Particle's drift in self-similar billiards

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In memory of Bill Parry.

Abstract

We study a particle moving at unit speed in a channel made by connected self-similar billiard tables that grow in size by a factor $r > 1$ from left to right (this model was recently introduced in physics literature [1, 2]). Let $q(T)$ denote the position of the particle at time T. Our main result is the existence of an asymptotic distribution of $q(T)/T$ as $T \to \infty$ and $\{\ln T/\ln r\} \to \rho$ for some $0 \le \rho < 1$.

1 Introduction.

A billiard is a mechanical model in which a point particle moves in a container $\mathcal{D}$ and bounces off its boundary $\partial\mathcal{D}$. This is a Hamiltonian system preserving a smooth Liouville measure. The corresponding return map constructed on $\partial\mathcal{D}$ (also called the collision map) preserves a smooth measure, too.

If the billiard table $\mathcal{D}$ is unbounded and spatially isotropic, as is a periodic Lorentz gas, then billiard dynamics represents a mechanical system *in equilibrium*. The billiard particle in a planar periodic Lorentz gas with finite horizon exhibits a diffusive behavior without drift [6, 13]. If the horizon is infinite, the diffusion becomes abnormal [5, 18], but the drift is still absent.

In order to induce a non-equilibrium steady state with some transport of mass (manifested by the particle's drift), one can apply a constant external force on the particle [11, 12, 16]. Then the drift may be observed and the invariant measure (steady state) may become singular. Though one has to prevent an indefinite acceleration (heat-up) of the particle by introducing

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a thermostat. For example, Gaussian thermostat [11, 12, 16] keeps the kinetic energy of the particle constant; the corresponding equations of motion (between collisions) read

$$\dot{\mathbf{q}} = \mathbf{v} \qquad \dot{\mathbf{v}} = \mathbf{e} - \langle \mathbf{e}, \mathbf{v} \rangle \langle \mathbf{v}, \mathbf{v} \rangle^{-1} \mathbf{v}, \tag{1.1}$$

where $\mathbf{q}$ is the position and $\mathbf{v}$ the velocity of the particle, $\mathbf{e}$ is the (constant) external field, and $\langle \cdot, \cdot \rangle$ denotes the scalar product of vectors in $\mathbb{R}^2$. It is easy to see that $\langle \mathbf{v}, \dot{\mathbf{v}} \rangle = 0$, thus $\|\mathbf{v}\|^2 = \text{const}$.

Planar periodic Lorentz gases with finite horizon where the particle moves in a small external field $\mathbf{e}$ according to (1.1) were studied in [11, 12]. It was shown that the system had a unique (singular) invariant measure, $\mu_{\mathbf{e}}$, with smooth conditional densities on unstable manifolds (i.e., SRB measure), the average speed of the particle was $\mu_{\mathbf{e}}(\mathbf{v}) = \mathbf{D}\mathbf{e} + o(\mathbf{e})$, where $\mathbf{D}$ was the diffusion matrix corresponding to the unperturbed system (with $\mathbf{e} = 0$).

The Gaussian thermostatted dynamics (1.1) can be described by Hamiltonian formalism, as was first noticed in [14]. A general theorem by Wojtkowski [19, 20] states that the billiard table can be transformed by a conformal mapping to the so called torsion free connection (called the Weyl connection) so that the trajectories of (1.1) are mapped onto geodesic lines (trajectories of the Weyl flow), and the specular character of reflections at the boundary is preserved. The unit cell of the periodic Lorentz gas is then transformed into a distorted (asymmetric) domain, see below.

Now consider a unit cell of a periodic Lorentz gas (with finite horizon) and impose periodic boundary conditions in the y direction but not in the x direction. Then one gets the so-called Lorentz channel [15], a 1D chain of identical connected cells. Suppose the particle moves in the Lorentz channel according to (1.1) under a small horizontal external field $\mathbf{e} = (e, 0)$, $e > 0$. Then Wojtkowski's transformation maps the Lorentz channel onto a chain of connected self-similar dispersing billiard tables that grow in size by a factor $r = \exp(e) > 1$, see [3], from left to right.

Such a channel of self-similar billiard tables was recently independently introduced by Barra, Gilbert, and Romo [1], who studied the resulting dynamics heuristically and numerically. They made several interesting conjectures on the existence of a singular SRB measure, on the asymptotic drift of the particle, and on the relation between Lyapunov exponents and the entropy production rates. Some of their conjectures actually follow from the results of [11, 12] if one makes use of Wojtkowski's theorem [19, 20], see the latest papers [2, 3].

Here we obtain rigorous results related to some other conjectures made in [1, 2], specifically those concerned with the asymptotic drift of the particle in the Barra-Gilbert-Romo (BGR) channel.

2 Statement of the result

To define a BGR channel of self-similar billiard tables we first describe its fundamental cell $\mathcal{D}_0$, see Fig. 1.

We fix an $r > 1$ (the scaling factor, see below). The cell $\mathcal{D}_0$ is made of a trapezoid with unequal vertical sides of length $d\sqrt{3/r}$ and $d\sqrt{3r}$, respectively, and equal top and bottom sides, here d is the (horizontal) distance between the vertical sides. Our cell $\mathcal{D}_0$ is the trapezoid minus five disks: one of radius R centered on the intersection of the diagonals, two disks of radius $R\sqrt{r}$ centered on the right hand side vertices, and two disks of radius $R/\sqrt{r}$ centered on the left hand side vertices.

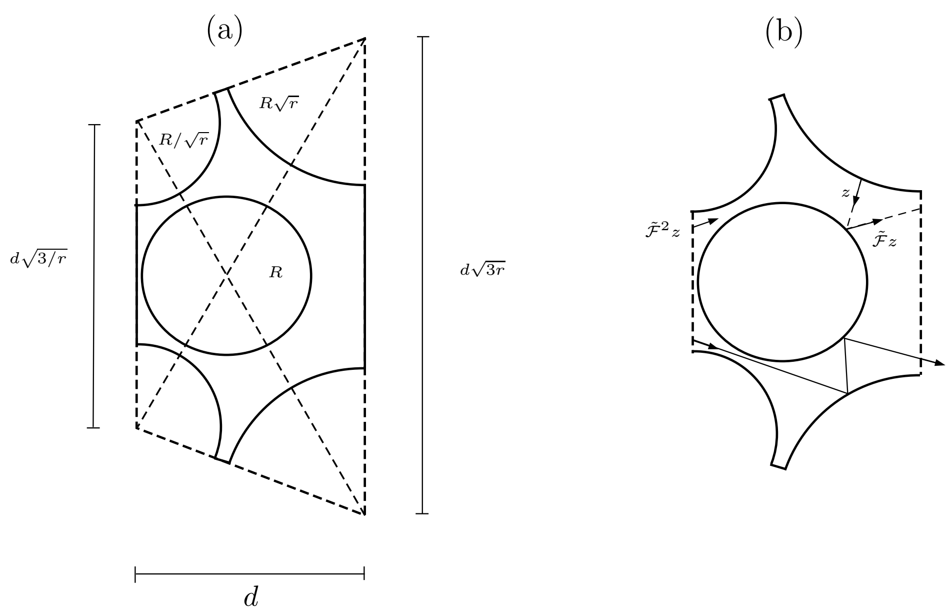

Figure 1: The cell $\mathcal{D}_0$ and the action of $\tilde{\mathcal{F}}$ on $\tilde{\Omega}$.

Thus our cell is a billiard table bounded by one full circle, four circular arcs, and four short line segments connecting the endpoints of the arcs. We assume that R is large enough to ensure the finite horizon condition (meaning that every billiard trajectory collides with the circular part of the boundary), but not too large to prevent the disks from overlapping. This imposes some restrictions on d and R, see details in [1, Appendix A].

