ON DECAY OF CORRELATIONS IN ANOSOV FLOWS.

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1. Statement of results. There is some disagreement about the meaning of the phrase 'chaotic flow.' However, there is no doubt that mixing Anosov flows provide an example of such systems. Anosov systems were introduced and extensively studied in the classical memoir of Anosov ([A]). Among other things he proved the following fact known now as Anosov alternative for flows: either every strong stable and strong unstable manifold is everywhere dense or the flow g^t is a suspension over an Anosov diffeomorphism by a constant roof function. If the first alternative holds g^t is mixing with respect to every Gibbs measure (see [PP2]).

Therefore the natural question is to estimate the rate of mixing. This is certainly one of the simplest questions concerning correlation decay in continuous time systems. Nevertheless the only results obtained until recently dealt with the case when the system discussed had an additional algebraic structure. The easier case of Anosov diffeomorphisms can be treated by the methods of thermodynamic formalism of Sinai, Ruelle and Bowen ([B2]). Namely one uses Markov partitions to construct an isomorphism between the diffeomorphism and a subshift of a finite type and then proves that all such subshifts are exponentially mixing. This method would succeed also for flows if any suspension over a subshift of a finite type had exponentially decaying correlations. However already the simplest example–suspensions with locally constant roof functions never have such a property ([R1]). One can use the above observation to produce examples of Axiom A flows with arbitrary slow correlation decay. It became clear therefore that some additional geometric properties should be taken into account. In a recent work Chernov ([Ch1], [Ch2]) has employed uniform non-integrability condition to get subexponential estimate for correlation functions for geodesic flows on surfaces of variable negative curvature. His method relies on the the technique of Markov approximation developed in [Ch1].

The aim of this paper is to combine geometric considerations of Chernov with the thermodynamic formalism approach. The later method seems to be more appropriate than Markov approximations since it gives simple enough description of the resonances ([P], [R2]) and hence one can hope to obtain the asymptotic expansion of the error term even though this problem seems to be much more difficult than just obtaining upper bound. In fact in this paper we show that under the condition introduced by Chernov correlations do decay exponentially as was conjectured in [Ch2]. More precisely we prove the following statement. Let F be a Hölder continuous potential and μ the Gibbs measure for F. Denote $\rho_{A,B}(t) = \int A(x)B(g^t x)\, d\mu(x)$, $\bar\rho_{A,B}(t) = \rho_{A,B}(t) - \int A(x)\, d\mu(x) \int B(x)\, d\mu(x)$.

THEOREM 1. *Let (M, g^t) be a geodesic flow on the unit tangent bundle M over a negatively curved C^7 surface. Then for any Hölder continuous (of the class C^α) potential F there exist constants C_1, C_2 such that for any pair of C^5 functions $A(x)$ and $B(x)$ $|\bar\rho_{A,B}(t)| \leq C_1 e^{-C_2 t}\|A\|_5\|B\|_5$.*

The most interesting examples of potentials are Sinai-Bowen-Ruelle potential $R(x) = \frac{\partial}{\partial t}|_{t=0} \ln\det(dg^t|e_u)$ which yields the Lesbegue measure and $F \equiv 0$ which corresponds to the measure of maximal entropy (see [M], [BMar], [PP1] for applications of the later measure to geometric problems).

Our method can also be generalized to higher dimensions. In fact we use only C^1-smoothness of the Anosov splitting of geodesic flow in two dimensions ([HP]) and Federer property of the conditionals of Gibbs measures (see Section 7). Actually we prove the following statement.

THEOREM 2. *Let g^t be a C^5-Anosov flow on a compact manifold M. Assume that stable and unstable foliations are of class C^1 . Then for Sinai-Bowen-Ruelle measure (F=R) there exist constants C_1, C_2 such that for any pair of C^5 functions $A(x)$ and $B(x)$ $|\bar\rho_{A,B}(t)| \leq C_1 e^{-C_2 t}\|A\|_5\|B\|_5$.*

COROLLARY 1. *Under the conditions of theorems 1 or 2 given $\tilde\alpha > 0$ there exist constants $C_1(\tilde\alpha)$, $C_2(\tilde\alpha)$ such that $\forall A, B \in C^{\tilde\alpha}(M)$*

$$|\bar\rho_{A,B}(t)| \leq C_1(\tilde\alpha) e^{-C_2(\tilde\alpha)t}\|A\|_{\tilde\alpha}\|B\|_{\tilde\alpha}.$$

PROOF: Take $\tilde A$, $\tilde B$ such that $\tilde A, \tilde B \in C^5(M)$, $\|A - \tilde A\|_0 \leq e^{-\tilde\alpha\gamma t}\|A\|_{\tilde\alpha}$ $\|B - \tilde B\|_0 \leq e^{-\tilde\alpha\gamma t}\|B\|_{\tilde\alpha}$, $\|A\|_5 \leq e^{5\gamma t}\|A\|_0$, $\|B\|_5 \leq e^{5\gamma t}\|B\|_0$. Then $\bar\rho_{A,B}(t) =$

$\bar\rho_{\tilde A,\tilde B}(t)+\delta(t)$ where $\delta(t)\leq$ Const $e^{-\tilde\alpha\gamma t}$. From the other hand $|\bar\rho_{\tilde A,\tilde B}(t)|\leq C_1\|A\|_{\tilde\alpha}\|B\|_{\tilde\alpha}e^{-C_2 t}e^{10\gamma t}$. Taking $\gamma=\frac{C_2}{10+\tilde\alpha}$ we obtain that $C_2(\tilde\alpha)=\frac{C_2\tilde\alpha}{10+\tilde\alpha}$ satisfies the requirement of the corollary.■

REMARK. The smoothness assumption on the flow g^t are not optimal and are made to simplify the exposition. It's easy to see that Theorems 1 and 2 remain true if $g^t\in C^{2+\epsilon}$. We conjecture, however, that the result should hold for $C^{1+\epsilon}$ flows.

We can also further weaken our assumptions and still get some consequences. The smoothness assumption amounts to that the temporal distance function $\varphi(x,y)$ (see Section 5) is of class C^1. (The temporal distance is used to measure non-integrability of non-smooth distributions. Roughly speaking it is obtained from the commutators by replacing infinitesimal increments by finite ones.) For Anosov flows we know that $\varphi(x,y)$ satisfies the intermediate value theorem. Surprisingly enough this simple observation implies quite rapid decay of correlations.

THEOREM 3. *Let g^t be an arbitrary topologically mixing C^∞ Anosov flow, F be an arbitrary Holder continuous potential and $A(x), B(x)$ be $C^\infty(M)$ functions. Then $\bar\rho_{A,B}(t)$ is rapidly decreasing in the sense of Schwartz.*

Note by contrast that in Ruelle's counterexamples φ assumes only finite number of values. We conjecture that Theorem 3 is true for any Axiom A flow such that the range of φ has a positive Hausdorff dimension. This would get us quite close to description of all Axiom A flows with slow decay of correlations (see [D] for more discussion on this subject).

The plan of the paper is the following. In Sections 2-4 we recall how to reduce our problem to the estimation of the spectral radii of a certain one-parameter family of transfer-operators $\mathcal{L}_\xi$. This procedure is due to Pollicott (see [P], [R2]) using earlier developments by Sinai, Bowen and Ruelle. Here we describe briefly this reduction. We take the Laplace transform of the correlation function and write it as a double integral over space and time. So when the space variable is fixed the integration is over the flow orbit. We now take a Markov section (that is some special cross section of the flow, see Section 3 for precise definitions). Let $\hat\sigma$ be the first return map and τ be the first return time. We chop the orbits on the pieces between consecutive hits of our Markov section. A simple calculation shows that the corresponding part of the integral can be expressed in terms of the operators $(L_\xi h)(x)=e^{i\xi\tau(x)}h(\hat\sigma x)$ (h is defined on the Markov section). The Markov property implies that these operators preserve the subspace of functions which are constant along the local stable leaves of our cross section. The transfer operator $\mathcal{L}_\xi$ is just the adjoint of L_ξ on this space restricted to the space of the densities (with respect to conditionals of μ_F). So it is clear the the spectra of $\mathcal{L}_\xi$ play an important role in our consideration. We study the spectra in Sections 5-8. In Section 5 we introduce uniform non-integrability condition (UNI) and explain that it is quite similar to certain non-degeneracy condition in the theory of oscillatory integral operators. (It is often useful to view transfer operators as integral operator with δ-type kernels.) In Section 6 we show what C^1 smoothness of Anosov splitting is a natural weaker version of (UNI). The proof of the main spectral bound is contained in Sections 7 and 8. Section 9 is devoted to the proof of Theorem 3. Most of the steps in the proof are completely analogous to ones in the proofs of Theorems 1 and 2. In such cases we leave the proof to the reader. Four Appendixes contain some more technical results. The calculations presented are pretty standard but since the details are spread in many different places we decided for the convenience of the reader to collect all the proofs at the end of the paper. We do not claim, however that our proofs in the Appendixes are shortest possible. The readers familiar with the subject should have no difficulty to do all the calculations by themselves. The others may wish to look through the Appendixes to get an idea of the proof and then try to fill the details consulting the paper in case any problems arise.

2.Symbolic dynamics. As it was explained in the introduction we will use Markov section to model our flow by some symbolic dynamical system. In this section we recall basic facts about such systems and also introduce our notations. For proofs and more information on the subject see [B2], [PP].

For a $n \times n$ matrix A whose entries are zeroes and ones we denote by $\Sigma_A = \{\{\omega_i\}_{i=-\infty}^{+\infty} : A_{\omega_i \omega_{i+1}} = 1\}$ the configuration space of a subshift of a finite type. Sometimes we omit A and write Σ instead of Σ_A. The shift σ acts on Σ by $(\sigma\omega)_i = \omega_{i+1}$. The one-sided shift (Σ_A^+, σ) is defined in the same way but the index set where is the set of non-negative integers. For $\theta < 1$ we consider the distance $d_b^\theta(\omega^1, \omega^2) = \theta^k$ where $k = \max\{j : \omega_i^1 = \omega_i^2 \text{ for } |i| \leq j\}$ (the subscript b stands for 'base'). We write $C_\theta(\Sigma)$ for the space of $d_b^\theta-$Lipschitz functions and $C_\theta^+(\Sigma)$ for the subspace of functions depending only on the future coordinates $\omega_0, \omega_1 \dots \omega_n \dots$. We can identify $C_\theta^+(\Sigma)$ with $C_\theta(\Sigma^+)$. We use the notation $L(h)$ for the Lipschitz constant of h and $h_n(\omega) = \sum_{i=0}^{n-1} h(\sigma^i \omega)$.

Functions f_1 and f_2 are called cohomologous ($f_1 \sim f_2$) if there is a function f_3 such that $f_1(\omega) = f_2(\omega) + f_3(\omega) - f_3(\sigma\omega)$. For any $f \in C_\theta(\Sigma)$ there exists a function $\tilde{f} \in C^+_{\sqrt{\theta}}(\Sigma)$ such that $f \sim \tilde{f}$. If $\bar{\omega}, \tilde{\omega}$ are points in Σ and $\bar{\omega}_0 = \tilde{\omega}_0$ we define their local product $[\bar{\omega}, \tilde{\omega}]$ by

$$[\bar{\omega}, \tilde{\omega}]_j = \begin{cases} \bar{\omega}_j, & j \le 0 \\ \tilde{\omega}_j & j \ge 0 \end{cases}$$

We assume that σ is topologically mixing (that is all entries of some power of A are positive). The pressure functional is defined by

$$Pr(f) = \sup_{\tilde{\nu}} \int f(\omega)\, d\tilde{\nu} + h_{\tilde{\nu}}(\sigma)$$

where the supremum is taken over the set of $\sigma-$invariant probability measures and $h_{\tilde{\nu}}(\sigma)$ is the measure theoretic entropy of σ with respect to $\tilde{\nu}$. A measure ν is called the equilibrium state or the Gibbs measure with the potential f if $\int f(\omega)\, d\nu + h_\nu(\sigma) = Pr(f)$. For $C_\theta(\Sigma)$ potentials Gibbs measures exist and are unique. It is clear that cohomologous functions have the same Gibbs measure. Take $f \in C^+_\theta(\Sigma)$ and let ν be its Gibbs measure. To describe ν it is enough to specify its projection to Σ^+. To this end consider the transfer operator $\mathcal{L}_f : C_\theta(\Sigma^+) \to C_\theta(\Sigma^+)$

$$(\mathcal{L}_f h)(\omega) = \sum_{\sigma\varpi=\omega} e^{f(\varpi)} h(\varpi). \tag{1}$$

Some useful properties of this operator are listed below. First of all the n-th power of $\mathcal{L}$ is a transfer operator for σ^n

$$(\mathcal{L}^n_f h)(\omega) = \sum_{\sigma^n\varpi=\omega} e^{f_n(\varpi)} h(\varpi).$$

The structure of the spectrum of the transfer operator is described by Ruelle-Perron-Frobenius Theorem.

