Ergod. Th. & Dynam. Sys. (2012), **32**, 643–651

doi:10.1017/S0143385711000678

Geodesics in Margulis spacetimes

WILLIAM M. GOLDMAN† and FRANÇOIS LABOURIE‡

† *Department of Mathematics, University of Maryland, College Park, MD 20742, USA (e-mail: wmg@math.umd.edu)*

‡ *Laboratoire de Mathématiques, Université Paris-Sud, Orsay F-91405 Cedex; CNRS, Orsay cedex, F-91405, France (e-mail: francois.labourie@math.u-psud.fr)*

(*Received* 27 *January* 2011 *and accepted in revised form* 14 *August* 2011)

Dedicated to the memory of Dan Rudolph

Abstract. Let M^3 be a Margulis spacetime whose associated complete hyperbolic surface Σ^2 has a compact convex core. Generalizing the correspondence between closed geodesics on M^3 and closed geodesics on Σ^2, we establish an orbit equivalence between recurrent spacelike geodesics on M^3 and recurrent geodesics on Σ^2. In contrast, no timelike geodesic recurs in either forward or backward time.

1. *Introduction*

A *Margulis spacetime* is a complete flat affine 3-manifold M^3 with free non-abelian fundamental group Γ. It necessarily carries a unique parallel Lorentz metric. Parallelism classes of timelike geodesics form a non-compact complete hyperbolic surface Σ^2. This complete hyperbolic surface is naturally associated to the flat 3-manifold M^3 and we regard M^3 as an *affine deformation* of Σ^2. This paper relates the dynamics of the geodesic flow of the flat affine manifold M^3 to the dynamics of the geodesic flow on the hyperbolic surface Σ^2.

We restrict ourselves to the case that Σ^2 has compact convex core (that is, Σ^2 has finite type and no cusps). Equivalently, the Fuchsian group Γ_0 corresponding to $\pi_1(\Sigma^2)$ is *convex cocompact.* In particular, Γ_0 is finitely generated and contains no parabolic elements. Under this assumption, every free homotopy class of an essential closed curve in Σ^2 contains a unique closed geodesic. Since Σ^2 and M^3 are homotopy-equivalent, free homotopy classes of essential closed curves in M correspond to free homotopy classes of essential closed curves in Σ^2. Every essential closed curve in M^3 is likewise homotopic to a unique closed geodesic in M^3.

In her thesis [**4, 8**], Charette studied the next case of dynamical behavior: geodesics spiralling around closed geodesics both in forward and backward time. She proved bispiralling geodesics in M^3 exist, and correspond to bispiralling geodesics in Σ^2.

This paper extends the above correspondence between geodesics on Σ^2 and M^3 to recurrent geodesics.

A geodesic (either in Σ^2 or in M^3) is *recurrent* if and only if it (together with its velocity vector) is recurrent in *both* directions. These correspond to recurrent points for the corresponding geodesic flows as in Katok and Hasselblatt [**17**, §3.3]. (Our meaning of the term 'recurrent' agrees with the term 'non-wandering' used by Eberlein [**12**].) Under our hypotheses on Σ^2, a geodesic on Σ^2 is recurrent if and only if the corresponding orbit of the geodesic flow is precompact.

THEOREM 1. *Let M^3 be a Margulis spacetime whose associated complete hyperbolic surface Σ has compact convex core.*

- *The recurrent part of the geodesic flow for Σ^2 is* topologically orbit-equivalent *to the recurrent spacelike part of the geodesic flow of M^3.*
- *The set of recurrent spacelike geodesics in a Margulis spacetime is the closure of the set of periodic geodesics.*
- *No timelike geodesic recurs.*

A semiconjugacy between these flows was observed by Fried [**13**].

This paper is the sequel to [**15**], which characterizes properness of affine deformations by positivity of a marked Lorentzian length spectrum, the *generalized Margulis invariant*. A crucial step in the proof that properness implies positivity is the construction of sections of the associated flat affine bundle, called *neutralized sections.* A further modification of neutralized sections produces an orbit equivalence between recurrent geodesics in Σ and recurrent geodesics in M.

It follows that the set of recurrent spacelike orbits of the geodesic flow is a Smale hyperbolic set in $\mathsf{T}M$.