The ratio of the vertical sides of our cell is $r > 1$. Now we attach to $\mathcal{D}_0$ a bigger cell, $\mathcal{D}_1$, identical in shape to $\mathcal{D}_0$ but scaled by r; we glue the right side of $\mathcal{D}_0$ with the (equal in size) left side of $\mathcal{D}_1$. Similarly, we attach to $\mathcal{D}_0$ a smaller cell, $\mathcal{D}_{-1}$, scaled by r^{-1}, gluing the left side of $\mathcal{D}_0$ with the right side of $\mathcal{D}_1$. Repeating this procedure gives a chain of self-similar cells $\mathcal{D}_i$, $i \in \mathbb{Z}$, and we call $\mathcal{D} = \cup_{i\in\mathbb{Z}}\mathcal{D}_i$ the Barra-Gilbert-Romo (BGR) channel (Fig. 2).

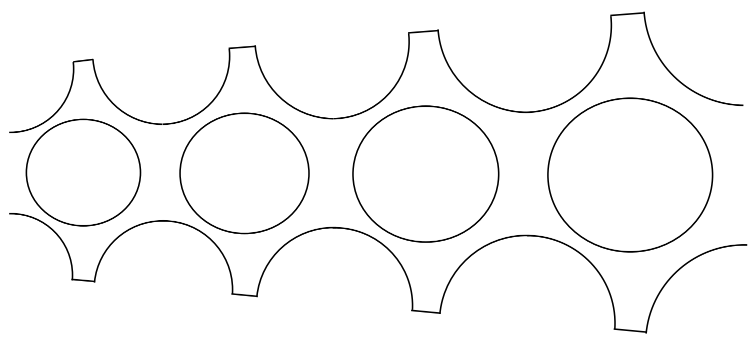

Figure 2: The Barra-Gilbert-Romo (BGR) channel.

Observe that the size of $\mathcal{D}_i$ is proportional to r^i, so our cells grow exponentially from left to right, and the negative 'half' of the chain $\cup_{i\leq 0}\mathcal{D}_i$ is actually bounded. Also note that each pair of adjacent circular arcs in the neighboring cells $\mathcal{D}_i$ and $\mathcal{D}_{i+1}$ have the same center and radius, thus their union is a (bigger) circular arc. Furthermore, every circular arc is perpendicular to the adjacent (top or bottom) side of the cell. This all implies that our billiard table essentially has no corner points (they can be eliminated by a standard unfolding scheme [13, Section 1.2]), and our dynamics equivalent to a dispersing billiard with smooth boundary.

We consider a particle moving in $\mathcal{D}$ at unit speed and bouncing off $\partial\mathcal{D}$. Note that the common vertical edge of every pair of neighboring cells $\mathcal{D}_i$ and $\mathcal{D}_{i+1}$ is not a part of $\partial\mathcal{D}$, thus the particle is free to move from cell to cell all across the channel $\mathcal{D}$. We suppose that the initial position $q(0)$ of the particle is uniformly distributed within $\mathcal{D}_0$ and its initial velocity $v(0)$ is uniformly distributed on the unit circle. Let $(q(t), v(t))$ denote the state of the particle at time t.

Theorem 1. *There is $\varepsilon_0 > 0$ such that for $1 < r < 1 + \varepsilon_0$ the following holds. Suppose that $T_n \to \infty$ so that the fractional part $\{\frac{\ln T_n}{\ln r}\} \to \rho \in [0, 1)$. Then the distribution of $\frac{q(T_n)}{T_n}$ converges to a limit (which may depend on ρ).*

Our theorem states that the limit distribution of $\frac{q(T_n)}{T_n}$, as $n \to \infty$, exists but it does not specify how (and *if*) it depends on ρ. In particular, it may be constant, i.e. $\frac{q(T)}{T}$ may simply converge to a limit as $T \to \infty$.

However, our explicit formulas in Section 5 suggest that the limit of $\frac{q(T_n)}{T_n}$ has a non-trivial dependence on ρ. In addition, recent computer simulations [4] reveal that the ratio $\frac{q(T)}{T}$ does not converge to a limit but evolves periodically, in accordance with our theorem (more precisely, $\frac{q(T)}{T}$ changes periodically with respect to the variable $\ln T$; and its period is $\ln r$).

3 Collision map

The self-similar structure of the BGR channel allows us to reduce the dynamics of the particle in $\mathcal{D}$ to the motion of a (model) particle in the fundamental cell $\mathcal{D}_0$. Precisely, if the real particle (q, v) moves in $\mathcal{D}_i$, our model particle $(\tilde{q}, \tilde{v})$ moves in $\mathcal{D}_0$ so that

$$\tilde{q} = q_\infty + (q - q_\infty)/r^i, \qquad \tilde{v} = v/r^i, \tag{3.1}$$

where $q_\infty = (-d/(r-1), 0)$ is the accumulation point of $\mathcal{D}_i$ as $i \to -\infty$. We denote by π the projection (3.1) of the phase space $\mathcal{M} = \mathcal{D} \times S^1$ of the real particle on the phase space $\tilde{\mathcal{M}} = \mathcal{D}_0 \times \mathbb{R}^2$ of the model particle. Then we have $\tilde{\Phi}^t \circ \pi = \pi \circ \Phi^t$, where $\Phi^t \colon \mathcal{M} \to \mathcal{M}$ and $\tilde{\Phi}^t \colon \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}$ denote the corresponding phase flows.

The motion of the model particle in $\mathcal{D}_0$ is governed by the following rules. Let Γ_{L} and Γ_{R} denote the left and right (vertical) sides of $\mathcal{D}_0$, respectively. When the particle hits Γ_{R} at a point (d, y) with velocity v, it instantly reappears on Γ_{L} at the point $(0, y/r)$ with velocity v/r. When it hits Γ_{L} at a point $(0, y)$ with velocity v, it reappears on Γ_{R} at the point (d, yr) with velocity vr. These are 'periodic' boundary conditions with rescaling of the y coordinate and the velocity.

Let Ω denote the cross-section of the phase space $\mathcal{M}$ consisting of pair $z = (q, v)$ where q lies either on the boundary $\partial\mathcal{D}$ or on a common vertical side of some neighboring cells $\mathcal{D}_i$ and $\mathcal{D}_{i+1}$ and (for $q \in \partial\mathcal{D}$) v is the 'outgoing' (postcollisional) velocity vector. We call Ω the (extended) collision space and denote by $\mathcal{F}\colon \Omega \to \Omega$ the corresponding return map, or the (extended) collision map.

Then $\tilde{\Omega} = \pi(\Omega)$ is a cross-section (the collision space) for the flow $\tilde{\Phi}$; it consists of points $z = (q, v)$ where $q \in \partial\mathcal{D}_0$ and v points inside $\mathcal{D}_0$. We denote by $\tilde{\mathcal{F}}\colon \tilde{\Omega} \to \tilde{\Omega}$ the corresponding return map; note that $\tilde{\mathcal{F}}^n \circ \pi = \pi \circ \mathcal{F}^n$. The action of $\tilde{\mathcal{F}}$ is illustrated in Fig. 1 (b). It is clearly independent of the speed $\|v\|$ of the model particle, so we may for simplicity normalize all the velocity vectors in the space $\tilde{\Omega}$.