PROPOSITION 1. (RUELLE). *There exist a positive function $\hat{h} \in C_\theta(\Sigma^+)$ and a measure $\hat{\nu}$ on Σ^+ such that*
a) $\mathcal{L}_f \hat{h} = e^{Pr(f)} \hat{h}$;
b) $\mathcal{L}^*_f \hat{\nu} = e^{Pr(f)} \hat{\nu}$ $\mathcal{L}^*_f$ being the adjoint to $\mathcal{L}_f$;
c) there exist constants C_3, ε_1 such that for all $h \in C_\theta(\Sigma^+)$ for all n

$$\|e^{-nPr(f)} \mathcal{L}^n_f h - \hat{\nu}(h)\hat{h}\|_\theta \le C_3(1-\varepsilon_1)^n \|h\|_\theta.$$

d) The measure $\nu = \hat{h}\hat{\nu}$ is σ invariant, moreover it is the projection of $f-$ Gibbs measure on Σ^+.
(A good estimate for ε_1 was given in a recent paper by Liverani [L].)
REMARK. It is clear from this statement that the constants C_3, ε_1 can be chosen to depend continuously on f which we always assume in the sequel. $\mathcal{L}_f$ is called normalized if $\mathcal{L}_f 1 = e^{Pr(f)}1$. We can always normalize $\mathcal{L}$ by replacing f by $f(\omega) + \ln \hat{h}(\omega) - \ln \hat{h}(\sigma\omega)$. In this case $\mathcal{L}^*\nu = e^{Pr(f)}\nu$. Normalized operators satisfy the following useful identity. Let $w = w_1 w_2 \dots w_n$ be an admissible word (that is $A_{w_i w_{i+1}} = 1$). The map $\varpi(\omega) = w\omega$ is defined on a subset of the space Σ_A^+. On this subset the following equation holds:

$$\frac{d\nu(\varpi)}{d\nu(\omega)} = \exp\left[f_n(\varpi) - nPr(f)\right]. \tag{2}$$

Let $\tau \in C_\theta(\Sigma)$ be a positive function. Consider the space

$$\Sigma^\tau = \Sigma \times \mathbf{R}/\{(\omega, s) \sim (\sigma\omega, s + \tau(\omega))\}$$

with the distance $d^\theta((\omega^1, s_1), (\omega^2, s_2)) = d_b^\theta(\omega^1, \omega^2) + |s_1 - s_2|^\theta$. Elements of Σ^τ will be denoted by q. The suspension flow with the roof function τ is defined by $G^t(\omega, s) = (\omega, s + t)$. The pressure and Gibbs measures for G^t are defined in the same way as it was done for σ. These measures can be described as follows. Let $F(q) \in C_\theta(\Sigma^\tau)$ and μ be the corresponding Gibbs measure. Denote $f(\omega) = \int\limits_0^{\tau(\omega)} F(\omega, s)ds$. Then $d\mu(q) = \frac{1}{\nu(\tau)} d\nu(\omega)ds$ where ν is the Gibbs measure with the potential $f(\omega) - Pr_G(F)\tau(\omega)$ and $Pr_\sigma(f - Pr_G(F)\tau) = 0$. For the study of G^t the so called complex Ruelle-Perron-Frobenius theorem is handy (see Section 4).
PROPOSITION 2. (POLLICOTT, HAYDN, RUELLE) *a) The spectral radius $r(\mathcal{L}_{f+i\tau}) \le e^{Pr(f)}$ and $r(\mathcal{L}_{f+is\tau}) = e^{Pr(f)}$ for some real $s \neq 0$ if and only if G^t is not weak-mixing;*
b) the specter of $\mathcal{L}_{f+i\tau}$ in the annulus $\{\theta e^{Pr(f)} < |z| \le e^{Pr(f)}\}$ consists of isolated eigenvalues of finite multiplicity;
c) the leading eigenvalue $\lambda(s)$ of $\mathcal{L}_{f+is\tau}$ is analytic near 0 and $\lambda'(0) = i\lambda(0)\nu(\tau)$ (ν being the Gibbs measure for f).
3. Anosov flows. In this section we provide a background about Anosov flows and symbolic dynamics associated with them.

Recall that a flow g^t on a compact Riemann manifold M is called Anosov if there exists a continuous dg^t-invariant splitting of the tangent bundle $TM = E_u \oplus E_0 \oplus E_s$ such that
1) $E_0(x)$ is generated by the tangent vector to the flow;
2) There exist constants $C_4,\ C_5 > 0$ such that

$$\forall v \in E_s(x)\ \forall t > 0 : \quad \|dg^t v\| \leq C_4 e^{-C_5 t}\|v\|$$

$$\forall v \in E_u(x)\ \forall t > 0 : \quad \|dg^{-t} v\| \leq C_4 e^{-C_5 t}\|v\|.$$

For Anosov flows there always exists an adapted metric for which C_4 can be taken to be 1 (possibly on the expense of replacing C_5 by a smaller constant). We will assume that our metric is the adapted one. The fields E_u and E_s are always integrable. The corresponding integral manifolds are called the strong unstable manifold of x $W^{su}(x)$ and the strong stable manifold of x $W^{ss}(x)$ respectively. Unstable manifold $W^u(x)$ and stable manifold $W^s(x)$ of x are g^t-orbits of $W^{su}(x)$ and $W^{ss}(x)$ respectively. The local versions of these objects are sometimes useful. The local strong stable manifold $W^{ss}_{loc}(x)$ is the set of points $\{y \in W^{ss}(x) : \forall t > 0 \text{ dist}(g^t x, g^t y) \leq \varepsilon\}$. $W^{su}_{loc}(x), W^s_{loc}(x)$ and $W^u_{loc}(x)$ can be defined in a similar fashion. If ε is small enough one can find a neighborhood $O(\text{diag})$ of the diagonal in $M \times M$ such that for $(x, y) \in O$ the intersection $W^u_{loc}(x) \cap W^{ss}_{loc}(x)$ consists of a single point which is denoted $[x, y]$. A set Π is called parallelogram if it can be represented as $\Pi = \{[x, y] : x \in U(\Pi), y \in S(\Pi)\}$ where $U(\Pi) \in W^{su}_{loc}(x)$ and $S(\Pi) \in W^{ss}(x)$ are admissible sets i. e. $U(\Pi) = \text{Cl}(\text{Int}U(\Pi))$, $S(\Pi) = \text{Cl}(\text{Int}S(\Pi))$ (the closure and the interior are taken in the induced topology of the corresponding local manifolds). Π has the natural partition by local leaves of the unstable (respectively strong stable) foliation. The element of this partition containing x will be denoted $W^u_\Pi(x)$ (respectively $W^s_\Pi(x)$). We introduce a coordinate system (u, s) on Π so that points of $U(\Pi)$ have coordinates $(u, 0)$, points of $S(\Pi)$ have coordinates $(0, s)$ and $(u, s) = [(u, 0), (0, s)]$.
Let $\mathcal{P}$ be a collection of parallelograms: $\mathcal{P} = \{\Pi_i\}$. Put $\Pi = \bigcup\limits_i \Pi_i$, $U = \bigcup\limits_i U(\Pi_i)$ and $W^*_\Pi = \bigvee\limits_i W^*_{\Pi_i}$ that is $W^*_\Pi(x) = W^*_{\Pi_i}(x)$ if $x \in \Pi_i$. $\mathcal{P}$ is called a Markov section if the first return map $\hat{\sigma} : \Pi \to \Pi$ has the following properties: $\hat{\sigma}(W^s_\Pi(x)) \subset W^s_\Pi(\hat{\sigma}x)$ and $\hat{\sigma}^{-1}(W^u_\Pi(x)) \subset W^u_\Pi(\hat{\sigma}^{-1}x)$. The existence of Markov sections for Anosov flows was proven by Bowen and Ratner ([B1], [Rt]).
Markov sections allow us to construct a symbolic representation of our flow as follows. If $\mathcal{P}$ is a Markov section consider the matrix A with the following

entries

$$A_{ij} = \begin{cases} 1, & \text{if } \hat{\sigma}(\mathrm{Int}\Pi_i) \cap \mathrm{Int}\Pi_j \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

and let $\tau : \Pi \to \mathbf{R}_+$ be the first return time: $\hat{\sigma}x = g^{\tau(x)}x$. The map $\zeta : \Sigma_A \to \Pi$ given by $\zeta(\omega) = \bigcap\limits_{i=-\infty}^{+\infty} \hat{\sigma}^{-i}\Pi_{\omega_i}$ which is well-defined due to the Markovness of $\mathcal{P}$ is a surjective semiconjugacy between σ and $\hat{\sigma}$. If g^t is a topologically mixing Anosov flow one can choose such a Markov section that σ is topologically mixing. Write $\tau(\omega) = \tau(\zeta(\omega))$ and let $G^t : \Sigma^\tau \to \Sigma^\tau$ be the suspension flow with the roof function τ. We can extend ζ to a semiconjugacy between G^t and g^t by $\zeta(\omega, s) = g^t(\zeta(\omega))$. Then $\zeta([\bar{\omega}, \tilde{\omega}]) = [\zeta(\bar{\omega}), \zeta(\tilde{\omega})]$. If $F \in C^\alpha(M)$ consider $F(q) = F(\zeta(q))$. $F(q)$ belongs to the space $C_\theta(\Sigma^\tau)$ (the constant $\theta < 1$ depends on the Hölder exponent α). We will need the fact that a measure μ on M is the Gibbs measure for $F(x)$ iff its pullback on Σ^τ is the Gibbs measure for $F(q)$.

4. The reduction to the main estimate. In this section we describe the plan of the proof of theorem 1. All steps are pretty standard except step IV which contains new estimates of certain oscillatory integrals depending on a parameter running over the unit ball in some Banach space.

I) Correlation density. In this subsection we recall one useful expression for the Laplace transform $\hat{\rho}_{A,B}(\xi) = \int\limits_0^\infty \rho_{A,B}(t)e^{-\xi t}dt$ of the correlation function

$$\rho_{A,B}(t) = \int\limits_{\Sigma^\tau} A(q)B(G^t q)\, d\mu_F(q).$$

Starting from this point we write $\xi = a + ib$. The expression 'for small a means 'there exist $a_0 > 0$ such that for $|a| \leq a_0$.' The phrase 'for large b should be understood similarly.

Proposition 3. *Let $\tau \in C^+_\theta(\Sigma)$, $F \in C_{\theta^2}(\Sigma)$. Then there exist constants ε_2, C_6, C_7, K, and linear operators $\mathcal{Q}_n(\xi), \mathcal{R}_n(\xi) : C_\theta(\Sigma^\tau) \to C^+_\theta(\Sigma)$ such that uniformly for small a and large b*

a) $\|\mathcal{Q}_n(\xi)A\|_0 \leq C_6(1-\varepsilon_2)^n \|A\|_\theta |b|$, $\|\mathcal{R}_n(\xi)A\|_0 \leq C_6(1-\varepsilon_2)^n \|A\|_\theta |b|$;

b) $L(\mathcal{Q}_n(\xi)A) \leq C_7 K^n \|A\|_\theta |b|^2$, $L(\mathcal{R}_n(\xi)A) \leq C_7 K^n \|A\|_\theta |b|^2$;

c)

$$\hat{\rho}_{A,B}(\xi) = \hat{\rho}^*_{A,B}(\xi) +$$

$$\sum_{j,k=0}^{\infty} \int \left[\mathcal{L}^{j+k}_{f-(Pr(F)+\xi)\tau} \left(1 - \mathcal{L}_{f-(Pr(F)+\xi)\tau}\right)^{-1} \mathcal{Q}_j(\xi)A\right] \mathcal{R}_k(\xi)B\, d\nu \tag{3}$$

where $f \sim \int\limits_0^{\tau(\omega)} F(\omega, s)\, ds$, $\mathcal{L}_{f-Pr(F)\tau}1 = 1$, $\mathcal{L}^*_{f-Pr(F)\tau}\nu = \nu$ *and* $\frac{\hat{\rho}^*_{A,B}}{\|A\|_0\|B\|_0}$ *is uniformly bounded (for small* $a's$*) (*$\mathcal{L}_\star$ *is defined by formula (1)).*

This statement was essentially proven in [P] with further refinements given in [R2] except both authors did not need estimates a) and b). For the convenience of the reader we reproduce their proof and check the above bounds in Appendix 1. From Propositions 1-3 one sees in particular that $\hat{\rho}$ has a simple pole at 0. The residue is equal to $\mu_F(A)\mu_F(B)$. (This is clear from the fact that $\rho_{A,B}(t) \sim \mu_F(A)\mu_F(B)$ but it can also be verified directly using the formulae for $\mathcal{Q}$ and $\mathcal{R}$ (see [P], [R2]).)

Now if (M, g^t) is an Anosov flow and $\mathcal{P} = \{\Pi_i\}$ is a Markov section we can view $C^\alpha(M)$ and $C^\alpha(\Pi)$ as subspaces of $C_\theta(\Sigma^\tau)$ and $C_\theta(\Sigma)$ respectively. Then $C^\alpha(U)$ is identified with a subspace of $C^+_\theta(\Sigma)$ since if $h(u,s)$ does not depend on s, $h(\zeta\omega)$ does not depend on $\{\omega_j\}$, $j < 0$ by the definition of ζ. The transfer operator then acts as follows

$$(\mathcal{L}_f h)(u) = \sum_{\sigma v = u} e^{f(v)} h(v) \tag{4}$$

where $\sigma : U \to U$ means the composition of the first return map $\hat{\sigma}$ and the canonical projection $p : \Pi \to U$. If the Anosov splitting is C^1 and $f \in C^\alpha(U)$ then $\mathcal{L}_f$ preserves $C^\alpha(U)$. Moreover we have the following statement.

Proposition 4. *Let* $F(q)$ *in proposition 1 be of the form* $F_M \circ \zeta$, $F_M \in C^\alpha(M)$ *then* $\mathcal{Q}_n(\xi)$ *and* $\mathcal{R}_n(\xi)$ *map* $C^\alpha(M)$ *to* $C^\alpha(U)$ *and there exist constants* $C_8, C_9, \varepsilon_3 \bar{K}$ *such that for small* $a's$

a) $\|\mathcal{Q}_n(\xi)A\|_0 \le C_8(1-\varepsilon_3)^n\|A\|_\alpha|b|$, $\|\mathcal{R}_n(\xi)A\|_0 \le C_8(1-\varepsilon_3)^n\|A\|_\alpha|b|$;

b) $G(\mathcal{Q}_n(\xi)A) \le C_9\bar{K}^n\|A\|_\alpha|b|^2$, $G(\mathcal{R}_n(\xi)A) \le C_9\bar{K}^n\|A\|_\alpha|b|^2$, $G(h)$ *being the Holder constant for* h.

Proposition 4 follows easily from the explicit expressions for $\mathcal{Q}_n$ and $\mathcal{R}_n$ presented in Appendix 1. Thus we are lead to study the spectra of $\mathcal{L}_{ab} = \mathcal{L}_{f-(Pr(F)+\xi)\tau}$ on the space of Holder functions. Now it may be worthwhile to recall Ruelle-Perron-Frobenius theorem in this setting. Without the loss of generality we may assume that $\|(\sigma')^{-1}\| \le \varepsilon_4 < 1$.