Null geodesics not parallel to a point in the limit set Λ of Γ_0 do not recur. In this paper, we do not discuss the recurrence of null geodesics parallel to a point of Λ.

2. *Geodesics on affine manifolds*

An *affinely flat manifold* is a smooth manifold with a distinguished atlas of local coordinate systems whose charts map to an affine space E such that the coordinate changes are restrictions of affine automorphisms of E. Denote the group of affine automorphisms of E by $\mathsf{Aff}(\mathsf{E})$. This structure is equivalent to a flat torsion-free affine connection. The affine coordinate atlas globalizes to a *developing map*

$$\tilde{M} \xrightarrow{\mathsf{dev}} \mathsf{E},$$

where $\tilde{M} \to M$ denotes a universal covering space of M. The coordinate changes globalize to an affine holonomy homomorphism

$$\pi_1(M) \xrightarrow{\rho} \mathsf{Aff}(\mathsf{E}),$$

where $\pi_1(M)$ denotes the group of deck transformations of $\tilde{M} \to M$. The developing map is equivariant with respect to ρ.

Denote the vector space of translations $\mathsf{E} \to \mathsf{E}$ by V. The action of V by translations on E defines a trivialization of the tangent bundle $\mathsf{T}M \cong M \times \mathsf{V}$. In these local coordinate charts, a geodesic is a path

$$p \longmapsto p + t\mathsf{v},$$

where $p \in \mathsf{E}$ and $\mathsf{v} \in \mathsf{V}$ is a vector. In terms of the trivialization, the geodesic flow is

$$\begin{aligned} \mathsf{E} \times \mathsf{V} &\xrightarrow{\tilde{\psi}_t} \mathsf{E} \times \mathsf{V}, \\ (p, \mathsf{v}) &\longmapsto (p + t\mathsf{v}, \mathsf{v}), \end{aligned}$$

for $t \in \mathbb{R}$. Clearly, this $\mathbb{R}$-action commutes with $\mathsf{Aff}(\mathsf{E})$.

Geodesic completeness implies that dev is a diffeomorphism. Thus the universal covering $\tilde{M}$ is affinely isomorphic to the affine space E and $M \cong \mathsf{E}/\Gamma$, where $\Gamma := \rho(\pi_1(M))$ is a discrete group of affine transformations acting properly and freely on E.

3. *Flat Lorentz* 3-*manifolds*

Let $\mathsf{Aff}(\mathsf{E}) \xrightarrow{\mathsf{L}} \mathsf{GL}(\mathsf{V})$ denote the homomorphism given by the *linear part,* that is, $\mathsf{L}(\gamma) = A$, where

$$p \xrightarrow{\gamma} A(p) + b.$$

The differential of γ at *any* point p identifies with its linear part $\mathsf{L}(\gamma)$ via the identification $\mathsf{T}M \cong M \times \mathsf{V}$.

Any $\mathsf{L}(\Gamma)$-invariant non-degenerate inner product $\langle\ ,\ \rangle$ on V defines a Γ-invariant flat pseudo-Riemannian structure on E which descends to $M = \mathsf{E}/\Gamma$. In particular, affine manifolds with $\mathsf{L}(\Gamma) \subset \mathsf{O}(n-1, 1)$ are precisely the *flat Lorentzian manifolds,* and the underlying affine structures their Levi-Civita connections.

For this reason, we henceforth fix the invariant Lorentzian inner product on V, and hence the (parallel) flat Lorentzian structure on E. The group $\mathsf{Isom}(\mathsf{E})$ of Lorentzian isometries is the semidirect product of the group V of translations of E with the orthogonal group $\mathsf{O}(n-1, 1)$ of linear isometries. The linear part $\mathsf{Isom}(\mathsf{E}) \xrightarrow{\mathsf{L}} \mathsf{O}(n-1, 1)$ defines the projection homomorphism for the semidirect product. For $l \in \mathbb{R}$, define

$$\mathsf{S}_l := \{\mathsf{v} \in \mathsf{V} \mid \langle \mathsf{v}, \mathsf{v} \rangle = l\}.$$

When $l > 0$, S_l is a Riemannian submanifold of constant curvature $-l^{-2}$, and when $l < 0$, it is a Lorentzian submanifold of constant curvature l^{-2}. In particular, S_{-1} is a disjoint union of two isometrically embedded copies of hyperbolic $n-1$-space H^{n-1} and S_1 is the *de Sitter space*, a model space of Lorentzian curvature $+1$.