When the original particle moving in the channel $\mathcal{D}$ crosses from one cell $\mathcal{D}_i$ into the neighboring cell $\mathcal{D}_{i+1}$ or $\mathcal{D}_{i-1}$, our model particle appears on Γ_{L} or Γ_{R}, respectively. Accordingly, we define a function Δ on $\tilde{\Omega}$ such that

$$\Delta(q, v) = \begin{cases} +1 & \text{if } q \in \Gamma_{\mathrm{L}} \\ -1 & \text{if } q \in \Gamma_{\mathrm{R}} \\ 0 & \text{elsewhere} \end{cases}$$

Let

$$I_n = \sum_{i=1}^{n} \Delta \circ \tilde{\mathcal{F}}^i.$$

Observe that the original particle, after n reflections (n iterations of $\mathcal{F}$), will be exactly in the cell $\mathcal{D}_{I_n}$.

As we said, Wojtkowski's theorem [19, 20] allows us to transform the trajectories of the flow Φ^t into those of the Gaussian thermostatted particle in a periodic Lorentz channel with finite horizon under a small external field $\mathbf{e} = (e, 0)$ (whose value is determined by r, precisely $e = \ln r$, see [3]). Even though Wojtkowski's transformation does not necessarily preserve convexity, the images of the curved boundaries of $\mathcal{D}_0$ will remain convex when ε_0 in Theorem 1 is small enough. While this transformation does not synchronize time between collisions, it certainly establishes a conjugacy between the corresponding collision maps. Thus the map $\tilde{\mathcal{F}}\colon \tilde{\Omega} \to \tilde{\Omega}$ has all the same properties as the collision map of the thermostatted particle studied in [11, 12, 8].

In particular, the map $\tilde{\mathcal{F}}$ has a unique SRB measure, μ (invariant probability measure whose conditional densities on unstable manifolds are smooth), which is ergodic, mixing, Bernoulli, and positive on open sets. This measure enjoys exponential decay of correlations, satisfies the central limit theorem, and has other strong statistical properties [11, 12, 8]. For $r = 1$, we recover the billiard map $\tilde{\mathcal{F}}_1$ on a (symmetric) fundamental cell of the periodic Lorentz gas that preserves a smooth measure μ_1.

If $\gamma \subset \tilde{\Omega}$ is a sufficiently smooth unstable curve and ρ a sufficiently smooth probability density on it, we call $\ell = (\gamma, \rho)$ a standard pair, see precise definitions in [9, Section 4] or [13, Chapter 7] (as usual, we only consider homogeneous stable and unstable curves, on which we can control distortions, see [8, page 216] or [13, Chapter 5]). We denote by $\mathbb{P}_\ell$ the measure on γ with density ρ. For any function $A\colon \tilde{\Omega} \to \mathbb{R}$ we put $\mathbb{E}_\ell(A) = \int_\gamma A\, d\mathbb{P}_\ell$. We say that $\ell = (\gamma, \rho)$ is proper if $\operatorname{length}(\gamma) > \delta_0$, where $\delta_0 > 0$ is a small but fixed constant.

A standard family [13, Chapter 7] is a (countable or uncountable) collection $\mathcal{G} = \{\ell_\alpha\} = \{(\gamma_\alpha, \rho_\alpha)\}$, $\alpha \in \mathfrak{A}$, of standard pairs with a probability factor measure $\lambda_{\mathcal{G}}$ on the index set $\mathfrak{A}$. Such a family induces a probability measure $\mathbb{P}_{\mathcal{G}}$ on the union $\cup_\alpha \gamma_\alpha$ (and thus on $\tilde{\Omega}$), and we write $\mathbb{E}_{\mathcal{G}}(A) = \int_{\tilde{\Omega}} A\, d\mathbb{P}_{\mathcal{G}}$. To control the size of curves γ_α in a standard family $\mathcal{G}$, we use

$$\mathcal{Z}_{\mathcal{G}} \colon = \sup_{\varepsilon > 0} \frac{\mathbb{P}_{\mathcal{G}}(L_{\mathcal{G}} < \varepsilon)}{\varepsilon} = \sup_{\varepsilon > 0} \frac{\int \mathbb{P}_{\ell_\alpha}\big(x \in \gamma_\alpha \colon L_{\mathcal{G}}(x) < \varepsilon\big)\, d\lambda_{\mathcal{G}}(\alpha)}{\varepsilon},$$

where $L_{\mathcal{G}}(x)$ denotes the distance from $x \in \gamma_\alpha$ to the closer endpoint of the curve γ_α. A standard family $\mathcal{G}$ is proper if $\mathcal{Z}_{\mathcal{G}} \leq C_0$, where C_0 is a large constant (so that any proper standard pair makes a proper standard family). If a family $\mathcal{G}$ is not proper, but $\mathcal{Z}_{\mathcal{G}} < \infty$ then its image $\tilde{\mathcal{F}}^n \mathcal{G}$ will be proper for $n \geq C_1 \ln \mathcal{Z}_{\mathcal{G}}$, where $C_1 > 0$ is a large constant. In particular, if a standard pair $\ell = (\gamma, \rho)$ is not proper, then its image $\tilde{\mathcal{F}}^n \ell$ will be a proper standard family for $n \geq C_1 \big|\ln |\gamma|\big|$.

For any proper standard family $\mathcal{G}$ the iterations of the measure $\mathbb{P}_{\mathcal{G}}$ under $\tilde{\mathcal{F}}$ weakly converge to μ, so that

$$\mathbb{E}_{\mathcal{G}}(A \circ \tilde{\mathcal{F}}^n) \to \mu(A),$$

and the convergence is exponentially fast for Hölder continuous functions. (For billiards, this fact was proved in [13, Section 7.5], and in our case the same argument applies, cf. [8].) Moreover, it is proved in [11] that the thermostatted Lorentz particle has a non-zero drift, thus

$$\bar{\Delta} \colon = \mu(\Delta) > 0.$$

More precisely,

$$\bar{\Delta} = De + o(e) = D(r-1) + o(r-1)$$

where $D = \frac{1}{2}\sum_{i=-\infty}^{\infty} \mu_1\big((\Delta \circ \mathcal{F}_1^i)\Delta\big)$ is half the sum of autocorrelations of the function Δ in the unperturbed (classical billiard) system.

The functions I_n have the following standard statistical properties:

Proposition 1 (Central Limit Theorem, see [8]). *For any proper standard family $\mathcal{G}$ the sequence $n^{-1/2}(I_n - n\bar{\Delta})$ converges in distribution to a normal random variable with respect to the measure $\mathbb{P}_{\mathcal{G}}$.*

Proposition 2 (Large Deviations). *For any constant $0 < a < \bar{\Delta}$ and for any proper standard family $\mathcal{G}$ we have $\mathbb{P}_{\mathcal{G}}(I_n \le an) \le c_1\theta_1^n$ for some constants $c_1 > 0$ and $\theta_1 \in (0,1)$, which depend on a.*

In what follows we have many exponential bounds similar to the one above, and we will denote by $c_i > 0$ and $\theta_i \in (0,1)$ various constants whose values are not important.

Proposition 3 (Moderate Deviations). *For any constant $\frac{1}{2} < b < \frac{2}{3}$ and for any proper standard family $\mathcal{G}$ we have $\mathbb{P}_{\mathcal{G}}(|I_n - n\bar{\Delta}| > n^b) \le c_2\theta_2^{n^{2b-1}}$.*

For the proofs of the last two propositions, see [9, Sections A.3–A.4].