Proposition 5. *a) Let* $f \in C^\alpha(U)$ *and* $\mathcal{L}_f$ *be defined by formula (3) then there exist a positive function* $\hat{h} \in C^\alpha(U)$ *and a measure* $\hat{\nu}$ *on* U *such that*

i) $\mathcal{L}_f\hat{h} = e^{Pr(f)}\hat{h}$;

ii) $\mathcal{L}^*_f\hat{\nu} = e^{Pr(f)}\hat{\nu}$;

iii) There exist C_{10}, ε_5 such that $\forall h \in C^\alpha(U)$ $\forall n$

$$\|e^{-nPr(f)}\mathcal{L}_f^n h - \hat{\nu}(h)\hat{h}\|_\alpha \leq C_{10}\|h\|_\alpha \varepsilon_5^n;$$

iv) The measure $\nu = h\hat{\nu}$ is $\hat{\sigma}$ invariant;
b) If g^t is topologically mixing then for real $s \neq 0$ $r(\mathcal{L}_{f+is\tau}) < e^{Pr(f)}$;
c) The specter of $\mathcal{L}_{f+is\tau}$ in the annulus $\{\varepsilon_4^\alpha e^{Pr(f)} < |z| \leq e^{Pr(f)}\}$ consists of isolated eigenvalues of finite multiplicity.

II) Smoothing. (This is a technical step. The point is that we want to prove Theorem 1 for F being only Holder continuous. The way we do it is the following. We give a proof for $F \in C^1(M)$ and show at the same time that all the constants in Theorem 1 depend continuously on F in Holder norm. The reader who is only interested in the case $F \in C^1(M)$ can safely skip this subsection and assume in that follows that $f^{(b)} \equiv f$.) We have to study the spectra of $\mathcal{L}_{f-[Pr(F)+\xi]\tau}$. This operator fails to preserve $C^1(U)$ if $f \notin C^1(U)$. However the contribution of f to $\mathcal{L}_{ab}$ is 'small' comparing to the term $b\tau(u)$ which has C^1-norm of the order of $|b|$. Consider a smooth approximation of f denoted by $f^{(b)}$ which is obtained from f by means of averaging over the ball of radius $\frac{1}{\sqrt{|b|.}}$ This function has the following properties

1) $\|f - f^{(b)}\|_0 \leq G(f)(\frac{1}{\sqrt{|b|}})^\alpha$, $G(f)$ being the Holder constant of f;
2)$\|f^{(b)}\|_1 \leq C_{11}\sqrt{|b|}$.
3) $f^{(b)} \to f$ in $C^{\alpha'}(U)$, as $b \to \infty$ for any $\alpha' < \alpha$.

Denote by λ_{ab} the largest eigenvalue of $\mathcal{L}_{f^{(b)}-(Pr(F)+a)\tau}$ and let h_{ab} be the corresponding eigenvector normalized by the condition $\sup_u h_{ab} = 1$. We now estimate $\frac{\partial}{\partial u}h_{ab}$. We have

$$\frac{\frac{\partial}{\partial u}h_{ab}(u)}{\lambda_{ab}} = \sum_{\sigma v = u}\left\{e^{[f^{(b)}-(Pr(F)+a)\tau](v)}\frac{\partial h}{\partial v}\frac{\partial v}{\partial u} + \frac{\partial}{\partial u}\left(e^{[f^{(b)}-(Pr(F)+a)\tau](v)}\right)h(v)\right\}.$$

Since λ_{ab} depends continuously on a and $\frac{1}{|b|}$ and $\lambda_{00} = 1$ we conclude that λ_{ab} is close to 1 for small a and large b. By compactness of the family $\{h_{ab}\}$ in $C^{\alpha'}$ $\inf_U |h(U)|$ is uniformly bounded from below and we prove the following inequality.

Lemma 1. *For small a and large b $\|\frac{\partial}{\partial u}\ln h\| \leq C_{12}\sqrt{|b|}$.*

III) Ionescu-Tulcea-Marinescu inequalities. As we already saw it is more convenient to work with the normalized operator. Denote by

$$\hat{\mathcal{L}}_{ab}h(u) = \frac{1}{\lambda_{ab}h_{ab}(u)}\left[\tilde{\mathcal{L}}_{ab}(hh_{ab})\right](u) \tag{5}$$

where

$$\tilde{\mathcal{L}}_{ab} = \mathcal{L}_{f^{(b)}-(Pr(F)+a-ib)\tau}.$$

This is also transfer operator with the potential $f^{(ab)} + ib\tau$ where

$$f^{(ab)}(u) = f^{(b)}(u) - (Pr(F)+a)\,\tau(u) + \ln h_{ab}(u) - \ln h_{ab}(\sigma u) - \ln \lambda_{ab}.$$

We will compare $\hat{\mathcal{L}}_{ab}$ with the operator $\hat{\mathcal{M}}_{ab}$ defined by

$$(\hat{\mathcal{M}}_{ab}h)(u) = \sum_{\sigma v = u} e^{f^{(ab)}(v)}h(v).$$

$\hat{\mathcal{M}}$ is a Markov operator, that is $\hat{\mathcal{M}}1 = 1$.
We recall some *a priori* estimates which ensure that for fixed a, b and h the set $\{\hat{\mathcal{L}}^n_{ab}h\}$ is precompact in C^0-topology.
Lemma 2. *There exist constants $C_{13}, C_{14}, C_{15}, \varepsilon_6$ so that uniformly for small a's*
a) $|\hat{\mathcal{L}}^n_{ab}h|(u) \le (\hat{\mathcal{M}}^n_{ab}|h|)(u)$;
b) $|\frac{\partial}{\partial u}((\hat{\mathcal{L}}^n_{ab}h)(u))| \le C_{13}\left[\varepsilon_6^n(\hat{\mathcal{M}}^n_{ab}|h'|(u)) + |b|(\hat{\mathcal{M}}_{ab}|h|(u))\right]$ *in particular*
c) $\|\hat{\mathcal{L}}^n_{ab}h\| \le C_{13}\left[b\|h\|_0 + \varepsilon_6^n\|h'\|_0\right]$.
d) Let $h \in C^{\alpha'}(U)$, $\alpha' < \alpha$ *then*

$$\|\mathcal{L}^n_{ab}h\|_0 \le C_{14}\lambda^n_{ab}\|h\|_0$$

and

$$G(\mathcal{L}^n_{ab}h) \le C_{15}\lambda^n_{ab}\left(|b|\|h\|_0 + \varepsilon_4^{n\alpha'}G(h)\right)$$

Proof: a) is trivial since we just estimate every term by its absolute value. Let us prove b)

$$\left|\frac{\partial}{\partial u}(\hat{\mathcal{L}}^n_{ab}h)\right|(u) = \left|\sum_{\sigma^n v = u} e^{[f_n^{(ab)}+ib\tau_n](v)}\left\{\frac{\partial h}{\partial v}\frac{\partial v}{\partial u} + h\frac{\partial}{\partial u}\left(f_n^{(ab)} + ib\tau_n\right)\right\}\right| \le$$

$$\varepsilon_4^n\left(\hat{\mathcal{M}}^n_{ab}\left|\frac{\partial}{\partial u}h\right|\right)(u) + \left(|b|\cdot\|\frac{\partial}{\partial u}\tau_n(v)\|_0 + \|\frac{\partial}{\partial u}f_n^{(ab)}(v)\|_0\right)(\hat{\mathcal{M}}^n_{a,b}|h|)(u).$$

Hence b) follows from the following simple result

LEMMA 3. *Given $f \in C^1(U)$ there is a constant C_{16} independent on n such that for any inverse branch $v(u)$ of $u = \sigma^n v$ we have*

$$\|\frac{\partial}{\partial u} h_n(v)\|_0 \leq C_{16} \|\frac{\partial h}{\partial v}(v)\|_0.$$

PROOF:

$$\frac{\partial}{\partial u} h_n(v) = \sum_{j=1}^{n} \frac{\partial}{\partial u} h(\sigma^{-j} u) = \sum_{j=1}^{n} \frac{\partial h}{\partial \sigma^{-j} u} \frac{\partial \sigma^{-j} u}{\partial u}$$

and since $\frac{\partial \sigma^{-j} u}{\partial u}$ decays exponentially the claim is proven. ■

c) is immediate consequence of b). d) can be established by very similar calculations. ■

IV) THE MAIN ESTIMATE. Lemma 2 tells us that if we introduce the norm $\|h\|_{(b)} = \max(\|h\|_0, \frac{\|h'\|_0}{|b|})$ then $\|\hat{\mathcal{L}}_{ab}^n\|_{(b)}$ is uniformly bounded for all n and large b's. This estimate suggest that we have a chance to get uniform in $|b|$ bounds using this norm.

LEMMA 4. *There exist ε_7, n_0 such that if $\|h\|_{(b)} \leq 1$ then*

$$\int |\mathcal{L}_{ab}^{n_0 N} h|^2 d\nu \leq (1 - \varepsilon_7)^N$$

ν being the invariant measure for $\mathcal{L}_{f - Pr(F)\tau}$.

The proof of Lemma 4 is given in Sections 5-8.

COROLLARY 2. *There exist constants C_{17}, C_{18}, β_1 so that if $\|h\|_{(b)} \leq 1$ then*

$$|\mathcal{L}_{ab}^{C_{17} \ln |b|} h|(u) \leq \frac{C_{18}}{|b|^{\beta_1}}.$$

PROOF:

$$\left|\hat{\mathcal{L}}_{ab}^N h\right|(u) = \left|\hat{\mathcal{L}}_{ab}^{N-\tilde{N}} (\hat{\mathcal{L}}_{ab}^{\tilde{N}} h)\right|(u) \leq \hat{\mathcal{M}}_{ab}^{N-\tilde{N}} (\left|\hat{\mathcal{L}}_{ab}^{\tilde{N}} h)\right|)(u) \quad \text{(Lemma 2)}$$

$$= \left(\hat{\mathcal{M}}_{a0}^{N-\tilde{N}} \left(\exp\left[(f^{(ab)} - f^{(a0)})_{N-\tilde{N}} \circ \sigma^{N-\tilde{N}}\right] \left|\hat{\mathcal{L}}_{ab}^{\tilde{N}} h\right|\right)\right)(u) \quad \text{(defenition of } \hat{\mathcal{M}}_{ab})$$

$$\leq \hat{\mathcal{M}}_{a0}^{N-\tilde{N}} \left(\exp\left[2(f^{(ab)} - f^{(a0)}) \circ \sigma^{N-\tilde{N}}\right]\right)(u) \hat{\mathcal{M}}_{ab}^{N-\tilde{N}} (|\mathcal{L}_{ab}^{\tilde{N}} h|^2)(u) \quad \text{(Couchy} - \text{Shwartz)}$$

Now we apply Ruelle-Perron-Frobenius Theorem

$$\hat{\mathcal{M}}_{ab}^{N-\tilde{N}} (|\hat{\mathcal{L}}_{ab}^{\tilde{N}} h|^2)(u) \leq \nu(|\hat{\mathcal{L}}_{ab}^{\tilde{N}} h|^2) + C_{10} \|\mathcal{L}_{ab}^{\tilde{N}} h\|_{C^1(U)} \varepsilon_5^{N-\tilde{N}} \leq$$

$$(1-\varepsilon_8)^{\tilde N}+C_{19}\varepsilon_5^{N-\tilde N}|b| \tag{6}$$

where the second term in the last inequality is estimated by Lemma 2. On the other hand

$$\left(\hat{\mathcal{M}}_{a0}^{N-\tilde N}\left(\exp\left[(f^{(ab)}-f^{(a0)})_{N-\tilde N}\circ\sigma^{N-\tilde N}\right]\right)\right)(u)=(\mathcal{L}_{f^{(a0)}+2(f^{(ab)}-f_{(a0)})}1)(u)\le$$
$$C_{20}\exp[(N-\tilde N)Pr(f^{(a0)}+2(f^{(ab)}-f_{(a0)}))].$$

Since Pr depends analytically on a, $\frac{1}{|b|}$ and $Pr(f^{(a0)})=0$ the last expression is bounded by $C_{20}\exp\left[(N-\tilde N)C_{21}(|a|+\frac{1}{|b|})\right]$. Collecting all terms together we obtain

$$|\hat{\mathcal{L}}_{ab}^N h|(u)\le\left\{C_{20}\exp\left[(N-\tilde N)C_{21}(|a|+\frac{1}{|b|})\right]\left((1-\varepsilon_8)^{\tilde N}+C_{10}\varepsilon_5^{N-\tilde N}\right)\right\}^{\frac12}.$$

So if we choose $\tilde N=C_{22}\ln|b|$ and $C_{17}\gg C_{22}$ the RHS of the last inequality has the required decay for small a and large b. ■

V) A PRIORI BOUNDS FOR $\hat\rho$. Estimates of the previous step enable us to get the following inequalities.

COROLLARY 3. *Let $\alpha'<\alpha$ (where α is the Holder exponent for f), then for small a and large b there exist constants C_{23},C_{24},β_2 so that*
a) $\|\hat{\mathcal{L}}_{ab}^{C_{23}\ln|b|}\|_{\alpha'}\le\frac{1}{|b|^{\beta_2}}$;
b) $\|(1-\hat{\mathcal{L}}_{ab})^{-1}h\|_0\le C_{24}\ln|b|\|h\|_{\alpha'}$.

In case $f\in C^1(U)$, $\alpha'=1$ Corollary 3 follows immediately from Lemma 2 and Corollary 2. The general case is treated by smoothing. See Appendix 2.

COROLLARY 4. *Let $A,B\in C^{\alpha'}(M)$ then $\hat\rho_{A,B}(\xi)$ has an analytic continuation to $\{|\Re\xi|\le a_0,\ |\Im\xi|\ge b_0\}$ and*

$$|\hat\rho_{A,B}(\xi)|\le C_{25}|\Im\xi|^2\ln|\Im\xi|\|A\|_{\alpha'}\|B\|_{\alpha'}.$$

Corollary 4 is derived from Corollary 3 by direct but lengthy calculations. For details consult Appendix 2.

VI) INTEGRATION BY PARTS. We now come to the case when A and B are smooth. Denote by ∂_t the differentiation along the orbits of g^t. Write $\rho_{A,B}(t)$ as

$$\rho_{A,B}(t)=\sum_{j=0}^{3}(\frac{\partial}{\partial t})^j\rho(0)\frac{t^j}{j!}+\int_0^t\frac{(t-s)^4}{4!}\rho_{A,\,\partial_t^4 B}(s)\,ds.$$

Laplace transform of the last term decays near the imaginary axis not slower than $\frac{C_{25}\ln|\xi|}{|\xi|^4}$ and has a pole of the forth order at 0. Therefore the application of the inversion formula for Laplace transform and the change of the contour of the integration prove Theorems 1 and 2.

