Trace coordinates on Fricke spaces of some simple hyperbolic surfaces

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1 Introduction

The work of Fricke-Klein [20] develops the deformation theory of hyperbolic structures on a surface Σ in terms of the space of representations of its fundamental group $\pi = \pi_1(\Sigma)$ in $\mathsf{SL}(2,\mathbb{C})$. This leads to an algebraic structure on the deformation spaces. Here we expound this theory from a modern viewpoint.

We emphasize the close relationship between algebra and geometry. In particular algebraic properties of 2×2 matrices are applied to hyperbolic geometry in low dimensions. Our main object of interest is the *deformation space* of hyperbolic structures on a fixed compact surface-with-boundary Σ. The points of this deformation space correspond to equivalence classes of *marked* hyperbolic structures on $\mathsf{int}(\Sigma)$ where the ends are either cusps (complete ends of finite area) or are collar neighborhoods of closed geodesics. Such deformation spaces have been named *Fricke spaces* by Bers-Gardiner [3]. When Σ is closed, then the uniformization theorem identifies hyperbolic structures with conformal structures and the Fricke space is commonly identified with the *Teichmüller space* of marked *conformal* structures on Σ.

Hyperbolic structures are a special case of *locally homogeneous geometric structures* modelled on a homogeneous space of a Lie group G. These structures were first systematically defined by Ehresmann [15], and they determine representations of the fundamental group $\pi_1(\Sigma)$ in G. Equivalence classes of structures determine *equivalence classes* of representations, and the first part of this chapter deals with the algebraic problem of determining the moduli space of equivalence classes of pairs of unimodular 2×2 matrices.

Our starting point is the following well-known yet fundamental fact when π is a free group $\mathbb{F}_2$ of rank two. This fact may be found in the book of Fricke and Klein [20] and the even earlier paper of Vogt [74]. Perhaps much was known at the time about invariants of 2×2 matrices among the early practitioners of what has since become known as "classical invariant theory". Now this algebraic work is contained in the powerful general theory developed by Procesi [65] and others, which in a sense completes the work begun in the 19th century.

Procesi's theorem implies that the ring of invariants on the space of representations $\pi \xrightarrow{\rho} \mathsf{SL}(2,\mathbb{C})$ is generated by *characters*

$$\rho \xmapsto{t_\gamma} \mathsf{tr}\big(\rho(\gamma)\big),$$

where $\gamma \in \pi$, and hence we call this ring the *character ring.* We begin by proving the elementary fact that character ring $\mathfrak{R}_1$ of a cyclic group is the polynomial ring $\mathbb{C}[\mathsf{tr}]$ generated by the *trace function* $\mathsf{SL}(2,\mathbb{C}) \xrightarrow{\mathsf{tr}} \mathbb{C}$. From this we proceed to the basic fact, that the character ring $\mathfrak{R}_2$ of the rank two free group $\mathbb{F}_2$ is a polynomial ring on *three variables:*

Theorem A (Vogt [74], Fricke [19]). Let $\mathsf{SL}(2,\mathbb{C})\times\mathsf{SL}(2,\mathbb{C}) \xrightarrow{f} \mathbb{C}$ be a regular function which is invariant under the diagonal action of $\mathsf{SL}(2,\mathbb{C})$ by conjugation. There exists a polynomial function $F(x,y,z) \in \mathbb{C}[x,y,z]$ such that

$$f(\xi,\eta) = F(\mathsf{tr}(\xi),\mathsf{tr}(\eta),\mathsf{tr}(\xi\eta)).$$

Furthermore, for all $(x,y,z) \in \mathbb{C}^3$, there exists $(\xi,\eta) \in \mathsf{SL}(2,\mathbb{C}) \times \mathsf{SL}(2,\mathbb{C})$ such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \mathsf{tr}(\xi) \\ \mathsf{tr}(\eta) \\ \mathsf{tr}(\xi\eta) \end{bmatrix}.$$

Conversely, if $x^2+y^2+z^2-xyz \neq 4$ and $(\xi,\eta),(\xi',\eta') \in \mathsf{SL}(2,\mathbb{C}) \times \mathsf{SL}(2,\mathbb{C})$ satisfy

$$\begin{bmatrix} \mathsf{tr}(\xi) \\ \mathsf{tr}(\eta) \\ \mathsf{tr}(\xi\eta) \end{bmatrix} = \begin{bmatrix} \mathsf{tr}(\xi') \\ \mathsf{tr}(\eta') \\ \mathsf{tr}(\xi'\eta') \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

then $(\xi',\eta') = g.(\xi,\eta)$ for some $g \in G$.

Algebro-geometrically, Theorem A asserts that the $\mathsf{SL}(2,\mathbb{C})$-character variety V_2 of a free group of rank two equals $\mathbb{C}^3$. This will be our basic algebraic tool for describing moduli spaces of structures on the surface Σ and their automorphisms arising from transformations of Σ.

The condition $x^2+y^2+z^2-xyz \neq 4$ also means that the matrix group $\langle\xi,\eta\rangle$ acts *irreducibly* on $\mathbb{C}^2$. That is, $\langle\xi,\eta\rangle$ preserves no proper nonzero linear subspace of $\mathbb{C}^2$. The condition that ξ,η generate an irreducible representation is crucial in several alternate descriptions of $\mathsf{SL}(2,\mathbb{C})$-representations of $\mathbb{F}_2$. In particular, it is equivalent to the condition that the $\mathsf{PGL}(2,\mathbb{C})$-orbit is closed in $\mathsf{Hom}(\mathbb{F}_2,\mathsf{SL}(2,\mathbb{C}))$. This condition is in turn equivalent to the orbit being *stable* in the sense of Geometric Invariant Theory.

A more geometric description involves the action of the subgroup $\langle\xi,\eta\rangle \subset \mathsf{SL}(2,\mathbb{C})$ on hyperbolic 3-space H^3. The group $\mathsf{PSL}(2,\mathbb{C})$ acts by orientation-preserving isometries of H^3. An *involution,* that is, an element $g \in \mathsf{PSL}(2,\mathbb{C})$

having order two, is reflection in a unique geodesic $\mathsf{Fix}(g) \subset \mathsf{H}^3$. Denote the space of such involutions by Inv.

Theorem B (Coxeter extension). Suppose that $\xi, \eta \in \mathsf{SL}(2,\mathbb{C})$ generate an irreducible representation and let $\zeta = \eta^{-1}\xi^{-1}$ so that

$$\xi\eta\zeta = \mathbb{I}.$$

Then there exists a unique triple of involutions

$$\iota_{\xi\eta}, \iota_{\eta\zeta}, \iota_{\zeta\xi} \in \mathsf{Inv}$$

such that the corresponding elements $\mathbb{P}(\xi), \mathbb{P}(\eta), \mathbb{P}(\zeta) \in \mathsf{PSL}(2,\mathbb{C})$ satisfy:

$$\begin{aligned} \mathbb{P}(\xi) &= \iota_{\zeta\xi}\iota_{\xi\eta} \\ \mathbb{P}(\eta) &= \iota_{\xi\eta}\iota_{\eta\zeta} \\ \mathbb{P}(\zeta) &= \iota_{\eta\zeta}\iota_{\zeta\xi}. \end{aligned}$$

From Theorem A follows the identification of the Fricke space of the three-holed sphere in terms of trace coordinates as $\big(-\infty, -2\big]^3$. The three trace parameters correspond to the three boundary components of Σ. From Theorem B follows the identification of the Fricke space of the three-holed sphere with the space of (mildly degenerate) right-angled hexagons in the hyperbolic plane H^2. (Right-angled hexagons are allowed to degenerate when some of the alternate edges covering boundary components degenerate to ideal points.)

The condition $x^2+y^2+z^2-xyz \neq 4$ means that $\langle \xi, \eta \rangle$ defines an irreducible representation on $\mathbb{C}^2$. This is equivalent to the condition that

$$\mathsf{tr}[\xi, \eta] \neq 2.$$

Thus the commutator trace plays an important role, partially because the fundamental group of the one-holed torus admits free generators X, Y such that the boundary component corresponds to $[X, Y]$. In particular trace coordinates identify the Fricke space of the one-holed torus with

$$\{(x,y,z) \in \big(2, \infty) \mid x^2 + y^2 + z^2 - xyz \leq 0\}$$

where the boundary trace equals

$$\mathsf{tr}[\xi, \eta] = x^2 + y^2 + z^2 - xyz \leq -2.$$

The trace coordinates are related to Fenchel-Nielsen coordinates. Similar descriptions of the Fricke spaces of the two-holed cross-surface (projective plane) and the one-holed Klein bottle are also given.

The character variety of $\mathbb{F}_3$ is more complicated. Let X_1, X_2, X_3 be free generators. The traces of the words

$$X_1, X_2, X_3, X_1X_2, X_1X_3, X_2X_3, X_1X_2X_3, X_1X_3X_2$$

generate the $\mathsf{SL}(2,\mathbb{C})$-character ring of $\mathbb{F}_3$. We denote these functions by

$$x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}, x_{132}$$

respectively. However, the character ring is not a polynomial ring on these generators, due to the trace identities expressing the *triple traces* x_{123} and x_{132} as the roots of a monic quadratic polynomial whose coefficients are polynomials in the *single traces* x_i and *double traces* x_{ij}:

$$\begin{aligned} x_{123} + x_{132} &= x_{12}x_3 + x_{13}x_2 + x_{23}x_1 - x_1x_2x_3 \\ x_{123}\ x_{132} &= (x_1^2 + x_2^2 + x_3^2) + (x_{12}^2 + x_{23}^2 + x_{13}^2) - \\ &\quad (x_1x_2x_{12} + x_2x_3x_{23} + x_3x_1x_{13}) + x_{12}x_{23}x_{13} - 4. \end{aligned}$$

Furthermore the character variety is a hypersurface in $\mathbb{C}^7$ which is a double branched covering of $\mathbb{C}^6$. In particular its coordinate ring, the *character ring,* is the quotient

$$\mathfrak{R}_3 := \mathbb{C}[x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}]/\mathfrak{I}$$

by the principal ideal $\mathfrak{I}$ generated by the polynomial

$$\begin{aligned} \Phi(x_1, x_2, x_3, &x_{12}, x_{13}, x_{23}, x_{123}) := \\ &x_1x_2x_3x_{123} + x_{12}x_{13}x_{23} \\ &\quad - x_1x_2x_{12} - x_1x_3x_{13} - x_2x_3x_{23} \\ &\quad\quad - x_1x_{23}x_{123} - x_2x_{13}x_{123} - x_3x_{12}x_{123} \\ &\quad\quad\quad + x_1^2 + x_2^2 + x_3^2 + x_{12}^2 + x_{13}^2 + x_{23}^2 \\ &\quad\quad\quad\quad + x_{123}^2 - 4 \end{aligned}$$

We use this description to discuss the Fricke spaces of the 4-holed sphere $\Sigma_{0,4}$ and the 2-holed torus $\Sigma_{1,2}$. In these cases, the generators X_i and their products correspond to curves on the surface, and we pay special attention to the elements corresponding to the boundary $\partial\Sigma$.

In particular we describe the homomorphisms on character rings induced by the orientable double coverings of the 2-holed cross-cap $C_{0,2}$

$$\Sigma_{0,4} \longrightarrow C_{0,2}$$

and the 1-holed Klein bottle $C_{1,1}$

$$\Sigma_{1,2} \longrightarrow C_{1,1}$$

respectively.

Finally we end with the important observation (see Vogt [74]) that the $\mathsf{SL}(2,\mathbb{C})$-character ring $\mathfrak{R}_n$ of a free group $\mathbb{F}_n$ where $n \geq 4$, is generated by traces of words of length ≤ 3.

This chapter began as an effort [30], to provide a self-contained exposition of Theorem A. Later it grew to include several results on hyperbolic geometry, which were used, for example in [29] but with neither adequate proofs nor references to the literature. In this version, we have tried to give a leisurely and elementary description of basic results on moduli of hyperbolic structures using trace coordinates.

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Notation and terminology. We mainly work over the field $\mathbb{C}$ of complex numbers and its subfield $\mathbb{R}$ of real numbers. Denote the ring of rational integers by $\mathbb{Z}$. We denote projectivization by $\mathbb{P}$, so that if V is a $\mathbb{C}$-vector space (respectively an $\mathbb{R}$-vector space), then $\mathbb{P}(V)$ denotes the set of all complex (respectively real) lines in V. Similarly if $V \xrightarrow{\xi} W$ is a linear transformation between vector spaces V, W, denote the corresponding projective transformation by $\mathbb{P}(\xi)$, wherever it is defined. For example the *complex projective line* $\mathbb{CP}^1 = \mathbb{P}(\mathbb{C}^2)$. The noncommutative field of Hamilton quaternions is denoted $\mathbb{H}$. The set of positive real numbers is denoted $\mathbb{R}_+$.

Denote the algebra of 2×2 matrices over $\mathbb{C}$ by $\mathsf{M}_2(\mathbb{C})$.

The trace and determinant functions are denoted tr and det respectively. Denote the transpose of a matrix A by $A^\dagger$.

Let k be a field (either $\mathbb{R}$ or $\mathbb{C}$). Denote the multiplicative group of k (the group of nonzero elements) by k^*.

Let $n > 0$ be an integer. The *general linear group* is denoted $\mathsf{GL}(n, \mathsf{k})$; for example $\mathsf{GL}(2, \mathbb{C})$ is the group of all invertible 2×2 complex matrices. We also denote the group of scalar matrices

$$\mathsf{k}^*\mathbb{I} \subset \mathsf{GL}(n, \mathsf{k})$$

by k^*. The *special linear group* consists of all matrices in $\mathsf{GL}(n,\mathsf{k})$ having determinant one, and is denoted $\mathsf{SL}(n,\mathsf{k})$. The *projective linear groups* $\mathsf{PGL}(n,\mathsf{k})$ (and respectively $\mathsf{PSL}(n,\mathsf{k})$) are the quotients of $\mathsf{GL}(n,\mathsf{k})$ (respectively $\mathsf{SL}(n,\mathsf{k})$) by the central subgroup $\{\lambda\mathbb{I} \mid \lambda \in \mathsf{k}^*\}$ of scalar matrices, which we also denote k^*.

If A, B are matrices, then their multiplicative commutator is denoted $[A,B] := ABA^{-1}B^{-1}$ and their additive commutator (their *Lie product*) is denoted $\mathsf{Lie}(A,B) := AB - BA$.

If A is a transformation, denote its set of fixed points by $\mathsf{Fix}(A)$. Denote the relation of conjugacy in a group by $\sim$. Denote free product of two groups A, B by $A * B$. If $a_1, \dots, a_n$ are elements of a group, then $\langle a_1, \dots a_n\rangle$ denotes the subgroup generated by $a_1, \dots, a_n$. The presentation of a group with generators $g_1, \dots g_m$ and relations $r_1(g_1, \dots g_m), \dots r_n(g_1, \dots g_m)$ is denoted

$$\langle g_1, \dots g_m \mid r_1, \dots r_n\rangle.$$

Denote the free group of rank n by $\mathbb{F}_n$. Denote the symmetric group on n letters by $\mathfrak{S}_n$.

Denote the (real) hyperbolic n-space by H^n.

We briefly summarize the topology of surfaces.

A compact surface with n boundary components will be called *n-holed.* If M is a closed surface, then the complement in M of n open discs will be called an *"n-holed M."* For example a 1-holed sphere is a disc and a 2-holed sphere is an annulus.

We adopt the following notation for topological types of connected compact surfaces, beginning with orientable surfaces. $\Sigma_{g,n}$ denotes the n-holed (orientable) surface of genus g. Thus $\Sigma_{0,0}$ is a sphere, $\Sigma_{1,0}$ is a torus, $\Sigma_{0,1}$ is a disc and $\Sigma_{0,2}$ is an annulus.

The connected sum operation # satisifies:

$$\Sigma_{g_1,n_1} \# \Sigma_{g_2,n_2} \approx \Sigma_{g_1+g_2,n_1+n_2}.$$

Other basic facts about orientable surfaces involve the Euler characteristic and the fundamental group:

$$\chi(\Sigma_{g,n}) = 2 - 2g - n$$

and if $n > 0$, the fundamental group $\pi_1(\Sigma_{g,n})$ is free of rank $2g + n - 1$.

For non-orientable surfaces, our starting point is the topological surface $C_{0,0}$ homeomorphic to the real projective plane, which J. H. Conway has proposed calling a *cross-surface.* We denote the n-holed $k+1$-fold connected sum of cross-surfaces by $C_{k,n}$. Thus the Möbius band is represented by $C_{0,1}$ and the Klein bottle by

$$C_{1,0} \approx C_{0,0} \# C_{0,0}$$

The operation of connected sum satisfies:

$$\Sigma_{g,n_1} \# C_{k,n_2} \approx C_{2g+k,n_1+n_2}$$
$$C_{k_1,n_1} \# C_{k_2,n_2} \approx C_{k_1+k_2+1,n_1+n_2}.$$

The Euler characteristic and the fundamental group satisfy:

$$\chi(C_{k,n}) = 1 - n - k$$

and $\pi_1(C_{k,n})$ is free of rank $n+k$ if $n > 0$.

The orientable double covering space of $C_{g,n}$ is $\Sigma_{g,2n}$.

2 Traces in $\mathsf{SL}(2,\mathbb{C})$

The purpose of this section is an elementary and relatively self-contained proof of Theorem A. This basic result explicitly describes the $\mathsf{SL}(2,\mathbb{C})$-character variety of a rank-two free group as the affine space $\mathbb{C}^3$, parametrized by the traces of the free generators X, Y and the trace of their product XY. Apparently due to Vogt [74], it is also in the work of Fricke [19] and Fricke-Klein [20].

We motivate the discussion by starting with the simpler case of conjugacy classes of single elements, that is *cyclic groups* (free groups of *rank one*). In this case the $\mathsf{SL}(2,\mathbb{C})$-character variety V_1 is the *affine line* $\mathbb{C}^1$, parametrized by the trace.

2.1 Cyclic groups

Theorem 2.1.1. *Let* $\mathsf{SL}(2,\mathbb{C}) \xrightarrow{f} \mathbb{C}$ *be a polynomial function invariant under inner automorphisms of* $\mathsf{SL}(2,\mathbb{C})$. *Then there exists a polynomial* $F(t) \in \mathbb{C}[t]$ *such that* $f(g) = F(\mathsf{tr}(g))$. *Conversely, if* $g, g' \in \mathsf{SL}(2,\mathbb{C})$ *satisfy*

$$\mathsf{tr}(g) = \mathsf{tr}(g') \neq \pm 2,$$

then $g' = hgh^{-1}$ *for some* $h \in \mathsf{SL}(2,\mathbb{C})$.

Proof. Suppose f is an invariant function. For $t \in \mathbb{C}$, define

$$\xi_t := \begin{bmatrix} t & -1 \\ 1 & 0 \end{bmatrix}$$

and define $F(t)$ by

$$F(t) = f(\xi_t).$$

Suppose that $t \neq \pm 2$ and $\mathsf{tr}(g) = t$. Then g and ξ_t each have distinct eigenvalues

$$\lambda_{\pm} = \frac{1}{2}\left(t \pm (t^2 - 4)^{1/2}\right)$$

and $hgh^{-1} = \xi_t$ for some $h \in \mathsf{SL}(2, \mathbb{C})$. Thus

$$f(g) = f(h^{-1}\xi_t h) = f(\xi_t) = F(t)$$

as desired. If $t = \pm 2$, then by taking Jordan normal form, either $g = \pm\mathbb{I}$ or g is conjugate to ξ_t. In the latter case, $f(g) = F(t)$ follows from invariance. Otherwise g lies in the closure of the $\mathsf{SL}(2, \mathbb{C})$-orbit of ξ_t and $f(g) = f(\xi_t) = F(t)$ follows by continuity of f.

The converse direction follows from Jordan normal form as already used above. □

The map

$$\mathsf{SL}(2, \mathbb{C}) \xrightarrow{\mathsf{tr}} \mathbb{C}$$

is a *categorical* quotient map in the sense of algebraic geometry, although it fails to be a quotient map in the usual sense. The discrepancy occurs at the critical level sets $\mathsf{tr}^{-1}(\pm 2)$. The critical values of tr are ± 2, and the restriction of tr to the regular set

$$\mathsf{tr}^{-1}(\mathbb{C} \setminus \{\pm 2\})$$

is a quotient map (indeed a holomorphic submersion). The critical level set $\mathsf{tr}^{-1}(2)$ consists of all unipotent matrices, and these are conjugate to the one-parameter subgroup

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

where $t \in \mathbb{C}$. For $t \neq 0$, these matrices comprise a single orbit. This orbit does *not* contain the identity matrix $\mathbb{I}$ (where $t = 0$), although its closure does. Any regular function cannot separate a non-identity unipotent matrix from $\mathbb{I}$. Thus $\mathsf{tr}^{-1}(2)$ contains two orbits: the non-identity unipotent matrices, and the identity matrix $\mathbb{I}$. Similar remarks apply to the other critical level set $\mathsf{tr}^{-1}(-2) = -\mathsf{tr}^{-1}(2)$.

For example,

$$\begin{aligned} \mathsf{SL}(2, \mathbb{C}) &\longrightarrow \mathbb{C} \\ \xi &\longmapsto \mathsf{tr}(\xi^2) \end{aligned}$$

is an invariant function and can be expressed in terms of $\mathsf{tr}(\xi)$ by:

$$\mathsf{tr}(\xi^2) = \mathsf{tr}(\xi)^2 - 2 \tag{2.1.1}$$

which follows from the Cayley-Hamilon theorem (see (2.2.2) below) by taking traces.