The subset $\mathsf{T}_l(M)$ consists of tangent vectors v such that $\langle v, v \rangle = l$ is invariant under the geodesic flow. Indeed, using parallel translation, these bundles trivialize over the universal covering E:

$$\mathsf{T}_l(\mathsf{E}) \xrightarrow{\cong} \mathsf{E} \times \mathsf{S}_l.$$

Abels–Margulis–Soifer [**2, 3**] proved that if a discrete group of Lorentz isometries acts properly on a Minkowski space E, and $\mathsf{L}(\Gamma)$ is Zariski dense in $\mathsf{O}(n-1, 1)$, then $n = 3$. Consequently, every complete flat Lorentz manifold is a flat Euclidean affine fibration over a complete flat Lorentz 3-manifold. Thus we henceforth restrict to $n = 3$.

Let M^3 be a complete affinely flat 3-manifold. By Fried and Goldman [**14**], either Γ is solvable or $\mathsf{L} \circ h$ embeds Γ as a discrete subgroup in (a conjugate of) the orthogonal group

$$\mathsf{SO}(2, 1) \subset \mathsf{GL}(3, \mathbb{R}).$$

The cases when Γ is solvable are easily classified (see [**14**]) and we assume we are in the latter case. In that case, M^3 is a complete flat Lorentz 3-manifold.

In the early 1980s, Margulis, answering a question of Milnor [**22**], constructed the first examples [**19, 20**], which are now called *Margulis spacetimes*. Explicit geometric constructions of these manifolds have been given by Drumm [**9, 10**] and his coauthors [**4–7, 11**]. For an excellent survey of this subject, see Abels [**1**].

Since the hyperbolic plane H^2 is the symmetric space of $\mathsf{SO}(2, 1)$, Γ acts properly and discretely on H^2. Since M^3 is aspherical, its fundamental group $\pi_1(M^3) \cong \Gamma$ is torsion-free, so Γ acts freely as well. Therefore the quotient $\mathsf{H}^2/\mathsf{L}(\Gamma)$ is a complete hyperbolic surface Σ^2. Furthermore, by Mess [**21**], Σ is non-compact. (See Goldman and Margulis [**16**] and Labourie [**18**] for alternative proofs.) Furthermore, every non-compact complete hyperbolic surface occurs for a Margulis spacetime (Drumm [**9**]).

The points of Σ^2 correspond to parallelism classes of (unoriented) timelike geodesics on M^3 as follows. It suffices to identify H^2 with the parallelism classes of (unoriented) timelike geodesics in E, equivariantly respecting $\mathsf{Isom}(\mathsf{E}) \xrightarrow{\mathsf{L}} \mathsf{SO}(2, 1)$. The velocity of a unit-speed timelike geodesic in E is a $\tilde{\psi}$-orbit in

$$\mathsf{T}_{-1}\mathsf{E} \cong (\mathsf{E} \times \mathsf{S}_{-1}).$$

The two components of S_{-1} correspond to future-pointing timelike geodesics and past-pointing timelike geodesics respectively. Points in S_{-1} correspond to points in H^2 (the projectivization of S_{-1}) together with an orientation of H^2. The geodesic flow $\tilde{\psi}$ gives $\mathsf{T}_{-1}\mathsf{E}$, the structure of a principal $\mathbb{R}$-bundle over the quotient. The quotient identifies with an affine bundle over $\mathsf{S}_{-1} \cong \mathsf{H}^2 \times \{\pm 1\}$, whose associated vector bundle is the tangent bundle, as follows: the fiber over the line spanned by a fixed timelike vector v is the affine space quotient of the space of lines parallel to v; the associated vector space is $\mathsf{V}/(\mathsf{v}) \cong (\mathsf{v})^{\perp}$. The tangent space to S_{-1} at v is $\mathsf{v}^{\perp}$ proving the claim.