5. An example. In this section we demonstrate the idea of the proof of the main estimate (Corollary 2) on the simplest example. Namely we consider the case then (M, g^t) is a geodesic flow on the unit tangent bundle over a negatively curved surface and μ_F is the Lesbegue measure. In this case τ and f are smooth (of class $C^{1+\gamma}$ [HP]) so $f^{(b)} = f$, $f^{(ab)} = f^{(a)}$. Also ν is absolutely continuous so that $d\nu = g(u)\,du$.

The important role in our consideration is played by the axiom of the uniform non-integrability (UNI) introduced by Chernov in [Ch2] where it was used to prove subexponential decay of correlations in the above setting. Here we recall this property. Let $x, y \in M$. Denote by p_{xy} the natural projection of $W^u_{loc}(x)$ to $W^u_{loc}(y)$ along the leaves of W^{ss}. We can introduce on $W^u_{loc}(x)$ and $W^u_{loc}(y)$ coordinate systems (u, t) in such a way that $g^t(u_0, t_0) = (u_0, t_0 + t)$, the curves $\{t = t_0\}$ are leaves of the strong unstable foliation and $u \circ p_{xy} = p_{xy} \circ u$. Let γ be the image of $W^{su}_{loc}(x)$. In our coordinate system γ is a graph of a function $t = T(u)$. Let $u(y)$ be $u-$coordinate of y and $u(x)$ be $u-$coordinate of x. Define

$$\varphi(x, y) = T(u(y)) - T(u(x)).$$

Denote by $x_1, y_1 \in W^u(y)$ the points with the coordinates $x_1 = (u(x), T(u(x)))$, $y_1 = (u(y), T(u(y)))$. The condition (UNI) reads as follows

$$C_{26} < \frac{\varphi(x, y)}{|xx_1||x_1y_1|} < C_{27}. \qquad (UNI)$$

The importance of the function φ is clear from the following simple observation. Let γ_1 be a curve in $W^u(x)$ given by the equation $t = T_1(u)$ and γ_2 be its image in $W^u(y)$. Assume that $\gamma_2 = \{(u, t) : t = T_2(u)\}$, then

$$\varphi(x, y) = T_2(u(y)) - T_2(u(x)) - T_1(u(y)) + T_1(u(x)). \qquad (7)$$

Another useful property of φ is the following. Let W_1 and W_2 be two pieces of local unstable manifolds and $p : W_1 \to W_2$ be the projection along strong stable leaves. For $u_1, u_2 \in W_1$ set $\Phi(u_1, u_2) = \varphi(u_1, pu_2)$ then

$$\Phi(u_1, u_2) + \Phi(u_2, u_3) = \Phi(u_1, u_3). \qquad (8)$$

We now show how to employ this condition for the study of the spectral properties of our transfer operators. In the proof of corollary 1 we used Lemma 4 only for $N = C_{22} \ln |b|$ with some constant C_{22}. So it is enough to prove the following bound.

LEMMA 5. *There exist constants C_{28}, β_3 such that for small $a's$ and large $b's$ and any h with $\|h\|_{(b)} \leq 1$ the following inequality holds*

$$\int \left| (\hat{\mathcal{L}}_{a,b}^{C_{28} \ln |b|} h)(u) \right|^2 \, d\nu(u) \leq \frac{1}{|b|^{\beta_3}}.$$

Before proving this statement let us point out the analogue with some estimates in oscillatory integral theory. We can formally write

$$(\hat{\mathcal{L}}_{ab} h)(u) = \int g^{(a)}(v) \delta(u - \sigma v) e^{ib\tau(v)} h(v) \, dv$$

where $g^{(a)}$ is some real valued function. So let us recall a similar result from the theory of oscillatory operators with smooth kernels. Let

$$(K_b h)(u) = \int k(u, v) e^{ibT(u,v)} h(v) dv.$$

Then under appropriate hypotheses, for example, if

$$\left| \frac{\partial^2}{\partial u \partial v} T \right| \geq c \tag{9}$$

one can prove power decay of the spectral radius (see [St]). To this end one considers the L^2-norm

$$\int |Kh|^2(u) \, du = \int \int \tilde{k}(u, v) h(u) \overline{h(v)} \, du dv$$

where the kernel $\tilde{k}(u, v)$ is given by

$$\tilde{k}(u, v) = \int k(u, w) \overline{k(v, w)} e^{ib[T(u,w) - T(v,w)]} \, dv.$$

Away from the diagonal one can estimate this expression integrating by parts since by (9)

$$\left| \frac{\partial}{\partial w} [T(u, w) - T(v, w)] \right| \geq c|u - v| \tag{10}$$

We apply the same strategy here:

$$|\hat{\mathcal{L}}_{ab}h|^2(u) = \sum_{\sigma v_1=\sigma v_2=u} \exp\left(f^{(a)}(v_1)+f^{(a)}(v_2)+ib[\tau(v_1)-\tau(v_2)]\right) h(v_1)\overline{h(v_2)}.$$

However, for the first power of $\hat{\mathcal{L}}_{ab}$ we can not gain much since even though if we could show that off-diagonal terms can be neglected the diagonal contribute by the amount independent on $|b|$. Therefore we have to consider higher powers of $\hat{\mathcal{L}}_{ab}$ so that the pairs (v_1, v_2) become uniformly distributed on $U \times U$. Let us now give the formal proof.

PROOF: $U = \bigcup_i U_i$ It is enough to bound the integral over U_1

$$\int_{U_1} \left|\hat{\mathcal{L}}^N_{a,b}h\right|^2 \, d\nu(u) =$$

$$\sum_{v_1,v_2}\int_{U_1} \exp\left(f_N^{(a)}(v_1)+f_N^{(a)}(v_2)+ib[\tau_N(v_1)-\tau_N(v_2)]\right) h(v_1)\overline{h(v_2)}g(u)d(u),$$

where the sum is taken over all the branches $(v_1(u), v_2(u))$ of σ^{-N}. Decompose this sum into two parts. Define $d(v_1,v_2) = \inf_u \operatorname{dist}(\hat{\sigma}_N(v_2(u)), \hat{\sigma}_N(v_2(u))$. Let $I_1(\delta)$ be the sum over all pairs (v_1, v_2) with $d(v_1,v_2) < \delta$ and $I_2(\delta)$ be the remaining part. Then

$$I_1(\delta) \le \sum_{d(v_1,v_2)<\delta} \int_{U_1} e^{f_N^{(a)}(v_1)+f_N^{(a)}(v_2)} g(u)\, du.$$

LEMMA 6. *There exist constants C_{29} and β_4 such that*

$$\sum_{d(v_1,v_2)<\delta} \int e^{f_N^{(a)}(v_1(u))+f_N^{(a)}(v_2(u))} g(u)\, du \le C_{29}\delta^{\beta_4}.$$

This lemma is proven in Appendix 3. Here we give heuristic arguments. We know that $\hat{\sigma} : \Pi \to \Pi$ is exponentially mixing with respect to ν. So the sum above up to exponentially small correction equals to the probability (with respect to ν) that if points v_1 and v_2 are chosen independently the distance between the projections of $\hat{\sigma}^N v_1$ and $\hat{\sigma}^N v_2$ to $S(\Pi)$ are within distance δ from each other.

To estimate $I_2(\delta)$ we need the following elementary estimate from the real analysis ([St]).

LEMMA 7. (VAN DER CORPUT LEMMA). *Let*

$$I = \int_J e^{ib\psi(u)} r(u)\, du$$

where the integration is performed over a segment J*. Assume that* $\psi \in C^{1+\gamma}(J)$*,* $\|\psi\|_{1+\gamma} \leq c_1$*,* $|\psi'(u)| \geq c_2$*, where* $\frac{1}{b} \leq c_2 \leq 1$*,* $\|r\|_0 \leq \epsilon$ *and* $\|r(u)\|_1 \leq \epsilon D$ *then*

$$|I| \leq \epsilon \mathrm{Const}(c_1) \left[\frac{D+1}{|b| c_2} + \frac{1}{|b|^\gamma c_2^2} \right].$$

If $\gamma = 2$ the lemma follows from integration by parts. The general case requires additional smoothing. See Appendix 4.
We now apply Lemma 7 to estimate

$$\int \exp\left(f_N^{(a)}(v_1(u)) + f_N^{(a)}(v_2) + ib[\tau_N(v_1) - \tau_N(v_2)] \right) h(v_1)\overline{h(v_2)} g(u)\, du$$

if $d(v_1, v_2) \geq \delta$. Set $\psi(u) = \tau_N(v_1(u)) - \tau_N(v_2)(u)$,

$$r(u) = \exp\left(f_N^{(a)}(v_1(u)) + f_N^{(a)}(v_2) \right) h(v_1)\overline{h(v_2)} g(u).$$

Then $\epsilon = \sup_u e^{f_N^{(a)}(v_1(u)) + f_N^{(a)}(v_2(u))}$, c_1 is uniformly bounded in N and

$$D = C_{30}(1 + |b|)\|h\|_{(b)} \| \frac{dv}{du} \|_0 \leq C_{30}(1 + |b| \varepsilon_4^N)$$

by Lemma 3. We have to bound $|\frac{\partial}{\partial u}\psi|$ from below. By (7)

$$[\tau_N(v_1(u_1)) - \tau_N(v_1(u_2))] - [\tau_N(v_2(u_1)) - \tau_N(v_2(u_2)]$$
$$= \varphi(\hat{\sigma}^N v_2(u_2), \hat{\sigma}^N v_1(u_1)) \tag{11}$$

and condition (UNI) implies that $c_2 \geq C_{31}\delta$ (cf. (10)). Hence

$$\int \exp\left(f_N^{(a)}(v_1(u)) + f_N^{(a)}(v_2) + ib[\tau_N(v_1) - \tau_N(v_2)] \right) h(v_1)\overline{h(v_2)} g(u)\, du \leq$$

$$C_{32} \sup_u e^{f_N^{(a)}(v_1(u)) + f_N^{(a)}(v_2(u))} \left[\frac{1 + |b|\varepsilon^N}{|b|\delta^2} + \frac{1}{|b|^\gamma \delta} \right].$$

Take some $u_0 \in U_1$. By Lemma 3

$$\sup_u e^{f_N^{(a)}(v_1(u))+f_N^{(a)}(v_2(u))} \leq C_{33} e^{f_N^{(a)}(v_1(u_0))+f_N^{(a)}(v_2(u_0))}.$$

Now specify $N = C_{28} \ln |b|$, $\delta = \frac{1}{|b|^{\beta_5}}$, where $\beta_5 < \min(\gamma, \frac{1}{2})$. Then $I_1(\delta) \leq \frac{C_{34}}{|b|^{\beta_6}}$ and

$$I_2(\delta) \leq \frac{C_{35}}{|b|_7^{\beta}} \sum_{(v_1,v_2)} e^{f_N^{(a)}(v_1(u_0))+f_N^{(a)}(v_2(u_0))}.$$

the last sum equals

$$\left(\sum_{\sigma^N v = u_0} e^{f_N^{(a)}(v)} \right)^2 = (\hat{\mathcal{M}}_a^N 1)^2 = 1$$

and Lemma 5 is proven. ∎

6. Description of the inductive procedure. In this section we begin with the proof of the main estimate without regularity assumptions of the last section. Comparing with Section 5 there are two additional difficulties to overcome. The first one is that where we had uniform estimate of $|\frac{\partial}{\partial u}(\tau_N(v_1(u)) - \tau_N(v_2(u)))|$ for most pairs (v_1, v_2) (Lemma 6). Now we know it only for some (v_1, v_2). More precisely let $d = \dim W^{(su)}$.

LEMMA 8. *There exist ε_9, N_0 and vectorfields $e_1(u), e_2(u) \dots e_d(u)$ such that $\|e_k(u)\| > \frac{1}{2}$ and for any $N \geq N_0$ there are two branches $v_1(u)$ and $v_2(u)$ of σ^{-N} such that*

$$\varepsilon_9 \leq |\partial_{e_1}(\tau_N(v_1(u)) - \tau_N(v_2(u)))| \leq 3\varepsilon_9$$

and for $k = 2 \dots d$

$$|\partial_{e_k}(\tau_N(v_1(u)) - \tau_N(v_2(u)))| \leq \frac{\varepsilon_9}{100\sqrt{d}}.$$

(Here 100 can be replaced by any constant grater than 2 and $\sqrt{d}$ is the diameter of the unit cube in $\mathbf{R}^d$.) Of course it is lower bound which is of primary interest here. The upper bound is added just for technical reasons.

PROOF: φ is C^1 function which is not identically zero on $\bigcup_i (\Pi_i \times \Pi_i)$. Take some (x_0, y_0) such that $\varphi(x_0, y_0) \neq 0$. Denote $U^{(N)} = \hat{\sigma}^N U$. As $N \to \infty$ $U^{(N)}$ fills Π densely. So we may assume that $x_0, y_0 \in U^{(n_0)}$ for some n_0. To fix our notation suppose that $x_0, y_0 \in \Pi_1$. Let $p_1 : U_1 \to W^u_{\Pi_1}(x_0)$ and $p_2 : U_1 \to W^u_{\Pi_1}(y_0)$ be the canonical isomorphisms. Let $\Phi(u_1, u_2) = \varphi(p_1 u_1, p_2 u_2)$. Denote $\bar{u} = p_2^{-1} y_0$. Since $\Phi(\bar{u}, \bar{u}) = 0$ and $\Phi(p_1^{-1} x_0, \bar{u}) \neq 0$, $\frac{\partial}{\partial u}\varphi(u, \bar{u})$ is not identically zero. Hence there exist an open set U_0 such that $\|\frac{\partial}{\partial u}\Phi(u, \bar{u})\| \geq 2\varepsilon_9$ for some ε_9. Choose a coordinate system in $z_1, z_2 \dots z_d$ in U_0 so that $\frac{\partial}{\partial z_1}\Phi(\cdot, \bar{u}) = 1$, $\frac{\partial}{\partial z_k}\Phi(\cdot, \bar{u}) = 0$, for $k = 2 \dots d$ and $\|\frac{\partial}{\partial z_j}\| \geq \frac{1}{\varepsilon_9}$. Let $\tilde{e}_k(u) = \varepsilon_9 \frac{\partial}{\partial z_k}$. Take n_1 so large that $\sigma^{n_1} U_0 = U$ and set $e_k = d\sigma^{n_1}\tilde{e}_k$. Recall that $W^u_{\Pi_1}(x_0) \in U^{(n_0)}$ and $W^u_{\Pi_1}(y_0) \in U^{(n_0)}$. Let $\tilde{v}_1(u)$ and $\tilde{v}_2(u)$ be corresponding branches of σ^{-n_0}. By (5) and (6) $\frac{\partial}{\partial z_1}[\tau_{n_0}(\tilde{v}_1(u)) - \tau_{n_0}(\tilde{v}_2(u)] = 1$, $\frac{\partial}{\partial z_k}[\tau_{n_0}(\tilde{v}_1(u)) - \tau_{n_0}(\tilde{v}_2(u)] = 0$, for $k = 2 \dots d$. Denote $V_1 = \tilde{v}_1(U_0)$, $V_2 = \tilde{v}_2(U_0)$. To complete the proof we need the following statement.