2.2 Two-generator groups.

We begin by recording the first (trivial) normalization for computing traces:

$$\mathsf{tr}(\mathbb{I}) = 2. \tag{2.2.1}$$

This will be the first of three properties of the trace function which enables the computation of traces of arbitrary words in elements of $\mathsf{SL}(2,\mathbb{C})$.

The Cayley-Hamilton theorem. If ξ is a 2×2-matrix,

$$\xi^2 - \mathsf{tr}(\xi)\xi + \mathsf{det}(\xi)\mathbb{I} = 0. \tag{2.2.2}$$

Suppose $\xi,\eta \in \mathsf{SL}(2,\mathbb{C})$. Multiplying (2.2.2) by ξ^{-1} and rearranging,

$$\xi + \xi^{-1} = \mathsf{tr}(\xi)\mathbb{I} \tag{2.2.3}$$

from which follows (using (2.2.1)):

$$\mathsf{tr}(\xi) = \mathsf{tr}(\xi^{-1}). \tag{2.2.4}$$

Multiplying (2.2.3) by η and taking traces, we obtain (switching ξ and η):

Theorem 2.2.1 (The Basic Identity). *Let $\xi,\eta \in \mathsf{SL}(2,\mathbb{C})$. Then*

$$\mathsf{tr}(\xi\eta) + \mathsf{tr}(\xi\eta^{-1}) = \mathsf{tr}(\xi)\mathsf{tr}(\eta). \tag{2.2.5}$$

As we shall see, the three identities (2.2.1),(2.2.3) and (2.2.4) apply to compute the trace of any word $w(\xi,\eta)$ for $\xi,\eta \in \mathsf{SL}(2,\mathbb{C})$.

Traces of reduced words: an algorithm. Here is an important special case of Theorem A. Namely, let $w(X,Y) \in \pi$ be a reduced word. Then

$$\begin{aligned} \mathsf{SL}(2,\mathbb{C}) \times \mathsf{SL}(2,\mathbb{C}) &\longrightarrow \mathbb{C} \\ (\xi,\eta) &\longmapsto \mathsf{tr}\big(w(\xi,\eta)\big) \end{aligned}$$

is an $\mathsf{SL}(2,\mathbb{C})$-invariant function on $\mathsf{SL}(2,\mathbb{C})\times\mathsf{SL}(2,\mathbb{C})$. Theorem A guarantees a polynomial

$$f_w(x,y,z) \in \mathbb{C}[x,y,z]$$

such that

$$\mathsf{tr}\big(w(\xi,\eta)\big) \;=\; f_w\big(\mathsf{tr}(\xi),\mathsf{tr}(\eta),\mathsf{tr}(\xi\eta)\big) \tag{2.2.6}$$

for all $\xi, \eta \in \mathsf{SL}(2,\mathbb{C})$. We describe an algorithm for computing $f_w(x,y,z)$. For notational convenience we write

$$\mathsf{tr}\big(w(\xi,\eta)\big) := f_{w(X,Y)}(x,y,z).$$

For example,

$$\begin{aligned} \mathsf{tr}(\mathbb{I}) &= 2, \\ \mathsf{tr}(\xi^{-1}) = \mathsf{tr}(\xi) &= x, \\ \mathsf{tr}(\eta^{-1}) = \mathsf{tr}(\eta) &= y, \end{aligned}$$

verifying assertion (2.2.6) for words w of length $\ell(w) \leq 1$. For symmetry, we write $Z = Y^{-1}X^{-1}$, so that X, Y, Z satisfy the relation

$$XYZ = \mathbb{I}.$$

(For a geometric interpretation of this presentation in terms of the three-holed sphere $\Sigma_{0,3}$, compare §3.2.) Write $\zeta = (\xi\eta)^{-1}$ so that $\xi\eta\zeta = \mathbb{I}$. Then

$$\begin{aligned} \mathsf{tr}(\xi\eta) = \mathsf{tr}(\eta\xi) = \mathsf{tr}(\xi^{-1}\eta^{-1}) = \\ \mathsf{tr}(\eta^{-1}\xi^{-1}) = \mathsf{tr}(\zeta) = \mathsf{tr}(\zeta^{-1}) = z, \end{aligned}$$

The reduced words of length two are

$$\begin{gathered} X^2, Y^2, XY, XY^{-1}, YX, YX^{-1}, \\ X^{-2}, Y^{-2}, X^{-1}Y^{-1}, X^{-1}Y, Y^{-1}X^{-1}, Y^{-1}X. \end{gathered}$$

As mentioned above, the trace of a square (2.1.1) follows immediately by taking the trace of (2.2.2). Thus:

$$\begin{aligned} \mathsf{tr}(\xi^2) &= x^2 - 2 \\ \mathsf{tr}(\eta^2) &= y^2 - 2 \\ \mathsf{tr}\big((\xi\eta)^2\big) &= z^2 - 2 \end{aligned}$$

Further applications of the trace identities imply:

$$\begin{aligned} \mathsf{tr}(\xi\eta^{-1}) &= xy - z \\ \mathsf{tr}\big(\eta(\xi\eta)\big) = \mathsf{tr}(\eta\zeta^{-1}) &= yz - x \\ \mathsf{tr}\big((\xi\eta)^{-1})\xi^{-1}\big) = \mathsf{tr}(\zeta\xi^{-1}) &= zx - y \end{aligned}$$

For example, taking $w(X,Y) = XY^{-1}$,

$$\mathsf{tr}(\xi\eta^{-1}) = \mathsf{tr}(\xi)\mathsf{tr}(\eta) - \mathsf{tr}(\xi\eta).$$

Furthermore

$$\begin{aligned}\mathsf{tr}(\xi\eta\xi^{-1}\eta) &= \mathsf{tr}(\xi\eta)\mathsf{tr}(\xi^{-1}\eta) - \mathsf{tr}(\xi^2) \\ &= z(xy-z) - (x^2-2) \\ &= 2 - x^2 - z^2 + xyz.\end{aligned} \tag{2.2.7}$$

An extremely important example is the *commutator word*

$$k(X,Y) := XYX^{-1}Y^{-1}.$$

Computation of its trace polynomial $\kappa = f_k$ follows easily from applying (2.2.5) to (2.2.7):

$$\begin{aligned}\mathsf{tr}(\xi\eta\xi^{-1}\eta^{-1}) &= \mathsf{tr}(\xi\eta\xi^{-1})\mathsf{tr}(\eta) - \mathsf{tr}(\xi\eta\xi^{-1}\eta) \\ &= y^2 - (2 - x^2 - z^2 + xyz) \\ &= x^2 + y^2 + z^2 - xyz - 2\end{aligned}$$

whence

$$\kappa(x,y,z) = f_k(x,y,z) = x^2 + y^2 + z^2 - xyz - 2. \tag{2.2.8}$$

Assume inductively that for all reduced words $w(X,Y) \in \pi$ with $\ell(w) < m$, there exists a polynomial $f_w(x,y,z) = \mathsf{tr}(w(\xi,\eta))$ satisfying (2.2.6). Suppose that $u(X,Y) \in \mathbb{F}_2$ is a reduced word of length $\ell(u) = m$.

The explicit calculations above begin the induction for $m \le 2$. Thus we assume $m > 2$.

Furthermore, we can assume that u is *cyclically reduced,* that is the initial symbol of u is not inverse to the terminal symbol of u. For otherwise

$$u(X,Y) = Su'(X,Y)S^{-1},$$

where S is one of the four symbols

$$X, Y, X^{-1}, Y^{-1}$$

and $\ell(u') = m - 2$. Then $u(X,Y)$ and $u'(X,Y)$ are conjugate and

$$\mathsf{tr}(u(X,Y)) = \mathsf{tr}(u'(X,Y)).$$

If $m > 2$, and u is cyclically reduced, then $u(X,Y)$ has a repeated letter, which we may assume to equal X. That is, we may write, after conjugating by a subword,

$$u(X,Y) = u_1(X,Y)u_2(X,Y)$$

where u_1 and u_2 are reduced words each ending in $X^{\pm 1}$. Furthermore we may assume that

$$\ell(u_1) + \ell(u_2) = \ell(u) = m,$$

so that $\ell(u_1) < m$ and $\ell(u_2) < m$. Suppose first that u_1 and u_2 both end in X. Then

$$u(X,Y) = \big(u_1(X,Y)X^{-1}\big)\, X \ \big(u_2(X,Y)X^{-1}\big)\, X$$

and each of

$$u_1(X,Y)X^{-1},\ u_2(X,Y)X^{-1}$$

has a terminal XX^{-1}, which we cancel to obtain corresponding reduced words $u_1'(X,Y), u_2'(X,Y)$ respectively with

$$\ell(u_i'), \ell(u_i)$$

for $i = 1, 2$ and

$$u(X,Y) = u_1(X,Y)u_2(X,Y) = u_1'(X,Y)Xu_2'(X,Y)X$$

in $\mathbb{F}_2$. Then

$$(u_1(X,Y)X^{-1})(u_2(X,Y)X^{-1})^{-1} = u_1'(X,Y)u_2'(X,Y)^{-1}$$

is represented by a reduced word $u_3(X,Y)$ satisfying $\ell(u_3) < m$. By the induction hypothesis, there exist polynomials

$$f_{u_1(X,Y)},\, f_{u_2(X,Y)},\, f_{u_3(X,Y)} \ \in\ \mathbb{C}[x,y,z]$$

such that, for all $\xi, \eta \in \mathsf{SL}(2,\mathbb{C})$, $i = 1, 2, 3$,

$$\mathsf{tr}\big(u_i(\xi,\eta)\big) = f_{u_i(X,Y)}\big(\mathsf{tr}(\xi), \mathsf{tr}(\eta), \mathsf{tr}(\xi\eta)\big).$$

By (2.2.5),

$$f_u = f_{u_1}f_{u_2} - f_{u_3}$$

is a polynomial in $\mathbb{C}[x,y,z]$. The cases when u_1 and u_2 both end in the symbols X^{-1}, Y, Y^{-1} are completely analogous. Since there are only four symbols, the only cyclically reduced words without repeated symbols are commutators of the symbols, for example $XYX^{-1}Y^{-1}$. Repeated applications of the trace identities evaluate this trace polynomial as $\kappa(x,y,z)$ defined in (2.2.8). The other commutators of distinct symbols also have trace $\kappa(x,y,z)$ by identical arguments.

Surjectivity of characters of pairs: a normal form. We first show that

$$\begin{aligned} \tau : \mathsf{SL}(2,\mathbb{C}) \times \mathsf{SL}(2,\mathbb{C}) &\longrightarrow \mathbb{C}^3 \\ (\xi,\eta) &\longmapsto \begin{bmatrix} \mathsf{tr}(\xi) \\ \mathsf{tr}(\eta) \\ \mathsf{tr}(\xi\eta) \end{bmatrix} \end{aligned}$$

is surjective. Let $(x,y,z) \in \mathbb{C}^3$. Choose $\mathfrak{z} \in \mathbb{C}$ so that

$$\mathfrak{z} + \mathfrak{z}^{-1} = z,$$

that is, $\mathfrak{z} = \frac{1}{2}(z \pm \sqrt{z^2-4})$. Let

$$\xi_x = \begin{bmatrix} x & -1 \\ 1 & 0 \end{bmatrix}, \; \eta_{(y,\mathfrak{z})} = \begin{bmatrix} 0 & \mathfrak{z}^{-1} \\ -\mathfrak{z} & y \end{bmatrix}. \tag{2.2.9}$$

Then $\tau(\xi_x, \eta_{(y,\mathfrak{z})}) = (x,y,z)$.

Next we show that every $\mathsf{SL}(2,\mathbb{C})$-invariant regular function

$$\mathsf{SL}(2,\mathbb{C}) \times \mathsf{SL}(2,\mathbb{C}) \xrightarrow{f} \mathbb{C}$$

factors through τ. To this end we need the following elementary lemma on symmetric functions:

Lemma 2.2.2. *Let R be an integral domain where 2 is invertible, and let $R' = R[\mathfrak{z}, \mathfrak{z}^{-1}]$ be the ring of Laurent polynomials over R. Let $R' \xrightarrow{\sigma} R'$ be the involution which fixes R and interchanges $\mathfrak{z}$ and $\mathfrak{z}^{-1}$. Then the subring of σ-invariants is the polynomial ring $R[\mathfrak{z} + \mathfrak{z}^{-1}]$.*

Proof. Let $F(\mathfrak{z}, \mathfrak{z}^{-1}) \in R[\mathfrak{z}, \mathfrak{z}^{-1}]$ be a σ-invariant Laurent polynomial. Begin by rewriting R' as the quotient of the polynomial ring $R[x,y]$ by the ideal generated by $xy-1$. Then σ is induced by the involution $\tilde{\sigma}$ of $R[x,y]$ interchanging x and y. Let $f(x,y) \in R[x,y]$ be a polynomial whose image in R' is F. Then there exists a polynomial $g(x,y)$ such that

$$f(x,y) - f(y,x) = g(x,y)(xy-1).$$

Clearly $g(x,y) = -g(y,x)$. Let

$$\tilde{f}(x,y) = f(x,y) - \frac{1}{2}\, g(x,y)(xy-1)$$

so that $\tilde{f}(x,y) = \tilde{f}(y,x)$. By the theorem on elementary symmetric functions,

$$\tilde{f}(x,y) = h(x+y, xy)$$

for some polynomial $h(u,v)$. Therefore $F(\mathfrak{z}, \mathfrak{z}^{-1}) = h(\mathfrak{z}+\mathfrak{z}^{-1}, 1)$ as desired. ☐

By definition $f(\xi, \eta)$ is a polynomial in the matrix entries of ξ and η; regard two polynomials differing by elements in the ideal generated by $\mathsf{det}(\xi) - 1$ and $\mathsf{det}(\eta) - 1$ as equal. Thus $f(\xi_x, \eta_{(y,\mathfrak{z})})$ equals a function $g(x,y,\mathfrak{z})$ which is a polynomial in $x, y \in \mathbb{C}$ and a Laurent polynomial in $\mathfrak{z} \in \mathbb{C}^*$, "where ξ_x and $\eta_{(y,\mathfrak{z})}$ were defined in (2.2.9).

Lemma 2.2.3. *Let $\xi, \eta \in \mathsf{SL}(2,\mathbb{C})$ such that $\kappa(\tau(\xi,\eta)) \neq 2$. Then there exists $h \in \mathsf{SL}(2,\mathbb{C})$ such that*

$$h \cdot (\xi, \eta) = (\xi^{-1}, \eta^{-1}).$$

Proof. Let $(x, y, z) = \tau(\xi, \eta)$. By the commutator trace formula (2.2.8),

$$\mathsf{tr}[\xi, \eta] = \kappa(x, y, z)$$

where $[\xi, \eta] = \xi\eta\xi^{-1}\eta^{-1}$.

Let $L = \xi\eta - \eta\xi$. (Compare §4 of Jørgensen [41] or Fenchel [16].) Then

$$\mathsf{tr}(L) = \mathsf{tr}(\xi\eta) - \mathsf{tr}(\eta\xi) = 0.$$

Furthermore for any 2×2 matrix M, the characteristic polynomial

$$\lambda_M(t) := \mathsf{det}(t\mathbb{I} - M) = t^2 - \mathsf{tr}(M)t + \mathsf{det}(M).$$

Thus

$$\begin{aligned}
\mathsf{det}(L) &= \mathsf{det}([\xi, \eta] - \mathbb{I})\mathsf{det}(\eta\xi) \\
&= \mathsf{det}([\xi, \eta] - \mathbb{I}) \\
&= -\lambda_{[\xi,\eta]}(1) \\
&= -2 + \mathsf{tr}[\xi, \eta] \\
&= -2 + \kappa(x, y, z) \neq 0.
\end{aligned}$$

Choose $\mu \in \mathbb{C}^*$ such that $\mu^2\mathsf{det}(L) = 1$ and let $h = \mu L \in \mathsf{SL}(2, \mathbb{C})$.

Since $\mathsf{tr}(h) = 0$ and $\mathsf{det}(h) = 1$, the Cayley-Hamilton Theorem

$$\lambda_M(M) = 0$$

implies that $h^2 = -\mathbb{I}$. Similarly

$$\begin{aligned}
\mathsf{det}(h\xi) &= \mathsf{det}(h) = 1 \\
\mathsf{det}(h\eta) &= \mathsf{det}(h) = 1,
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{tr}(h\xi) &= \mu(\mathsf{tr}((\xi\eta)\xi) - \mathsf{tr}((\eta\xi)\xi)) \\
&= \mu(\mathsf{tr}(\xi(\eta\xi)) - \mathsf{tr}((\eta\xi)\xi)) = 0
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{tr}(h\eta) &= \mu(\mathsf{tr}((\xi\eta)\eta) - \mathsf{tr}((\eta\xi)\eta)) \\
&= \mu(\mathsf{tr}((\xi\eta)\eta) - \mathsf{tr}(\eta(\xi\eta)) = 0
\end{aligned}$$

so $(h\xi)^2 = (h\eta)^2 = -\mathbb{I}$. Thus

$$h\xi h^{-1}\xi = -h\xi h\xi = \mathbb{I}$$

whence $h\xi h^{-1} = \xi^{-1}$. Similarly $h\eta h^{-1} = \eta^{-1}$, concluding the proof of the lemma. ☐

Apply Lemma 2.2.3 to $\xi = \xi_x$ and $\eta = \eta_{(y,\mathfrak{z})}$ as above to obtain h such that conjugation by h maps

$$\xi \longmapsto \xi^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & x \end{bmatrix}$$

and

$$\eta \longmapsto \eta^{-1} = \begin{bmatrix} y & -1/\mathfrak{z} \\ \mathfrak{z} & 0 \end{bmatrix}.$$

If

$$u = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

then

$$uh\xi(uh)^{-1} = u\xi^{-1}u^{-1} = \begin{bmatrix} x & -1 \\ 1 & 0 \end{bmatrix} = \xi$$

and

$$uh\eta(uh)^{-1} = u\eta^{-1}u^{-1} = \begin{bmatrix} 0 & \mathfrak{z} \\ -1/\mathfrak{z} & y \end{bmatrix}.$$

Thus

$$\begin{aligned} g(x,y,\mathfrak{z}) &= f(\xi,\eta) \\ &= f(uh\xi(uh)^{-1}, uh\eta(uh)^{-1}) \\ &= g(x,y,\mathfrak{z}^{-1}). \end{aligned}$$

Lemma 2.2.2 implies that

$$g(x,y,\mathfrak{z}) = F(x,y,\mathfrak{z}+1/\mathfrak{z}) \tag{2.2.10}$$

for some polynomial $F(x,y,z) \in \mathbb{C}[x,y,z]$, whenever $\kappa(x,y,\mathfrak{z}+1/\mathfrak{z}) \neq 2$. Since this condition defines a nonempty Zariski-dense open set, (2.2.10) holds on all of $\mathbb{C}^2 \times \mathbb{C}^*$ and

$$f(\xi,\eta) = F(\mathsf{tr}(\xi), \mathsf{tr}(\eta), \mathsf{tr}(\xi\eta))$$

as claimed.

Injectivity of $\mathsf{SL}(2,\mathbb{C})$-characters of pairs. Finally we show that if $(\xi,\eta), (\xi',\eta') \in H$ satisfy

$$\begin{bmatrix} \mathsf{tr}(\xi) \\ \mathsf{tr}(\eta) \\ \mathsf{tr}(\xi\eta) \end{bmatrix} = \begin{bmatrix} \mathsf{tr}(\xi') \\ \mathsf{tr}(\eta') \\ \mathsf{tr}(\xi'\eta') \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \tag{2.2.11}$$

and $\kappa(x,y,z) \neq 2$, then (ξ,η) and (ξ',η') are $\mathsf{SL}(2,\mathbb{C})$-equivalent. By §2.2, the triple

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \mathsf{tr}(\xi) \\ \mathsf{tr}(\eta) \\ \mathsf{tr}(\xi\eta) \end{bmatrix}$$

determines the character function

$$\begin{aligned} \pi &\longrightarrow \mathbb{C} \\ w(X,Y) &\longmapsto \mathsf{tr}\big(w(\xi,\eta)\big) = f_w(x,y,z). \end{aligned}$$

Let ρ and ρ' denote the representations $\pi \longrightarrow \mathsf{SL}(2,\mathbb{C})$ taking X, Y to ξ, η and ξ', η' respectively and let χ, χ' denote their respective characters. Then our hypothesis (2.2.11) implies that $\chi = \chi'$.

2.3 Injectivity of the character map: the general case.

The conjugacy of representations (one of which is irreducible) having the same character follows from a general argument using the Burnside theorem. I am grateful to Hyman Bass [1] for explaining this to me.

Suppose ρ and ρ' are irreducible representations on $\mathbb{C}^2$. Burnside's Theorem (see Lang [47], p.445) implies the corresponding representations (also denoted ρ, ρ' respectively) of the group algebra $\mathbb{C}\pi$ into $\mathsf{M}_2(\mathbb{C})$ are surjective. Since the trace form

$$\begin{aligned} \mathsf{M}_2(\mathbb{C}) \times \mathsf{M}_2(\mathbb{C}) &\longrightarrow \mathbb{C} \\ (A,B) &\longmapsto \mathsf{tr}(AB) \end{aligned}$$

is nondegenerate, the kernel K of $\mathbb{C}\pi \xrightarrow{\rho} \mathsf{M}_2(\mathbb{C})$ consists of all

$$\sum_{\alpha\in\pi} a_\alpha \alpha \in \mathbb{C}\pi$$

such that

$$\begin{aligned} 0 &= \mathsf{tr}\bigg(\Big(\sum_{\alpha\in\pi} a_\alpha \rho(\alpha)\Big)\rho(\beta)\bigg) \\ &= \sum_{\alpha\in\pi} a_\alpha \,\mathsf{tr}\big(\rho(\alpha\beta)\big) \\ &= \sum_{\alpha\in\pi} a_\alpha \,\chi(\alpha\beta) \end{aligned}$$

for all $\beta \in \pi$. Thus the kernels of both representations of $\mathbb{C}\pi$ are equal, and ρ and ρ' respectively induce algebra isomorphisms

$$\mathbb{C}\pi/K \longrightarrow \mathsf{M}_2(\mathbb{C}),$$

denoted $\tilde{\rho}, \tilde{\rho}'$.