These properties imply that $I_n = n\bar{\Delta} + \mathcal{O}(\sqrt{n})$ grows linearly in n. On the other hand, let $L(q,v)$ denote the free path length, i.e. the distance (in $\mathcal{D}_0$) from $q \in \tilde{\Omega}$ to the next collision (in the 'extended' sense as defined above) at the point $q' \in \partial\mathcal{D}_0$, where $(q',v') = \tilde{\mathcal{F}}(q,v)$. Then the time elapsed between the 0th and the nth collision of the original particle at $\partial\mathcal{D}$ will be

$$S_n = \sum_{k=0}^{n-1} r^{I_k} L \circ \tilde{\mathcal{F}}^k. \tag{3.2}$$

Thus we should expect that $S_n \sim r^{I_n}$, and the x-coordinate of the particle at the nth collision is also $q(S_n) \sim r^{I_n}$, which indicates that $q(S_n)$ should be asymptotically proportional to S_n. However, the terms in (3.2) grow exponentially, so the major contribution comes from the few last terms, which makes the limit distribution of S_n strongly dependent on that of the few last terms. (This makes it necessary to impose restrictions on $\ln T$ in Theorem 1.)

Lastly we recall the Growth Lemma (see [9, Section 4.4] or [13, Chapter 5] or [8]) for the map $\tilde{\mathcal{F}}$. Let $\mathcal{G} = \{\ell_\alpha\} = \{(\gamma_\alpha, \rho_\alpha)\}$, $\alpha \in \mathfrak{A}$, be a standard family. For $n \geq 1$ and $x \in \gamma_\alpha$ denote by $\mathcal{L}_n(x)$ the distance from $\tilde{\mathcal{F}}^n(x)$ to the closer endpoint of the corresponding component of $\tilde{\mathcal{F}}^n(\gamma_\alpha)$.

Proposition 4 ("Growth Lemma"). *There exists a constant $C_1 > 0$ such that for any proper standard family $\mathcal{G}$ and $n \geq 1$ we have $\mathbb{P}_{\mathcal{G}}(\mathcal{L}_n < \varepsilon) \leq C_1\varepsilon$ for all $\varepsilon > 0$. In addition, for every $1 \leq n_1 \leq n_2$*

$$\mathbb{P}_{\mathcal{G}}\Big(\max_{n_1 \leq i \leq n_2} \mathcal{L}_i < \delta_0\Big) \leq c_3 \theta_3^{n_2 - n_1}. \tag{3.3}$$

Lastly, for any standard pair $\ell = (\gamma, \rho)$ its image under $\tilde{\mathcal{F}}^n$ is a proper standard family for all $n \geq A|\ln \operatorname{length}(\gamma)| + B$, where $A, B > 0$ are constants determined by the shape of $\mathcal{D}_0$ alone.

4 Advance map

It is convenient to 'reduce' the collision map $\tilde{\mathcal{F}}$ in a somewhat unusual way. Consider

$$\tilde{\Omega}_{\mathrm{L}} = \{(q, v) \in \tilde{\Omega} \colon q \in \Gamma_{\mathrm{L}}\}$$

the part of the collision space restricted to the vertical left side of $\mathcal{D}_0$ (recall that the velocity vectors v always point into $\mathcal{D}_0$). Then the map $\tilde{\mathcal{F}}$ induced the first return (Poincaré) map $\tilde{\mathcal{F}}_{\mathrm{L}} \colon \tilde{\Omega}_{\mathrm{L}} \to \tilde{\Omega}_{\mathrm{L}}$, which preserves the measure μ (restricted to $\tilde{\Omega}_{\mathrm{L}}$) and is ergodic.

Furthermore, given $z \in \tilde{\Omega}$ we denote by

$$\mathcal{N}(z) = \min\{n \geq 1 \colon I_n(z) = 1\}$$

the first collision when the original particle starting in $\mathcal{D}_0$ crosses from $\mathcal{D}_0$ to $\mathcal{D}_1$. We call the map $\mathcal{R} \colon \tilde{\Omega}_{\mathrm{L}} \to \tilde{\Omega}_{\mathrm{L}}$ defined by

$$\mathcal{R}(z) = \tilde{\mathcal{F}}^{\mathcal{N}(z)}(z)$$

the *advance map*, as its iterations correspond to the instances when the original particle advances one cell further to the right. Observe that for every $m \geq 1$

$$\mathcal{R}^m(z) = \tilde{\mathcal{F}}^{\mathcal{N}_m(z)}(z), \quad \text{where} \quad \mathcal{N}_m(z) = \min\{n \geq 1 \colon I_n(z) = m\};$$

also note that $\mathcal{N}_m(z) = \sum_{i=0}^{m-1} \mathcal{N}(\mathcal{R}^i z)$.

It follows from the statistical properties of the map $\tilde{\mathcal{F}}$ that the function $N(z)$, and thus the map $\mathcal{R}(z)$, are defined almost everywhere on $\tilde{\Omega}_{\rm L}$ (with respect to the Lebesgue measure). Moreover, due large deviations, for any proper standard family $\mathcal{G}$ in $\tilde{\Omega}$ we have an exponential tail bound

$$\mathbb{P}_{\mathcal{G}}(\mathcal{N}(z) \geq n) \leq \mathbb{P}_{\mathcal{G}}(\mathcal{N}_{an}(z) \geq n) \leq c_1 \theta_1^n. \tag{4.1}$$

Unlike $\tilde{\mathcal{F}}_{\rm L}$, the map $\mathcal{R}$ is not one-to-one. For example, a point $z \in \tilde{\Omega}_{\rm L}$ may leave $\Gamma_{\rm L}$, enter $\mathcal{D}_0$, then (before crossing $\Gamma_{\rm R}$) bounce back to $\Gamma_{\rm L}$, move into $\mathcal{D}_{-1}$, then bounce back to $\Gamma_{\rm L}$ again, cross it at some other point $z' \neq z$, move into $\mathcal{D}_0$ and then keep moving to the right and cross $\Gamma_{\rm R}$; in that case $\mathcal{R}(z) = \mathcal{R}(z')$. Thus the inverse map $\mathcal{R}^{-1}$ may be multiple-valued. For a similar reason, many points $x \in \tilde{\Omega}$ have no preimages under $\mathcal{R}$.

Still the action of $\mathcal{R}$ agrees with the hyperbolic structure in $\tilde{\Omega}_{\rm L}$ in two important ways. First, $\mathcal{N}(z)$ is constant on stable manifolds of the map $\tilde{\mathcal{F}}$, thus $\mathcal{R}$ maps stable manifolds into stable manifolds. Second, $\mathcal{R}$ maps every unstable manifold onto a finite or countable union of (whole) unstable manifolds; thus the restriction of $\mathcal{R}^{-1}$ onto any unstable manifold W^u has several branches, each of which takes W^u into another unstable manifold.

Next we consider a decreasing sequence of subsets $\tilde{\Omega}_{\rm L} \supset \Lambda_1 \supset \Lambda_2 \supset \cdots$ defined by $\Lambda_n = \mathcal{R}^n(\tilde{\Omega}_{\rm L})$ and the 'attractor' $\Lambda = \cap_n \Lambda_n$. Observe each Λ_n (as well as Λ) will be a union of (whole) unstable manifolds of the map $\tilde{\mathcal{F}}$. We denote by $(\Lambda^*, \mathcal{R}^*)$ the natural extension of $(\Lambda, \mathcal{R})$, i.e. the set of sequences $Z = \{z_i\}$, $i \leq 0$, such that $z_i = \mathcal{R}(z_{i-1})$ and $z_0 \in \Lambda$, on which the map $\mathcal{R}\colon \Lambda \to \Lambda$ induces the left shift $\mathcal{R}^*\colon \Lambda^* \to \Lambda^*$. We endow Λ^* with a metric $\rho^*(Z, Z') = \sum_{i=0}^{\infty} \lambda^i \operatorname{dist}(z_{-i}, z'_{-i})$ for some fixed $\lambda \in (0,1)$.