Passing to the quotient by Γ,

$$\mathsf{T}_{-1}M \cong (\mathsf{E} \times \mathsf{H}^2)/\Gamma.$$

Since $\Gamma \xrightarrow{\mathsf{L}} \mathsf{SO}(2, 1)$ is a discrete embedding [**14**], $\mathsf{SO}(2, 1)$ acting properly on H^2 implies that Γ acts properly on H^2. The Cartesian projection $\mathsf{E} \times \mathsf{H}^2 \to \mathsf{H}^2$ induces a projection

$$\mathsf{T}_{-1}M \longrightarrow \mathsf{H}^2/\mathsf{L}(\Gamma) = \Sigma,$$

invariant under the restriction of the geodesic flow ψ to $\mathsf{T}_{-1}M$, which defines an E-bundle over Σ. Its fiber over the orbit $\Gamma\mathsf{v}$ of a fixed future-pointing unit-timelike vector v is the union of geodesics in $M = \mathsf{E}/\Gamma$ parallel to $\Gamma\mathsf{v}$. In particular, properness of the $\mathsf{L}(\Gamma)$-action on H^2 implies non-recurrence of timelike geodesics, the last statement in Theorem 1.

More generally, any $\mathsf{L}(\Gamma)$-invariant subset $\Omega \subset \mathsf{V}$ defines a subset $\mathsf{T}_\Omega(M) \subset \mathsf{T}M$ invariant under the geodesic flow. If Ω is an open set upon which $\mathsf{L}(\Gamma)$ acts properly, then the geodesic flow defines a proper $\mathbb{R}$-action on $\mathsf{T}_\Omega(M)$. In particular, every geodesic whose velocity lies in Ω is properly immersed and is neither positively nor negatively recurrent.

An important example is the following. The lines in S_0 form the *ideal boundary* (the circle-at-infinity), $\partial\mathsf{H}^2$, of H^2. The *limit set* of $\mathsf{L}(\Gamma)$ consists of endpoints of recurrent geodesic rays in Σ. Furthermore, $\Lambda_{\mathsf{L}(\Gamma)}$ is the unique minimal $\mathsf{L}(\Gamma)$-invariant closed subset of $\partial\mathsf{H}^2$. In particular, the set of fixed points of elements of $\mathsf{L}(\Gamma)$ is dense in $\Lambda_{\mathsf{L}(\Gamma)}$. Moreover, $\mathsf{L}(\Gamma)$ acts properly on the complement

$$\Omega := \mathsf{S}_0 \backslash \Lambda_{\mathsf{L}(\Gamma)}.$$

Applying the above discussion, no geodesic tangent to $\mathsf{T}_\Omega(M)$ recurs, that is, a lightlike recurrent geodesic ray must be parallel to $\Lambda_{\mathsf{L}(\Gamma)}$.

4. *From geodesics in Σ^2 to geodesics in M^3*

While timelike directions correspond to points of Σ^2, spacelike directions correspond to geodesics in H^2. The recurrent geodesics in Σ intimately relate to the recurrent spacelike geodesics on M^3.

Denote the set of oriented spacelike geodesics in E by $\mathscr{S}$. It identifies with the orbit space of the geodesic flow $\tilde{\psi}$ on $\mathsf{T}_{+1}\mathsf{E} \cong \mathsf{E} \times \mathsf{S}_{+1}$. The natural map $\mathscr{S} \xrightarrow{\Upsilon} \mathsf{S}_{+1}$ associating to a spacelike vector its direction is equivariant with respect to $\mathsf{Isom}(\mathsf{E}) \xrightarrow{\mathsf{L}} \mathsf{SO}(2, 1)$.