The composition $\tilde{\rho}' \circ \tilde{\rho}^{-1}$ is an automorphism of the algebra $\mathsf{M}_2(\mathbb{C})$, which must be induced by conjugation by $g \in \mathsf{GL}(2, \mathbb{C})$. (See, for example, Corollary 9.122, p.734 of Rotman [67].) In particular $\rho'(\gamma) = g\rho(\gamma)g^{-1}$ as desired.

Irreducibility. The theory is significantly different for reducible representations. Representations

$$\rho_1, \rho_2 \in \mathsf{Hom}(\pi, \mathsf{SL}(2, \mathbb{C}))$$

are *equivalent* $\Longleftrightarrow$ they define the same point in the character variety, that is, for all regular functions f in the character ring,

$$f(\rho_1) = f(\rho_2).$$

If both are irreducible, then ρ_1 and ρ_2 are conjugate. Closely related is the fact that the conjugacy class of an irreducible representation is closed. Here are several equivalent conditions for irreducibility of two-generator subgroups of $\mathsf{SL}(2, \mathbb{C})$:

Proposition 2.3.1. *Let $\xi, \eta \in \mathsf{SL}(2, \mathbb{C})$. The following are equivalent:*

(1) *ξ, η generate an irreducible representation on $\mathbb{C}^2$;*

(2) $\mathsf{tr}(\xi\eta\xi^{-1}\eta^{-1}) \neq 2$*;*

(3) $\mathsf{det}(\xi\eta - \eta\xi) \neq 0$*;*

(4) *The pair (ξ, η) is not $\mathsf{SL}(2, \mathbb{C})$-conjugate to a representation by upper-triangular matrices*

$$\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix},$$

where $a \in \mathbb{C}^*, b \in \mathbb{C}$;

(5) *Either the group $\langle \xi, \eta \rangle$ is not solvable, or there exists a decomposition*

$$\mathbb{C}^2 = L_1 \oplus L_2$$

into an invariant pair *of lines L_i such that one of ξ, η interchanges L_1 and L_2;*

(6) *$\{\mathbb{I}, \xi, \eta, \xi\eta\}$ is a basis for $\mathsf{M}_2(\mathbb{C})$.*

In the next section we will find a further condition (Theorem 3.2.2) involving extending the representation to a representation of the free product $\mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2$.

Proof. The equivalence (1) $\Longleftrightarrow$ (2) is due to Culler-Shalen [13]. For completeness we give the proof here.

To prove (2)$\Longrightarrow$(1), suppose that ρ is reducible. If ξ, η generate a representation with an invariant subspace of $\mathbb{C}^2$ of dimension one, this representation is conjugate to one in which ξ and η are upper-triangular. Denoting their diagonal entries by a, a^{-1} and b, b^{-1} respectively, the diagonal entries of $\xi\eta$ are $ab, a^{-1}b^{-1}$. Thus

$$\begin{aligned} x &= a + a^{-1}, \\ y &= b + b^{-1}, \\ z &= ab + a^{-1}b^{-1}. \end{aligned}$$

By direct computation, $\kappa(x, y, z) = 2$.

To prove (1)$\Longrightarrow$(2), suppose that $\kappa(x, y, z) = 2$. Let $\mathfrak{A} \subset \mathsf{M}_2(\mathbb{C})$ denote the linear span of $\mathbb{I}, \xi, \eta, \xi\eta$. Identities derived from the Cayley-Halmilton theorem (2.2.2) such as (2.2.3) imply that $\mathfrak{A}$ is a subalgebra of $\mathsf{M}_2(\mathbb{C})$. For example, ξ^2 equals the linear combination

$$\xi^2 = -\mathbb{I} + x\xi \tag{2.3.1}$$

and

$$\eta\xi = (z - xy)\mathbb{I} + y\xi + x\eta - \xi\eta. \tag{2.3.2}$$

The latter identity follows by writing

$$\xi^{-1}\eta + \eta^{-1}\xi = \mathsf{tr}(\xi^{-1}\eta)\mathbb{I} = (xy - z)\mathbb{I}$$

and summing

$$\begin{aligned} x\eta &= \xi\eta + \xi^{-1}\eta \\ y\xi &= \eta\xi + \eta^{-1}\xi \end{aligned}$$

to obtain:

$$\xi\eta + \eta\xi = (z - xy)\mathbb{I} + x\eta + y\xi$$

as desired.

In the basis of $\mathsf{M}_2(\mathbb{C})$ by elementary matrices, the map

$$\begin{aligned} \mathbb{C}^4 &\longrightarrow \mathsf{M}_2(\mathbb{C}) \\ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &\longmapsto x_1\mathbb{I} + x_2\xi + x_3\eta + x_4\xi\eta \end{aligned}$$

has determinant $2 - \kappa(x, y, z) = 0$ and is not surjective. Thus $\mathfrak{A}$ is a proper subalgebra of $\mathsf{M}_2(\mathbb{C})$ and the representation is reducible, as desired.

(2) $\iff$ (3) follows from the suggestive formula, valid for $\xi, \eta \in \mathsf{SL}(2,\mathbb{C})$,

$$\mathsf{tr}(\xi\eta\xi^{-1}\eta^{-1}) + \mathsf{det}(\xi\eta - \eta\xi) = 2, \tag{2.3.3}$$

whose proof is left as an exercise.

The equivalence (1)$\iff$ (4) is essentially the definition of reducibility. If $L \subset \mathbb{C}^2$ is an invariant subspace, then conjugating by a linear automorphism which maps L to the first coordinate line $\mathbb{C} \times \{0\}$ makes the representation upper triangular.

(4)$\iff$ (5) follows from the classification of solvable subgroups of $\mathsf{SL}(2,\mathbb{C})$: a solvable subgroup is either conjugate to a group of upper-triangular matrices, or is conjugate to a *dihedral representation,* where one of ξ, η is a diagonal matrix and the other is the *involution*

$$i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(where the coefficient i is required for unimodularity). A dihedral representation is one which interchanges an invariant pair of lines although the lines themselves are not invariant. For a descripton of these representations in terms of hyperbolic geometry, see §3.2.

(1)$\iff$ (6) follows from the Burnside lemma, and identities such as (2.3.1) and (2.3.2) to express products of $\mathbb{I}, \xi, \eta, \xi\eta$ with the generators ξ, η as linear combinations of $\mathbb{I}, \xi, \eta, \xi\eta$. $\square$

3 Coxeter triangle groups in hyperbolic 3-space

An alternate geometric approach to the algebraic parametrization using traces involves right-angled hexagons in H^3. Specifically, a marked 2-generator group corresponds to an ordered triple of lines in H^2, no two of which are asymptotic. This triple completes to a right-angled hexagon by including the three common orthogonal lines. We use this geometric construction to identify, in terms of traces, which representations correspond to geometric structures on surfaces. However, since the trace is only defined on $\mathsf{SL}(2,\mathbb{C})$, and not on $\mathsf{PSL}(2,\mathbb{C})$, we must first discuss the conditions which ensure that a representation into $\mathsf{PSL}(2,\mathbb{C})$ lifts to $\mathsf{SL}(2,\mathbb{C})$.

3.1 Lifting representations to $\mathsf{SL}(2,\mathbb{C})$.

The group of orientation-preserving isometries of H^3 identifies with $\mathsf{PSL}(2,\mathbb{C})$, which is doubly covered by $\mathsf{SL}(2,\mathbb{C})$. In general, a representation $\Gamma \longrightarrow \mathsf{PSL}(2,\mathbb{C})$ may or may not lift to a representation to $\mathsf{SL}(2,\mathbb{C})$. Clearly if Γ

is a free group, every representation lifts, since lifting each generator suffices to define a lifted representation. In general the obstruction to lifting a representation $\Gamma \longrightarrow \mathsf{PSL}(2,\mathbb{C})$ is a cohomology class $\mathfrak{o} \in H^2(\Gamma, \mathbb{Z}/2)$. Furthermore there exists a central $\mathbb{Z}/2$-extension $\hat{\Gamma} \longrightarrow \Gamma$ (corresponding to $\mathfrak{o}$) and a lifted representation $\hat{\Gamma}$ such that

$$\begin{array}{ccc} \hat{\Gamma} & \longrightarrow & \mathsf{SL}(2,\mathbb{C}) \\ \downarrow & & \downarrow \\ \Gamma & \longrightarrow & \mathsf{PSL}(2,\mathbb{C}) \end{array}$$

commutes. This lift is *not unique;* the various lifts differ by multiplication by homomorphisms

$$\Gamma \longrightarrow \{\pm\mathbb{I}\} = \mathsf{center}\big(\mathsf{SL}(2,\mathbb{C})\big)$$

which comprise the group

$$\mathsf{Hom}\big(\pi_1(\Sigma), \{\pm\mathbb{I}\}\big) \cong H^1(\Sigma; \mathbb{Z}/2).$$

The cohomology class in $H^2(\Gamma, \mathbb{Z}/2)$ may be understood in terms of *Hopf's formula* for the second homology of a group. (See, for example, Brown [6].) Consider a presentation $\Gamma = F/R$ where F is a finitely generated free group and $R \lhd F$ is a normal subgroup. A set $\{f_1, \dots, f_N\}$ of free generators for F corresponds to the generators of Γ and R corresponds to the *relations* among these generators. Then Hopf's formula identifies $H_2(\Gamma)$ with the quotient group

$$\big([F,F] \cap R\big)/[F,R],$$

where $[F,F] \lhd F$ is the commutator subgroup and $[F,R]$ is the (normal) subgroup of F generated by commutators $[f,r]$ where $f \in F$ and $r \in R$. Intuitively, $H_2(\Gamma)$ is generated by relations which are products of simple commutators $[a_1,b_1]\dots[a_g,b_g]$, where $a_i, b_i \in F$ are words in $f_1, \dots, f_N$. Such *commutator relations* correspond to maps of a closed orientable surface Σ_g into the classifying space $B\Gamma$ of Γ. If $\Gamma \xrightarrow{\rho} G$ is a homomorphism into G and $\tilde{G} \longrightarrow G$ is a central extension (such as a covering group of a Lie group), then the obstruction is calculated for each commutator relation

$$w = [a_1,b_1]\dots[a_g,b_g] \in [F,F] \cap R$$

corresponding to a 2-cycle z, as follows. (Here each $a_i, b_i \in F$ is a word in the free generators $f_1, \dots, f_N$.) Lift each generator $\rho(f_i)$ to $\tilde{\rho}(f_i) \in \tilde{G}$ and evaluate the word $w(f_1, \dots, f_N)$ on the lifts $\tilde{\rho}(f_i)$ to obtain an element in the kernel K of $\tilde{G} \longrightarrow G$ (since $w \in R$). Furthermore since $w \in [F,F]$ and two lifts differ by an element of $K \subset \mathsf{center}(\tilde{G})$, this element is independent of the chosen lift $\tilde{\rho}$. This procedure defines an element of

$$H^2(\Gamma, K) \;\cong\; \mathsf{Hom}\left(\frac{[F,F]\cap R}{[F,R]}, K\right)$$

which evidently vanishes if and only if ρ lifts. (Compare Milnor [57]). For more discussion of lifting homomorphisms to $\mathsf{SL}(2,\mathbb{C})$, compare Culler [12], Kra [46], Goldman [26] or Patterson [63]. According to Patterson [63], the first result of this type, due to H. Petersson [64], is that a Fuchsian subgroup of $\mathsf{PSL}(2,\mathbb{R})$ lifts to $\mathsf{SL}(2,\mathbb{R})$ if and only if it has no elements of order two.)

A representation $\Gamma \longrightarrow \mathsf{PSL}(2,\mathbb{C})$ is *irreducible* if one (and hence every) lift $\hat{\Gamma} \longrightarrow \mathsf{SL}(2,\mathbb{C})$ is irreducible.

3.2 The 3-holed sphere.

The basic building block for hyperbolic surfaces is the three-holed sphere $\Sigma_{0,3}$.

Geometric version of Theorem A. Theorem 1 has a suggestive interpretation in terms of the three-holed sphere $\Sigma_{0,3}$, or "pair-of-pants." Namely, the fundamental group

$$\pi_1(\Sigma_{0,3}) \cong \mathbb{F}_2$$

admits the *redundant geometric presentation*

$$\pi = \pi_1(\Sigma_{0,3}) = \langle X, Y, Z \mid XYZ = 1\rangle,$$

where X, Y, Z correspond to the three components of $\partial\Sigma_{0,3}$. Denoting the corresponding trace functions by lower case, for example

$$\begin{aligned} \mathsf{Hom}(\pi, G) &\xrightarrow{x} \mathbb{C} \\ \rho &\longmapsto \mathsf{tr}\big(\rho(X)\big), \end{aligned}$$

Theorem A asserts that the $\mathsf{SL}(2,\mathbb{C})$-character ring of π is the polynomial ring $\mathbb{C}[x,y,z]$.

Theorem 3.2.1. *The equivalence class of a flat $\mathsf{SL}(2,\mathbb{C})$-bundle over $\Sigma_{0,3}$ with irreducible holonomy is determined by the equivalence classes of its restrictions to the three components of $\partial\Sigma_{0,3}$. Furthermore any triple of isomorphism classes of flat $\mathsf{SL}(2,\mathbb{C})$-bundles over $\partial\Sigma_{0,3}$ whose holonomy traces satisfy*

$$x^2 + y^2 + z^2 - xyz \neq 4$$

extends to a flat $\mathsf{SL}(2,\mathbb{C})$-bundle over $\Sigma_{0,3}$.

The hexagon orbifold. Every irreducible representation ρ corresponds to a geometric object in H^3, a *triple of geodesics.* Any two of these geodesics

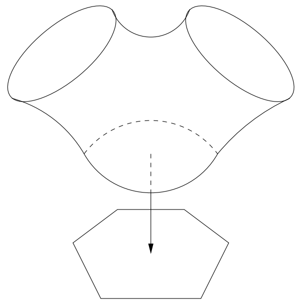

Figure 1. The three-holed sphere double covers a hexagon orbifold

admits a unique common perpendicular geodesic. These perpendiculars cut off a hexagon bounded by geodesic segments, with all six angles right angles. Such a *right hexagon* in H^3 is an alternate geometric object corresponding to ρ.

The surface $\Sigma_{0,3}$ admits an *orientation-reversing involution*

$$\Sigma_{0,3} \xrightarrow{\iota_{\mathsf{Hex}}} \Sigma_{0,3}$$

whose restriction to each boundary component is a reflection. The quotient Hex by this involution is a disc, combinatorially equivalent to a hexagon. The three boundary components map to three intervals $\partial_i(\mathsf{Hex})$, for $i = 1, 2, 3$, in the boundary $\partial\mathsf{Hex}$. The other three edges in $\partial\mathsf{Hex}$ correspond to the three arcs comprising the fixed point set $\mathsf{Fix}(\iota_{\mathsf{Hex}})$. The orbifold structure on Hex is defined by mirrors on these three arcs on $\partial\mathsf{Hex}$. The quotient map

$$\Sigma_{0,3} \xrightarrow{\Pi_{\mathsf{Hex}}} \mathsf{Hex}$$

is an orbifold covering-space, representing $\Sigma_{0,3}$ as the orientable double covering of the orbifold Hex.

The orbifold fundamental group is

$$\begin{aligned} \hat{\pi} := \pi_1(\mathsf{Hex}) &= \langle \iota_{YZ}, \iota_{ZX}, \iota_{XY} \mid \iota_{YZ}^2 = \iota_{ZX}^2 = \iota_{XY}^2 = 1 \rangle \\ &\cong \mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2. \end{aligned}$$

The covering-space $\Sigma_{0,3} \xrightarrow{\Pi_{\mathsf{Hex}}} \mathsf{Hex}$ induces the embedding of fundamental groups:

$$\begin{aligned} \pi_1(\Sigma_{0,3}) &\xrightarrow{(\Pi_{\mathsf{Hex}})_*} \pi_1(\mathsf{Hex}) \\ X &\longmapsto \iota_{ZX}\iota_{XY} \\ Y &\longmapsto \iota_{XY}\iota_{YZ} \\ Z &\longmapsto \iota_{YZ}\iota_{ZX}. \end{aligned}$$

Theorem 3.2.2. *Let $\pi \xrightarrow{\rho} \mathsf{PGL}(2,\mathbb{C})$ be an irreducible representation. Then there exists a unique representation $\hat{\pi} \xrightarrow{\hat{\rho}} \mathsf{PGL}(2,\mathbb{C})$ such that $\rho = \hat{\rho} \circ (\Pi_{\mathsf{Hex}})_*$.*

Every element of order two in $\mathsf{PGL}(2,\mathbb{C})$ is reflection about some geodesic. Therefore a representation $\hat{\rho}$ corresponds exactly to an ordered triple of geodesics in H^3. Denote this ordered triple of geodesics in H^3 corresponding to ρ by ι^ρ.

Corollary 3.2.3. *Irreducible representations $\pi \xrightarrow{\rho} \mathsf{PGL}(2,\mathbb{C})$ correspond to triples ι^ρ of geodesics in H^3, which share neither a common endpoint nor a common orthogonal geodesic.*

The proofs of Theorem 3.2.2 and Corollary 3.2.3 occupy the remainder of this section.

Involutions in $\mathsf{PGL}(2,\mathbb{C})$. We are particularly interested in projective transformations of $\mathbb{CP}^1$ of order two, which we call *involutions.* Such an involution is given by a matrix $\xi \in \mathsf{GL}(2,\mathbb{C})$ such that ξ^2 does act identically on $\mathbb{CP}^1$ but ξ does *not* act identically on $\mathbb{CP}^1$. Thus ξ is a matrix whose square is a scalar matrix but ξ itself is not scalar. Since $\mathsf{det}(\xi) \neq 0$, replacing ξ by

$$\mathsf{det}(\xi)^{-1/2}\xi$$

— for either choice of $\mathsf{det}(\xi)^{-1/2}$ — ensures that $\mathsf{det}(\xi) = 1$. Then the scalar matrix $\xi^2 = \pm\mathbb{I}$. If $\xi^2 = \mathbb{I}$, then $\mathsf{det}(\xi) = 1$ implies $\xi = -\mathbb{I}$, a contradiction. Hence $\xi^2 = -\mathbb{I}$, and ξ must have distinct reciprocal eigenvalues $\pm i$. Thus ξ is conjugate to

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

The corresponding projective transformation $\mathbb{P}(\xi)$ has two fixed points. The orbit of any point not in $\mathsf{Fix}\big(\mathbb{P}(\xi)\big)$ has cardinality two.

Proposition 3.2.4. *Let $\xi \in \mathsf{M}_2(\mathbb{C})$. The following conditions are equivalent:*

- $\mathbb{P}(\xi) \in \mathsf{Inv}$*;*
- *ξ is conjugate to* $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$*;*
- $\mathsf{det}(\xi) = 1$ *and* $\mathsf{tr}(\xi) = 0$*;*
- $\xi^2 = -\mathbb{I}$ *and* $\xi \neq \pm i\mathbb{I}$*;*
- $\xi^2 = -\mathbb{I}$ *and ξ is not a scalar matrix.*

The proof is left as an exercise. Denote the collection of such matrices by

$$\begin{aligned} \widetilde{\mathsf{Inv}} &:= \mathsf{SL}(2,\mathbb{C}) \cap \mathfrak{sl}(2,\mathbb{C}) \\ &= \{\xi \in \mathsf{M}_2(\mathbb{C}) \mid \mathsf{det}(\xi) = 1, \mathsf{tr}(\xi) = 0\}. \end{aligned}$$

Notice that $\widetilde{\mathsf{Inv}}$ is invariant under $\pm\mathbb{I}$, and the quotient

$$\mathsf{Inv} := \widetilde{\mathsf{Inv}}/\{\pm\mathbb{I}\} \subset \mathsf{PGL}(2,\mathbb{C})$$

consists of all projective involutions of $\mathbb{CP}^1$. It naturally identifies with the collection of unordered pairs of distinct points in $\mathbb{CP}^1$, that is, the quotient

$$\big(\mathbb{CP}^1 \times \mathbb{CP}^1 \setminus \Delta_{\mathbb{CP}^1}\big)/\mathfrak{S}_2$$

of the complement in $\mathbb{CP}^1 \times \mathbb{CP}^1$ of the diagonal

$$\Delta_{\mathbb{CP}^1} \subset \mathbb{CP}^1 \times \mathbb{CP}^1$$

by the symmetric group $\mathfrak{S}_2$. In §3.2, we interpret $\widetilde{\mathsf{Inv}}$ as the space of *oriented* geodesics in hyperbolic 3-space H^3.