LEMMA 9. *There exist n_2 such that for $n > n_2$ there is a branch $v(u)$ of σ^{-n} such that*

$$\|\frac{\partial}{\partial u}\tau_n(v(u))\| \leq \frac{\varepsilon_9}{200\sqrt{d}}.$$

PROOF: By the definition of φ

$$\tau_n(v(u_1)) - \tau_n(v(u_2)) = \varphi(\hat{\sigma}^n v(u_1), p_u \hat{\sigma}^n u_2). \tag{12}$$

$\frac{\partial}{\partial x}\varphi(x, y)$ depends continuously on y and vanish for $y = x$ (since $\varphi(x, y) = 0$ for $x \in W^{(u)}_{loc}(y) \bigcup W^{(s)}_{loc}(y)$). For large n $U^{(n)}$ fills Π densely so we can pick up $v(u)$ such that $\hat{\sigma}^n v(u)$ is very close to U and the statement follows by (12). ∎

Let $N_0 = n_0 + n_1 + n_2$. There exist two branches $v_1(u)$ and $v_2(u)$ such that $\sigma^{N-n_0-n_1} v_1 \subset V_1$, $\sigma^{N-n_0-n_1} v_2 \subset V_2$ and $|\partial_{\tilde{e}_k}(\tau_{N-n_0-n_1}(v_1)| \leq \frac{\varepsilon_9}{200\sqrt{d}}$, $|\partial_{\tilde{e}_k}(\tau_{N-n_0-n_1}(v_2)| \leq \frac{\varepsilon_9}{200\sqrt{d}}$. Then

$$\partial_{e_k}\left[\tau_N(v_1) - \tau_N(v_2)\right] =$$

$$\partial_{e_k}\left[\tau_{N-n_1}(v_1) - \tau_{N-n_1}(v_2)\right] \quad (\text{since } \hat{\sigma}^{N-n_1} v_1 \in W^s_\Pi(v_2))$$

$$= \partial_{\tilde{e}_k}\left[\tau_{N-n_1}(v_1) - \tau_{N-n_1}(v_2)\right] =$$

$$\partial_{\tilde{e}_k}\left[\tau_{n_0}(\sigma^{N-n_0-n_1} v_1) - \tau_{n_0}(\sigma^{N-n_0-n_1} v_2)\right] + \partial_{e_k}\left[\tau_{N-n_0-n_1}(v_1) - \tau_{N-n_0-n_1}(v_2)\right].$$

The first term is always less than $\frac{\varepsilon_9}{100\sqrt{d}}$ while the second one is $2\varepsilon_9$ or 0 depending on if $k = 1$ or $k > 1$. ∎

The second problem is that if ν is not absolutely continuous there is no integration by parts formula. Nonetheless we can still prove a weaker version of van der Corput lemma and use it to obtain the following inequality.

LEMMA 10. *There exist $\bar{n}, \varepsilon_{10}$ so that if $\|h\|_{(b)} \leq 1$ then*

$$\int |\hat{\mathcal{L}}_{ab}^{\bar{n}} h|^2 \, d\nu \leq 1 - \varepsilon_{10}. \tag{13}$$

This lemma however does not suffice to obtain Corollary 1 because if we try to repeat its proof using Lemma 10 instead of lemma 4 the term $|b|(1-\varepsilon_5)^{N-\tilde{N}}$ in (6) still force us take N of the order of $\ln |b|$ and this would lead only to the bound

$$\|\hat{\mathcal{L}}_{ab}^{\mathrm{Const}\ln|b|}\| \leq 1 - \epsilon$$

which is much less than we want. Therefore we have to iterate (13). For this we need a local version of Lemma 10. Denote by K_A the cone

$$K_A = \{h \in C^1(U) : \|\frac{\partial}{\partial u} \ln h\| \leq A\}.$$

LEMMA 10′. *There exist $\bar{n}, \varepsilon_{10}$ and E such that if $|h(u)| \leq H(u)$ and $\|h'(u)\| \leq E|b|H(u)$ for some $H \in K_{E|b|}$ then*

$$\int |\hat{\mathcal{L}}_{ab}^{\bar{n}} h|^2 \, d\nu \leq (1 - \varepsilon_{10}) \int H^2 \, d\nu.$$

(Lemma 10 is just a particular case when $H \equiv 1$. So Lemma 10′ tells us that Lemma 10 remains valid if we replace the constant function by a function which looks like a constant on the scale $\frac{1}{|b|}$.)

The only problem now is to find a suitable majorant for $\hat{\mathcal{L}}_{ab}^{k\bar{n}} h$. Fortunately it is provided in the proof of Lemma 10′.

LEMMA 10″. *There exist $\varepsilon, \bar{n}, E$ so that for given b there is a finite number $\mathcal{N}_1(b), \mathcal{N}_2(b) \dots \mathcal{N}_{l(b)}(b)$ of linear operators such that*

a) $\mathcal{N}_j(b)$ preserves $K_{E|b|}$;

b) For $H \in K_{E|b|}$

$$\int |\mathcal{N}_j H|^2 \, d\nu \leq (1 - \varepsilon_{10}) \int H^2 \, d\nu;$$

c) If $|h(u)| \leq H(u)$, $\|h'(u)\| \leq E|b|H(u)$ for some $H \in K_{E|b|}$ then there exist $j = j(h, H)$ such that $|\hat{\mathcal{L}}_{ab}^{\bar{n}} h(u)| \leq (\mathcal{N}_j(H))(u)$ and $\|(\hat{\mathcal{L}}_{ab}^{\bar{n}} h)'(u)\| \leq E|b|(\mathcal{N}_j H)(u)$.

Lemma $10''$ clearly implies Lemma 4. Indeed denote $h^{(k)}=\hat{\mathcal{L}}_{ab}^{k\bar{n}}h$. Let $H^{(0)}\equiv 1$ and set $H^{(k+1)}=\mathcal{N}_{j(h^{(k)},H^{(k)})}H^{(k)}$. Then by induction $h^{(k+1)}\leq H^{(k+1)}$, $\|(h^{(k+1)})'(u)\|\leq E|b|H^{(k+1)}(u)$ and

$$\nu(\left(H^{(k+1)}\right)^2)\leq(1-\varepsilon_{10})\nu(|H^{(k)}|^2)\leq(1-\varepsilon_{10})^{k+1}.$$

Therefore

$$\nu(|h^{(k+1)}|^2)\leq\nu(\left(H^{(k+1)}\right)^2)\leq(1-\varepsilon_{10})^{k+1}.$$

In Section 7 we define $\mathcal{N}_j$. Lemma $10''$ is proven in Section 8.

7. Construction of $\mathcal{N}_j's$. Take a cutoff function $\Delta(x):\mathbf{R}^d\to\mathbf{R}$ such that

a) $\Delta(x)\geq 0$;

b) $\Delta(x)\equiv 0$ for $|x|\geq 1$;

c) $\Delta(x)=1$ for $|x|\leq\frac{1}{2}$.

If R is a cube centered at x_0 with side $2a$ let $\Delta_R(x)=\Delta(\frac{x-x_0}{a})$. Recall U_0, $z_1,z_2\dots z_d$ constructed in the proof of Lemma 8. Divide U_0 into cubes

$$Z_{\vec{l}}=\{\frac{l_i\varepsilon_{11}}{|b|}\leq z_i\leq\frac{(l_i+1)\varepsilon_{11}}{|b|}\}\tag{14}$$

where ε_{11} will be specified below. Denote $Y^1_{\vec{l}}=\tilde{v}_1(U_0)$, $Y^2_{\vec{l}}=\tilde{v}_2(U_0)$ where $\tilde{v}_j(u)$ were defined in the proof of Lemma 8. Let J be some subcollection of $\{Y^1_{\vec{l}}\}\bigcup\{Y^2_{\vec{l}}\}$. Write $Y(J)=\bigcup_J Y^i_{\vec{l}}$. Let $v_1(u)$ and $v_2(u)$ be two branches of $\sigma^{-\bar{n}}$ constructed in Lemma 8. Define the function

$$m_{\epsilon,J}(v)=\begin{cases}1, & \text{if } v\notin v_1(U)\bigcup v_2(U)\\ 1, & \text{if } \sigma^{\bar{n}-n_0-n_1}v\notin Y(J)\\ 1-\varepsilon\Delta_{Z_{\vec{l}}}(\sigma^{\bar{n}-n_0}v), & \text{if } \sigma^{\bar{n}-n_0-n_1}v\in Y^i_{\vec{l}}\subset Y(J)\end{cases}.$$

Define $\mathcal{N}_{ab}^{(J,\varepsilon_{12})}h=\hat{\mathcal{M}}_{ab}^{\bar{n}}(m_{\varepsilon_{12},J}h)$. Precise conditions on J's, $\varepsilon_{12},\bar{n},E$ will be given below. First we choose E (Lemma 11). After that we choose $\bar{n}$ and then ε_{12} (in the proof of Lemma 13). Given $E,\bar{n},\varepsilon_{12}$ the set of J's is specified by Lemma 12. Below we give some properties of $\mathcal{N}_{ab}^{(J,\varepsilon_{12})}$.

Proposition 6. *If $\bar{n}$ is large enough $\mathcal{N}_{ab}^{(J,\varepsilon_{12})}$ preserves $K_{E|b|}$.*

Proof: Direct calculation shows that the multiplication by $m_{J,\varepsilon_{12}}$ maps $K_{E|b|}$ to $K_{C_{36}E|b|}$ and by Lemma 2 $\hat{\mathcal{M}}_{ab}^{\bar{n}}:K_{C_{36}E|b|}\to K_{\varepsilon_4^{\bar{n}}C_{36}E|b|+C_{37}}$ Take $\bar{n}$ so large that $\varepsilon_4^{\bar{n}}C_{36}E|b|+C_{37}<E|b|$. ∎

LEMMA 11. *If $E, \bar{n}$ are large enough then for any (h, H) such that $H \in K_{E|b|}$ $|h(u)| \leq H(u)$ and $\|h'(u)\| \leq E|b|H(u)$ the following inequality holds*

$$\|(\hat{\mathcal{L}}_{ab}^{\bar{n}} h)'(u)\| \leq E|b|(\mathcal{N}_{ab}^{(J,\varepsilon_{12})} H)(u).$$

PROOF: By Lemma 2

$$\|(\hat{\mathcal{L}}_{ab}^{\bar{n}} h)'(u)\| \leq (C_{13}\varepsilon_4^{\bar{n}} E + 1)|b|(\hat{\mathcal{M}}_{ab}^{\bar{n}} H)(u) \leq \frac{(C_{13}\varepsilon_4^{\bar{n}} E + 1)|b|}{(1 - \varepsilon_{12})}(\mathcal{N}_{ab}^{(J,\varepsilon_{12})} H)(u).$$

Choose $E, \bar{n}$ so large that $\frac{(C_{13}\varepsilon_4^{\bar{n}} E+1)}{(1-\varepsilon_{12})} \leq E$. ∎

Before proceeding further recall another property of ν.

DEFINITION. A measure μ on a metric space (X, ρ) is called Federer measure if given N there exist a constant C_N such that for all x, r $\mu(B(x, Nr)) \leq C_N \mu(B(x, r))$.

PROPOSITION 7. *ν is a Federer measure.*

Under the conditions of theorem 2 (ν is SBR-measure) this is immediate corollary of absolute continuity. The proof under the conditions of theorem 1 ($d = 1$) is provided in Appendix 3.

DEFINITION. A set Y is called (r, N)-dense in X if the intersection of any ball $B(x, Nr)$ with Y contains a ball of radius r.

COROLLARY 6. *Given E, N there exist a constant $\epsilon = \epsilon(E, N)$ such that if W is (N, r)-dense in U and $H \in K_{\frac{E}{r}}$ then*

$$\int_W H^2 \, d\nu \geq \epsilon \int_U H^2 \, d\nu.$$

We say that J is dense if for any $\vec{l}$ there is a cube $Y_{\vec{l}'}^i \in J$ such that $\sigma^{n_0} Y_{\vec{l}'}^i$ is adjacent to $Z_{\vec{l}}$.