The identity component of $\mathsf{SO}(2, 1)$ simply acts transitively on the unit tangent bundle UH^2, and therefore we identify $\mathsf{SO}(2, 1)^0$ with UH^2 by choosing a basepoint u_0 in UH^2. Unit-spacelike vectors in S_{+1} correspond to oriented geodesics in H^2. Explicitly, if $v \in \mathsf{S}_{+1}$, then there is a one-parameter subgroup $a(t) \in \mathsf{SO}(2, 1)$, having v as a fixed vector, and such that

$$\det(v, v^-, v^+) > 0,$$

where v^+ is an expanding eigenvector of $a(t)$ (for $t > 0$) and v^- is the contracting eigenvector. Choose a basepoint $v_0 \in \mathsf{S}_{+1}$ corresponding to the orbit of u_0 under the geodesic flow on $\mathsf{U}\Sigma$. Geodesics in H^2 relate to spacelike directions by an equivariant mapping

$$\begin{aligned} \mathsf{UH}^2 &\longrightarrow \mathsf{S}_{+1}, \\ g(u_0) &\longmapsto g(v_0). \end{aligned}$$

The unit tangent bundle $\mathsf{U}\Sigma$ of Σ identifies with the quotient

$$\mathsf{L}(\Gamma)\backslash\mathsf{UH}^2 \cong \mathsf{L}(\Gamma)\backslash\mathsf{SO}(2, 1)^0,$$

where the geodesic flow ψ corresponds to the right-action of $a(-t)$ (see, for example, [**15**, §1.2]).

Observe that a geodesic in Σ^2 is recurrent if and only if the endpoints of any of its lifts to $\tilde{\Sigma} \approx \mathsf{H}^2$ lie in the limit set $\Lambda_{\mathsf{L}(\Gamma)}$ of $\mathsf{L}(\Gamma)$. If the convex core of Σ^2 is compact, then the union $\mathsf{U}_{\mathsf{rec}}\Sigma$ of recurrent ϕ-orbits is compact.

LEMMA 2. *There exists an orbit-preserving map*

$$\mathsf{U}_{\mathsf{rec}}\Sigma \xrightarrow{\widehat{\mathbf{N}}} \mathsf{T}_{+1}(M)$$

mapping ϕ-orbits injectively to recurrent ψ-orbits.

Proof. The associated flat affine bundle $\mathbb{E}_\Gamma$ over $\mathsf{U}\Sigma$ associated to the affine deformation Γ is defined as follows. The affine representation of Γ defines a diagonal action of Γ

on $\widetilde{\mathsf{U}\Sigma} \times \mathsf{E}$. Its total space is the quotient of the product $\widetilde{\mathsf{U}\Sigma} \times \mathsf{E}$ by the diagonal action of $\pi_1(\mathsf{U}\Sigma)$:

$$\pi_1(\mathsf{U}\Sigma) \longrightarrow \pi_1(\Sigma) \longrightarrow \mathsf{Isom}(\mathsf{E}).$$

Similarly, the flat vector bundle V_Γ over $\mathsf{U}\Sigma$ is the quotient of $\widetilde{\mathsf{U}\Sigma} \times \mathsf{V}$ by the diagonal action

$$\pi_1(\mathsf{U}\Sigma) \longrightarrow \pi_1(\Sigma) \longrightarrow \mathsf{Isom}(\mathsf{E}) \xrightarrow{\mathsf{L}} \mathsf{SO}(2, 1).$$

According to [**15**], the *neutral section* of V_Γ is a $\mathsf{SO}(2, 1)$-invariant section which is parallel with respect to the geodesic flow on $\mathsf{U}\Sigma$, and arises from the graph of the $\mathsf{SO}(2, 1)$-equivariant map

$$\mathsf{U}\tilde{\Sigma} \cong \mathsf{UH}^2 \longrightarrow \mathsf{V}$$

with image S_{+1}, the space of unit-spacelike vectors in V.

Here is the main construction of [**15**]. To every section σ of $\mathbb{E}_\Gamma$ continuously differentiable along ϕ, associate the function

$$F_\sigma := \langle \nabla_\phi \sigma, \nu \rangle$$

on $\mathsf{U}\Sigma$. (Here the covariant derivative of a section of $\mathbb{E}_\Gamma$ along a vector field ϕ in the base is a section of the associated vector bundle V_Γ.) Different choices of section σ yield cohomologous functions F_σ. (Recall that two functions f_1, f_2 are *cohomologous,* written $f_1 \sim f_2$, if

$$f_1 - f_2 = \phi g$$

for a function g which is differentiable with respect to the vector field ϕ [**17**, §2.2]).