Involutions and the complex projective line. Denote by $\overline{\mathsf{Inv}}$ the closure of Inv in the projective space $\mathbb{P}(\mathfrak{sl}(2,\mathbb{C}))$. The complement $\overline{\mathsf{Inv}}\setminus\mathsf{Inv}$ corresponds to $\mathbb{CP}^1$ embedded as the diagonal $\Delta_{\mathbb{CP}^1}$ in the above description. For example the elements of $\overline{\mathsf{Inv}}$ corresponding to $0,\infty \in \mathbb{CP}^1$ are the respective lines

$$\begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ * & 0 \end{bmatrix} \subset \mathsf{M}_2(\mathbb{C}).$$

The closure corresponds to the full quotient space

$$\big(\mathbb{CP}^1 \times \mathbb{CP}^1\big)/\mathfrak{S}_2.$$

An element $\xi \in \mathsf{PGL}(2,\mathbb{C}) \setminus \{\mathbb{I}\}$ stabilizes a unique element $\iota_\xi \in \overline{\mathsf{Inv}}$. If ξ is semisimple ($\#\mathsf{Fix}(\xi) = 2$), then ι_ξ is the unique involution with the same fixed points. Otherwise ξ is parabolic ($\#\mathsf{Fix}(\xi) = 1$), and ι_ξ corresponds to the line

$$\mathsf{Fix}\big(\mathsf{Ad}(\xi)\big) = \mathsf{Ker}\big(\mathbb{I} - \mathsf{Ad}(\xi)\big) \subset \mathfrak{sl}(2,\mathbb{C}),$$

the Lie algebra centralizer of ξ in $\mathfrak{sl}(2,\mathbb{C})$. Further discussion of semisimple elements in $\mathsf{SL}(2,\mathbb{C})$ and $\mathsf{PSL}(2,\mathbb{C})$ is given in §3.2.

Here is an elegant matrix representation. If $\xi \in \mathsf{SL}(2,\mathbb{C})$ is semisimple and $\neq \mathbb{I}$, then the two lifts of $\iota_\xi \in \mathsf{Inv}$ to $\widetilde{\mathsf{Inv}} \subset \mathsf{SL}(2,\mathbb{C})$ differ by $\pm\mathbb{I}$. Since ξ is semisimple, its *traceless projection*

$$\xi' := \xi - \frac{1}{2}\mathsf{tr}(\xi)\mathbb{I}$$

satisfies

- $\mathsf{tr}(\xi') = 0$;
- ξ' commutes with ξ;
- $\det(\xi') \neq 0$ (semisimplicity).

Choose $\delta \in \mathbb{C}^*$ such that

$$\delta^2 = \mathsf{det}(\xi') = \frac{4 - \mathsf{tr}(\xi)^2}{4}.$$

Then $\delta^{-1}\xi' \in \widetilde{\mathsf{Inv}}$ and represents the involution ι_ξ centralizing ξ:

$$\widetilde{\iota_\xi} \; = \; \pm \frac{2}{\sqrt{4 - \mathsf{tr}(\xi)^2}} \left(\xi - \frac{\mathsf{tr}(\xi)}{2} \mathbb{I} \right). \tag{3.2.1}$$

This formula will be used later in (4.2.3).

3-dimensional hyperbolic geometry. The group $\mathsf{GL}(2, \mathbb{C})$ acts by orientation-preserving isometries on *hyperbolic* 3*-space* H^3. The kernel of the action equals the center of $\mathsf{GL}(2, \mathbb{C})$, the group $\mathbb{C}^*$ of nonzero scalar matrices. The quotient

$$\mathsf{PGL}(2, \mathbb{C}) := \mathsf{GL}(2, \mathbb{C})/\mathbb{C}^*$$

acts effectively on H^3. The restriction of the quotient homomorphism

$$\mathsf{GL}(2, \mathbb{C}) \longrightarrow \mathsf{PGL}(2, \mathbb{C})$$

to $\mathsf{SL}(2, \mathbb{C}) \subset \mathsf{GL}(2, \mathbb{C})$ defines an isomorphism

$$\mathsf{PSL}(2, \mathbb{C}) \xrightarrow{\cong} \mathsf{PGL}(2, \mathbb{C}).$$

The projective line $\mathbb{CP}^1$ identifies naturally with the ideal boundary $\partial\mathsf{H}^3$. The center of $\mathsf{SL}(2, \mathbb{C})$ consists of $\pm\mathbb{I}$, which is the kernel of the actions on H^3 and $\mathbb{CP}^1$. The only element of order two in $\mathsf{GL}(2, \mathbb{C})$ is $-\mathbb{I}$, and an element of even order $2k$ in $\mathsf{PGL}(2, \mathbb{C})$ corresponds to an element of order $4k$ in $\mathsf{GL}(2, \mathbb{C})$. Elements of odd order $2k + 1$ in $\mathsf{PGL}(2, \mathbb{C})$ have two lifts to $\mathsf{SL}(2, \mathbb{C})$, one of order $2k + 1$ and the other of order $2(2k + 1)$.

We use the *upper-half-space model* of H^3 as follows. The algebra $\mathbb{H}$ of *Hamilton quaternions* is the $\mathbb{R}$-algebra generated by $1, i, \mathsf{j}$ subject to the relations

$$i^2 = \mathsf{j}^2 = -1, i\mathsf{j} + \mathsf{j}i = 0.$$

$\mathbb{H}$ contains the smaller subalgebra $\mathbb{C}$ having basis $\{1, i\}$. Define

$$\mathsf{H}^3 := \{z + u\mathsf{j} \in \mathbb{H} \mid z \in \mathbb{C}, u \in \mathbb{R}, u > 0\}$$

where

$$\xi = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathsf{GL}(2, \mathbb{C})$$

acts by

$$z + u\mathsf{j} \longmapsto \big(a(z + u\mathsf{j}) + b\big)\big(c(z + u\mathsf{j}) + d\big)^{-1}.$$

$\mathsf{PSL}(2,\mathbb{C})$ is the group of orientation-preserving *isometries* of H^3 with respect to the *Poincaré metric*

$$u^{-2}\big(|dz|^2 + du^2\big)$$

of constant curvature -1. The restriction of $\mathsf{GL}(2,\mathbb{C})$ to $\partial\mathsf{H}^3$ identifies with the usual projective action of $\mathsf{PGL}(2,\mathbb{C})$ on

$$\partial\mathsf{H}^3 := \mathbb{CP}^1 = \mathbb{C}\cup\{\infty\},$$

the space of complex lines (1-dimensional linear subspaces) in $\mathbb{C}^2$.

Oriented geodesics in H^3 correspond to ordered pairs of distinct points in $\mathbb{CP}^1$, via their endpoints. Unoriented geodesics correspond to unordered pairs. For example geodesics with an endpoint at ∞ are represented by vertical rays $z + \mathbb{R}_+\mathsf{j}$, where $z \in \mathbb{C}$ is the other endpoint. The unit-speed parametrization of this geodesic is

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathsf{H}^3 \\ t &\longmapsto z + e^t\mathsf{j} \end{aligned}$$

Distinct $z_1, z_2 \in \mathbb{C}$ span a geodesic in H^3 whose unit-speed parametrization is:

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathsf{H}^3 \\ t &\longmapsto \frac{z_1 + z_2}{2} + \frac{z_2 - z_1}{2}\Big(\tanh(t) + \operatorname{sech}(t)\mathsf{j}\Big). \end{aligned}$$

A geodesic $l \subset \mathsf{H}^3$ corresponds uniquely to the involution $\iota = \iota_l \in \mathsf{PSL}(2,\mathbb{C})$ for which

$$l = \mathsf{Fix}(\iota).$$

For example, if $z_1, z_2 \in \mathbb{C}$, the involution in $\mathsf{PGL}(2,\mathbb{C})$ fixing z_1, z_2 is given by the pair of matrices

$$\pm\frac{i}{z_1 - z_2}\begin{bmatrix} z_1 + z_2 & -2z_1z_2 \\ 2 & -(z_1+z_2)\end{bmatrix} \in \mathsf{SL}(2,\mathbb{C}). \tag{3.2.2}$$

If $z_2 = \infty$, the corresponding matrices are:

$$\pm i\begin{bmatrix} -1 & 2z_1 \\ 0 & 1\end{bmatrix} \in \mathsf{SL}(2,\mathbb{C}). \tag{3.2.3}$$

Compare Fenchel [16] for more details.

Let $\xi \in \mathsf{SL}(2,\mathbb{C})$ be *non-central:* $\xi \neq \pm\mathbb{I}$. Then the following conditions are equivalent:

- ξ has two distinct eigenvalues;
- $\mathsf{tr}(\xi) \neq \pm 2$;
- the corresponding collineation of $\mathbb{CP}^1$ has two fixed points;

- the correponding orientation-preserving isometry of H^3 leaves invariant a unique geodesic ℓ_ξ, each of whose endpoints is fixed;
- a unique involution ι_ξ centralizes ξ.

(In the standard terminology, the corresponding isometry of H^3 is either *elliptic* or *loxodromic.*)

We shall say that ξ is *semisimple.* Otherwise ξ is *parabolic:* it has a repeated eigenvalue (necessarily ± 1, because $\mathsf{det}(\xi) = 1$), and fixes a *unique* point on $\mathbb{CP}^1$.

Suppose $\xi \in \mathsf{SL}(2,\mathbb{C})$ and the corresponding isometry $\mathbb{P}(\xi) \in \mathsf{PSL}(2,\mathbb{C})$ leaves invariant a geodesic $l \subset \mathsf{H}^3$. Then the restriction $\mathbb{P}(\xi)|_l$ is an isometry of $l \approx \mathbb{R}$.

Any isometry of $\mathbb{R}$ is either a translation of $\mathbb{R}$, a reflection in a point of $\mathbb{R}$, or the identity. We distinguish these three cases as follows. For concreteness choose coordinates so that l is represented by the imaginary axis $\mathbb{R}_+\mathsf{j} \subset \mathsf{H}^3$ in the upper-half-space model. The endpoints of l are $0, \infty$:

- $\mathbb{P}(\xi)|_l$ **acts by translation:** Then $\mathbb{P}(\xi)$ is *loxodromic,* represented by
$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \tag{3.2.4}$$
where $\lambda \in \mathbb{C}^*$ is a nonzero complex number and $|\lambda| \neq 1$. The fixed point set is
$$\mathsf{Fix}\big(\mathbb{P}(\xi)\big) = \{0, \infty\}.$$
The restriction of $\mathbb{P}(\xi)$ to l is translation along l by distance $2\log|\lambda|$, in the direction from 0 (its repellor) to ∞ (its attractor) if $|\lambda| > 1$. (If $|\lambda| < 1$, then 0 is the attractor and ∞ is the repellor.)
- $\mathbb{P}(\xi)|_l$ **acts identically:** Now $\mathbb{P}(\xi)$ is *elliptic* and is represented by the diagonal matrix (3.2.4), except now $|\lambda| = 1$. If $\lambda = e^{i\theta}$, then $\mathbb{P}(\xi)$ represents a rotation through angle 2θ about l. In particular if $\lambda = \pm i$, then $\mathbb{P}(\xi)$ is the involution fixing l. Although $\mathbb{P}(\xi)$ has order two in $\mathsf{PGL}(2,\mathbb{C})$, its matrix representatives in $\mathsf{SL}(2,\mathbb{C})$ each have order 4. (Compare Proposition 3.2.4.)
- $\mathbb{P}(\xi)|_l$ **acts by reflection:** In this case $\mathbb{P}(\xi)$ interchanges the two endpoints $0, \infty$ and is necessarily of order two. Its restriction $\mathbb{P}(\xi)|_l$ to l fixes the point $p = \mathsf{Fix}\big(\mathbb{P}(\xi)\big) \cap l$, and is reflection in p. The corresponding matrix is:
$$\begin{bmatrix} 0 & -\lambda \\ \lambda^{-1} & 0 \end{bmatrix}$$
where $\lambda \in \mathbb{C}^*$ and $p = |\lambda|\mathsf{j}$ is the fixed point of $\mathbb{P}(\xi)_l$. Necessarily $\mathbb{P}(\xi) \in \mathsf{Inv}$ and
$$\mathsf{Fix}\big(\mathbb{P}(\xi)\big) = \{\pm i\lambda\}.$$

Dihedral representations. The following lemma is crucial in the proof of Theorem 3.2.2.

Lemma 3.2.5. *Suppose that* $\xi \in \mathsf{SL}(2,\mathbb{C}) \setminus \{\pm\mathbb{I}\}$ *and* $\iota \in \mathsf{Inv}$*.*

(1) *Suppose that* $\#\mathsf{Fix}\big(\mathbb{P}(\xi)\big) = 2$*. Let* $\ell_\xi \subset \mathsf{H}^3$ *denote the unique* ξ*-invariant geodesic (the geodesic with endpoints* $\mathsf{Fix}\big(\mathbb{P}(\xi)\big)$*). Then*

$$\iota\xi\iota = \xi^{-1} \tag{3.2.5}$$

if and only if ι *preserves* ℓ_ξ *and its restriction acts by reflection. In that case* ι *interchanges the two elements of* $\mathsf{Fix}\big(\mathbb{P}(\xi)\big)$*.*

(2) *Suppose that* $\#\mathsf{Fix}\big(\mathbb{P}(\xi)\big) = 1$*. Then* $\iota\xi\iota = \xi^{-1}$ *if and only if* $\mathsf{Fix}\big(\mathbb{P}(\xi)\big) \subset \mathsf{Fix}(\iota)$*.*

Proof. Consider first the case that ξ is semisimple, that is, when $\#\mathsf{Fix}\big(\mathbb{P}(\xi)\big) = 2$. Let $\ell_\xi \subset \mathsf{H}^3$ be the ξ-invariant geodesic with endpoints $\mathsf{Fix}\big(\mathbb{P}(\xi)\big)$. Then ι interchanges the two elements of $\mathsf{Fix}\big(\mathbb{P}(\xi)\big)$. In terms of the linear representation, ξ preserves a decomposition into eigenspaces

$$\mathbb{C}^2 = L_1 \oplus L_2$$

where each line $L_i \subset \mathbb{C}^2$ corresponds to a fixed point in $\mathbb{CP}^1$, and ι interchanges L_1 and L_2.

When $\#\mathsf{Fix}\big(\mathbb{P}(\xi)\big) = 1$, the corresponding matrix has a unique eigenspace, which we take to be the first coordinate line. Then ξ is represented by the upper-triangular matrix

$$\pm \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}$$

and ι is also represented by an upper-triangular matrix, of the form

$$\pm \begin{bmatrix} i & w \\ 0 & -i \end{bmatrix}.$$

Rewrite (3.2.5) as:

$$(\iota\xi)^2 = \mathbb{I}$$

so that $\iota\xi \in \mathsf{Inv}$. Thus ξ factors as the product of two involutions

$$\xi = \iota(\iota\xi).$$

Conversely if $\iota, \iota' \in \mathsf{Inv}$, then the product $\xi := \iota\iota'$ satisfies (3.2.5). ∎

Geometric interpretation of the Lie product. These ideas provide an elegant formula for the common orthogonal of the invariant axes of elements of $\mathsf{PSL}(2,\mathbb{C})$. Suppose $\xi,\eta \in \mathsf{SL}(2,\mathbb{C})$. Then the *Lie product*

$$\mathsf{Lie}(\xi,\eta) := \xi\eta - \eta\xi \tag{3.2.6}$$

has trace zero, and vanishes if and only if ξ,η commute. Furthermore (2.3.3) and Proposition 2.3.1, (1) $\iff$ (2) imply that $\mathsf{Lie}(\xi,\eta)$ is invertible if and only if $\langle\xi,\eta\rangle$ acts irreducibly (as defined in §3). Suppose $\langle\xi,\eta\rangle$ acts irreducibly, so that $\mathsf{Lie}(\xi,\eta)$ defines an element $\lambda \in \mathsf{PSL}(2,\mathbb{C})$. Since $\mathsf{tr}\big(\mathsf{Lie}(\xi,\eta)\big) = 0$, the isometry λ has order two, that is, lies in Inv.

Now

$$\begin{aligned}
\mathsf{tr}\big(\xi\,\mathsf{Lie}(\xi,\eta)\big) &= \mathsf{tr}\big(\xi(\xi\eta)\big) - \mathsf{tr}\big(\xi(\eta\xi)\big)\\
&= \mathsf{tr}\big(\xi(\xi\eta)\big) - \mathsf{tr}\big((\xi\eta)\xi\big)\\
&= 0
\end{aligned}$$

which implies that $\xi\lambda$ also has order two, that is, $\lambda\xi\lambda = \xi^{-1}$. Lemma 3.2.5 implies that λ acts by reflection on the invariant axis ℓ_ξ. Similarly λ acts by reflection on the invariant axis ℓ_η. Hence the fixed axis ℓ_λ is orthogonal to both ℓ_ξ and ℓ_η:

Proposition 3.2.6. *If $\xi,\eta \in \mathsf{GL}(2,\mathbb{C})$, then the Lie product $\mathsf{Lie}(\xi,\eta)$ represents the common orthogonal geodesic $\perp(\ell_{\mathbb{P}(\xi)},\ell_{\mathbb{P}(\eta)})$ to the invariant axes $\ell_{\mathbb{P}(\xi)},\ell_{\mathbb{P}(\eta)}$ of $\mathbb{P}(\xi)$ and $\mathbb{P}(\eta)$ respectively.*

Compare Marden [53] and the references given there.

Geometric proof of Theorem 3.2.2.

Proof of Theorem 3.2.2. Abusing notation, write X,Y,Z for

$$\rho(X),\rho(Y),\rho(Z) \in \mathsf{PGL}(2,\mathbb{C})$$

respectively. We seek respective involutions

$$\rho(\iota_{XY}),\ \rho(\iota_{YZ}),\ \rho(\iota_{ZX}),$$

which we respectively denote ρ_{XY}, ρ_{YZ}, ρ_{ZX}. These involutions will be the ones fixing the respective pairs. For example we take ρ_{XY} to be the involution fixing $\perp(l_X,l_Y)$, and similarly for ρ_{YZ} and ρ_{ZX}.

Suppose that $\langle X,Y\rangle \subset \mathsf{SL}(2,\mathbb{C})$ acts irreducibly on $\mathbb{C}^2$. Write $Z = Y^{-1}X^{-1}$ so that

$$XYZ = \mathbb{I}.$$

Since $\langle X,Y\rangle$ acts irreducibly, none of X,Y,Z act identically.

Let $\rho_{XY} \in \mathsf{Inv}$ be the unique involution such that

$$\begin{aligned} \rho_{XY} X \rho_{XY} &= X^{-1} \\ \rho_{XY} Y \rho_{XY} &= Y^{-1} \end{aligned} \tag{3.2.7}$$

respectively. (By Proposition 3.2.6, it is represented by the Lie product $\mathsf{Lie}(X,Y)$.) The involution ρ_{XY} will be specified by its fixed line $l_{XY} = \mathsf{Fix}(\rho_{XY})$, which is defined as follows: If both X, Y are semisimple, then Lemma 3.2.5 implies ρ_{XY} is the involution fixing the unique common orthogonal geodesic to the invariant axes of X or Y. If both X, Y are parabolic, then ρ_{XY} is the involution in the geodesic bounded by the fixed points of X, Y. Finally consider the case when one element is semisimple and the other element is parabolic. Then l_{XY} is the unique geodesic, for which one endpoint is the fixed point of the parabolic element, and which is orthogonal to the invariant axis of the semisimple element.

Similarly define lines l_{YZ}, l_{ZX} with respective involutions $\rho_{YZ}, \rho_{ZX} \in \mathsf{Inv}$. The triple $(\rho_{XY}, \rho_{YZ}, \rho_{ZX})$ defines the homomorphism $\hat{\rho}$ of Theorem 3.2.2.

Claim: $X = \rho_{ZX}\rho_{XY}$. To this end we show $X\rho_{XY}$ equals ρ_{ZX}. First, $X\rho_{XY}$ fixes $\mathsf{Fix}(X)$, since both X and ρ_{XY} fix $\mathsf{Fix}(X)$. By (3.2.7),

$$\rho_{XY} XY \rho_{XY} = X^{-1}Y^{-1} = X^{-1}(XY)^{-1}X.$$

Equivalently,

$$\rho_{XY} Z^{-1} \rho_{XY} = X^{-1}Y^{-1} = X^{-1}ZX,$$

which implies

$$(X\rho_{XY})Z^{-1}(X\rho_{XY})^{-1} = Z.$$

Now Lemma 3.2.5 (1) implies that $X\rho_{XY}$ preserves $\mathsf{Fix}(Z)$ and its restriction to the corresponding line is a reflection. Thus $X\rho_{XY}$ is itself an involution ℓ_Z with $\mathsf{Fix}(X\rho_{XY})$ orthogonal to the axis of Z.

Since $X\rho_{XY}$ fixes $\mathsf{Fix}(X)$, it follows that $X\rho_{XY} = \rho_{ZX}$ as claimed. Similarly $Y\rho_{YZ} = \rho_{XY}$ and $Z\rho_{ZX} = \rho_{YZ}$, completing the proof of Theorem 3.2.2. $\square$

3.3 Orthogonal reflection groups.

An algebraic proof of Theorem 3.2.2 involves three-dimensional inner product spaces and is described in Goldman [26]. This proof exploits the isomorphism $\mathsf{PSL}(2,\mathbb{C}) \longrightarrow \mathsf{SO}(3,\mathbb{C})$.

The 3-dimensional orthogonal representation of $\mathsf{PSL}(2,\mathbb{C})$. Let $W = \mathbb{C}^2$ with a nondegenerate symplectic form ω. The *symmetric square* $\mathsf{Sym}^2(W)$ is a 3-dimensional vector space based on monomials $e \cdot e, e \cdot f, f \cdot f$, where e, f is a basis of W, and $x \cdot y$ denotes the *symmetric product* of x, y (the image of the tensor product $x \otimes y$ under symmetrization). $\mathsf{Sym}^2(W)$ inherits a symmetric inner product defined by:

$$(u_1 \cdot u_2, v_1 \cdot v_2) \longmapsto \frac{1}{2}\big(\omega(u_1, v_1)\omega(u_2, v_2) + \omega(u_1, v_2)\omega(u_2, v_1)\big).$$

If $e, f \in W$ is a symplectic basis for W, the corresponding inner product for $\mathsf{Sym}^2(W)$ has matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1/2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

with respect to the above basis of $\mathsf{Sym}^2(W)$. In particular the inner product is nondegenerate. Every $\xi \in \mathsf{SL}(2, \mathbb{C})$ induces an isometry of $\mathsf{Sym}^2(W)$ with respect to this inner product. This correspondence defines a local isomorphism

$$\mathsf{SL}(2, \mathbb{C}) \xrightarrow{\mathsf{Sym}^2} \mathsf{SO}(3, \mathbb{C})$$

with kernel $\{\pm \mathbb{I}\}$ and a resulting *isomorphism* $\mathsf{PSL}(2, \mathbb{C}) \longrightarrow \mathsf{SO}(3, \mathbb{C})$.