LEMMA 12. *Given $E, \varepsilon_{12}, \bar{n}$ there exist ε_{10} such that if J is dense, $H \in K_{E|b|}$ then*

$$\int (\mathcal{N}_{ab}^{(J,\varepsilon_{12})} H)^2 \, d\nu \leq (1 - \varepsilon_{10}) \int H^2 \, d\nu.$$

PROOF:

$$(\mathcal{N}_{ab}^{(J,\varepsilon_{12})} H)^2(u) = (\hat{\mathcal{M}}_{ab}^{\bar{n}}(m_{J,\varepsilon_{12}} H))^2(u) \leq (\hat{\mathcal{M}}_{ab}^{\bar{n}} m_{J,\varepsilon_{12}}^2)(u)(\hat{\mathcal{M}}_{ab}^{\bar{n}} H^2)(u).$$

For fixed $\bar{n}$ there exist ε_{13} such that if $m_{J,\varepsilon_{12}}(v_1(u)) = 1-\varepsilon_{12}$ or $m_{J,\varepsilon_{12}}(v_2(u)) = 1-\varepsilon_{12}$ then $(\hat{\mathcal{M}}^{\bar{n}}_{ab} m_{J,\varepsilon_{12}}) \leq (1-\varepsilon_{13})$. Let W be set of such u's. If J is dense then W is $(\frac{1}{\varepsilon_{14}^2}, \frac{\varepsilon_{14}}{|b|})$-dense for some ε_{14}. Hence

$$\nu(\mathcal{N}^{(J,\varepsilon_{12})}_{ab} H)^2 \leq \nu(\hat{\mathcal{M}}^{\bar{n}}_{ab} H^2) - \varepsilon_{13} \int\limits_W (\hat{\mathcal{M}}^{\bar{n}}_{ab} H^2)\, d\nu \leq (1-\varepsilon_{15}\varepsilon_{13})\nu(\hat{\mathcal{M}}^{\bar{n}}_{ab} H^2)$$

by Corollary 6 and Proposition 6. Now

$$\hat{\mathcal{M}}^{\bar{n}}_{ab} h = \hat{\mathcal{M}}^{\bar{n}}_{a0}(e^{(f^{(ab)}-f^{(a0)})_{\bar{n}}\circ\sigma^{\bar{n}}} h) \leq C_{38}(|a| + \frac{1}{|b|})\hat{\mathcal{M}}^{\bar{n}}_{a0} h$$

where C_{38} depends only on $\bar{n}$. Hence

$$\nu(\hat{\mathcal{M}}^{\bar{n}}_{ab} H^2) \leq (1-\varepsilon_{13}\varepsilon_{15})\left(1 + C_{38}(|a| + \frac{1}{|b|})\right)\nu(\hat{\mathcal{M}}^{\bar{n}}_{a0}((H^{(k)})^2) \leq$$

$$(1-\varepsilon_{13}\varepsilon_{15})\left(1 + C_{38}(|a| + \frac{1}{|b|})\right)\nu\left((H^{(k)})^2\right).$$

If a is small enough and b is large enough the above factor is less than 1. ■

8. End of the proof of lemma $10''$. It remains to show that if $|h| \leq H$, $\|h'\| \leq E|b|H$ for $H \in K_{E|b|}$ then for ε_{12} small enough there exist dense J so that

$$|\hat{\mathcal{L}}^{\bar{n}}_{ab} h|(u) \leq (\mathcal{N}^{(J,\varepsilon_{12})}_{ab} H)(u).$$

Let

$$\gamma^{(1)}_{\varepsilon}(u) = \frac{|e^{(f^{(ab)}_{\bar{n}}+ib\tau_{\bar{n}})(v_1(u))} h(v_1(u)) + e^{(f^{(ab)}_{\bar{n}}+ib\tau_{\bar{n}})(v_2(u))}(v_2(u))|}{(1-\varepsilon)e^{(f^{(ab)}_{\bar{n}})(v_1(u))} H(v_1(u)) + e^{(f^{(ab)}_{\bar{n}})(v_2(u))} H(v_2)},$$

$$\gamma^{(2)}_{\varepsilon}(u) = \frac{|e^{(f^{(ab)}_{\bar{n}}+ib\tau_{\bar{n}})(v_1(u))} h(v_1(u)) + e^{(f^{(ab)}_{\bar{n}}+ib\tau_{\bar{n}})(v_2(u))}(v_2(u))|}{e^{(f^{(ab)}_{\bar{n}})(v_1(u))} H(v_1(u)) + (1-\varepsilon)e^{(f^{(ab)}_{\bar{n}})(v_2(u))} H(v_2)}.$$

Denote $V_{\vec{l}} = \sigma^{n_1} Z_{\vec{l}}$, $X^i_{\vec{l}} = \{v : \ v = v_i(u)$ for some u and $\sigma^{n-n_0-n_1} v \in Y^i_{\vec{l}}\}$.

Lemma 13. *The following statement holds provided that $\varepsilon_{12}, \varepsilon_{11}$ (see (14)) are small enough. Let cubes $Z_{\vec{l}^{I}}$, $Z_{\vec{l}^{II}}$ and $Z_{\vec{l}^{III}}$ be obtained from each other by the smallest possible shift in z_1–direction, i.e. $l^{III}_1 = l^{II}_1 + 1 = l^{I}_1$ and $l^{I}_k = l^{II}_k = l^{III}_k$ for $k = 2\dots d$. Then there exist $i \in \{I, II, III\}$, $j \in \{1, 2\}$ such that for all $u \in V_{\vec{l}^i}$ $\gamma^j_{\varepsilon_{12}}(u) \leq 1$.*

Clearly Lemma 13 implies Lemma 10″ since one can take

$$J = J(h, H) = \{Y_{\vec{l}}^j : \forall u \in X_{\vec{l}}\, \gamma_{\varepsilon_{12}}^j(u) \leq 1\}$$

To prove Lemma 13 we need several elementary bounds.
LEMMA 14. *Let h, H satisfy $|h| \leq H$, $\|h'\| \leq E|b|H$, $H \in K_{E|b|}$. If ε_{12} is small enough then for all $\vec{l}, j$*
a) for all $v_1, v_2 \in X_{\vec{l}}^j$

$$\frac{1}{2} \leq \frac{H(v_1)}{H(v_2)} \leq 2;$$

b) either

$$\forall v \in X_{\vec{l}}^j\ |h(v)| \leq \frac{3}{4} H(v) \qquad (A)$$

$$or\ \ \forall u \in X_{\vec{l}}\, |h(v)| \geq \frac{1}{4} H(v) \qquad (B).$$

PROOF: a) is immediate since the logarithmic derivative of H is at most $E|b|$ and the diameter of $X_{\vec{l}}^j$ is less than $\frac{\varepsilon_{11}\sqrt{d}}{|b|}$.
b) Assume that there is $v_0 \in X_{\vec{l}}^j$ such that $|h(v_0)| \geq \frac{3}{4} H(v)$. Then $\forall v \in X_{\vec{l}}^j$

$$|h(v)| \geq |h(v_0)| - E|b| \sup_{X_{\vec{l}}^j}(H)\, \mathrm{diam}(X_{\vec{l}}^j) \geq \frac{3}{4} H(v_0) - 2E|b|H(v_0)\frac{\varepsilon_{11}\sqrt{d}}{|b|} \geq$$

$$(\frac{3}{4} - 2E\varepsilon_{11}\sqrt{d})H(v_0) \geq \frac{1}{2}(\frac{3}{4} - 2E\varepsilon_{11}\sqrt{d})H(v)$$

so (B) is satisfied if $\varepsilon_{11} \leq \frac{1}{16E\sqrt{d}}$. ∎
LEMMA 15. *Let*

$$\tilde{\psi}(u) = \mathrm{Arg}(\exp[ib\tau_N(v_1(u)) - ib\tau_N(v_2(u))]).$$

Then there exist constants $\varepsilon_{16}, \varepsilon_{17}$ such that $\forall u^{I} \in V_{\vec{l}^{I}}$, $u^{III} \in V_{\vec{l}^{III}}$

$$\varepsilon_{16} \leq |\tilde{\psi}(u^{I}) - \tilde{\psi}(u^{III})| \leq \varepsilon_{17}$$

and ε_{17} can be made as small as we wish by decreasing ε_{11}
(The point of the upper bound is of course to make sure that this difference is not a multiple of 2π.)

PROOF: Consider coordinates $\bar{z}_1 \ldots \bar{z}_d$ on $V_{\vec{l}^j}$ such that $\bar{z}_k(u) = z_k(\sigma^{\bar{n}-n_1}v_1(u))$ (i.e. $\bar{z}_k$ is the pushforward of z_k.) Consider $\tilde{u}^{III}$ such that $\bar{z}_1(\tilde{u}^{III}) = \bar{z}_1(u^{III})$, $\bar{z}_k(\tilde{u}^{III}) = \bar{z}_k(u^I)$ for $k = 2\ldots d$. Since $|\tilde{\psi}(u^I) - \tilde{\psi}(\tilde{u}^{III})| = |\frac{\partial}{\partial\bar{z}_1}[\tau_{\bar{n}}(v_1(u)) - \tau_{\bar{n}}(v_2(u))](\cdot)||u^I - \tilde{u}^{III}|$ Lemma 8 implies that

$$\varepsilon_9\varepsilon_{11} \leq |\tilde{\psi}(u^I) - \tilde{\psi}(\tilde{u}^{III})| \leq C_{39}\varepsilon_9\varepsilon_{11}.$$

Likewise

$$|\tilde{\psi}(u^{III}) - \tilde{\psi}(\tilde{u}^{III})| \leq \frac{\varepsilon_9}{100\sqrt{d}}|u^{III} - \tilde{u}^{III}| \leq \varepsilon_9\varepsilon_{11}.\blacksquare$$

PROPOSITION 8. *$\forall N, \epsilon$ there exist $\delta = \delta(N,\epsilon) > 0$ such that if in $\triangle ABC$ $\angle A \geq \epsilon$ and $|AB| \geq \frac{|AC|}{N}$ then*

$$|BC| \leq |AB| + (1-\delta)|AC|.$$

PROOF OF LEMMA 13: If for some $i \in \{I, II, III\}$, $j \in \{1,2\}$ the alternative (A) of Lemma 14 holds there is nothing to prove (since we can take $\varepsilon_{12} \leq \frac{1}{4}$). So we assume that inequality (B) is satisfied for all $v \in X^j_{\vec{l}}$. Denote

$$\psi(u) = \operatorname{Arg}(e^{ib\tau_N(v_1(u))}h(v_1(u))) - \operatorname{Arg}(e^{ib\tau_N(v_2(u))}h(v_2(u))).$$

By assumption (B)

$$\|\frac{\partial}{\partial u}\ln h(v)\|(u) = \frac{\|h'\|}{|h(v)|}\|\frac{\partial v}{\partial u}\| \leq 4E|b|\varepsilon_4\bar{n}$$

and so $\forall u^I \in V_{\vec{l}^I}$, $u^{III} \in V_{\vec{l}^{III}}$

$$|\psi(u^I) - \psi(u^{III})| \geq \varepsilon_{16} - \|\frac{\partial}{\partial u}\ln h(v)\|\operatorname{diam}(V^j_{\vec{l}}) \leq \varepsilon_{16} - C_{40}\varepsilon_4^{\bar{n}}.$$

Thus if $\bar{n}$ is large enough

$$|\psi(u^I) - \psi(u^{III})| \geq \frac{\varepsilon_{16}}{2}$$

and so either $\forall u^I \in V_{\vec{l}^I}$

$$|\psi(u^I)| \geq \frac{\varepsilon_{16}}{4}$$

or $\forall u^{I\!I\!I} \in V_{\vec{l}^{I\!I\!I}}$

$$|\psi(u^{I\!I\!I})| \geq \frac{\varepsilon_{16}}{4}.$$

Assume to fix our notation that the first inequality is true. Take some $u_0 \in V_{\vec{l}^{I}}$. There are two cases. If $H(v_1(u_0)) \geq H(v_2(u_0))$ then by Lemma 14 $\forall u \in V_{\vec{l}^{I}}$ $H(v_1(u)) \geq 4H(v_2(u))$. Also $\forall v_1, v_2 \in U$

$$\frac{1}{C_{41}} \leq \frac{\exp f_{\bar{n}}^{(ab)}(v_1)}{\exp f_{\bar{n}}^{(ab)}(v_2)} \leq C_{41}$$

where $C_{41} = \exp\left[2\bar{n}\|f_{\bar{n}}^{(ab)}\|_0\right]$. therefore Proposition 8 implies that $\gamma_{\varepsilon_{12}}^{(2)} \leq 1$ where $\varepsilon_{12} = \epsilon(4C_{41}, \frac{\varepsilon_{17}}{4})$. Likewise if $H(v_1(u_0)) < H(v_2(u_0))$ then $\gamma_{\varepsilon_{12}}^{(1)} \leq 1$. ■

9. Proof of theorem 3. In this section we give the proof of theorem 3. Some steps of the proof are word-by-word repetitions of the proof of Theorems 1 and 2. In this case we give only the statement leaving the proof to the reader (who may also consult [D] for details). We find it convenient to change our notation slightly in this section. Namely we shall write σ only for the map $\Sigma^+ \to \Sigma^+$ and shall use $\hat{\sigma}$ for the map $\Sigma \to \Sigma$ to keep up with notation in the proof of Theorem 1 and 2. This change is only effective in Section 9. Unlike Theorems 1 and 2 we have to work with $C_\theta(\Sigma^+)$ since $\mathcal{L}_\star$ does not preserve spaces $C^\alpha(U)$. We define $\hat{\mathcal{L}}_{ab}$ as before but without smoothing (i.e. $f^{(b)} \equiv f$). We analogue of Lemma 2 is the following estimate.

Proposition 9.

$$L(\hat{\mathcal{L}}_{ab}^n h) \leq C_{42}(\|h\|_0 + |b|\theta^n L(h)).$$

We prove now an analogue of lemma 8.