Similarly, for each $n \geq 1$ we denote by Λ_n^* the set of finite sequences $Z = \{z_i\}$, $-n \leq i \leq 0$, such that $z_i = \mathcal{R}(z_{i-1})$ for $i > -n$ and $z_0 \in \Lambda_n$. We endow Λ_n^* with a metric $\rho_n^*(Z, Z') = \sum_{i=0}^{n} \lambda^i \operatorname{dist}(z_{-i}, z'_{-i})$.

For each $m < n$ we have a natural projection π_m^* from Λ_n^* (and Λ^*) into Λ_m^*, which is defined by discarding all coordinates z_i, $i < -m$. Then $\{\pi_m^*(\Lambda_n^*)\}_{n=m}^{\infty}$ is a decreasing sequence of sets shrinking to $\pi_m^*(\Lambda^*)$; moreover $\pi_m^*(\Lambda_n^*)$ lies in a $\mathcal{O}(\lambda^n)$-neighborhood of $\pi_m^*(\Lambda^*)$. Note that $[\pi_m^*]^{-1} \rho_m^*$ is a pseudo-metric on Λ^* that uniformly converges to ρ^*. Every unstable manifold $W^u \subset \Lambda_m$ (or $W^u \subset \Lambda$) can be naturally lifted to finitely or countably many unstable manifolds in Λ_m^* (resp., in Λ^*).

Next we establish a (weaker) analogue of the first part of Growth Lemma (Proposition 4) for the map $\mathcal{R}$. Given a standard family $\mathcal{G}$ on $\tilde{\Omega}$, we denote by

$L_m(x)$ the distance from $\mathcal{R}^m(x)$ to the closer endpoint of the corresponding component of $\mathcal{R}^m(\gamma_\alpha)$.

Proposition 5 (Weak Growth Lemma). *(a) There exists a constant C_2 such that for all $\varepsilon > 0$ for any proper standard family $\mathcal{G}$ and $m \geq 1$ we have $\mathbb{P}_{\mathcal{G}}(L_m < \varepsilon) \leq C_2 \varepsilon^{0.9}$.*

(b) Moreover for proper standard familty $\mathcal{G} = \{\ell_\alpha\}$ for any m and any $m(\alpha)$ such that $\frac{m}{2} \leq m(\alpha) \leq \frac{3m}{2}$ we have

$$\mathbb{P}_{\mathcal{G}}(L_{m(\alpha)} < \varepsilon) \leq C_2 \varepsilon^{0.9}$$

for all $\varepsilon > 0$.

Clearly it suffices to prove (b). We consider two cases:

Case I: $m \leq \varepsilon^{-0.1}$. By (4.1) we have $\mathbb{P}_{\mathcal{G}}(\mathcal{N}_m \geq \varepsilon^{-0.1}) \leq c_1 \theta_1^{\varepsilon^{-0.1}} \ll \varepsilon^{0.9}$, and by Proposition 4

$$\mathbb{P}_{\mathcal{G}}\Big(\min_{k \leq \varepsilon^{-0.1}} \mathcal{L}_k \leq \varepsilon\Big) \leq C_1 \varepsilon \, \varepsilon^{-0.1} = C_1 \varepsilon^{0.9},$$

thus $\mathbb{P}_{\mathcal{G}}(L_m < \varepsilon) \leq (C_1 + 1)\, \varepsilon^{0.9}$.

Case II: $m > \varepsilon^{-0.1}$. Our goal is to find proper standard pairs $\ell_\beta = (\gamma_\beta, \rho_\beta)$ such that

(a) each γ_β is a component of $\tilde{\mathcal{F}}^{n_\beta}(\mathcal{G})$ for some $n_\beta > 0$, with density ρ_β induced by $\tilde{\mathcal{F}}^{n_\beta} \mathbb{P}_{\mathcal{G}}$;

(b) their preimages $\tilde{\mathcal{F}}^{-n_\beta}(\gamma_\beta)$ are disjoint pieces of the family $\mathcal{G}$;

(c) their total $\mathbb{P}_{\mathcal{G}}$-measure is $\geq 1 - \varepsilon^{0.9}$;

(d) on each $\tilde{\mathcal{F}}^{-n_\beta}(\gamma_\beta)$ we have $\mathcal{N}_{m(\alpha)} > n_\beta$ and $m(\alpha) - I_{n_\beta} \in [0, \varepsilon^{-0.1}]$.

Then we can apply the argument of Case I to each proper standard pair ℓ_β, sum up the resulting estimates, and obtain $\mathbb{P}_{\mathcal{G}}(L_{m(\alpha)} < \varepsilon) \leq (C_1 + 2)\, \varepsilon^{0.9}$.

Our construction of $\{\ell_\beta\}$ has inductive character. At the first step, we put $m_1 = m(\alpha)$ and $n_1 = m_1 / \bar{\Delta}$. It follows from Proposition 3 that

$$\mathbb{P}_{\mathcal{G}}\big(|I_{n_1} - m_1| > n_1^{0.6}\big) = \mathcal{O}\big(\theta_2^{n_1^{0.2}}\big) \ll \varepsilon^{0.9} \tag{4.2}$$

and

$$\mathbb{P}_{\mathcal{G}}\big(|\mathcal{N}_{m_1}-n_1|>n_1^{0.6}\big)=\mathcal{O}\big(\theta_2^{n_1^{0.1}}\big)\ll\varepsilon^{0.9}, \tag{4.3}$$

Due (3.3), there is a family of proper standard pairs $\ell'_\beta=(\gamma'_\beta,\rho'_\beta)$, each being a component of $\tilde{\mathcal{F}}^{n_\beta}(\mathcal{G})$ for some

$$n_\beta\in[n_1-n_1^{0.65},n_1-n_1^{0.6}], \tag{4.4}$$

whose preimages under $\tilde{\mathcal{F}}^{-n_\beta}$ are disjoint, and whose total $\mathbb{P}_{\mathcal{G}}$-measure is $\geq 1-c_3\theta_3^{n_1^{0.65}-n_1^{0.6}}$.

Observe that I_{n_β} is constant on the preimage $\gamma''_\beta=\tilde{\mathcal{F}}^{-n_\beta}(\gamma'_\beta)$ of every γ'_β (because I_{n_β} is the cell number where $\mathcal{F}^{n_\beta}(\gamma''_\beta)$ lies). Curves on which $\mathcal{N}_{m_1}\leq n_\beta$ can be discarded due to (4.3), then we have $I_{n_\beta}<m_1$ on every (not yet discarded) curve. Curves on which $I_{n_\beta}<m_1-2n_1^{0.65}$ can be discarded due to Proposition 3 and (4.4), then we have $m_1-I_{n_\beta}\in[0,2n_1^{0.65}]$. Now curves on which $m_1-I_{n_\beta}\in[0,\varepsilon^{-0.1}]$ are 'good', we include them in our 'target' family $\ell_\beta=(\gamma_\beta,\rho_\beta)$. On the remaining curves $m_1-I_{n_\beta}\in[\varepsilon^{-0.1},2n_1^{0.65}]$, and we will deal with them next.

At the second step we apply the above procedure to each remaining proper standard pair $\ell'_\beta=(\gamma'_\beta,\rho'_\beta)$ (which was not discarded or added to the 'target' family). Precisely, on each ℓ'_β we denote $m_2=m_1-I_{n_\beta}$ (observe that $0<m_2\leq C_3m_1^{0.65}$ for some constant $C_3>0$), put $n_2=m_2/\bar{\Delta}$, and then repeat our procedure word for word, only changing index 1 to index 2. In the course of this construction, some images of ℓ'_β will be discarded, some added to our 'target' family, and some will remain for the third step; the latter will start by setting $m_3=m_2-I_{n_\beta}$ (note again, as before, that $m_3\leq C_3m_2^{0.65}$), then setting $n_3=m_3/\bar{\Delta}$, etc.