If $\xi \in \mathsf{SL}(2, \mathbb{C})$, then

$$\mathsf{tr}\big(\mathsf{Sym}^2(\xi)\big) = \mathsf{tr}(\xi)^2 - 1. \tag{3.3.1}$$

For example, the diagonal matrix

$$\xi = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$$

induces the diagonal matrix

$$\mathsf{Sym}^2(\xi) = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^{-2} \end{bmatrix}$$

and

$$\mathsf{tr}\big(\mathsf{Sym}^2(\xi)\big) = \lambda^2 + 1 + \lambda^{-2} = (\lambda + \lambda^{-1})^2 - 1.$$

Alternatively, this is the adjoint representation of $\mathsf{SL}(2, \mathbb{C})$ on its Lie algebra $\mathfrak{sl}(2) \cong \mathsf{Sym}^2(W)$. Here the standard basis of $\mathbb{C}$ is

$$e = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and the monomials correspond to

$$e \cdot e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad e \cdot f = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad f \cdot f = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The inner product corresponds to the *trace form*

$$(X, Y) \longmapsto \frac{1}{2}\,\mathsf{tr}(XY)$$

which is 1/8 the Killing form on $\mathfrak{sl}(2, \mathbb{C})$.

3-dimensional inner product spaces. Let e_1, e_2, e_3 denote the standard basis of $\mathbb{C}^3$. A 3×3 symmetric matrix B determines an inner product $\mathbb{B}$ on $\mathbb{C}^3$ by the usual rule:

$$(v, w) \xmapsto{\mathbb{B}} v^\dagger B w.$$

We suppose that $\mathbb{B}$ is nonzero on e_1, e_2, e_3; in fact, let's normalize $\mathbb{B}$ so that its basic values are 1:

$$\mathbb{B}(e_i, e_i) = 1 \text{ for } i = 1, 2, 3.$$

In other words, the diagonal entries satisfy $B_{11} = B_{22} = B_{33} = 1$.

Let $R_i = R_i^{(B)}$ denote the orthogonal reflection in e_i defined by $\mathbb{B}$:

$$v \xmapsto{R_i} v - 2\,\mathbb{B}(v, e_i)e_i$$

with corresponding matrix:

$$R_i \;:=\; \mathbb{I} - 2e_i(e_i)^\dagger B.$$

$(e_i(e_i)^\dagger B$ is the 3×3 matrix with the same i-th row as B and the other two rows zero.) Since

$$\mathbb{I} = \mathbb{B}(e_i, e_i) = (e_i)^\dagger B e_i \qquad \text{(matrix multiplication)},$$

$$\begin{aligned}
R_i^\dagger B R_i - B &= \big(\mathbb{I} - 2Be_i(e_i)^\dagger\big)B\big(\mathbb{I} - 2e_i(e_i)^\dagger B\big) - B \\
&= -2Be_i(e_i)^\dagger B - 2Be_i(e_i)^\dagger B + 4Be_i(e_i)^\dagger Be_i(e_i)^\dagger B \\
&= -2Be_i(e_i)^\dagger B - 2Be_i(e_i)^\dagger B + 4Be_i(e_i)^\dagger B \\
&= 0
\end{aligned}$$

so R_i *is orthogonal with respect to* $\mathbb{B}$.

Thus the matrix B determines a triple of involutions $R_1^{(B)}, R_2^{(B)}, R_3^{(B)}$ in the *orthogonal group of* $\mathbb{B}$:

$$\mathsf{O}(\mathbb{C}^3, \mathbb{B}) := \{\xi \in \mathsf{GL}(2, \mathbb{C}) \mid \xi^\dagger B \xi = B\}.$$

In other words, B defines a representation $\hat\rho := \hat\rho^{(B)}$ of the free product

$$\hat\pi := \mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2$$

in $\mathsf{O}(\mathbb{C}^3, \mathbb{B})$, taking the free generators $\iota_{XY}, \iota_{YZ}, \iota_{ZX}$ of $\hat{\pi}$ into $R_1^{(B)}, R_2^{(B)}, R_3^{(B)}$ respectively. The restriction $\rho := \rho^{(B)}$ of $\hat{\rho}^{(B)}$ to the index-two subgroup

$$\mathbb{Z} * \mathbb{Z} \cong \pi \subset \hat{\pi}$$

(compare §3.2) assumes values in the subgroup

$$\mathsf{SO}(\mathbb{C}^3, \mathbb{B}) := \mathsf{SL}(3, \mathbb{C}) \cap \mathsf{O}(\mathbb{C}^3, \mathbb{B}).$$

When $\mathbb{B}$ is nondegenerate, then $\mathsf{SO}(\mathbb{C}^3, \mathbb{B}) \cong \mathsf{SO}(3, \mathbb{C})$ (a specific isomorphism corresponds to an orthonormal basis for $\mathbb{B}$). There are exactly 4 lifts $\tilde{\rho}$ of ρ to the double covering-space

$$\mathsf{SL}(2, \mathbb{C}) \longrightarrow \mathsf{SO}(3, \mathbb{C}) \xrightarrow{\cong} \mathsf{SO}(\mathbb{C}^3, \mathbb{B})$$

To see this, for each generator X, Y, Z of π, its ρ-image has exactly two lifts, differing by $\pm\mathbb{I}$. Lifting the generators to $\widetilde{\rho(X)}, \widetilde{\rho(Y)}, \widetilde{\rho(Z)}$ respectively, exactly half of the eight choices satisfy

$$\widetilde{\rho(X)}\widetilde{\rho(Y)}\widetilde{\rho(Z)} = \mathbb{I}, \tag{3.3.2}$$

(as desired), and for the other four choices the product equals $-\mathbb{I}$.

Choose one of the four lifts satisfying (3.3.2), and denote it $\tilde{\rho}$. If $i \neq j$, the trace of $R_i R_j \in \mathsf{SL}(3, \mathbb{C})$ equals $4(B_{ij})^2 - 1$. For example, take $i = 1, j = 2$:

$$\begin{aligned}\widetilde{\rho(Z)} = R_1 R_2 &= \begin{bmatrix} -1 & -2B_{12} & -2B_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2B_{12} & -1 & -2B_{23} \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4(B_{12})^2 - 1 & -2B_{12} & 4B_{23}B_{12} - 2B_{13} \\ -2B_{12} & -1 & -2B_{23} \\ 0 & 0 & 1 \end{bmatrix}\end{aligned}$$

has trace $4(B_{12})^2 - 1$.

This calculation gives another proof of surjectivity in Theorem 1 (as in [26]). For $(x, y, z) \in \mathbb{C}^3$, the matrix

$$B = \begin{bmatrix} 1 & z/2 & y/2 \\ z/2 & 1 & x/2 \\ y/2 & x/2 & 1 \end{bmatrix} \tag{3.3.3}$$

defines a bilinear form $\mathbb{B}$ and a representation $\rho^{(B)}$ as above.

The corresponding $\mathsf{SL}(2, \mathbb{C})$-traces of the $\tilde{\rho}$-images of the generators X, Y, Z of π satisfy:

$$\begin{aligned} \mathsf{tr}\big(\tilde{\rho}(X)\big) &= \pm 2B_{23} \\ \mathsf{tr}\big(\tilde{\rho}(Y)\big) &= \pm 2B_{13} \\ \mathsf{tr}\big(\tilde{\rho}(Z)\big) &= \pm 2B_{12} \end{aligned}$$

because (using (3.3.1)):

$$\mathsf{tr}\big(\tilde\rho(X)\big)^2 = 1 + \mathsf{tr}\Big(\mathsf{Sym}^2\big(\tilde\rho(X)\big)\Big) = 1 + \mathsf{tr}(R_2R_3) = 4(B_{23})^2$$
$$\mathsf{tr}\big(\tilde\rho(Y)\big)^2 = 1 + \mathsf{tr}\Big(\mathsf{Sym}^2\big(\tilde\rho(Y)\big)\Big) = 1 + \mathsf{tr}(R_3R_1) = 4(B_{31})^2$$
$$\mathsf{tr}\big(\tilde\rho(Z)\big)^2 = 1 + \mathsf{tr}\Big(\mathsf{Sym}^2\big(\tilde\rho(Z)\big)\Big) = 1 + \mathsf{tr}(R_1R_2) = 4(B_{12})^2$$

Now adjust the lifts as above to arrange a representation $\pi \xrightarrow{\tilde\rho} \mathsf{SL}(2,\mathbb{C})$ with

$$\mathsf{tr}\big(\tilde\rho(X)\big) = x$$
$$\mathsf{tr}\big(\tilde\rho(Y)\big) = y$$
$$\mathsf{tr}\big(\tilde\rho(Z)\big) = z$$

as desired.

When $\kappa(x,y,z) = 2$, the matrix B is singular and we obtain *reducible* representations. There are two cases, depending on whether $\mathsf{rank}(B) = 2$ or $\mathsf{rank}(B) = 1$. (Since $B \neq 0$, its rank cannot be zero.)

3.4 Real characters and real forms.

A *real character* $(x,y,z) \in \mathbb{R}^3$ corresponds to a representation of a rank-two free group in one of the two *real forms* $\mathsf{SU}(2), \mathsf{SL}(2,\mathbb{R})$ of $\mathsf{SL}(2,\mathbb{C})$. This was first stated and proved in general by Morgan-Shalen [59]. Geometrically, $\mathsf{SU}(2)$-representations are those which fix a point in H^3, and $\mathsf{SL}(2,\mathbb{R})$-representations are those which preserve a plane $\mathsf{H}^2 \subset \mathsf{H}^3$ as well as an orientation on the plane.

Theorem 3.4.1. *Let* $(x,y,z) \in \mathbb{R}^3$ *and*

$$\kappa(x,y,z) := x^2 + y^2 + z^2 - xyz - 2.$$

Let $\pi \xrightarrow{\rho} \mathsf{SL}(2,\mathbb{C})$ *be a representation with character* (x,y,z)*. Suppose first that* $\kappa(x,y,z) \neq 2$.

- *If* $-2 \leq x,y,z \leq 2$ *and* $\kappa(x,y,z) < 2$*, then* $\rho(\pi)$ *fixes a unique point in* H^3 *and is conjugate to a* $\mathsf{SU}(2)$*-representation.*
- *Otherwise* $\rho(\pi)$ *preserves a unique plane in* H^3 *and its restriction to that plane preserves orientation.*

If $\kappa(x,y,z) = 2$*, then* ρ *is reducible and one of the following must occur:*

- $\rho(\pi)$ *acts identically on* H^3*, in which case* $\rho(\pi) \subset \{\pm\mathbb{I}\}$ *is a* central representation.

- *$\rho(\pi)$ fixes a line in H^3, in which case $-2 \le x, y, z \le 2$ and ρ is conjugate to a representation taking values in $\mathsf{SO}(2) = \mathsf{SU}(2) \cap \mathsf{SL}(2,\mathbb{R})$.*
- *$\rho(\pi)$ acts by transvections along a unique line in H^3, in which case*

  $$x, y, z \in \mathbb{R} \setminus \big(-2, 2\big).$$

  Then ρ is conjugate to a representation taking values in $\mathsf{SO}(1,1) \subset \mathsf{SL}(2,\mathbb{R})$.
- *$\rho(\pi)$ fixes a unique point on $\partial_\infty \mathsf{H}^3$.*

Recall that $\mathsf{SO}(1,1)$ is isomorphic to the multiplicative group $\mathbb{R}^*$ of nonzero real numbers, and is conjugate to the subgroup of $\mathsf{SL}(2,\mathbb{R})$ consisting of diagonal matrices.

Corollary 3.2.3 associates to a generic representation ρ an ordered triple ι^ρ of geodesics in H^3. When ρ is irreducible, the corresponding cases for ι^ρ are the following:

- If ρ fixes a unique point $p \in \mathsf{H}^3$, then the three lines are distinct and intersect in p. Conversely if the three lines are concurrent, then ρ is conjugate to an $\mathsf{SU}(2)$-representation.
- If ρ preserves a unique plane P, then the three lines are distinct. There are two cases:
  - The three lines are orthogonal to P;
  - The three lines lie in P.

The first case, when the lines are orthogonal to P, occurs when $\kappa(x,y,z) < 2$. In this case the corresponding involutions preserve orientation on P. The second case, when the lines lie in P, occurs when $\kappa(x,y,z) > 2$. In that case the involutions restrict to reflections in geodesics in P which reverse orientation.

Real symmetric 3×3 matrices. We deduce these facts from the classification given in Theorem 1. First assume that (x,y,z) is an *irreducible* character, that is, $\kappa(x,y,z) \neq 2$. Theorem 1 implies that (x,y,z) is the character of the representation ρ given by (2.2.9), and all such characters are $\mathsf{PGL}(2,\mathbb{C})$-conjugate.

The matrix B defining the bilinear form $\mathbb{B}$ in (3.3.3) satisfies:

$$4\,\mathsf{det}(B) \;=\; 2 - \kappa(x,y,z)$$

so $\mathbb{B}$ is degenerate if and only if $\kappa(x,y,z) = 2$, in which case ρ is reducible.

Suppose that $\mathbb{B}$ is nondegenerate, so that either $\kappa(x,y,z) > 2$ or $\kappa(x,y,z) < 2$. Suppose first that $\kappa(x,y,z) > 2$. Then $\mathsf{det}(B) < 0$. Since the diagonal entries of B are positive, $\mathbb{B}$ is indefinite of signature $(2,1)$. In particular, it cannot be negative definite. In this case the triple of lines corresponding to ρ

are all coplanar. The corresponding involutions reverse orientation on P and act by reflections of P in the three geodesics respectively.

When $\kappa(x,y,z) < 2$, there are two cases: either $\mathbb{B}$ is positive definite (signature $(3,0)$) or indefinite (signature $(1,2)$). The restriction of $\mathbb{B}$ to the coordinate plane spanned by e_i and e_j is given by the 2×2 symmetric matrix

$$\begin{bmatrix} 1 & B_{ij} \\ B_{ij} & 1 \end{bmatrix}$$

which is positive definite if and only if $-1 < B_{ij} < 1$. Thus $\mathbb{B}$ is positive definite if and only if $-2 < x,y,z < 2$.

Otherwise $\mathbb{B}$ is indefinite and ρ corresponds to a representation in $\mathsf{SO}(1,2)$. In this case the triple of lines in H^3 are all orthogonal to the invariant plane P in H^3. The corresponding three involutions preserve orientation on P and act by symmetries about points in P.

The two-dimensional normal form. Another approach to finding a representation with given traces involves a direct computation with the explicit normal form (2.2.9) as follows. Let (x,y,z) be as above. First solve $z = 2\cos(\theta)$ to obtain representative matrices:

$$\xi_x := \begin{bmatrix} x & -1 \\ 1 & 0 \end{bmatrix}, \; \eta_{y,\theta} := \begin{bmatrix} 0 & e^{-i\theta} \\ -e^{i\theta} & y \end{bmatrix},$$

with a slight change of notation from (2.2.9).

A Hermitian form on $\mathbb{C}^2$ is given by a Hermitian 2×2-matrix H. A complex 2×2 matrix H is Hermitian $\Longleftrightarrow H = \bar{H}^\dagger$. The corresponding Hermitian form on $\mathbb{C}^2$ is:

$$(u,v) \longmapsto \bar{v}^\dagger H u,$$

where $u,v \in \mathbb{C}^2$. A linear transformation $\mathbb{C}^2 \xrightarrow{\xi} \mathbb{C}^2$ preserves $H \Longleftrightarrow \bar{\xi}^\dagger H \xi = H$.

The ξ_x-invariant Hermitian forms comprise the real vector space with basis

$$\begin{bmatrix} 2 & x \\ x & 2 \end{bmatrix}, \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

and the $\eta_{y,\theta}$-invariant Hermitian forms comprise the real vector space with basis

$$\begin{bmatrix} 2 & -y\sec(\theta) \\ -y\sec(\theta) & 2 \end{bmatrix}, \begin{bmatrix} 0 & i - y\tan(\theta) \\ -i - y\tan(\theta) & 0 \end{bmatrix}.$$

The ρ-invariant Hermitian forms thus comprise the intersection of these two vector spaces, the vector space with basis

$$H = \begin{bmatrix} 2\sin(\theta) & x\sin(\theta) - i(y + xz/2) \\ x\sin(\theta) + i(y + xz/2) & 2\sin(\theta) \end{bmatrix}.$$

This Hermitian matrix is definite since its determinant is positive:

$$\mathsf{det}(H) = 4\sin^2(\theta) - x^2\sin^2(\theta) - (y - xz/2)^2 = 2 - \kappa(x, y, z) > 0$$

(since $\sin^2(\theta) = 1 - (z/2)^2$ and $\kappa(x, y, z) < 2$).

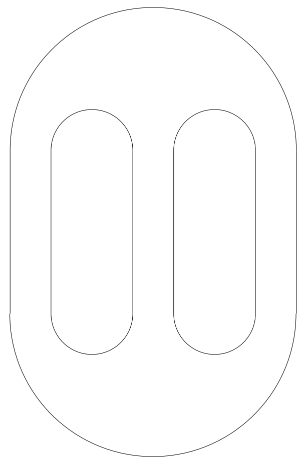

Figure 2. A ribbon graph for a three-holed sphere

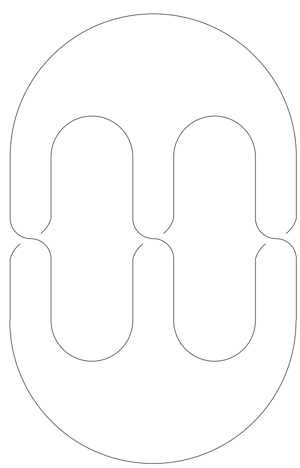

Figure 3. A ribbon graph for a one-holed torus

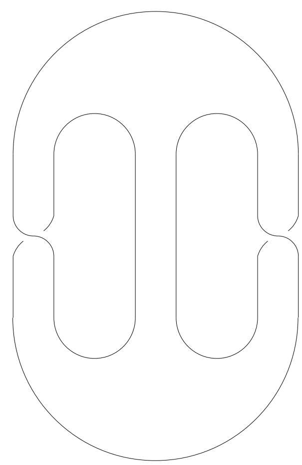

Figure 4. A ribbon graph for a two-holed cross-surface

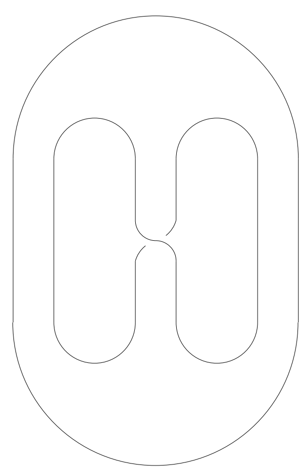

Figure 5. A ribbon graph for a one-holed Klein bottle

4 Hyperbolic structures on surfaces of $\chi = -1$

We apply this theory to compute, in trace coordinates, the deformation spaces of hyperbolic structures on compact connected surfaces Σ with $\chi(\Sigma) = -1$. Equivalently, such surfaces are characterized by the condition that $\pi_1(\Sigma)$ is a free group of rank two. There are four possibilities:

- Σ is homeomorphic to a three-holed sphere (a "pair-of-pants" or "trinion") $\Sigma_{0,3}$;
- Σ is homeomorphic to a one-holed torus $\Sigma_{1,1}$;
- Σ is homeomorphic to a one-holed Klein bottle $C_{1,1}$;
- Σ is homeomorphic to a two-holed projective plane $C_{0,2}$.

Each of these surfaces can be realized as a ribbon graph with three bands connecting two 2-cells. The number of boundary components and the orientability can be read off from the parities of the number of twists in the three bands.

4.1 Fricke spaces.

The *Fricke space* $\mathfrak{F}(\Sigma)$ is the space of isotopy classes of marked hyperbolic structures on Σ with $\partial\Sigma$ geodesic. The group of isometries of H^2 equals $\mathsf{PGL}(2,\mathbb{R})$, which embeds in $\mathsf{PSL}(2,\mathbb{C})$. Its identity component $\mathsf{PSL}(2,\mathbb{R})$ consists of the isometries of H^2 which *preserve an orientation* on H^2. The *holonomy map* embeds $\mathfrak{F}(\Sigma)$ in the deformation space

$$\mathsf{Hom}\big(\pi_1(\Sigma), \mathsf{PGL}(2,\mathbb{R})\big)//\mathsf{PGL}(2,\mathbb{R}).$$

Since $\partial\Sigma \neq \emptyset$, $\pi_1(\Sigma)$ is a free group, and the problem of lifting a representation of $\pi_1(\Sigma)$ to $\mathsf{GL}(2,\mathbb{R})$ is unobstructed.