Restrict the affine bundle $\mathbb{E}_\Gamma$ to $\mathsf{U}_{\mathsf{rec}}\Sigma$. Goldman *et al* [**15**, Lemma 8.4] guarantees the existence of a *neutralized section*, that is, a section N of $(\mathbb{E}_\Gamma)|_{\mathsf{U}_{\mathsf{rec}}\Sigma}$ satisfying

$$\nabla_\phi N = f\nu,$$

for some function f.

Although the following lemma is well known, we could not find a proof in the literature. For completeness, we supply a proof in the appendix. □

LEMMA 3. *Let X be a compact space equipped with a flow ϕ. Let $f \in C(X)$, such that, for all ϕ-invariant measures μ on X,*

$$\int f \, d\mu > 0.$$

Then f is cohomologous to a positive function.

Since Γ acts properly, [**15**, Proposition 8.1] implies that $\int F_\sigma \, d\mu \neq 0$ for all ϕ-invariant probability measures μ on $\mathsf{U}_{\mathsf{rec}}\Sigma$. Since the set of invariant measures is connected, $\int F_\sigma \, d\mu$ is either positive for all ϕ-invariant probability measures μ on $\mathsf{U}_{\mathsf{rec}}\Sigma$ or negative for all ϕ-invariant probability measures μ on $\mathsf{U}_{\mathsf{rec}}\Sigma$. Conjugating by $-I$ if necessary, we may assume that $\int F_\sigma d\mu > 0$. Lemma 3 implies $F_\sigma + \phi g > 0$ for some function g. Write

$$\widehat{N} = N + g\nu.$$

$\widehat{N}$ remains neutralized, and $\nabla_\phi \widehat{N}$ vanishes nowhere.

Let $\widetilde{\mathsf{U}_{\mathsf{rec}}\Sigma}$ be the preimage of $\mathsf{U}_{\mathsf{rec}}\Sigma$ in UH^2. Then $\widehat{N}$ determines a Γ-equivariant map

$$\widetilde{\mathsf{U}_{\mathsf{rec}}\Sigma} \xrightarrow{\tilde{N}} \mathsf{E}.$$

Each $\tilde{\phi}$-orbit injectively maps to a spacelike geodesic. The map

$$\begin{aligned} \mathsf{U}_{\mathsf{rec}}\Sigma &\xrightarrow{\widehat{\mathbf{N}}} (\mathsf{E} \times \mathsf{S}_{+1})/\Gamma, \\ x &\longmapsto [(\widehat{N}(x), \nu(x))] \end{aligned}$$

is the desired orbit equivalence $\mathsf{U}_{\mathsf{rec}}\Sigma \longrightarrow \mathsf{T}_{+1}(M)$.

LEMMA 4. *Any spacelike recurrent geodesic parallel to a geodesic γ in the image of $\widehat{\mathbf{N}}$ coincides with γ.*

Proof. Let $t \overset{g}{\longmapsto} \phi_t(v)$ be an orbit in $\mathsf{U}_{\mathsf{rec}}\Sigma$. A geodesic ξ parallel to $\widehat{\mathbf{N}}(g)$ determines a parallel section u of V along g. Since g recurs, the resulting parallel section is a bounded invariant parallel section along the closure of g. By the Anosov property, such a section is along ν, and, therefore, up to reparametrization, $\gamma = \widehat{\mathbf{N}}(g)$. □

PROPOSITION 5. *$\widehat{\mathbf{N}}$ is injective and its image is the set of recurrent spacelike geodesics.*

Proof. An orbit of the geodesic flow ϕ recurs if and only if the corresponding Γ-orbit in the space $\mathscr{S}$ of spacelike geodesics in E recurs. Similarly a ϕ-orbit in $\mathsf{T}_{+1}(M)$ recurs if and only if the corresponding $\mathsf{L}(\Gamma)$-orbit in S_{+1} recurs. The map $\mathscr{S} \xrightarrow{\Upsilon} \mathsf{S}_{+1}$ recording the direction of a spacelike geodesic is L-equivariant. If the Γ-orbit of $g \in \mathscr{S}$ corresponds to a recurrent spacelike geodesic in M, then the $\mathsf{L}(\Gamma)$-orbit of $\Upsilon(g)$ corresponds to a recurrent ϕ-orbit in $\mathsf{U}\Sigma$.