In finitely many steps we arrive at $2n_k^{0.65}<\varepsilon^{-0.1}$, thus no curves will be left, and our construction will stop (observe that $k=\mathcal{O}(\ln\ln m)$). The total measure of all discarded curves will be $\mathcal{O}\big(\theta_3^{n_1^{0.1}}\big)\ll\varepsilon^{0.9}$. This completes the proof of Proposition 5. □

Corollary 6. *Let A and B be the constants of Proposition 4. Then for any standard pair $\ell=(\gamma,\rho)$ and $m>2A|\ln\operatorname{length}(\gamma)|+2B$ we have $\mathbb{P}_\ell(L_m<\varepsilon)\leq C_2\varepsilon^{0.9}$ for all $\varepsilon>0$.*

Proof. Let $m_0=A|\ln\operatorname{length}(\gamma)|+B$. By Proposition 4, $\mathcal{G}=\tilde{\mathcal{F}}^{m_0}\ell$ is a proper standard family and so we can apply Proposition 5(b) with $m(\alpha)=m-I_{m_0}$

(observe that the last expression depends only on which curve in $\mathcal{G}$ our points lands on). $\square$

In other words, short unstable curves grow under the iterations of $\mathcal{R}$ exponentially fast into standard families that consist of predominantly long unstable curves. Such properties are instrumental in the construction of SRB measures for hyperbolic maps with singularities. We turn to that next.

For a standard family $\mathcal{G}$, we denote by $\mathcal{G}_n = \mathcal{R}^n(\mathcal{G})$ its image and by $\mathbb{P}_{\mathcal{G}_n} = \mathcal{R}^n(\mathbb{P}_{\mathcal{G}})$ the corresponding measure on $\mathcal{G}_n \subset \Lambda_n$. The latter naturally induces a measure $\mathbb{P}^*_{\mathcal{G}_n}$ on the 'extended set' Λ^*_n as follows: let Π_n be the map $\Lambda \to \Lambda^*_n$ defined by $[\Pi_n(z)]_i = \mathcal{R}^{i+n}z$; then we set $P^*_{\mathcal{G}_n} = \Pi_n(P_{\mathcal{G}})$. All these measures have absolutely continuous distributions on unstable manifolds. Lastly note that each $\mathcal{R}$-invariant measure ν on Λ can be naturally lifted to a $\mathcal{R}^*$-invariant measure ν^* on Λ^*.

Proposition 7. *For every proper standard pair $\mathcal{G}$, the Cesaro averages $\frac{1}{n}\sum_{i=0}^{n-1}\mathbb{P}_{\mathcal{G}_i}$ weakly converge, as $n \to \infty$, to a unique $\mathcal{R}$-invariant SRB measure ν on Λ. It is ergodic. It is either mixing or cyclically permutes $K \geq 2$ components so that $\mathcal{R}^K$ is mixing on each one. Moreover, for each fixed $m \geq 1$ the Cesaro averages $\frac{1}{n}\sum_{i=0}^{n-1}\pi^*_m(\mathbb{P}^*_{\mathcal{G}_i})$ weakly converge to the measure $\pi^*_m(\nu^*)$.*

Proof. Our first observation is that the map $\tilde{\mathcal{F}}_{\rm L}\colon \tilde{\Omega}_{\rm L} \to \tilde{\Omega}_{\rm L}$ is ergodic since it is a first return map of an ergodic transformation. It may not be mixing, though, but due to general results [17, 21] it is either mixing or cyclically permutes $K \geq 2$ components of $\tilde{\Omega}_{\rm L}$ (each has measure $1/K$), and then $\tilde{\mathcal{F}}_{\rm L}^K$ is mixing on each component.

Furthermore, a useful *coupling lemma* proved for dispersing billiards in [9, Appendix A], see also [13, Chapter 7], can be easily adapted to the map $\tilde{\mathcal{F}}_{\rm L}$ and give a valuable extra information. Namely, for any two standard pairs $\tilde{\ell} = (\tilde{\gamma}, \tilde{\rho})$ and $\tilde{\tilde{\ell}} = (\tilde{\tilde{\gamma}}, \tilde{\tilde{\rho}})$ in $\tilde{\Omega}_{\rm L}$ there is a measure preserving map (*coupling map*)

$$\zeta\colon \big(\tilde{\gamma} \times [0,1], \mathbb{P}_{\tilde{\ell}} \times \mathrm{Leb}\big) \to \big(\tilde{\tilde{\gamma}} \times [0,1], \mathbb{P}_{\tilde{\tilde{\ell}}} \times \mathrm{Leb}\big)$$

and a measurable map

$$\Upsilon\colon \tilde{\gamma} \times [0,1] \to \mathbb{N}$$

(called *coupling time map*) such that if $\zeta(\tilde{x}, \tilde{s}) = (\tilde{\tilde{x}}, \tilde{\tilde{s}})$, then there is $m = m(\tilde{x}, \tilde{\tilde{x}}) \in [0, K-1]$ such that the two points

$$\tilde{\mathcal{F}}_{\rm L}^{\Upsilon(\tilde{x},\tilde{s})+m}\tilde{x} \qquad \text{and} \qquad \tilde{\mathcal{F}}_{\rm L}^{\Upsilon(\tilde{x},\tilde{s})}\tilde{\tilde{x}} \tag{4.5}$$

belong to the same stable manifold of the map $\tilde{\mathcal{F}}_{\mathrm{L}}$ (if $\tilde{\mathcal{F}}_L$ is mixing, then m is always equal to 0).

This allows us to show that for any standard pairs $\tilde{\ell}, \tilde{\tilde{\ell}}$ and a continuous function A on $\tilde{\Omega}_{\mathrm{L}}$

$$\frac{1}{n}\sum_{j=0}^{n-1}\int A(\tilde{\mathcal{F}}_L^j x)\, d\mathbb{P}_{\tilde{\ell}}(x) - \frac{1}{n}\sum_{j=0}^{n-1}\int A(\tilde{\mathcal{F}}_L^j x)\, d\mathbb{P}_{\tilde{\tilde{\ell}}}(x) \to 0. \tag{4.6}$$

Observe that since the two points (4.5) belong to the same stable manifold, the points $\mathcal{R}^{\tilde{m}}\tilde{x}$ and $\mathcal{R}^{\tilde{\tilde{m}}}\tilde{\tilde{x}}$ belong to the same stable manifold (of the map $\tilde{\mathcal{F}}_{\mathrm{L}}$) for some $\tilde{m}, \tilde{\tilde{m}} \geq 0$. Thus the argument proving (4.6) also shows that

$$\frac{1}{n}\sum_{j=0}^{n-1}\int A(\mathcal{R}^j x)\, d\mathbb{P}_{\tilde{\ell}}(x) - \frac{1}{n}\sum_{j=0}^{n-1}\int A(\mathcal{R}^j x)\, d\mathbb{P}_{\tilde{\tilde{\ell}}}(x) \to 0. \tag{4.7}$$

In turn, (4.7) implies that for any two standard families $\tilde{\mathcal{G}}$ and $\tilde{\tilde{\mathcal{G}}}$

$$\frac{1}{n}\sum_{j=0}^{n-1}\int A(\mathcal{R}^j x)\, d\mathbb{P}_{\tilde{\mathcal{G}}}(x) - \frac{1}{n}\sum_{j=0}^{n-1}\int A(\mathcal{R}^j x)\, d\mathbb{P}_{\tilde{\mathcal{G}}}(x) \to 0. \tag{4.8}$$