The various lifts are permuted by the group $H^1(\Sigma;\mathbb{Z}/2)$, which is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ when $\chi(\Sigma) = -1$. In terms of trace coordinates on the $\mathbb{R}$-locus of the character variety this action is given by:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} x \\ -y \\ -z \end{bmatrix}, \begin{bmatrix} -x \\ y \\ -z \end{bmatrix}, \begin{bmatrix} -x \\ -y \\ z \end{bmatrix}.$$

Theorem 4.1.1. *Using trace coordinates of the boundary, the Fricke space of the three-holed sphere $\Sigma_{0,3}$ identifies with the quotient of the four octants*

$$\begin{aligned}(-\infty,-2]\times(-\infty,-2]\times(-\infty,-2] &\coprod (-\infty,-2]\times[2,\infty)\times[2,\infty)\\ &\coprod [2,\infty)\times[2,\infty)\times(-\infty,-2]\\ &\coprod [2,\infty)\times(-\infty,-2]\times[2,\infty)\subset\mathbb{R}^3\end{aligned}$$

by $H^1(\Sigma, \mathbb{Z}/2)$*. The octant*

$$(-\infty, -2] \times (-\infty, -2] \times (-\infty, -2]$$

defines a slice for the $H^1(\Sigma, \mathbb{Z}/2)$*-action.*

The proof will be given in §4.3.

Theorem 4.1.2. *The Fricke space of the one-holed torus* $\Sigma_{1,1}$ *identifies with the quotient of*

$$\kappa^{-1}\big((-\infty, -2]\big) = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 - xyz \leq 0\}$$

by $H^1(\Sigma, \mathbb{Z}/2)$*. The region*

$$\{(x, y, z) \in (2, \infty)^3 \mid x^2 + y^2 + z^2 - xyz \leq 0\}$$

is a connected component of $\kappa^{-1}\big((-\infty, -2]\big)$*, defines a slice for the* $H^1(\Sigma, \mathbb{Z}/2)$*-action, and hence identifies with the Fricke space of* Σ*.*

The proof will be given in §4.4.

4.2 Two-dimensional hyperbolic geometry

We take for our model of the *hyperbolic plane* H^2 the subset of H^3 comprising quaternions $z + u\mathrm{j}$, where $u > 0$ and $z \in \mathbb{R}$. A matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathsf{GL}(2, \mathbb{C})$$

determines a projective transformation of $\mathbb{CP}^1 = \partial\mathsf{H}^3$ which extends to an orientation-preserving isometry of H^3. This isometry preserves H^2 if and only if it is a scalar multiple of a real matrix. As usual, normalize $A \in \mathsf{GL}(2, \mathbb{C})$ by dividing by a square root $\sqrt{\mathsf{det}(A)} \in \mathbb{C}^*$. An element of

$$\mathsf{SL}(2, \mathbb{C}) \ \cap \ \mathbb{C}^*\mathsf{GL}(2, \mathbb{R})$$

is either:

- a real matrix of determinant 1, or
- a purely imaginary matrix iA' where $A' \in \mathsf{GL}(2, \mathbb{R})$ satisfies $\mathsf{det}(A') = 1$.

In the first case the corresponding orientation-preserving isometry of H^3 preserves orientation on H^2, and in the second case its restriction *reverses orientation* on H^2. The traces of their representative matrices in $\mathsf{SL}(2, \mathbb{C})$ distinguish these cases. We emphasize that these representatives are only determined up to ± 1. Suppose that $A \in \mathsf{PSL}(2, \mathbb{C})$ preserves a plane $P \subset \mathsf{H}^3$ and let $\tilde{A} \in \mathsf{SL}(2, \mathbb{C})$ be a lift of A. Then:

- The restriction of A to P preserves orientation $\Longleftrightarrow \mathsf{tr}(\tilde{A}) \in \mathbb{R}$;
- The restriction of A to P reverses orientation $\Longleftrightarrow \mathsf{tr}(\tilde{A}) \in i\mathbb{R}$.

Observe that an element $A \in \mathsf{SL}(2,\mathbb{C})$ whose trace is both real and purely imaginary — that is, equals zero — is an involution in a line $\ell := \mathsf{Fix}(A)$. The A-invariant planes fall into two types: those which contain ℓ, upon which A reverses orientation, and those which are orthogonal to ℓ, upon which A preserves orientation.

The hyperbolic plane and involutions of H^3. The proof of Theorem 4.1.1 requires an algebraic representation of half-planes in H^2. Given an orientation on H^2, and an oriented geodesic $\ell \subset \mathsf{H}^2$, there is a well-determined half-plane bounded by ℓ, defined as follows. Let $x \in \ell$. Choose the unit vector v_ℓ tangent to ℓ at x determined by the orientation of ℓ. The choice of half-plane $\mathfrak{H} \subset \mathsf{H}^2 \setminus \ell$ is determined by the normal vector ν to ℓ at x pointing outward from $\mathfrak{H}$. Choose $\mathfrak{H}$ so that the basis $\{v_\ell, \nu\} \subset T_x\mathsf{H}^2$ is positively oriented.

The points of H^2 identify with geodesics in H^3 which are orthogonal to H^2; the endpoints of such geodesics are complex-conjugate elements of $\mathbb{CP}^1 \setminus \mathbb{RP}^1$. An involution which interchanges these endpoints is given by matrices $\pm I_{x+\mathrm{j}u}$, where the 2×2 real matrix

$$I_{x+\mathrm{j}u} := \frac{1}{u}\begin{bmatrix} x & -(x^2+u^2) \\ 1 & -x \end{bmatrix} \tag{4.2.1}$$

has determinant 1 and trace 0. (Apply (3.2.2), taking $z_1 = x + iu$ and $z_2 = x - iu$.) The space of such matrices has two components, depending on the signs of the off-diagonal elements. A matrix

$$A \in \mathsf{SL}(2,\mathbb{R}) \cap \mathfrak{sl}(2,\mathbb{R})$$

equals $I_{x+\mathrm{j}u}$, for some $x + \mathrm{j}u \in \mathsf{H}^2$ if and only if $A_{12} < 0 < A_{21}$, in which case

$$u = (A_{21})^{-1},\ x = (A_{21})^{-1}A_{11}.$$

The above inequalities determine one of the two sheets of the two-sheeted hyperboloid in $\mathfrak{sl}(2,\mathbb{R})$ defined by the unimodularity condition $\mathsf{det}(A) = 1$.

This gives a convenient form of the Klein hyperboloid model for H^2, as the quadric in $\mathfrak{sl}(2,\mathbb{R}) \cong \mathbb{R}^3$ defined by

$$1 = \mathsf{det}(A) = -\frac{1}{2}\mathsf{tr}(A^2).$$

Next we represent oriented geodesics by involutions. A geodesic in H^2 determines an involution by (3.2.2) and (3.2.3), where z_1, z_2 are distinct points in $\mathbb{RP}^1$. Such a matrix is purely imaginary, has trace zero and determinant one. Multiplying by i, we obtain an element A of $\mathfrak{sl}(2,\mathbb{R})$ which has determinant -1. Such a matrix has well-defined 1-dimensional eigenspaces with eigenvalues $\pm i$. These eigenspaces determine the respective fixed points in $\mathbb{RP}^1$. Replacing

A by $-A$ interchanges the $\pm i$-eigenspaces. In this way, we identify the space of oriented geodesics in H^2 with

$$\{A \in \mathfrak{sl}(2,\mathbb{R}) \mid \mathsf{det}(A) = -1\}.$$

This is just the usual hyperboloid model. Think of $\mathfrak{sl}(2,\mathbb{R})$ as a 3-dimensional real inner product space under the inner product

$$\langle A, B\rangle := \frac{1}{2}\mathsf{tr}(AB).$$

The corresponding quadratic form relates to the *determinant* by

$$\langle A, A\rangle = \frac{1}{2}\mathsf{tr}(A^2) = -\mathsf{det}(A).$$

This quadratic form is readily seen to have signature $(2,1)$ since its value on

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

equals $a^2 + bc$. Then H^2 corresponds to one component of the two-sheeted hyperboloid (say the one with $b < 0 < c$)

$$\{v \in \mathfrak{sl}(2,\mathbb{R}) \mid \langle v, v\rangle = -1\}$$

and the space of oriented geodesics corresponds to *de Sitter space*

$$\mathsf{dS}^2_1 := \{v \in \mathfrak{sl}(2,\mathbb{R}) \mid \langle v, v\rangle = 1\}.$$

A vector $v \in \mathsf{dS}^2_1$ determines a half-plane $\mathcal{H}(v)$ by:

$$\mathcal{H}(v) := \{w \in \mathsf{H}^2 \mid \langle w, v\rangle \geq 0\}. \tag{4.2.2}$$

In particular, $\mathcal{H}(-v)$ is the half-plane complementary to $\mathcal{H}(v)$.

For example, the half-plane corresponding to

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

consists of all $x + u\mathsf{j}$ where $x \geq 0$, as can easily be verified using (4.2.1).

The main criterion for disjointness of half-planes is the following lemma, whose proof is an elementary exercise and left to the reader. (Recall that two geodesics in H^2 are *ultraparallel* if and only if they admit a common orthogonal geodesic; equivalently distances between their respective points have a positive lower bound.)

Lemma 4.2.1. *Let $v_1, v_2 \in \mathsf{dS}^2_1$ determine geodesics $\ell_1, \ell_2 \subset \mathsf{H}^2$ and half-planes $\mathcal{H}_i := \mathcal{H}(v_i)$ with $\partial\mathcal{H}_i = \ell_i$. The following conditions are equivalent:*

- $|\langle v_1, v_2\rangle| > 1$*;*
- *The invariant geodesics ℓ_1 and ℓ_2 are ultraparallel.*

In this case, the following two further conditions are equivalent:

- $\langle v_1, v_2\rangle > 1$*;*
- *Either* $\mathcal{H}_1 \subset \mathcal{H}_2$ *or* $\mathcal{H}_2 \subset \mathcal{H}_1$*.*

Contrariwise, the following two conditions are equivalent:

- $\langle v_1, v_2\rangle < -1$*;*
- *Either* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *are disjoint or their complements are disjoint.*

Hyperbolic isometries. An element $A \in \mathsf{SL}(2,\mathbb{R})$ is *hyperbolic* if it satisfies any of the following equivalent conditions:

- $\mathsf{tr}(A) > 2$ or $\mathsf{tr}(A) < -2$;
- A has distinct real eigenvalues;
- The isometry of H^2 defined by A has exactly two fixed points on $\partial\mathsf{H}^2$;
- The isometry of H^2 defined by A leaves invariant a (necessarily unique) geodesic ℓ_A, upon which it acts by a nontrivial translation.

A geodesic ℓ in H^2 is specified by its *reflection* ρ_ℓ, an isometry of H^2 whose fixed point set equals ℓ. If $v \in \mathsf{dS}^2_1$ is a vector corresponding to ℓ, then ρ_ℓ is the restriction to H^2 of the orthogonal reflection in $\mathsf{SO}(2,1)$ fixing v:

$$u \xmapsto{\rho_v} -u + 2\langle u, v\rangle v.$$

(3.2.1) implies the following useful formula for the invariant axis of a hyperbolic element:

Lemma 4.2.2. *Let* A *be hyperbolic. Then*

$$\widehat{A} \;:=\; \frac{2A - \mathsf{tr}(A)\mathbb{I}}{\sqrt{\mathsf{tr}(A)^2 - 4}} \;\in\; \widetilde{\mathsf{Inv}} \cap \mathsf{SL}(2,\mathbb{R}) \tag{4.2.3}$$

defines the reflection in the invariant axis of A*.*

Notice that

$$\widehat{-A} \;=\; \widehat{A^{-1}} \;=\; -\widehat{A}$$

so that A and A^{-1} determine complementary half-planes.

For example

$$A = \begin{bmatrix} e^{l/2} & 0 \\ 0 & e^{-l/2} \end{bmatrix}$$

represents translation along a geodesic (the imaginary axis in H^2) by distance $l > 0$ from 0 to ∞. The corresponding reflection is

$$\widehat{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

which determines the half-plane:

$$\mathcal{H}(\widehat{A}) = \{x + u\mathsf{j} \in \mathsf{H}^2 \mid x \geq 0, u > 0\}$$

as above.

4.3 The three-holed sphere.

We now show that a representation corresponding to a character $(x, y, z) \in \mathbb{R}^3$ satisfying $x, y, z < -2$ is the holonomy representation of a hyperbolic structure on a three-holed sphere. We find matrices X, Y, Z of the desired type and compute the corresponding reflections $\widehat{X}, \widehat{Y}, \widehat{Z}$. Then we show that the corresponding half-planes are all disjoint (after possibly replacing $\widehat{X}, \widehat{Y}, \widehat{Z}$ by their negatives). From this we construct a developing map for a hyperbolic structure on Σ. For details on geometric structures on manifolds and their developing maps, see Goldman [27, 29, 31] or Thurston [73].

Lemma 4.3.1. *Suppose $X, Y, Z \in \mathsf{SL}(2, \mathbb{C})$ satisfy $XYZ = \mathbb{I}$ and have real traces $x, y, z < -2$ respectively. Then the inner products*

$$\langle \widehat{X}, \widehat{Y} \rangle, \langle \widehat{Y}, \widehat{Z} \rangle, \langle \widehat{Z}, \widehat{X} \rangle$$

are all < -1.

Proof. The proof breaks into a series of calculations. By symmetry it suffices to prove $\langle \widehat{X}, \widehat{Y} \rangle < -1$. By the definition (4.2.3)

$$\begin{aligned} \langle \widehat{X}, \widehat{Y} \rangle &= \frac{1}{2}\mathsf{tr}\left(\frac{2X - x\mathbb{I}}{\sqrt{x^2 - 4}} \, \frac{2Y - y\mathbb{I}}{\sqrt{y^2 - 4}} \right) \\ &= \frac{\mathsf{tr}\big(4XY - 2xY - 2yX + xy\mathbb{I} \big)}{2\sqrt{(x^2 - 4)(y^2 - 4)}} \\ &= \frac{2z - xy}{\sqrt{(x^2 - 4)(y^2 - 4)}} \end{aligned} \tag{4.3.1}$$

since $\mathsf{tr}(XY) = z$, $\mathsf{tr}(Y) = y$, $\mathsf{tr}(X) = x$ and $\mathsf{tr}(\mathbb{I}) = 2$. Because

$$x, y, z < -2 \implies 2z - xy < 0,$$

the calculation above implies

$$\langle \widehat{X}, \widehat{Y} \rangle < 0. \tag{4.3.2}$$

Now

$$x^2 + y^2 + z^2 - xyz - 4 \; > \; 4 + 4 + 4 - 8 - 4 \; = \; 0$$

implies that

$$(2z - xy)^2 - (x^2 - 4)(y^2 - 4) \; = \; 4(x^2 + y^2 + z^2 - xyz - 4) \; > \; 0$$

and

$$\left(\frac{2z - xy}{\sqrt{(x^2-4)(y^2-4)}}\right)^2 > 1. \tag{4.3.3}$$

Thus (4.3.1) and (4.3.3) imply

$$\langle \widehat{X}, \widehat{Y} \rangle^2 > 1$$

whence (4.3.2) implies:

$$\langle \widehat{X}, \widehat{Y} \rangle < -1$$

as claimed. □

Conclusion of proof of Theorem 4.1.1. Thus the half-planes $\mathcal{H}_{\widehat{X}}, \mathcal{H}_{\widehat{Y}}, \mathcal{H}_{\widehat{Z}}$ are either all disjoint or their complements are all disjoint. Replacing $\widehat{X}, \widehat{Y}, \widehat{Z}$ by their negatives if necessary, assume that the complements to $\mathcal{H}_{\widehat{X}}, \mathcal{H}_{\widehat{Y}}, \mathcal{H}_{\widehat{Z}}$ are pairwise disjoint.

The intersection

$$\Delta_\infty := \mathcal{H}_{\widehat{X}} \cap \mathcal{H}_{\widehat{Y}} \cap \mathcal{H}_{\widehat{Z}}$$

is bounded by the three geodesics

$$\ell_X = \partial \mathcal{H}_{\widehat{X}}, \quad \ell_Y = \partial \mathcal{H}_{\widehat{Y}}, \quad \ell_Z = \partial \mathcal{H}_{\widehat{Z}}$$

and three segments of $\partial \mathsf{H}^2$. When some of $\rho(X), \rho(Y), \rho(Z)$ are parabolic, then these segments degenerate into ideal points. If a, b are lines or ideal points, denote their common orthogonal segment by $\perp (a, b)$. Define:

$$\begin{aligned}
\sigma_{XY} &:= \perp (\ell_X, \ell_Y) \\
\sigma_{YZ} &:= \perp (\ell_Y, \ell_Z) \\
\sigma_{ZX} &:= \perp (\ell_Z, \ell_X).
\end{aligned}$$

Let $\mathsf{Hex}_\rho \subset \Delta_\infty$ denote the right hexagon bounded by $\sigma_{XY}, \sigma_{YZ}, \sigma_{ZX}$ and segments of ℓ_X, ℓ_Y, ℓ_Z as in Figure 6. Map the abstract hexagon Hex of §3.2 to Hex_ρ so that

$$\begin{aligned}
\partial_1(\mathsf{Hex}) &\longmapsto \ell_X \\
\partial_2(\mathsf{Hex}) &\longmapsto \ell_Y \\
\partial_3(\mathsf{Hex}) &\longmapsto \ell_Z
\end{aligned}$$

and the other three edges of $\partial\mathsf{Hex}$ map homeomorphically to $\sigma_{XY}, \sigma_{YZ}, \sigma_{ZX}$ respectively. This mapping embeds Hex into H^2.

A fundamental domain for the action of

$$\pi_1(\Sigma) = \langle X, Y, Z \mid XYZ = 1 \rangle$$

on the universal covering surface

$$\tilde{\Sigma} := \Big(\mathsf{Hex} \times \hat{\pi}\Big)/\sim$$

is the union

$$\Delta := \mathsf{Hex} \cup \iota_{XY}(\mathsf{Hex}).$$

We shall extend the embedding $\mathsf{Hex} \hookrightarrow \mathsf{H}^2$ to a local diffeomorphism (a *developing map*)

$$\tilde{\Sigma} \longrightarrow \mathsf{H}^2$$

which is $\pi_1(\Sigma)$-equivariant with respect to the action ρ on H^2. Pull back the hyperbolic structure from H^2 to obtain a $\pi_1(\Sigma)$-invariant hyperbolic structure on Σ, as desired.

Extend $\mathsf{Hex} \hookrightarrow \mathsf{H}^2$ to $\Delta \longrightarrow \mathsf{H}^2$ as follows. Map the reflected image of Hex to $\iota_{XY}\mathsf{Hex}_\rho$. Then

$$X = \iota_{ZX}\iota_{XY}$$

identifies the two sides of Δ corresponding to σ_{ZX} and $\iota_{XY}\sigma_{ZX}$, and

$$Y = \iota_{XY}\iota_{YZ}$$

identifies the two sides of Δ corresponding to $\iota_{XY}\sigma_{YZ}$ and σ_{YZ}. (Compare Figure 6.) This defines a hyperbolic structure on Σ with geodesic boundary developing to ℓ_X, ℓ_Y and ℓ_Z.

This completes the proof that every character $(x, y, z) \in \big(-\infty, 2]^3$ is the holonomy of a hyperbolic structure on $\Sigma_{0,3}$.

$\square$

4.4 The one-holed torus.

Now consider the case $\Sigma \approx \Sigma_{1,1}$. Present $\pi = \pi_1(\Sigma)$ as freely generated by X, Y corresponding to simple closed curves which intersect transversely in one point. Then the boundary $\partial\Sigma$ corresponds to the commutator $K = [X, Y]$ and we obtain the presentation

$$\pi \;=\; \langle X, Y, Z, K \;\mid\; XYZ = \mathbb{I}, \quad XY = KYX \rangle$$

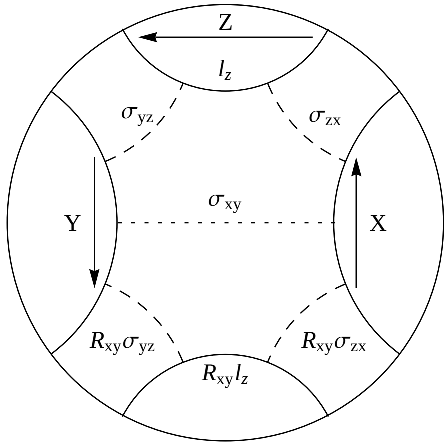

Figure 6. Fundamental domain for hyperbolic structure on $\Sigma_{0,3}$.

The corresponding trace functions are

$$\begin{aligned} x([\rho]) &:= \mathsf{tr}\big(\rho(X)\big) \\ y([\rho]) &:= \mathsf{tr}\big(\rho(Y)\big) \\ z([\rho]) &:= \mathsf{tr}\big(\rho(Z)\big) \\ k([\rho]) &:= \mathsf{tr}\big(\rho(K)\big) \\ &= \kappa(x,y,z) = x^2 + y^2 + z^2 - xyz - 2, \end{aligned}$$

which we denote by x, y, z, k (without reference to ρ) when the context is clear.

The goal of this section is to prove Theorem 4.1.2.

Lemma 4.4.1. *Suppose $(x, y, z) \in \mathbb{R}^3$ satisfies $x^2 + y^2 + z^2 - xyz < 4$. Then either:*

- $(x, y, z) \in [-2, 2]^3$ *or*
- $|x|, |y|, |z| > 2$.

In the first case (x, y, z) is the character of an $\mathsf{SU}(2)$-representation as in Theorem 3.4.1.