Lemma 16. *There exist $\varepsilon_{18} > 0, C_{43}$ such that for any $\epsilon \leq \varepsilon_{18}$ for any $n > C_{43}\ln(\frac{1}{\epsilon})$ there are two branches $w^1(\omega)$ and $w^2(\omega)$ of σ^{-n} and two points ω' and $\omega'' \in \Sigma^+$ such that*

$$\frac{\epsilon}{2} \leq \left|\left[\tau_n(w^1(\omega')) - \tau_n(w^1(\omega''))\right] - \left[\tau_n(w^2(\omega')) - \tau_n(w^2(\omega''))\right]\right| \leq 2\epsilon.$$

Proof: $\zeta^{-1}(\Pi_i)$ is the cylinder $\mathcal{C}_i = \{\omega: \ \omega_0 = i\}$. Since φ is not identically 0 on $\bigcup_i(\Pi_i \times \Pi_i)$ by the Intermediate Value Theorem there exist Π_i such that $\varphi(\Pi_i \times \Pi_i)$ contains an interval $[0, \varepsilon_{18}]$. If $\epsilon \leq \varepsilon_{18}$ there are two points $\bar{\omega}$ and

$\tilde{\omega}$ such that $\varphi(\bar{\omega}, \tilde{\omega}) = \epsilon$. Let $\omega' = p_u \bar{\omega}$, $\omega'' = p_u \tilde{\omega}$ (following the proof of Theorem 1 we write p_u for the natural projection $p_u : \Sigma \to \Sigma^+$). Recall the expression of φ through τ ([PP]). If $\omega^{(1)}$ and $\omega^{(2)}$ are two points such that $\omega^{(1)}_j = \omega^{(2)}_j$ for $j \leq 0$ define

$$\Delta(\omega^{(1)}, \omega^{(2)}) = \sum_{k=1}^{\infty} [\tau(\sigma^{-k}\omega^{(2)}) - \tau(\sigma^{-k}\omega^{(1)})].$$

Then $W^{wu}_{loc}((\omega^{(1)}, 0)) = \{(\omega^{(2)}, t) : \ \omega^{(1)}_j = \omega^{(2)}_j \text{ for } j \leq 0 \text{ and } t = -\Delta(\omega^{(1)}, \omega^{(2)})\}$. Thus

$$\varphi(\bar{\omega}, \tilde{\omega}) = \Delta(\bar{\omega}, [\bar{\omega}, \tilde{\omega}]) - \Delta([\tilde{\omega}, \bar{\omega}], \tilde{\omega}).$$

Let $w^1(\omega) = \bar{\omega}_{-n} \bar{\omega}_{-(n-1)} \dots \bar{\omega}_{-1} \omega$, $w^2(\omega) = \tilde{\omega}_{-n} \tilde{\omega}_{-(n-1)} \dots \tilde{\omega}_{-1} \omega$. Then

$$\left| \left[\tau_n(w^1(\omega')) - \tau_n(w^1(\omega'')) \right] - \left[\tau_n(w^2(\omega')) - \tau_n(w^2(\omega'')) \right] - \varphi(\bar{\omega}, \tilde{\omega}) \right| \leq C_{44} \varepsilon_{20}^n$$

and the lemma follows. ■

Denote $\|h\|_{(b)} = \max(\|h\|_0, \frac{L(h)}{|b|})$.

LEMMA 17. *There exist C_{45}, C_{46}, β_7 such that if $\|h\|_{(b)} \leq 1$ then*

$$\nu_a(|\hat{\mathcal{L}}_{ab}^{C_{45} \ln |b|} h|) \leq 1 - \frac{C_{46}}{|b|^{\beta_7}}.$$

PROOF: Denote $N = C_{45} \ln |b|$. Consider two cases. The easier one is if there exist $\omega^{(0)}$ such that $|h(\omega^{(0)})| \leq \frac{1}{2}$ because then we can just bound $\nu_a(|\hat{\mathcal{L}}_{ab}^{C_{45} \ln |b|} h|)$ by $\nu_a(|h|)$. Indeed then $|h(\omega)| \leq \frac{3}{4}$ for ω in the ball $b(\omega^{(0)}, \frac{1}{2|b|})$ centered at $\omega^{(0)}$ and of radius $\frac{1}{2|b|}$. But $\nu_a(b(\omega^{(0)}, \frac{1}{2|b|})) \geq \frac{C_{47}}{|b|^{\beta_8}}$ and we are done. So assume that $\inf |h| > \frac{1}{2}$. Choose $\epsilon = (\frac{1}{|b|})^2$ in Lemma 16 and let

$$\gamma' = |e^{(f_N^{(ab)} + ib\tau_N)(\omega^1(\omega'))} h(\omega^1(\omega')) + e^{(f_N^{(ab)} + ib\tau_N)(\omega^2(\omega'))} h(\omega^2(\omega'))|,$$

$$\gamma'' = |e^{(f_N^{(ab)} + ib\tau_N)(\omega^1(\omega''))} h(\omega^1(\omega'')) + e^{(f_N^{(ab)} + ib\tau_N)(\omega^2(\omega''))} h(\omega^2(\omega''))|.$$

We claim that for some β_9 $\gamma' \leq 1 - \frac{1}{|b|^{\beta_9}}$ or $\gamma'' \leq 1 - \frac{1}{|b|^{\beta_9}}$. In view of Proposition 8 and the fact that $\exp[f_N^{(ab)}(\omega)] \geq \frac{1}{|b|^{\beta_{10}}}$ it is enough to prove that

$$\left| \operatorname{Arg}\left(e^{ib\tau_N(\omega^1(\omega'))} h(\omega^1(\omega'))\right) - \operatorname{Arg}\left(e^{ib\tau_N(\omega^1(\omega'))} h(\omega^1(\omega'))\right) \right| \geq \frac{1}{|b|^4} \qquad (A)$$

or

$$\left|\operatorname{Arg}\left(e^{ib\tau_N(\omega^1(\omega''))}h(\omega^1(\omega''))\right)-\operatorname{Arg}\left(e^{ib\tau_N(\omega^1(\omega''))}h(\omega^1(\omega''))\right)\right|\geq\frac{1}{|b|^4}\quad(B).$$

Assume to the contrary that both (A) and (B) are false. We also have

$$\left|\operatorname{Arg}(h(\omega^1(\omega')))-\operatorname{Arg}(h(\omega^1(\omega')))\right|\leq 2L(h)\theta^N\quad(\text{since }|h|>\frac{1}{2})$$

$$\leq 2|b|\theta^N.$$

Similarly

$$\left|\operatorname{Arg}(h(\omega^1(\omega'')))-\operatorname{Arg}(h(\omega^1(\omega'')))\right|\leq 2|b|\theta^N$$

So if N is large enough (i.e. C_{45} is large) this implies that

$$\left|\operatorname{Arg}\left(e^{ib\tau_N(\omega^1(\omega''))}\right)-\operatorname{Arg}\left(e^{ib\tau_N(\omega^1(\omega''))}\right)\right|\geq\frac{3}{|b|^4}.$$

But by Lemma 16 this difference is between $\frac{1}{2|b|}$ and $\frac{1}{|b|}$. Hence either (A) or (B) is true. ∎

Corollary 7. *There exist $C_{48}, C_{49}, \beta_{11}$ such that if $\|h\|_{(b)}\leq 1$ then*

$$|\hat{\mathcal{L}}_{ab}^{C_{48}\ln|b|}h|\leq 1-\frac{C_{49}}{|b|^{\beta_{11}}}.$$

Proof:

$$|\hat{\mathcal{L}}_{ab}^{N}h|(\omega)=|\hat{\mathcal{L}}_{ab}^{N-\tilde{N}}(\hat{\mathcal{L}}_{ab}^{\tilde{N}}h)|(\omega)\leq\left(\hat{\mathcal{L}}_{a0}^{N-\tilde{N}}|\hat{\mathcal{L}}_{ab}^{\tilde{N}}h|\right)(\omega)\leq$$

$$\nu_a(\hat{\mathcal{L}}_{ab}^{\tilde{N}}h)+C_{50}|b|\theta^{N-\tilde{N}}$$

as in the proof of Corollary 2. Take $\tilde{N}=C_{45}\ln|b|$ and choose $C_{48}\gg C_{45}$. ∎

Corollary 8. *There exist C_{51}, β_{12} so that*

$$\|\hat{\mathcal{L}}_{ab}^{C_{51}\ln|b|}\|_{(b)}\leq 1-|b|^{-\beta_{12}}.$$

(This follows immediately from corollary 7 and lemma 15.)

Corollary 9. There exist $C_{52}, C_{53}, \beta_{13}, \beta_{14}$ such that if $A, B\in C^{\alpha}(M)$ and $|a|\leq C_{52}|b|^{-\beta_{13}}$ then

$$|\hat{\rho}_{A,B}(a+ib)|\leq C_{53}|b|^{\beta_{14}}\|A\|_{\alpha}\|B\|_{\alpha}.$$

(Repeat the calculations of corollary 5.)
If $A, B \in C^{N+\alpha}(M)$ then

$$|\hat{\rho}_{A,B}(a+ib)| = \frac{1}{|a+ib|^N} |(\widehat{\frac{\partial}{\partial t}}^N \rho_{A,B})| =$$

$$\frac{1}{|a+ib|^N} |\hat{\rho}_{(\frac{\partial}{\partial t})^N A,\, B}(a+ib)| \leq C_{53}|b|^{\beta_{14}-N} \|A\|_{N+\alpha}\|B\|_{\alpha}.$$

Thus for $A, B \in C^{\infty}(M)$ $\hat{\rho}$ decays faster than any power of b in the region $\{|a| \leq C_{52}|b|^{\beta_{13}}\}$. Now the Cauchy formula implies that $\left(\frac{\partial}{\partial b}\right)^N \hat{\rho}(ib)$ also decays faster than any power and theorem 3 is proven. ■

Appendix 1. Correlation density. In this section we recall the expression for Laplace transform of the correlation function. Our exposition follows closely [P], [R2].

Consider the suspension flow G^t with the roof function τ. We assume that $\tau \in C^+_\theta(\Sigma)$ which is true in the case when τ comes from the construction described in the previous section. Let μ be the Gibbs measure for the potential $F \in C_{\theta^2}(\Sigma^\tau)$. We can decompose the mean value $\bar{F}(\omega) = \int\limits_0^{\tau(\omega)} F(\omega, s)\, ds$ as $\bar{F}(\omega) = f(\omega) + H(\omega) - H(\sigma\omega)$ with $f(\omega) \in C^+_\theta(\Sigma)$. μ can be written as $d\mu(q) = \frac{1}{C} d\nu(\omega)\, ds$ where C is the normalization constant and $\nu(\omega)$ is the Gibbs measure for $f(\omega)$

Let $A, B \in C_\theta(\Sigma^\tau)$ and $\rho_{A,B}(t) = \int\limits_{\Sigma^\tau} A(q)B(G^t q)\, d\mu(q)$ be the correlation function. Consider its Laplace transform

$$\hat{\rho}(\xi) = \int\limits_0^{\infty} e^{-\xi t} \int\limits_{\Sigma^\tau} A(q)B(G^t q)\, d\mu(q) dt$$

$$= \int\limits_{\Sigma^\tau} A(\omega, s) \sum_{n=0}^{\infty} \int\limits_0^{\tau(\sigma^n\omega)} B(\sigma^n\omega, \bar{s}) e^{-\xi(\tau_n(\omega)+\bar{s}-s)} ds d\bar{s} d\mu - \int\limits_{\Sigma^\tau} A(\omega, s) \int\limits_0^{s} B(\omega, \bar{s}) e^{-\xi(\bar{s}-s)} ds d\bar{s} d\mu$$

$$= \hat{\rho}_I(\xi) + \hat{\rho}_{II}(\xi),$$

where $\hat{\rho}_{II}(\xi)$ is an entire function bounded as long as $Re\xi$ is bounded. Denote by $\hat{A}(\omega, \xi) = \int\limits_0^{\tau(\omega)} A(\omega, s) e^{\xi s}\, ds$ the Laplace transform of A then

$$\hat{\rho}_I(\xi) = \frac{1}{\text{Const}} \int\limits_{\Sigma} \hat{A}(\omega, \xi) \sum_{n=0}^{\infty} e^{-\xi\tau_n(\omega)} \hat{B}(\sigma^n\omega, -\xi)\, d\nu(\omega).$$

Note that

$$\|\hat{A}(\omega,\xi)\|_0 \le C_{54}\|A\|_\theta, \quad \|\hat{B}(\omega,\xi)\|_0 \le C_{54}\|B\|_\theta, \tag{15}$$

$$L(\hat{A}(\omega,\xi)) \le C_{55}\|A\|_\theta|b|, \quad L(\hat{B}(\omega,\xi)) \le C_{55}\|B\|_\theta|b|. \tag{16}$$

We now utilize the following decomposition.

PROPOSITION 10. *Every $h \in C_\theta(\Sigma)$ can be decomposed as $h = \sum_{j=0}^{\infty} h_j$ where*

1) $\|h\|_0 = C_{56}\|h\|_\theta \varepsilon_{20}^j$;
2) $L(h) \le C_{57}K_1^n L(h)$;
3) $h_j(\sigma^{-j}\omega) \in C_\theta^+(\Sigma)$

PROOF: For any symbol i choose a backward sequence $\omega^{(i)} = \{\omega_j^{(i)}\}_{j\le 0}$ such that $\omega_0^{(i)} = i$. For $\omega \in \Sigma$ define $\omega_{(N)}$ by

$$(\omega_{(N)})_j = \begin{cases} \omega_j & j \ge -N \\ \omega_{j-N}^{(\omega_{-N})} & j \le -N \end{cases}$$

Choose some N_0 and define by induction $h_{(0)} = h(\omega_{(0)})$, $h_{(k)}(\omega) = h_{(k-1)}(\omega) + (h - h_{(k-1)})(\omega_{(N_0 k)})$. Then

$$\|h - h_{(k)}\|_0 \le \left(L(h) + L(h_{(k-1)})\right)\theta^{N_0 k},$$

$$L(h_{(k)}) \le 2L(h_{(k-1)}) + L(h) \le \left(2^{(k+1)} - 1\right)L(h).$$

Thus is $\theta^{N_0 k} < \frac{1}{2}$ $\|h - h_{(k)}\|_0$ decays exponentially. Let $h_0 = h_{(0)}$, $h_{jN_0} = h_{(j)} - h_{(j-1)}$ and $h_j = 0$ if j is not a multiple of N_0. ■

So write $\hat{A}(\omega,\xi) = \sum_{j=0}^{\infty} \hat{A}_j(\omega)$, $\hat{B}(\omega,-\xi) = \sum_{j=0}^{\infty} \hat{B}_j(\omega)$, and let $\bar{A}_j(\omega) = \hat{A}(\sigma^{-j}\omega)$, $\bar{B}_j(\omega) = \hat{B}(\sigma^{-j}\omega)$ then $\bar{A}_j \in C_\theta^+(\Sigma)$, $\bar{B}_j \in C_\theta^+(\Sigma)$ and

$$\|\bar{A}_j\|_0 \le C_{56}\|A(\omega,\xi)\|_0\varepsilon_{20}^j, \quad \|\bar{B}_j\|_0 \le C_{56}\|B(\omega,-\xi)\|_0\varepsilon_{20}^j, \tag{17}$$

$$L(\bar{A}_j) \le C_{57}L(A(\omega,\xi))K_2^j, \quad L(\bar{B}_j) \le C_{57}L(B(\omega,\xi))K_2^j. \tag{18}$$