Now take any standard family $\mathcal{G}$ and let ν be a limit point of Cesaro averages $\frac{1}{n}\sum_{i=0}^{n-1}\mathcal{R}^i\big(\mathbb{P}_{\mathcal{G}}\big)$. Then ν is invariant under $\mathcal{R}$ and absolutely continuous with respect to unstable leaves, hence it corresponds to a standard family $\mathcal{G}(\nu)$. Applying (4.8) with $\tilde{\mathcal{G}} = \mathcal{G}$ and $\tilde{\tilde{\mathcal{G}}} = \mathcal{G}(\nu)$ we prove that in fact

$$\frac{1}{n}\sum_{j=0}^{n-1}\int A(\mathcal{R}^j x)\, d\mathbb{P}_{\mathcal{G}}(x) \to \nu(A). \tag{4.9}$$

In particular, ν is a unique $\mathcal{R}$-invariant measure with smooth densities on unstable leaves.

Moreover using the fact that the image an unstable curve is a union of unstable curves it is not difficult to deduce from (4.9) that

$$\frac{1}{n}\sum_{j=0}^{n-1}\int \phi(x)A(\mathcal{R}^j x)\, d\mathbb{P}_{\mathcal{G}}(x) \to \mathbb{P}_{\mathcal{G}}(\phi)\nu(A) \tag{4.10}$$

first for any piecewise constant function ϕ, and then for any bounded measurable function ϕ. In particular

$$\frac{1}{n}\sum_{j=0}^{n-1}\int \phi(x)A(\mathcal{R}^j x)\,d\nu(x) \to \nu(\phi)\nu(A),$$

and thus $\mathcal{R}$ is ergodic with respect to ν.

Lastly, we address the mixing properties of the map $\mathcal{R}$. It will be mixing if $\mathcal{R}^k$ is ergodic for every $k \geq 2$, see [21], otherwise the return times to the base of Young's tower will have a common multiple K, see [21, Lemma 5], and then $\mathcal{R}$ will cyclically permute K components, on each of which $\mathcal{R}^K$ will be mixing. This proves the first part of Proposition 7.

The second part (involving natural extensions) follows from the first, because for every continuous function A the convergence of averages $\frac{1}{n}\sum_j \mathbb{E}_{\mathcal{G}}(A\circ \mathcal{R}^j)$ implies the convergence of $\frac{1}{n}\sum_j \mathbb{E}_{\mathcal{G}}(A\circ\mathcal{R}^{j-m})$ for every $m \geq 0$. $\square$

We note that if $K \geq 2$, then all periodic points of $\mathcal{R}$ have periods proportional to K (see [21]) and this fact can be used to check mixing of $\mathcal{R}$. Given a cell $\mathcal{D}_0$, if one can find two periodic points for the map $\mathcal{R}$ with incommensurate (mutually prime) periods then $\mathcal{R}$ is in fact mixing. We believe that this fact can be used to prove that the advance map for BGR channel is in fact mixing but we do not pursue this point here since mixing is not used in the proof of our main result.

In fact the foregoing analysis gives more precise conclusions. Namely, if the measure ν is mixing, we can replace Cesaro averages with just iterations of $\mathbb{P}_{\mathcal{G}}$. If $K \geq 2$, we first need to average the first K iterations of our measure: $\mathbb{P}_{\bar{\mathcal{G}}} = \frac{1}{K}\sum_{i=0}^{K-1}\mathbb{P}_{\mathcal{G}_i}$, this gives us a 'well balanced' initial measure whose iterations will converge to ν. (It is clear that the average measure is also supported by a standard family, which we denote by $\bar{\mathcal{G}}$.)

Corollary 8. *For every proper standard pair $\mathcal{G}$ the measure $\mathbb{P}_{\bar{\mathcal{G}}_n}$ weakly converges, as $n \to \infty$, to ν, and for each fixed $m \geq 1$ the measure $\pi_m^*(\mathbb{P}^*_{\bar{\mathcal{G}}_n})$ weakly converges to $\pi_m^*(\nu^*)$.*

Lastly, it is easy to generalize (4.1) as follows: for any proper standard family $\mathcal{G}$ and any $m, n \geq 1$ we have

$$\mathbb{P}_{\mathcal{G}}(\mathcal{N}(\mathcal{R}^m z) \geq n) \leq \mathbb{P}_{\mathcal{G}}(\mathcal{N}_{an}(\mathcal{R}^m z) \geq n) \leq c_4\theta_4^n \tag{4.11}$$

and due to Proposition 7 we also have

$$\nu(\mathcal{N}(\mathcal{R}^m z) \geq n) \leq \nu(\mathcal{N}_{an}(\mathcal{R}^m z) \geq n) \leq c_4 \theta_4^n. \tag{4.12}$$

(Here c_4 and θ_4 do not depend on m because the estimate in Proposition 5 is uniform.)

5 Proof of Theorem 1

First, for every initial point $z \in \mathcal{D}_0 \times S^1$ denote by $\hat{\tau}(z)$ the first time the trajectory $\Phi^t(z)$ crosses $\tilde{\Omega}_{\mathrm{L}}$ and by $\hat{\pi}(z) = \Phi^{\hat{\tau}(z)}(z)$ the crossing point. Then

$$\left| q\big(\Phi^T(z)\big) - q\big(\Phi^T(\hat{\pi}(z))\big) \right| \leq \psi(z),$$

where $\psi(z)$ does not depend on T. Thus the limit distribution of $q(T)/T$ will not be affected if we replace each z with $\hat{\pi}(z)$; therefore we replace the initial uniform distribution on $\mathcal{D}_0 \times S^1$ with its image on $\tilde{\Omega}_{\mathrm{L}}$, i.e. with a smooth probability distribution, μ_0, on $\tilde{\Omega}_{\mathrm{L}}$. Similarly, given a $k \geq 1$ we have

$$\left| q\big(\Phi^T(z)\big) - q\big(\Phi^T(\mathcal{R}^k(z))\big) \right| \leq \psi_k(z),$$

where $\psi_k(z)$ does not depend on T. Hence the limit distribution of $q(T)/T$ will not be affected if we replace each z with $\mathcal{R}^k(z)$; thus we can replace μ_0 with the average

$$\bar{\mu}_0 = \frac{1}{K} \sum_{i=0}^{K-1} \mathcal{R}^i(\mu_0).$$

This is also a smooth probability distribution on $\tilde{\Omega}_{\mathrm{L}}$, so it can be represented by a proper standard family $\mathcal{G}$ in a usual way (e.g., one can foliate $\tilde{\Omega}_{\mathrm{L}}$ with unstable manifolds of the map $\tilde{\mathcal{F}}$), hence $\bar{\mu}_0 = \mathbb{P}_{\mathcal{G}}$. Now, due to Corollary 8, the measure $\mathcal{R}^n(\bar{\mu}_0) = \mathbb{P}_{\mathcal{G}_n}$ converges to the SRB measure ν.

Next consider the similarity transformation $\mathcal{S}_r$ of the phase space $\mathcal{M}$ defined by

$$\mathcal{S}_r(q, v) = (q_\infty + (q - q_\infty)r, v),$$

cf. (3.1); observe that $\mathcal{S}_r \circ \Phi^t = \Phi^{rt} \circ \mathcal{S}_r$. We can always choose the coordinate frame so that $q_\infty = 0$, this will simplify our formulas.