Proof. Rewriting the hypothesis as

$$(x^2 - 4)(y^2 - 4) > (2z - xy)^2,$$

it follows that $(x^2-4)(y^2-4) > 0$. By symmetry $(y^2-4)(z^2-4) > 0$ and $(z^2-4)(x^2-4) > 0$ as well. Thus none of $|x|$, $|y|$, $|z|$ equal 2, and either

$$x^2-4, \quad y^2-4, \quad z^2-4$$

are all positive or all negative. If

$$|x|, |y|, |z| < 2$$

then (x, y, z) is an $\mathsf{SU}(2)$-character as in Theorem 3.4.1; otherwise

$$|x|, |y|, |z| > 2$$

as desired. This completes the proof of Lemma 4.4.1. □

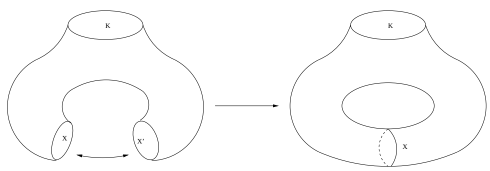

Figure 7. The one-holed torus as an identification space. The identification map Y conjugates X to a boundary element of $\pi_1(\Sigma')$, but with the opposite orientation.

Denote by $X \subset \Sigma$ a simple closed curve corresponding to the generator $X \in \pi_1(\Sigma)$. The surface-with-boundary $\Sigma' := \Sigma|X$ obtained by *splitting* Σ along X is homeomorphic to a three-holed sphere. Denoting the quotient map by $\Sigma' \xrightarrow{\Pi} \Sigma$, the three components of $\partial\Sigma'$ are the connected preimage $\partial' := \Pi^{-1}(\partial\Sigma)$ and the two components $X_\pm$ of the preimage $\Pi^{-1}(X)$. Choose arcs from the basepoint to X_+ and represent the boundary generators of $\pi_1(\Sigma')$ by the elements ∂', X_+, X_- subject to the relation $X_- X_+ \partial' = \mathbb{I}$. The quotient map induces a monomorphism

$$\begin{aligned}
\pi_1(\Sigma') &\overset{\Pi_*}{\hookrightarrow} \pi_1(\Sigma) \\
X_+ &\longmapsto X \\
X_- &\longmapsto Y X^{-1} Y^{-1} \\
\partial' &\longmapsto \partial = X^{-1} Y X Y^{-1}.
\end{aligned}$$

Compare Figure 7.

Lemma 4.4.2. *The composition $\rho \circ \Pi_*$ is the holonomy representation of a hyperbolic structure on $\Sigma' \approx \Sigma_{0,3}$.*

Proof. By Theorem 4.1.1, it suffices to show that the images of the boundary generators $\partial', X_-, X_+ \in \pi_1(\Sigma')$ under $\rho \circ \Pi_*$ have trace ≤ -2. By Lemma 4.4.1,

$$\mathsf{tr}\big(\Pi_* \circ \rho(X_-)\big) = \mathsf{tr}\big(\Pi_* \circ \rho(X_+)\big) = x$$

is either > 2 or < -2. In the former case, replace $\rho(X)$ by $-\rho(X)$ to assume that $x < -2$. Now by assumption

$$\mathsf{tr}\big(\Pi_* \circ \rho(\partial')\big) = \mathsf{tr}\big(\rho(K)\big) = k \leq -2$$

so that all three boundary generators of $\pi_1(\Sigma')$ have trace ≤ -2, as desired. ☐

Conclusion of proof of Theorem 4.1.2. Thus we obtain a hyperbolic structure on Σ' with geodesic boundary. Choosing a developing map with holonomy $\Pi_* \circ \rho$, the isometry $\rho(Y)$ realizes the identification of ξ_- with ξ_+ for which Π is the quotient map. The resulting quotient space is homeomorphic to Σ and inherits a hyperbolic structure from the one on Σ' and the identification. Therefore ρ is the holonomy representation of a hyperbolic structure on Σ. This concludes the proof of Theorem 4.1.2. ☐

Compare Goldman [29] for a different proof.

The algebraic methods discussed here easily imply several other qualitative geometric facts:

Proposition 4.4.3. *Suppose that $x, y, z \in \mathbb{R}$ satisfy $\kappa(x,y,z) \leq -2$ and $(x,y,z) \neq (0,0,0)$. Then (x,y,z) is the character of a representation $\pi \xrightarrow{\rho} \mathsf{SL}(2,\mathbb{R})$ and $\rho(X)$ and $\rho(Y)$ are hyperbolic elements whose axes cross.*

Proof. Since $\kappa(x,y,z) \leq -2 < 2$, Lemma 4.4.1 applies. The representation ρ with character (x,y,z) is conjugate to either an $\mathsf{SU}(2)$-representation or an $\mathsf{SL}(2,\mathbb{R})$-representation. Since

$$\mathsf{tr}\big(\rho([X,Y])\big) \leq -2,$$

the only possibility for an $\mathsf{SU}(2)$-representation occurs if $\kappa(x,y,z) = -2$. Then ρ is conjugate to the quaternion representation

$$\rho(X) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \rho(Y) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},$$

and $(x,y,z) = (0,0,0)$, a contradiction. (Compare §2.6 of Goldman [29].) Thus (x,y,z) corresponds to an $\mathsf{SL}(2,\mathbb{R})$-representation ρ. Lemma 4.4.1 implies that $\rho(X)$ and $\rho(Y)$ are both hyperbolic. Proposition 3.2.6 implies that the involution fixing the common orthogonal $\perp(\ell_{\rho(X)}, \ell_{\rho(Y)})$ of their respective invariant axes is given by the Lie product $\mathsf{Lie}\big(\rho(X),\rho(Y)\big)$. In particular their axes cross if and only if $\perp(\ell_{\rho(X)}, \ell_{\rho(Y)})$ is orthogonal to the real plane $\mathsf{H}^2 \subset \mathsf{H}^3$, that is, if $\mathsf{Lie}\big(\rho(X),\rho(Y)\big)$ defines an orientation-preserving involution of H^2.

This occurs precisely when the matrix $\mathsf{Lie}\big(\rho(X), \rho(Y)\big)$ has positive determinant. By (2.3.3), the Lie product $\mathsf{Lie}\big(\rho(X), \rho(Y)\big)$ has positive determinant if and only if the commutator trace

$$\mathsf{tr}([\rho(X), \rho(Y)]) < 2,$$

as assumed. The proof of Proposition 4.4.3 is complete. □

4.5 Fenchel-Nielsen coordinates.

In an influential manuscript written in the early 20th century but only recently published, Fenchel and Nielsen [17] gave geometric coordinates for Fricke space. We briefly relate these coordinates to trace coordinates for the surfaces of Euler characteristic -1.

Pants decompositions. Decompose Σ into a union of three-holed spheres ("pants")

$$P_1, \dots, P_l$$

along a system of N disjoint simple curves

$$\gamma_1, \dots, \gamma_N \subset \mathsf{int}(\Sigma).$$

Let $\partial_1, \dots, \partial_n$ denote components of $\partial\Sigma$. For a given marked hyperbolic structure on Σ, the curves γ_i, ∂_j may be taken to be simple closed geodesics. Theorem A implies that the isometry type of the complementary subsurfaces P_k are determined by the lengths of the three closed geodesics representing ∂P_k. The resulting map

$$\begin{aligned} \mathfrak{F}(\Sigma) &\xrightarrow{F} (\mathbb{R}_+)^N \times (\mathbb{R}_{\geq 0})^n \\ \langle M \rangle &\longmapsto \ell_M(\gamma_i) \times \ell_M(\partial_j) \end{aligned}$$

which associates to a hyperbolic surface M the lengths of the geodesics γ_i, ∂_j is onto. Its fibers correspond to the various ways in which the subsurfaces P_k are identified along interior curves γ_i.

Choose a section σ of the map F as follows. Each interior curve γ bounds two subsurfaces, which we denote P' and P''. The corresponding boundary curves are denoted $\gamma' \subset P'$ and $\gamma'' \subset P''$ respectively. The *twist parameter* $\tau_i \in \mathbb{R}$ represents the displacement between points on the marked surfaces P, P' corresponding to the section σ. This realizes the Fenchel-Nielsen map F as a principal $\mathbb{R}^N$-bundle over $\mathfrak{F}(\Sigma)$. Wolpert [77, 78, 79] shows that, when Σ is closed and orientable, the Fenchel-Nielsen coordinates on $\mathfrak{F}(\Sigma)$ are *canonical* or *Darboux* coordinates for the symplectic structure arising from the Weil-Petersson Kähler form on Teichmüller space, and indeed F is a moment map for a completely integrable system on $\mathfrak{F}(\Sigma)$.

In the orientable case, $\Sigma \approx \Sigma_{g,n}$, then

$$N = 3(g-1) + n. \tag{4.5.1}$$

Since $\chi(\Sigma) = 2 - 2g + n$ and each P_k has Euler characteristic -1, the number l of subsurfaces P_k equals

$$l = -\chi(\Sigma) = 2 + 2g + n.$$

Consider the set S of pairs (α, C), where α is one of the $N + n$ curves ∂_i, γ_j and $C \subset P_k$ is a collar neighborhood of α inside P_k for some k. Each C lies in exactly one P_k and each subsurface P_k contains exactly three pairs (α, C), the cardinality of S equals $3l$. Furthermore the number of collars C equals $2N + n$, since each γ_i is two-sided in Σ and each ∂_j is one-sided. Computing the cardinality of S in two ways:

$$2N + n = 3l = 3(2g - 2 + n)$$

implies (4.5.1).

The nonorientable case, say $\Sigma \approx C_{k,n}$, reduces to the orientable case by cutting along a disjoint family of simple loops: k of them reverse orientation and $N = k + 2 - n$ preserve orientation. This follows easily from the classification of surfaces: $C_{k,n}$ can be obtained from the planar surface $\Sigma_{0,k+n}$ by attaching k cross-caps (copies of $C_{0,k}$) to k of the components of $\Sigma_{0,k+n}$. In the nonorientable surface $\Sigma \approx C_{k,n}$ are k disjoint orientation-reversing simple loops $s_1, \dots, s_k$ so that the surface Σ' obtained by splitting Σ along $s_1, \dots, s_k$ identifies to Σ. Denote the resulting quotient map by

$$\Sigma' \approx \Sigma_{0,n+k} \xrightarrow{\phi} \Sigma \approx C_{k,n}.$$

Let $s_i' \subset \Sigma'$ denote the preimage $\phi^{-1}(s_i)$. Given a hyperbolic structure on Σ', there is a uniqe way of extending this hyperbolic structure to Σ as follows. As usual, assume that each $s_i' \subset \Sigma'$ is a closed geodesic. Choose $\epsilon > 0$ sufficiently small so that all the ϵ-collars $N_\epsilon(s_i')$ of s_i' are disjoint. Denote the complement of these collars by

$$\Sigma'' := \Sigma' \setminus \bigcup_{i=1}^{k} N_\epsilon(s_i').$$

Represent the geodesic s_i' as the quotient $\tilde{s}/\langle\xi\rangle$, where $\xi \in \mathsf{PSL}(2,\mathbb{R})$ is hyperbolic and $\tilde{s} \subset \mathsf{H}^2$ is the ξ-invariant geodesic. That is, ξ is a transvection along the geodesic $\tilde{s}$. Let $\sqrt{\xi}$ denote the unique *glide-reflection* whose square is ξ; it is the composition of reflection in $\tilde{s}$ with the transvection of displacement $\ell(\xi)/2$ where $\ell(\xi)$ is the displacement of ξ. If a matrix representative of ξ has trace $x > 2$, then $x = 2\cosh\big(\ell(\xi)/2\big)$ and a matrix representing the glide-reflection $\sqrt{\xi}$ equals

$$\frac{1}{\sqrt{x-2}}\big(\xi - \mathbb{I}\big).$$

Let $N_\epsilon(\tilde{s}) \subset \mathsf{H}^2$ be the tubular neighborhood of width ϵ about $\tilde{s}$. The quotient $N_\epsilon(\tilde{s})/\langle\sqrt{\xi}\rangle$ is a cross-cap bounded by a hypercycle (equidistant curve). The union

$$\Sigma'' \bigcup N_\epsilon(\tilde{s})/\langle\sqrt{\xi}\rangle$$

is the desired hyperbolic structure on Σ.

Fenchel-Nielsen coordinates on $\mathbf{\Sigma_{1,1}}$. We relate the Fricke trace coordinates to Fenchel-Nielsen coordinates as follows. We suppose that the boundary $\partial\Sigma$ has length $b \geq 0$, the case $b = 0$ corresponding to the complete finite-area structure (where the holonomy around $\partial\Sigma$ is parabolic).

Suppose that $X \subset \Sigma$ has length $l > 0$, and has holonomy represented by:

$$\tilde{\rho}(X) := \begin{bmatrix} e^{l/2} & 0 \\ 0 & e^{-l/2} \end{bmatrix}$$

where $x = 2\cosh(l/2)$. Then

$$\mathsf{Fix}\big(\rho(X)\big) = \{0, \infty\}.$$

A fundamental domain for the cyclic group $\langle\rho(X)\rangle$ is bounded by the geodesics with endpoints $\pm e^{-l/2}$ and $\pm e^{l/2}$ respectively.

Normalize the twist parameter τ so that $\tau = 0$ corresponds to the case that the invariant axes $\ell_{\rho(X)}, \ell_{\rho(Y)}$ are orthogonal. In that case take $\mathsf{Fix}(\rho(Y)) = \pm 1$ and define

$$\tilde{\rho}_0(Y) := \begin{bmatrix} \cosh(\mu/2) & \sinh(\mu/2) \\ \sinh(\mu/2) & \cosh(\mu/2) \end{bmatrix}$$

where $y = 2\cosh(\mu/2)$.

A fundamental domain for the cyclic group $\langle\rho(Y)\rangle$ is bounded by the geodesics with endpoints $-e^{\pm\mu/2}$ and $e^{\pm\mu/2}$ respectively. For this representation,

$$z = \mathsf{tr}\big(\tilde{\rho}(X)\tilde{\rho}_0(Y)\big) = xy/2 = 2\cosh(l/2)\cosh(\mu/2).$$

Let $\tau \in \mathbb{R}$ be the twist parameter for Fenchel-Nielsen flow. (Compare Wolpert [77]). The orbit of the Fenchel-Nielsen twist deformation is defined by the representation

$$\rho(Y) \ := \ \rho_0(Y)\exp\Big(\,(\tau/2)\,\widehat{\tilde{\rho}(X)}\,\Big)$$

where

$$\widehat{\tilde{\rho}(X)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

defines the one-parameter subgroup

$$\exp\big((\tau/2)\, \widehat{\tilde{\rho}(X)} \big) \;=\; \begin{bmatrix} e^{\tau/2} & 0 \\ 0 & e^{-\tau/2} \end{bmatrix}.$$

Now $x = 2\cosh(l/2)$ is constant but

$$\begin{aligned} y \;&=\; \mathsf{tr}\big(\tilde{\rho}(Y)\big) \;=\; \mathsf{tr}\left(\tilde{\rho}_0(Y) \begin{bmatrix} e^{\tau/2} & 0 \\ 0 & e^{-\tau/2} \end{bmatrix}\right) \\ &=\; 2\cosh(l/2)\, \cosh(\tau/2). \end{aligned}$$

Similarly,

$$\begin{aligned} z \;&=\; \mathsf{tr}\Big(\tilde{\rho}(X)\, \tilde{\rho}(Y)\Big) = \mathsf{tr}\left(\begin{bmatrix} e^{l/2} & 0 \\ 0 & e^{-l/2} \end{bmatrix} \tilde{\rho}_0(Y) \begin{bmatrix} e^{\tau/2} & 0 \\ 0 & e^{-\tau/2} \end{bmatrix} \right) \\ &=\; 2\cosh(\mu/2)\cosh\big((l+\tau)/2\big). \end{aligned}$$

Now the commutator trace $\mathsf{tr}\big(\tilde{\rho}([X,Y])\big)$ equals

$$-2\cosh(b/2) = \kappa(x,y,z) = 2 - \sinh^2(l/2)\sinh^2(\mu/2)$$

whence

$$\cosh^2(\mu/2) = 1 - 4\,\mathrm{csch}^2(l/2)\sinh^2(b/4).$$

Therefore the Fricke trace coordinates are expressed in terms of Fenchel-Nielsen coordinates by:

$$\begin{aligned} x \;&=\; 2\cosh(l/2) \\ y \;&=\; 2\sqrt{1 - 4\,\mathrm{csch}^2(l/2)\sinh^2(b/4)}\, \cosh(\tau/2) \\ z \;&=\; 2\sqrt{1 - 4\,\mathrm{csch}^2(l/2)\sinh^2(b/4)}\, \cosh\big((\tau+l)/2\big). \end{aligned}$$

4.6 The two-holed cross-surface.

Following John H. Conway's suggestion, we call a surface homeomorphic to a real projective plane a *cross-surface.* Suppose that $\Sigma = C_{0,2}$ is a 2-holed cross-surface (Figure 4). Then $\pi_1(\Sigma)$ is freely generated by two orientation-reversing simple loops P, Q on the interior. These loops correspond to the two 1-handles in Figure 4. The two boundary components $\partial_\pm$ of Σ correspond to elements

$$R := P^{-1}Q^{-1}, \quad R' := QP^{-1},$$

obtaining a redundant geometric presentation of $\pi_1(\Sigma)$:

$$\pi \;=\; \langle P, Q, R, R' \;\mid\; PQR = PQ^{-1}R' = \mathbb{I}\rangle.$$

The characters of the generators of this presentation define a presentation of the character ring

$$\mathbb{C}[f_P, f_Q, f_R, f_{R'}] \,/\, \big(f_R + f_{R'} - f_P f_Q\big)$$

the relation being (2.2.5). Of course p, q, r (respectively p, q, r') are free generators for the character ring (a polynomial ring in three variables).

The Fricke space of Σ was computed by Stantchev [70]; compare also the forthcoming paper by Goldman-McShane-Stantchev-Tan [32]. For a given hyperbolic structure on Σ, the holonomy transformations $\rho(P)$ and $\rho(Q)$ reverse orientation, their traces are purely imaginary, and the traces of $\rho(R)$ and $\rho(R')$ are real. For this reason we write

$$\begin{aligned} ip = f_P &= \mathsf{tr}\big(\rho(P)\big) \in i\mathbb{R} \\ iq = f_Q &= \mathsf{tr}\big(\rho(Q)\big) \in i\mathbb{R} \\ r = f_R &= \mathsf{tr}\big(\rho(R)\big) \in \mathbb{R} \\ r' = f_{R'} &= \mathsf{tr}\big(\rho(R')\big) \in \mathbb{R} \end{aligned}$$

where $p, q, r, r' \in \mathbb{R}$ and

$$r' := r + pq \in \mathbb{R}.$$

By an analysis similar to that of $\Sigma_{0,3}$ and $\Sigma_{1,1}$, the Fricke space of Σ identifies with

$$\{(p, q, r) \in \mathbb{R}^3 \;\mid\; r \le -2,\; pq + r \ge 2\}.$$

Compare [70, 34, 32] for further details.

4.7 The one-holed Klein bottle.

Now suppose Σ is a one-holed Klein bottle. (Compare Figure 5.) Once again we choose free generators P, Q for π corresponding to the two 1-handles in Figure 5 which reverse orientation. The boundary component D corresponds to P^2Q^2 and, writing $R = (PQ)^{-1}$, we obtain a redundant geometric presentation

$$\pi \;=\; \langle P, Q, R, D \;\mid\; PQR = \mathbb{I}, D = P^2Q^2\rangle$$

and the character ring has presentation

$$\mathbb{C}[f_P, f_Q, f_R, f_D]/\big(f_D - (f_P f_Q f_R - f_P^2 - f_Q^2 + 4)\big)$$

Since P, Q reverse orientation on H^2, the functions f_P, f_Q are purely imaginary and f_R, f_D are real. Thus we write

$$\begin{aligned} ip = f_P &= \mathsf{tr}\big(\rho(P)\big) \in i\mathbb{R} \\ iq = f_Q &= \mathsf{tr}\big(\rho(Q)\big) \in i\mathbb{R} \\ r = f_R &= \mathsf{tr}\big(\rho(R)\big) \in \mathbb{R} \\ d = f_D &= \mathsf{tr}\big(\rho(D)\big) \in \mathbb{R} \end{aligned}$$

and the Fricke space of Σ identifies with

$$\{(p, q, r) \in \mathbb{R}^3 \ \mid \ p^2 + q^2 - pqr \geq 0\}.$$

See Stantchev [70] and [32] for details.

5 Three-generator groups and beyond

Let Σ be a compact connected surface-with-boundary. Suppose $\partial\Sigma \neq \emptyset$. Then the fundamental group $\pi_1(\Sigma)$ is free of rank 3 if and only if the Euler characteristic $\chi(\Sigma) = -2$. Such a surface is homeomorphic to one of the four topological types:

- A 4-holed sphere $\Sigma_{0,4}$;
- A 2-holed torus $\Sigma_{1,2}$;
- A 3-holed cross-surface (projective planes) $C_{0,3}$;
- A 2-holed Klein bottle $C_{1,2}$.

In this section we only consider the orientable topological types, namely $\Sigma_{0,4}$ and $\Sigma_{1,2}$. We relate their character varieties to those of nonorientable surfaces $C_{0,2}$ and $C_{1,1}$, discussed in §5.5 and §5.6.

5.1 The $\mathsf{SL}(2, \mathbb{C})$-character ring of $\mathbb{F}_3$.

Representations ρ of the free group $\langle X_1, X_2, X_3\rangle$ of rank three correspond to arbitrary triples

$$\big(\rho(X_1), \rho(X_2), \rho(X_3)\big) \ \in \ \mathsf{SL}(2, \mathbb{C})^3.$$

As before we consider the quotient space (in the sense of Geometric Invariant Theory) under the action of $\mathsf{SL}(2, \mathbb{C})$ by inner automorphisms, the *character variety.* Its coordinate ring is by definition the subring of invariants (the *character ring*)

$$\mathbb{C}[\mathsf{SL}(2, \mathbb{C})^3]^{\mathsf{PSL}(2,\mathbb{C})} \ \subset \ \mathbb{C}[\mathsf{SL}(2, \mathbb{C})^3]$$

of the induced *effective* $\mathsf{PSL}(2,\mathbb{C})$-action on the ring of coordinate ring $\mathbb{C}[\mathsf{SL}(2,\mathbb{C})^3]$.

