 R\pi_*$ denote the perverse version of the proper push-forward derived functor $R\pi_*$. Suppose temporarily that all the eigenvalues of Fr_q on $\mathcal{H}^i\,{}^p\pi_*(\overline{\mathbb{Q}}_\ell)_y$ are equal to $q^{i/2}$. We claim that the same then holds for $\mathcal{H}^i(X(w))_y$. It is enough to show that the perverse sheaf $IC := IC(X(w))[\ell(w)]$ is a subquotient of ${}^p\pi_*(\overline{\mathbb{Q}}_\ell)[\ell(w)]$, in the category $P^{\mathrm{Weil}}(X(w),\overline{\mathbb{Q}}_\ell)$. However, this follows from a more general result proved in Lemma 10.7 of [GH].

It remains to show that all the eigenvalues of Fr_q on the cohomology stalks $\mathcal{H}^i\,{}^p\pi_*(\overline{\mathbb{Q}}_\ell)_y$ are $q^{i/2}$, assuming that $R\pi_*(\overline{\mathbb{Q}}_\ell)$ has this property (as we have already proved). At this point, we have to bring in the results of [BBD] underlying the decomposition theorem. Namely, we note that since π is proper, the complex $R\pi_*(\overline{\mathbb{Q}}_\ell)$ is a **pure complex** in the sense of [BBD], §5.4. It follows from the proof of [BBD], Thm. 5.4.5, that the distinguished triangle

$$ {}^p\mathrm{H}^j R\pi_*(\overline{\mathbb{Q}}_\ell) \longrightarrow {}^p\tau^{\geq j} R\pi_*(\overline{\mathbb{Q}}_\ell) \longrightarrow {}^p\tau^{\geq j+1} R\pi_*(\overline{\mathbb{Q}}_\ell) $$

in $D_c^{b,\mathrm{Weil}}(X,\overline{\mathbb{Q}}_\ell)$ becomes a direct sum in $D_c^b(X,\overline{\mathbb{Q}}_\ell)$, i.e., after we forget the Galois structures. On taking cohomology stalks, we get short exact sequences of $\overline{\mathbb{Q}}_\ell$-spaces equipped with Weil structures

$$ 0 \longrightarrow \mathcal{H}^i({}^p\mathrm{H}^j\mathcal{K})_y \longrightarrow \mathcal{H}^i({}^p\tau^{\geq j}\mathcal{K})_y \longrightarrow \mathcal{H}^i({}^p\tau^{\geq j+1}\mathcal{K})_y \longrightarrow 0, $$

where for brevity we have written $\mathcal{K}$ in place of $R\pi_*(\overline{\mathbb{Q}}_\ell)$.

Note that for $j << 0$ the middle term is simply $\mathcal{H}^i R\pi_*(\overline{\mathbb{Q}}_\ell)_y$. Now an easy argument by ascending induction on j shows that for all j, the eigenvalues of Fr_q on $\mathcal{H}^i({}^p\mathrm{H}^j R\pi_*(\overline{\mathbb{Q}}_\ell))_y$ are equal to $q^{i/2}$. This applies in particular to ${}^p\pi_*(\overline{\mathbb{Q}}_\ell)_y = {}^p\mathrm{H}^0 R\pi_*(\overline{\mathbb{Q}}_\ell)_y$. This completes the proof of Theorem 2.0.1. □

5. Application to Kazhdan-Lusztig polynomials

The main application of Theorem 2.0.1 is the geometric description of the Kazhdan-Lusztig polynomials. Here we will give only the outline of the proof; there are certain elementary statements which we use without explicit mention, and details for those can be found in [KL2]. We will now switch to the notation of loc. cit. (for the most part):

$$\mathcal{B}_w := BwB/B$$

$$\mathcal{B}_{\overline{w}} := \overline{BwB/B}$$

$$\mathcal{B}^w := \overline{B}wB/B \;\cong\; \mathcal{B}_{w_0w}$$

$$\mathcal{A}^w := (\text{the open affine space of Borels opposite to } {}^w\overline{B}) \;\cong\; \mathcal{B}_w \times \mathcal{B}^w.$$

Theorem 5.0.4. *Let $y \leq w$ in W. Then*

$$P_{y,w}(q) = \sum_i \dim\, \mathcal{H}^{2i}_{{}^yB}(\mathcal{B}_{\overline{w}})\; q^i = \sum_i \dim\, IH^{2i}(\mathcal{B}_{\overline{w}} \cap \mathcal{B}^y, \overline{\mathbb{Q}}_\ell)\; q^i.$$