$\widehat{\mathbf{N}}$ is injective along orbits of the geodesic flow. Thus it suffices to prove that the restriction of Υ to the subset of Γ-recurrent geodesics in $\mathscr{S}$ is injective. Since the fibers of Υ are parallelism classes of spacelike geodesics, Lemma 4 implies injectivity of $\widehat{\mathbf{N}}$.

Finally, let g be a ψ-recurrent point in $\mathsf{T}_{+1}(M)$, corresponding to a spacelike recurrent geodesic γ in M. It corresponds to a recurrent Γ-orbit Γg in $\mathscr{S}$. Then $\Upsilon(\Gamma g)$ is a recurrent $\mathsf{L}(\Gamma)$-orbit in S_{+1}, and corresponds to a recurrent ϕ-orbit in $\mathsf{U}\Sigma$. The image of this ϕ-orbit under $\widehat{\mathbf{N}}$ is a spacelike recurrent geodesic in $\mathsf{T}_{+1}(M)$ parallel to γ. Now apply Lemma 4 again to conclude that g lies in the image of $\widehat{\mathbf{N}}$. □

The proof of Theorem 1 is complete.

Acknowledgements. We thank Mike Boyle, Virginie Charette, Suhyoung Choi, Todd Drumm, David Fried, and Gregory Margulis for helpful conversations. We are grateful to Domingo Ruiz for pointing out several corrections.

A. *Appendix. Cohomology and positive functions*

Let X be a smooth manifold equipped with a smooth flow ϕ. A function $g \in C(X)$ is *continuously differentiable along ϕ* if, for each $x \in X$, the function

$$t \mapsto g(\phi_t(x))$$

is a continuously differentiable map $\mathbb{R} \longrightarrow X$. Denote the subspace of $C(X)$ consisting of functions continuously differentiable along ϕ by $C_\phi(X)$. For $g \in C_\phi(X)$, denote its directional derivative by

$$\phi(g) := \frac{d}{dt}\bigg|_{t=0} g \circ \phi_t.$$

The proof of Lemma 3 will be based on two lemmas.

LEMMA A.1. *Let $f \in C_\phi(X)$. For any $T > 0$, define*

$$f_T(x) := \frac{1}{T}\int_0^T f(\phi_s(x))\, ds.$$

Then $f \sim f_T$.

Proof. We must show that there exists a function $g \in C_\phi(X)$ such that

$$f_T - f = \phi g.$$

By the fundamental theorem of calculus,

$$f \circ \phi_t = f + \int_0^t (\phi f \circ \phi_s)\, ds.$$

Writing

$$g = \frac{1}{T}\int_0^T \int_0^t (f \circ \phi_s)\, ds\, dt,$$

then

$$\begin{aligned} f_T - f &= \frac{1}{T}\int_0^T (f \circ \phi_t - f)\, dt \\ &= \frac{1}{T}\int_0^T \int_0^t \phi(f \circ \phi_s)\, ds\, dt \\ &= \phi g. \end{aligned}$$

as desired. □

LEMMA A.2. *Assume that for all ϕ-invariant measures μ,*

$$\int f\, d\mu > 0.$$

Then $f_T > 0$ for some $T > 0$.

Proof. Otherwise, sequences $\{T_m\}_{m \in \widehat{\mathbb{N}}}$ of positive real numbers and sequences $\{x_m\}_{m \in \widehat{\mathbb{N}}}$ of points in M exist such that

$$f_{T_n}(x_n) \le 0.$$

Using the flow ϕ_t, push forward the (normalized) Lebesgue measure

$$\frac{1}{T_m}\mu_{[0, T_m]}$$

on the interval $[0, T_m]$ to X, to obtain a sequence of probability measures μ_n on X such that

$$\int f\, d\mu_n \le 0.$$

As in [**15**, §7], a subsequence weakly converges to a ϕ-invariant measure μ for which

$$\int f \, d\mu \leq 0,$$

contradicting our hypotheses. □

Proof of Lemma 3. By Lemma A.1, $f \sim f_T$ for any $T > 0$, and Lemma A.2 implies that $f_T > 0$ for some T. □

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