For $z \in \tilde{\Omega}_{\mathrm{L}}$ and $n \geq 1$, let $\tau_n(z)$ denote the continuous time elapsed between the points z and $\mathcal{F}^{\mathcal{N}_n(z)}(z)$, i.e. the time it takes the trajectory

$\Phi^t(z)$ to reach the left side of the cell $\mathcal{D}_n$. Observe that

$$\tau_n(z) = \sum_{i=0}^{n-1} \tau_1(\mathcal{R}^i z)\, r^i = r^n \sum_{k=1}^{n} r^{-k} \tau_1(\mathcal{R}^{-k}\mathcal{R}^n z).$$

Now recall that Theorem 1 assumes that $\ln T_n = n \ln r + \rho \ln r + o(1)$, i.e. $T_n = r^{n+\rho+o(1)}$. Therefore

$$\begin{aligned} q(T_n(z))/T_n &= r^{-\rho}\, q\big(\mathcal{S}_r^{-n}(\Phi^{T_n}(z))\big) + o(1) \\ &= r^{-\rho}\, q\big(\Phi^{r^{-n}(T_n-\tau_n(z))}(\mathcal{S}_r^{-n}(\mathcal{F}^{\mathcal{N}_n(z)}(z)))\big) + o(1), \end{aligned}$$

because $\Phi^{\tau_n(z)}(z) = \mathcal{F}^{\mathcal{N}_n(z)}(z)$. Since $\mathcal{S}_r^{-n}(\mathcal{F}^{\mathcal{N}_n(z)}(z)) = \mathcal{R}^n(z)$, we have

$$q(T_n(z))/T_n = r^{-\rho}\, q\big(\Phi^{r^\rho}(\Phi^{-r^{-n}\tau_n(z)}(\mathcal{R}^n(z)))\big) + o(1). \tag{5.1}$$

Now a change of variable $\mathcal{R}^n(z) \mapsto z$ transforms (5.1) into

$$q(T_n \circ \mathcal{R}^{-n})/T_n = r^{-\rho}\, q(\Phi^{r^\rho} \circ \Phi^{-w_n}) + o(1), \tag{5.2}$$

where

$$w_n(z) = \sum_{k=1}^{n} r^{-k} \tau_1(\mathcal{R}^{-k} z) \tag{5.3}$$

is a function defined on Λ_n^* (we recall that Λ_n^* consists of sequences $Z = \{z_i\}_{i=-n}^{\infty}$, but here for the ease of notation we identify Z with $z = z_0$). Note also that our change of variable transforms $\mathbb{P}_{\mathcal{G}}$ into $\mathbb{P}_{\mathcal{G}_n}^*$ on Λ_n^*.

Next, due to our finite horizon assumption and (4.11),

$$\mathbb{P}_{\mathcal{G}}(\tau_1(\mathcal{R}^m z) \geq M) \leq c_4' \theta_4^M \tag{5.4}$$

for all $m, M > 0$ and some constant $c_4' > 0$, and a similar estimate holds if we replace $\mathbb{P}_{\mathcal{G}}$ with ν, according to (4.12). Thus the terms in the sum (5.3) decay exponentially in k, so the value of $w_n(z)$ is mostly determined by the first few pre-images of z. In fact (5.4) implies an exponential tail bound

$$\mathbb{P}_{\mathcal{G}_n}^*(w_n(z) \geq M) \leq c_5 \theta_5^M \tag{5.5}$$

uniformly in n, and a similar bound holds if we replace $\mathbb{P}_{\mathcal{G}_n}^*$ with $\pi_n^*(\mathbb{P}_{\mathcal{G}_k}^*)$ for any $k > n$, or with $\pi_n^*(\nu^*)$.

Also consider the 'limit' function

$$w(z) = \sum_{k=1}^{\infty} r^{-k} \tau_1(\mathcal{R}^{-k} z),$$

which is well defined a.e. on Λ^*. Our tail bounds imply

$$\nu^*\big(|w - w_n \circ \pi_n^*| \geq r^{-n} M\big) \leq c_6 \theta_6^M$$

for all $M > 0$ and $n \geq 1$, and a tail bound similar to (5.5):

$$\nu^*(w \geq M) \leq c_7 \theta_7^M.$$

Theorem 1 immediately follows from the next proposition:

Proposition 9. *Let A be a continuous function on $\mathbb{R}^2$ with compact support. Then, as $n \to \infty$, we have*

$$\mathbb{E}_{\mathcal{G}}\big(A(q(T_n)/T_n)\big) \to \int_{\Lambda^*} A\big(r^{-\rho}\, q(\Phi^{r^\rho - w})\big)\, d\nu^*. \tag{5.6}$$

The integral here determines the limit distribution of $q(T_n)/T_n$; observe its explicit dependence on ρ.

Proof of Proposition 9. According to (5.2),

$$\mathbb{E}_{\mathcal{G}}\big(A(q(T_n)/T_n)\big) = \int_{\Lambda_n^*} A\big(r^{-\rho}\, q(\Phi^{r^\rho - w_n})\big)\, d\mathbb{P}_{\mathcal{G}_n}^* + o(1). \tag{5.7}$$

Observe that the function w_n is piecewise continuous, has countably many domains of continuity, and may be unbounded. We will construct a nicer approximation to w_n as follows.

Our tail bounds imply that for any $\varepsilon > 0$ there is $m \geq 1$ such that

$$\nu^*\big(|w - w_m \circ \pi_m^*| > \varepsilon\big) < \varepsilon \qquad \text{and} \qquad \mathbb{P}_{\mathcal{G}_n}^*\big(|w_n - w_m \circ \pi_m^*| > \varepsilon\big) < \varepsilon$$

uniformly for all $n > m$. Furthermore, there exists $m_0 \geq 1$ (that may depend on m) such that

$$\nu^*\big(\mathcal{N}_m \circ \mathcal{R}^{-m} > m_0\big) < \varepsilon \qquad \text{and} \qquad \mathbb{P}_{\mathcal{G}_n}^*\big(\mathcal{N}_m \circ \mathcal{R}^{-m} > m_0\big) < \varepsilon$$

uniformly for all $n > m$. Now define a new function on Λ_m^*:

$$\hat{w}_m(z) = \begin{cases} w_m(z) & \text{if } \mathcal{N}_m(\mathcal{R}^{-m}z) \le m_0 \\ 0 & \text{if } \mathcal{N}_m(\mathcal{R}^{-m}z) > m_0 \end{cases} .$$

The above estimates show that we can replace both w and w_n in (5.6)–(5.7) with the new function $\hat{w}_m \circ \pi_m^*$, and the errors committed by this replacement can be made arbitrarily small by choosing an appropriate $\varepsilon > 0$.

Lastly, observe that the function $\hat{w}_m$ is bounded and has finitely many domains of continuity; more precisely, their coordinatewise projections onto $\tilde{\Omega}_{\mathrm{L}}$ are domains with piecewise smooth boundary consisting of singularity lines of the map $\tilde{\mathcal{F}}^{\pm m_0}$; thus the ν^*-measure of the boundary of these domains is zero. Now the weak convergence claimed in Corollary 8 implies

$$\int_{\Lambda_n^*} A\big(r^{-\rho}\, q(\Phi^{r^\rho - \hat{w}_m \circ \pi_m^*})\big)\, d\mathbb{P}_{\mathcal{G}_n}^* \to \int_{\Lambda^*} A\big(r^{-\rho}\, q(\Phi^{r^\rho - \hat{w}_m \circ \pi_m^*})\big)\, d\nu^*$$

(note that Φ^{r^ρ} is always continuous). This proves Proposition 9.

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