So $\hat{\rho}_I(\xi) = \frac{1}{\nu(\tau)}\sum_{jk}\hat{\rho}_{jk}(\xi)$, where

$$\hat{\rho}_{jk}(\xi) = \int_\Sigma \hat{A}_j(\omega)\sum_{n=0}^{\infty} e^{-\xi\tau_n(\omega)}\hat{B}_k(\omega)\,d\nu(\omega).$$

The rearrangement of this series performed below is valid at least for small $Re\xi$. By $\sigma-$invariance of the measure ν

$$\hat{\rho}_{jk}(\xi) = \int\limits_{\Sigma} \bar{A}_j(\omega) \sum_{n=0}^{\infty} e^{-\xi\tau_n(\sigma^j\omega)} \bar{B}(\sigma^{n+j+k}\omega)\, d\mu(\omega).$$

Denote

$$\tilde{A}_j(\omega,\xi) = \bar{A}_j(\omega,\xi)e^{-\xi\tau_j(\omega)}, \ \ \tilde{B}_j(\omega,\xi) = \bar{B}_j(\omega,\xi)e^{-\xi\tau_j(\omega)}$$

so that

$$\|\tilde{A}_j\| \le \|\bar{A}_j\|_0 e^{\varepsilon_{20} j}, \ \ \|\tilde{B}_j\| \le \|\bar{B}_j\|_0 e^{\varepsilon_{20} j}, \tag{19}$$

$$L(\tilde{A}_j) \le \left(C_{58}\|\bar{A}_j\|_0|b| + L(\bar{A}_j)\right) e^{\varepsilon_{20} j}, \tag{20}$$

$$L(\tilde{B}_j) \le \left(C_{58}\|\bar{B}_j\|_0|b| + L(\bar{B}_j)\right) e^{\varepsilon_{20} j}. \tag{21}$$

We have

$$\hat{\rho}_{jk}(\xi) = \sum_{n=j+k}^{\infty} \int\limits_{\Sigma} \tilde{A}_j(\omega) e^{-\xi\tau_n(\omega)} \tilde{B}_k(\sigma^n\omega)\, d\nu(\omega).$$

Since $\tilde{A}$ and $\tilde{B}$ depend only on the future the integration in the last expression may be taken over Σ^+ as well. Performing the change of variables $\varpi = \sigma^n\omega$ we obtain

$$\hat{\rho}_{jk}(\xi) = \sum_{n=j+k}^{\infty} \int\limits_{\Sigma^+} \sum_{\sigma^n\omega=\varpi} \tilde{B}_k(\varpi) \left[\tilde{A}_j(\omega) e^{-xi\tau_n(\omega)} \frac{d\nu(\omega)}{d\nu(\varpi)}\right] d\nu(\varpi).$$

Assuming that the corresponding transfer operator is normalized we get the following expression for the Jacobian (2):

$$\frac{d\nu(\omega)}{d\nu(\varpi)} = \exp[f_n(\omega) - Pr(F)\tau_n(\omega)].$$

Therefore

$$\hat{\rho}_{jk}(\xi) = \sum_{n=j+k}^{\infty} \int\limits_{\Sigma^+} \tilde{B}_k(\varpi) \left[\sum_{\sigma^n\omega=\varpi} \tilde{A}_j(\omega) e^{f_n(\omega)-[Pr(F)+\xi]\tau_n(\omega)}\right] d\nu(\varpi).$$

In terms of transfer-operators this can be rewritten as follows:

$$\hat\rho_{jk}(\xi) = \int\limits_{\Sigma^+} \left[\mathcal{L}^{j+k}_{f-[Pr(F)+\xi]\tau}(1-\mathcal{L}_{f-[Pr(F)+\xi]\tau})^{-1}\tilde A_j\right]\tilde B_k\, d\nu.$$

Let $\mathcal{Q}_j : A \to \tilde A_j$, $\mathcal{R}_j : B \to \tilde B_j$. Then bounds a) and b) of Proposition 3 follow immediately from (15)-(21).

Appendix 2. A priori bounds.

PROOF OF COROLLARY 3: Consider the following norm in $C^{\alpha'}(U)$

$$\|h\|_{(b,\alpha')} = \max(\|h\|_0, \frac{G(h)}{|b|}).$$

We prefer to work with this norm because we already saw that $\frac{1}{|b|}$ is a natural scale for the study of $\hat{\mathcal{L}}_{ab}$. Take $h \in C^{\alpha'}(U)$ with $\|h\|^{(b)} \le 1$ and decompose it $h = \tilde h + (h-\tilde h)$ where $\|h-\tilde h\| \le (\frac{1}{|b|})^{\alpha'}$ $\tilde h \in C^1(U)$ and $\|\frac{\partial}{\partial u}\tilde h\|_0 \le C_{59}|b|$. By Corollary 2

$$|\hat{\mathcal{L}}^{C_{17}\ln|b|}_{ab}\tilde h| \le C_{18}(\frac{1}{|b|})^{\beta_1}.$$

Since $\mathcal{L}_{ab}$ does not increase the norm of C^0 functions

$$|\hat{\mathcal{L}}^{C_{17}\ln|b|}_{ab} h| \le \frac{C_{60}}{|b|^{\beta_{15}}}$$

where $\beta_{15} = \min(\alpha', \beta_1)$. Recall the relation between $\hat{\mathcal{L}}_{ab}$ and $\tilde{\mathcal{L}}_{ab}$ (5). Since the operator of multiplication by h_{ab} is uniformly bounded in $\|\cdot\|_{(b,\alpha')}$ (in fact h_{ab} is almost constant on the scale $\frac{1}{|b|}$) we get the following estimate valid for small a and large b

$$\left|\tilde{\mathcal{L}}^{C_{17}\ln|b|}_{ab} h\right| \le C_{61}\left(\frac{1}{|b|}\right)^{\beta_{15}-C_{62}(|a|+\frac{1}{|b|})}.$$

Using analyticity of λ_{ab} in a and $\frac{1}{|b|}$ we obtain for small a and large b the following bound

$$\left|\tilde{\mathcal{L}}^{C_{17}\ln|b|}_{ab} h\right| \le C_{61}\frac{1}{|b|^{\beta_{16}}},$$

$\beta_{16} > 0$. Now

$$\left|\mathcal{L}^{C_{17}\ln|b|}_{ab}\right| = \left|\tilde{\mathcal{L}}^{C_{17}\ln|b|}_{ab}\left(\exp[(f^{(b)}_{C_{17}\ln|b|} - f_{C_{17}\ln|b|})\circ\sigma^{C_{17}\ln|b|}]h\right)\right| \le$$

$$|\tilde{\mathcal{L}}_{ab}^{C_{17}\ln|b|}h|+\left|\mathcal{L}_{ab}^{C_{17}\ln|b|}\right| = \left|\tilde{\mathcal{L}}_{ab}^{C_{17}\ln|b|}\left(\left(\exp[(f^{(b)}_{C_{17}\ln|b|} - f_{C_{17}\ln|b|})\circ\sigma^{C_{17}\ln|b|}] - 1\right)h\right)\right| \le$$

$$C_{61}\frac{1}{|b|^{\beta_{16}}} + C_{63}\ln|b|\left(\frac{1}{\sqrt{|b|}}\right) \le C_{64}\frac{1}{|b|^{\beta_{17}}}.$$

Now take $C_{23} \gg C_{17}$. Then

$$\left|\mathcal{L}_{ab}^{C_{23}\ln|b|}h\right| = \left|\mathcal{L}_{ab}^{(C_{23}-C_{17})\ln|b|}(\mathcal{L}_{ab}^{C_{17}\ln|b|}h)\right| \le \left|\mathcal{L}_{ab}^{C_{17}\ln|b|}h\right| \le C_{64}\frac{1}{|b|^{\beta_{17}}}.$$

From the other hand by Lemma 2.d)

$$G(\mathcal{L}_{ab}^{C_{23}\ln|b|}h) = G(\mathcal{L}_{ab}^{(C_{23}-C_{17})\ln|b|}(\mathcal{L}_{ab}^{C_{17}\ln|b|}h)) \le$$

$$\lambda_{ab}^{(C_{23}-C_{17})\ln|b|}C_{15}\left(|b|\|\mathcal{L}_{ab}^{C_{17}\ln|b|}h\|_0 + \varepsilon_4^{\alpha'(C_{23}-C_{17})\ln|b|}G(\mathcal{L}_{ab}^{C_{17}\ln|b|}h)\right)$$

So for small a and large b the following bounds holds

$$\|\mathcal{L}_{ab}^{C_{23}\ln|b|}\|_{(b,\alpha')} \le \frac{C_{65}}{|b|^{\beta_{21}}}.$$

This estimate clearly implies Corollary 3. ■

Corollary 4 follows from term-by-term summation in (3) using the following bound.

LEMMA 18. *Let $h \in C^{\alpha'}(U)$ $\alpha' < \alpha$, and $G(h) \le D|b|\|h\|_0$, $D > 1$. Then*

$$\|(1-\mathcal{L}_{ab})^{-1}h\|_0 \le C_{66}D^{\epsilon(a,b)}(\ln|b| + \ln D)\|h\|_0$$

where $\epsilon \to 0$ as $a \to 0$, $b \to \infty$.

PROOF: By Lemma 2.d)

$$\|\mathcal{L}_{ab}^N h\|_0 \le C_{14}\lambda_{ab}^N\|h\|_0,$$

$$G(\mathcal{L}_{ab}^N h) \le C_{15}(|b|\|h\|_0 + \varepsilon_4^{\alpha' N}G(h)).$$

Therefore if $N = C_{67}\ln D$ where C_{67} is large enough

$$\|\mathcal{L}_{ab}^N h\|_{(b)} \le 2\lambda_{ab}^{C_{67}\ln D}\|h\|_0.$$

Write

$$(1-\mathcal{L}_{ab})^{-1}h = \sum_{j=0}^{C_{67}\ln D}\mathcal{L}_{ab}^j h + \sum_{j=1}^{\infty}\mathcal{L}_{ab}^j(\mathcal{L}_{ab}^{C_{67}\ln D}h)$$

and estimate the first term by Lemma 2.d) and the second one by Corollary 3. ■

Appendix 3. Gibbs measures. In this section we collect some distortion properties of Gibbs measures for codimension 1 Anosov flows.

PROOF OF LEMMA 6: It is enough to fix u and v_1 and to bound

$$\sum_{v_2:\, d(v_1,v_2)\leq\delta} e^{f_N^{(a)}(v_2)}$$

There is a constant C_{68} such that if $d(v_1,v_2)<\delta$ then $\sigma^{N-n}v_1=\sigma^{N-n}v_2$ for $n\leq C_{68}\ln\frac{1}{\delta}$. But

$$\sum_{v_2:\,\sigma^{N-n_0}v_1=\sigma^{N-n_0}(v_2)} e^{f_N^{(a)}(v_2)}=\exp[f_{n_0}^{(a)}(v_1)]\sum_{\sigma^{N-n_0}v_2=\sigma^{N-n_0}v_1} e^{f_{N-n_0}^{(a)}(v_2)}=$$

$$\exp[f_{n_0}^{(a)}(v_1)]\left(\hat{\mathcal{L}}_{ab}^{N-n_0}1\right)(\sigma^{N-n_0}v_1)=\exp[f_{n_0}^{(a)}(v_1)]\blacksquare$$

Now we prove Proposition 7.

PROPOSITION. *Under conditions of theorem 1 given N there is a constant C_N such that if $I_1\subset I_2\subset U$ are two intervals and $|I_1|\geq\frac{|I_2|}{N}$ then $\nu(I_2)\leq C_N\nu(I_1)$.*

PROOF: Let $n_0=\max\{n:|\sigma^n I_2|\leq 1\}$. Then by Lemma 3 $\forall v_1,v_2\in I_2$ $\frac{1}{C_{69}}\leq\frac{(\sigma^{n_0})'(v_1)}{(\sigma^{n_0}v_2)}\leq C_{69}$ where the constant C_{69} does not depend on I_2. Therefore $|\sigma^{n_0}I_1|>C_{70}$ for some constant C_{70}. Since the measure of any open set is positive there is a constant C_{71} such that $\nu(\sigma^{n_0}I_1)>C_{71}$. But by (2) and Lemma 3

$$\frac{1}{C_{72}}\leq\frac{\nu(\sigma^{n_0}I_1)\nu(I_2)}{\nu(\sigma^{n_0}I_2)\nu(I_1)}\leq C_{72}.$$

The last two inequalities prove the proposition. ■

Appendix 4. The proof of van der Corput lemma. This section contains the proof of the following statement.

LEMMA 7. *Let*

$$I=\int e^{ib\psi(u)}r(u)\,du$$

Assume that $\psi\in C^{1+\gamma}(J)$, $\|\psi\|_{1+\gamma}\leq c_1$, $|\psi'(u)|\geq c_2$, where $\frac{1}{b}\leq c_2\leq 1$, $\|r\|_0\leq\epsilon$ and $\|r(u)\|_1\leq\epsilon D$ then

$$|I|\leq\epsilon\mathrm{Const}(c_1)\left[\frac{D+1}{|b|c_2}+\frac{1}{|b|^\gamma c_2^2}\right].$$

PROOF:

$$I = \frac{1}{ib} \int \frac{r(u)}{\psi'(u)}\, de^{ib\psi(u)}.$$

Take $\bar{\psi} \in C^1(J)$ so that $|\bar{\psi} - \psi'| \leq \frac{1}{|b|}$, $\|\bar{\psi}\|_1 \leq |b|^{1-\gamma}$ then $I = \bar{I} + \Delta I$ where

$$\bar{I} = \frac{1}{ib} \int \frac{r(u)}{\bar{\psi}'(u)}\, de^{ib\psi(u)}$$

and $|\Delta I| \leq \mathrm{Const}\frac{1}{|b|}$. Integrating by parts we obtain

$$\bar{I} = \frac{1}{ib} \left[e^{ib\psi(u)} \frac{r(u)}{\bar{\psi}(u)} |_J - \int e^{ib\psi(u)} \frac{\partial}{\partial u} \left[r(u)\bar{\psi}(u) \right] du \right].$$

The statement follows since

$$\| \frac{\partial}{\partial u} \left[r(u)\bar{\phi}(u) \right] \|_0 \leq \mathrm{Const} \left(\frac{D}{c_2} + \frac{|b|^{1-\gamma}}{c_2^2} \right). \blacksquare$$

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