Proof. First of all, for any $i \in \mathbb{Z}$ the cohomology stalk $\mathcal{H}^i_{{}^yB}(\mathcal{B}_{\overline{w}})$ is isomorphic to the global intersection cohomology group $IH^i(\mathcal{B}_{\overline{w}} \cap \mathcal{B}^y, \overline{\mathbb{Q}}_\ell)$; this is based on the B-equivariance of $IC(\mathcal{B}_{\overline{w}})$ and the existence of a contracting $\mathbb{G}_m$-action on $\mathcal{B}_{\overline{w}} \cap \mathcal{A}^y \cong \mathcal{B}_y \times (\mathcal{B}_{\overline{w}} \cap \mathcal{B}^y)$ which contracts $\mathcal{B}_{\overline{w}} \cap \mathcal{B}^y$ onto the point ${}^yB$ (see [KL2], §$1.4-1.5$, and Lemma 4.5). Therefore, we just need to check the first equality.

Recall that the $P_{y,w}$ are characterized as the unique family of polynomials in $\mathbb{Z}[q]$ satisfying

(i) $P_{w,w} = 1$;
(ii) $\deg P_{y,w} \leq (\ell(w) - \ell(y) - 1)/2$ if $y < w$;
(iii) $q_w q_y^{-1} P_{y,w}(q^{-1}) = \sum_{y \leq z \leq w} R_{y,z} P_{z,w}$.

By Theorem 2.0.1, it is clear that

$$\mathrm{Tr}(\mathrm{Fr}_q, IC(\mathcal{B}_{\overline{w}})_{{}^yB}) = \sum_i \dim\, \mathcal{H}^{2i}_{{}^yB}(\mathcal{B}_{\overline{w}})\; q^i,$$

and that it satisfies property (i). Property (ii) also follows easily (in light of Theorem 2.0.1, the degree bound is a restatement of the sharp constraints on the support of $\mathcal{H}^i IC(\mathcal{B}_{\overline{w}})$, in the very definition we gave of the intersection complex $IC_w = IC(\mathcal{B}_{\overline{w}})$). To check (iii) we use Poincare duality for intersection cohomology (a formal consequence of the fact that $IC(\mathcal{B}_{\overline{w}})$ is Verdier self-dual, up to a shift and Tate-twist). Namely, the Lefschetz trace formula implies that

$$\begin{aligned}\mathrm{Tr}(\mathrm{Fr}_q, IH^\bullet_c(\mathcal{B}_{\overline{w}} \cap \mathcal{A}^y)) &= \sum_{y \leq z \leq w} \sum_{z' \in (\mathcal{B}_z \cap \mathcal{A}^y)(\mathbb{F}_q)} \mathrm{Tr}(\mathrm{Fr}_q, \mathcal{H}^\bullet_{z'}(\mathcal{B}_{\overline{w}})) \\ &= \sum_{y \leq z \leq w} q^{\dim \mathcal{B}_y} R_{y,z}(q) \mathrm{Tr}(\mathrm{Fr}_q, \mathcal{H}^\bullet_z(\mathcal{B}_{\overline{w}})).\end{aligned}$$

By Poincaré duality we have

$$\mathrm{Tr}(\mathrm{Fr}_q, IH^\bullet_c(\mathcal{B}_{\overline{w}} \cap \mathcal{A}^y)) = q^{\dim(\mathcal{B}_{\overline{w}} \cap \mathcal{A}^y)} \mathrm{Tr}(\mathrm{Fr}_q^{-1}, IH^\bullet(\mathcal{B}_{\overline{w}} \cap \mathcal{A}^y)),$$

and [KL2], Lemma 4.5 gives us

$$\mathrm{Tr}(\mathrm{Fr}_q^{-1}, IH^\bullet(\mathcal{B}_{\overline{w}} \cap \mathcal{A}^y)) = \mathrm{Tr}(\mathrm{Fr}_q^{-1}, \mathcal{H}^\bullet_y(\mathcal{B}_{\overline{w}})).$$

Altogether, we now get

$$q^{\ell(w)-\ell(y)} \mathrm{Tr}(\mathrm{Fr}_q^{-1}, \mathcal{H}^\bullet_y(\mathcal{B}_{\overline{w}})) = \sum_{y \leq z \leq w} R_{y,z} \mathrm{Tr}(\mathrm{Fr}_q, \mathcal{H}^\bullet_z(\mathcal{B}_{\overline{w}})).$$

This shows that the family of polynomials $\mathrm{Tr}(\mathrm{Fr}_q, IC(\mathcal{B}_{\overline{w}})_{{}^yB})$ satisfies property (iii), and thereby completes the proof of the theorem. □

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