We saw in §2 that for a free group of rank two, the character variety is an affine space and the charcter ring is a polynomial ring. The situation in rank three is more complicated. The character variety V_3 is a six-dimensional hypersurface in $\mathbb{C}^7$, which admits a branched double covering onto the six-dimensional affine space $\mathbb{C}^6$.

Explicitly, the character ring $\mathfrak{R}_3$ is generated by eight trace functions

$$t_1, t_2, t_3, t_{12}, t_{23}, t_{13}, t_{123}, t_{132}$$

defined by

$$\begin{aligned} t_i(\rho) &:= \mathsf{tr}\big(\rho(X_i)\big) \\ t_{ij}(\rho) &:= \mathsf{tr}\big(\rho(X_iX_j)\big) \\ t_{ijk}(\rho) &:= \mathsf{tr}\big(\rho(X_iX_jX_k)\big) \end{aligned}$$

subject to two relations expressing the sum and product of traces of the length 3 monomials in terms of traces of monomials of length 1 and 2:

$$t_{123} + t_{132} = t_{12}t_3 + t_{13}t_2 + t_{23}t_1 - t_1t_2t_3 \tag{5.1.1}$$

$$\begin{aligned} t_{123}\ t_{132} = {} & (t_1^2 + t_2^2 + t_3^2) + (t_{12}^2 + t_{23}^2 + t_{13}^2) - \\ & (t_1t_2t_{12} + t_2t_3t_{23} + t_3t_1t_{13}) + t_{12}t_{23}t_{13} - 4. \end{aligned} \tag{5.1.2}$$

We call (5.1.1) the *Sum Relation* and (5.1.2) the *Product Relation* respectively. They imply that the *triple traces* t_{123} and t_{132} are the respective roots λ of the irreducible monic quadratic equation:

$$\lambda^2 - f_\Sigma\lambda + f_\Pi = 0$$

where the coefficients:

$$f_\Sigma := t_{12}t_3 + t_{23}t_1 + t_{13}t_2 - t_1t_2t_3$$

and

$$\begin{aligned} f_\Pi := {} & (t_1^2 + t_2^2 + t_3^2) \\ & + (t_{12}^2 + t_{23}^2 + t_{13}^2) \\ & - (t_1t_2t_{12} + t_2t_3t_{23} + t_3t_1t_{13}) \\ & + t_{12}t_{23}t_{13} - 4 \end{aligned}$$

are the polynomials appearing in the right-hand sides of (5.1.1) and (5.1.2) respectively.

V_3 is a hypersurface in $\mathbb{C}^7$. Eliminating t_{132} in (5.1.1) as

$$t_{132} = t_{12}t_3 + t_{13}t_2 + t_{23}t_1 - t_1t_2t_3 - t_{123}, \tag{5.1.3}$$

realizes V_3 as the hypersurface in $\mathbb{C}^7$ consisting of all

$$\big(t_1, t_2, t_3, t_{12}, t_{23}, t_{13}, t_{123}\big) \in \mathbb{C}^7$$

satisfying

$$\begin{aligned} t_{123}\,\big(t_{12}t_3 + t_{13}t_2 + t_{23}t_1 - t_1t_2t_3 - t_{123}\big) = (t_1^2 + t_2^2 + t_3^2) \,+\, (t_{12}^2 + t_{23}^2 \,+\, t_{13}^2) \\ -\,(t_1t_2t_{12} + t_2t_3t_{23} + t_3t_1t_{13}) \,+\, t_{12}t_{23}t_{13} \,-\, 4. \end{aligned}$$

***V_3* double covers $\mathbb{C}^6$.** The double covering of V_3 over $\mathbb{C}^6$ arises from the composition

$$V_3 \hookrightarrow \mathbb{C}^8 \xrightarrow{\Pi} \mathbb{C}^6 \tag{5.1.4}$$

where

$$\begin{aligned} \mathbb{C}^8 &\xrightarrow{\Pi} \mathbb{C}^6 \\ \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_{12} \\ t_{23} \\ t_{13} \\ t_{123} \\ t_{132} \end{bmatrix} &\longmapsto \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_{12} \\ t_{23} \\ t_{13} \end{bmatrix} \end{aligned}$$

is the coordinate projection.

Proposition 5.1.1. *The composition* (5.1.4)

$$\begin{aligned} V &\xrightarrow{\mathsf{t}} \mathbb{C}^6 \\ [\rho] &\longmapsto \begin{bmatrix} t_1(\rho) \\ t_2(\rho) \\ t_3(\rho) \\ t_{12}(\rho) \\ t_{23}(\rho) \\ t_{13}(\rho) \end{bmatrix} \end{aligned}$$

is onto. Furthermore it is a double covering branched along the discriminant hypersurface in $\mathbb{C}^6$ defined by

$$\begin{aligned} \big(t_{12}t_3 + t_{13}t_2 + t_{23}t_1 - t_1t_2t_3\big)^2 = 4\Big((t_1^2 + t_2^2 + t_3^2) \,+\, (t_{12}^2 + t_{23}^2 \,+\, t_{13}^2) \\ -\,(t_1t_2t_{12} + t_2t_3t_{23} + t_3t_1t_{13}) \,+\, t_{12}t_{23}t_{13} \,-\, 4\Big). \end{aligned}$$

The goal of this section is to prove the identities (5.1.1), (5.1.2) and Proposition 5.1.1.

Proof of the Sum Relation. To prove these identities, we temporarily introduce the following notation. Let $\xi = \rho(X_1)$, $\eta = \rho(X_2)$, $\zeta = \rho(X_3)$, so that (5.1.1) becomes:

$$\begin{aligned}\mathsf{tr}(\xi\eta\zeta) + \mathsf{tr}(\xi\zeta\eta) = \mathsf{tr}(\xi\eta)\mathsf{tr}(\zeta) \\ - \mathsf{tr}(\zeta)\mathsf{tr}(\xi)\mathsf{tr}(\eta) + \mathsf{tr}(\zeta\xi)\mathsf{tr}(\eta) \\ + \mathsf{tr}(\eta\zeta)\mathsf{tr}(\xi).\end{aligned} \tag{5.1.5}$$

To prove (5.1.5), apply the Basic Identity (2.2.5) three times:

$$\mathsf{tr}(\xi\eta\zeta) + \mathsf{tr}(\xi\eta\zeta^{-1}) = \mathsf{tr}(\xi\eta)\mathsf{tr}(\zeta) \tag{5.1.6}$$

$$\begin{aligned}\mathsf{tr}(\zeta^{-1}\xi\eta) + \mathsf{tr}(\zeta^{-1}\xi\eta^{-1}) &= \mathsf{tr}(\zeta^{-1}\xi)\mathsf{tr}(\eta) \\ &= (\mathsf{tr}(\zeta)\mathsf{tr}(\xi) - \mathsf{tr}(\zeta\xi))\mathsf{tr}(\eta)\end{aligned} \tag{5.1.7}$$

$$\begin{aligned}\mathsf{tr}(\eta^{-1}\zeta^{-1}\xi) + \mathsf{tr}(\eta^{-1}\zeta^{-1}\xi^{-1}) &= \mathsf{tr}(\eta^{-1}\zeta^{-1})\mathsf{tr}(\xi) \\ &= \mathsf{tr}(\eta\zeta)\mathsf{tr}(\xi)\end{aligned} \tag{5.1.8}$$

Now add (5.1.6), subtract (5.1.7) and add (5.1.8) to obtain:

$$\begin{aligned}\big(\mathsf{tr}(\xi\eta\zeta) + \mathsf{tr}(\xi\eta\zeta^{-1})\big) - \big(\mathsf{tr}(\zeta^{-1}\xi\eta) + \mathsf{tr}(\zeta^{-1}\xi\eta^{-1})\big) \\ + \big(\mathsf{tr}(\eta^{-1}\zeta^{-1}\xi) + \mathsf{tr}(\xi\zeta\eta)\big) \\ = \mathsf{tr}(\xi\eta)\mathsf{tr}(\zeta) - \big(\mathsf{tr}(\zeta)\mathsf{tr}(\xi) - \mathsf{tr}(\zeta\xi)\big)\mathsf{tr}(\eta) \\ + \mathsf{tr}(\eta\zeta)\mathsf{tr}(\xi)\end{aligned} \tag{5.1.9}$$

The right hand side of (5.1.9) is the right-hand side of (5.1.5). The left-hand side of (5.1.9) equals:

$$\begin{aligned}\mathsf{tr}(\xi\eta\zeta) + \big(\mathsf{tr}(\xi\eta\zeta^{-1}) - \mathsf{tr}(\zeta^{-1}\xi\eta)\big) \\ +\big(- \mathsf{tr}(\zeta^{-1}\xi\eta^{-1}) + \mathsf{tr}(\eta^{-1}\zeta^{-1}\xi)\big) \\ +\mathsf{tr}(\xi\zeta\eta) \;=\; \mathsf{tr}(\xi\eta\zeta) + \mathsf{tr}(\xi\zeta\eta),\end{aligned}$$

the left-hand side of (5.1.5), from which (5.1.5) follows.

Proof of the Product Relation. We derive this formula in several steps. Directly applying the Basic Identity (2.2.5):

$$\begin{aligned}\mathsf{tr}(\zeta\xi\zeta\eta) &= \mathsf{tr}(\zeta\xi)\mathsf{tr}(\zeta\eta) - \mathsf{tr}(\xi\eta^{-1}) \qquad (5.1.10)\\ &= \mathsf{tr}(\zeta\xi)\mathsf{tr}(\zeta\eta) - (\mathsf{tr}(\xi)\mathsf{tr}(\eta) - \mathsf{tr}(\xi\eta)) \\ &= \mathsf{tr}(\zeta\xi)\mathsf{tr}(\zeta\eta) - \mathsf{tr}(\xi)\mathsf{tr}(\eta) + \mathsf{tr}(\xi\eta)\end{aligned}$$

Apply a calculation similar to (2.2.7) to ξ, ζ^{-1}:

$$\mathsf{tr}(\xi\zeta^{-1}\xi^{-1}\zeta^{-1}) = \mathsf{tr}(\xi)\mathsf{tr}(\zeta)\mathsf{tr}(\zeta\xi) - \mathsf{tr}(\zeta\xi)^2 - \mathsf{tr}(\xi)^2 + 2 \tag{5.1.11}$$

$$\begin{aligned}
\mathsf{tr}(\xi\eta\zeta\xi\zeta\eta) &= \mathsf{tr}(\xi\eta)\mathsf{tr}(\zeta\xi\zeta\eta) \\
&\qquad - \mathsf{tr}(\xi\zeta^{-1}\xi^{-1}\zeta^{-1}) \\
&= \mathsf{tr}(\xi\eta)\left(\mathsf{tr}(\zeta\xi)\mathsf{tr}(\zeta\eta) - \mathsf{tr}(\xi)\mathsf{tr}(\eta) + \mathsf{tr}(\xi\eta)\right) \\
&\qquad - \big(\mathsf{tr}(\xi)\mathsf{tr}(\zeta)\mathsf{tr}(\zeta\xi) - \mathsf{tr}(\zeta\xi)^2 - \mathsf{tr}(\xi)^2 + 2\big) \\
&\qquad\qquad \text{(by (5.1.10) and (5.1.11))} \\
&= \mathsf{tr}(\xi\eta)\mathsf{tr}(\zeta\xi)\mathsf{tr}(\eta\zeta) \\
&\qquad - \mathsf{tr}(\xi)\mathsf{tr}(\eta)\mathsf{tr}(\xi\eta) - \mathsf{tr}(\zeta)\mathsf{tr}(\xi)\mathsf{tr}(\zeta\xi) \\
&\qquad + \mathsf{tr}(\xi\eta)^2 + \mathsf{tr}(\xi)^2 - 2.
\end{aligned} \tag{5.1.12}$$

Finally, applying (5.1.12) and the Commutator Identity (2.2.8) to η, ζ:

$$\begin{aligned}
\mathsf{tr}(\xi\eta\zeta)\mathsf{tr}(\xi\zeta\eta) &= \mathsf{tr}(\xi\eta\zeta\xi\zeta\eta) + \mathsf{tr}(\eta\zeta\eta^{-1}\zeta^{-1}) \\
&= \Big(\mathsf{tr}(\xi\eta)\mathsf{tr}(\zeta\xi)\mathsf{tr}(\eta\zeta) - \mathsf{tr}(\xi)\mathsf{tr}(\eta)\mathsf{tr}(\xi\eta) \\
&\quad - \mathsf{tr}(\zeta)\mathsf{tr}(\xi)\mathsf{tr}(\zeta\xi) + \mathsf{tr}(\xi\eta)^2 + \mathsf{tr}(\zeta\xi)^2 + \mathsf{tr}(\xi)^2 - 2\Big) \\
&\quad + \Big(\mathsf{tr}(\eta)^2 + \mathsf{tr}(\zeta)^2 + \mathsf{tr}(\eta\zeta)^2 - \mathsf{tr}(\eta)\mathsf{tr}(\zeta)\mathsf{tr}(\eta\zeta) - 2\Big)
\end{aligned}$$

obtaining (5.1.2).

Proof that t is onto. Now we prove Proposition 5.1.1. For a more general treatment see Florentino [18].

Theorem 1 guarantees $\xi_1, \xi_2 \in \mathsf{SL}(2,\mathbb{C})$ such that

$$\begin{aligned}
\mathsf{tr}(\xi_1) &= t_1, \\
\mathsf{tr}(\xi_2) &= t_2, \\
\mathsf{tr}(\xi_1\xi_2) &= t_{12}.
\end{aligned} \tag{5.1.13}$$

We seek $\xi_3 \in \mathsf{SL}(2,\mathbb{C})$ such that

$$\begin{aligned}
\mathsf{tr}(\xi_3) &= t_3, \\
\mathsf{tr}(\xi_2\xi_3) &= t_{23}, \\
\mathsf{tr}(\xi_1\xi_3) &= t_{13}.
\end{aligned} \tag{5.1.14}$$

To this end, consider the affine subspace $\mathcal{W}$ of $\mathsf{M}_2(\mathbb{C})$ consisting of matrices ω satisfying

$$\begin{aligned} \mathsf{tr}(\omega) &= t_3, \\ \mathsf{tr}(\xi_2\omega) &= t_{23}, \\ \mathsf{tr}(\xi_1\omega) &= t_{13}. \end{aligned} \tag{5.1.15}$$

Since the bilinear pairing

$$\begin{aligned} \mathsf{M}_2(\mathbb{C}) \times \mathsf{M}_2(\mathbb{C}) &\longrightarrow \mathbb{C} \\ (\xi, \eta) &\longmapsto \mathsf{tr}(\xi\eta) \end{aligned}$$

is nondegenerate, each of the three equations in (5.1.15) describes an affine hyperplane in $\mathsf{M}_2(\mathbb{C})$. We first suppose that (t_1, t_2, t_{12}) describes an irreducible character, that is $\kappa(t_1, t_2, t_{12}) \neq 2$:

$$4 - t_1^2 - t_2^2 - t_{12}^2 + t_1 t_2 t_{12} \neq 0. \tag{5.1.16}$$

Our goal will be to find an element $\xi_3 \in \mathcal{W}$ such that $\mathsf{det}(\xi_3) = 1$.

Lemma 5.1.2. *There exist $\xi_1, \xi_2 \in \mathsf{SL}(2, \mathbb{C})$ satisfying* (5.1.13) *such that $\mathcal{W}$ is a (nonempty) affine line.*

Proof. Since $\kappa(t_1, t_2, t_{12}) \neq 2$, Proposition 2.3.1, (2) implies that the pair ξ_1, ξ_2 generates an irreducible representation.

We claim that $\{\mathbb{I}, \xi_1, \xi_2\}$ is a linearly independent subset of the 4-dimensional vector space $\mathsf{M}_2(\mathbb{C})$. Otherwise the nonzero element ξ_1 is a linear combination of ξ_2 and $\mathbb{I}$. Let $v \neq 0$ be an eigenvector of ξ_2. Then the line (v) spanned by v is invariant under ξ_1 as well, and hence under the group generated by ξ_1 and ξ_2. This contradicts irreducibility of the representation generated by ξ_1 and ξ_2.

Since $\{\mathbb{I}, \xi_1, \xi_2\}$ is linearly independent, the three linear conditions of (5.1.15) are independent. Hence $\mathcal{W} \subset \mathsf{M}_2(\mathbb{C})$ is an affine line. $\square$

Let $\omega_0, \omega_1 \in \mathcal{W}$ be distinct elements in this line. Then the function

$$\begin{aligned} \mathbb{C} &\longrightarrow \mathbb{C} \\ s &\longmapsto \mathsf{det}\big(s\omega_1 + (1-s)\omega_0\big) \end{aligned}$$

is polynomial of degree ≤ 2, and is thus onto unless it is constant.

We shall show that this map is onto, and therefore $\mathcal{W} \cap \mathsf{SL}(2, \mathbb{C}) \neq \emptyset$. The desired matrix ξ_3 will be an element of $\mathcal{W} \cap \mathsf{SL}(2, \mathbb{C})$.

Lemma 5.1.3. *Let $\omega_0, \omega_1 \in \mathsf{M}_2(\mathbb{C})$. Then*

$$\begin{aligned}\mathsf{det}\big(s\omega_1 + (1-s)\omega_0\big) &= \mathsf{det}\big(\omega_0 + s(\omega_1 - \omega_0)\big)\\ &= \mathsf{det}(\omega_0) + \ s\ \Big(\mathsf{tr}(\omega_0)\mathsf{tr}(\omega_1 - \omega_0) - \mathsf{tr}\big(\omega_0(\omega_1 - \omega_0)\big)\Big)\\ &\qquad + \ s^2\ \mathsf{det}(\omega_1 - \omega_0)\end{aligned}$$

Proof. Clearly

$$\mathsf{tr}\big(\omega_0 \ + \ s\,(\omega_1 - \omega_0)\big) = \mathsf{tr}(\omega_0) \ + \ s\,\mathsf{tr}(\omega_1 - \omega_0). \tag{5.1.17}$$

Now

$$\mathsf{det}(\omega) = \frac{\mathsf{tr}(\omega)^2 - \mathsf{tr}(\omega^2)}{2} \tag{5.1.18}$$

whenever $\omega \in \mathsf{M}_2(\mathbb{C})$. Now apply (5.1.18) to (5.1.17) taking

$$\omega = \omega_0 + s(\omega_1 - \omega_0)$$

□

Thus the restriction $\mathsf{det}|_{\mathcal{W}}$ is constant only if $\mathsf{det}(\omega_1 - \omega_0) = 0$.

Choose a solution ξ of $\xi + \xi^{-1} = t_{12}$. Work in the slice

$$\xi_1 = \begin{bmatrix} t_1 & -1 \\ 1 & 0 \end{bmatrix},\ \xi_2 = \begin{bmatrix} 0 & \xi \\ -\xi^{-1} & t_2 \end{bmatrix}.$$

The matrix $\omega_0 \in \mathsf{M}_2(\mathbb{C})$ defined by:

$$\omega_0 = \begin{bmatrix} t_3 & \big((t_{13} - t_1 t_3)\xi + t_{23}\big)\xi/(\xi^2 - 1) \\ \big((t_{13} - t_1 t_3) + t_{23}\xi\big)/(\xi^2 - 1) & 0 \end{bmatrix},$$

satisfies (5.1.15). Any other $\omega \in \mathcal{W}$ must satisfy

$$\begin{aligned}\mathsf{tr}\big(\omega - \omega_0\big) &= 0,\\ \mathsf{tr}\big(\xi_2(\omega - \omega_0)\big) &= 0,\\ \mathsf{tr}\big(\xi_1(\omega - \omega_0)\big) &= 0.\end{aligned} \tag{5.1.19}$$

Lemma 5.1.4. *Any solution $\omega - \omega_0$ of* (5.1.19) *is a multiple of*

$$\mathsf{Lie}(\xi_1, \xi_2) = \xi_1\xi_2 - \xi_2\xi_1 = \begin{bmatrix} \xi^{-1} - \xi & -t_2 + t_1\xi \\ -t_2 + \xi^{-1} t_1 & \xi - \xi^{-1} \end{bmatrix}.$$

Proof. The first equation in (5.1.19) asserts that $\omega - \omega_0$ lies in the subspace $\mathfrak{sl}(2)$, upon which the trace form is nondegenerate. The second and third equations assert that $\omega - \omega_0$ is orthogonal to ξ_1 and ξ_2. By (5.1.16), ξ_1, ξ_2 and $\mathbb{I}$ are linearly independent in $\mathsf{M}_2(\mathbb{C})$, so the solutions of (5.1.19) form a one-dimensional linear subspace. The Lie product

$$\mathsf{Lie}(\xi_1, \xi_2) = \xi_1\xi_2 - \xi_2\xi_1$$

is nonzero and lies in $\mathfrak{sl}(2)$. Furthermore, for $i = 1, 2$,

$$\mathsf{tr}(\xi_i\xi_1\xi_2) = \mathsf{tr}(\xi_i\xi_2\xi_1)$$

implies that $\mathsf{Lie}(\xi_1, \xi_2)$ is orthogonal to ξ_1 and ξ_2. The lemma follows. □

Parametrize $\mathcal{W}$ explicitly as $\omega = \omega_0 + s\,\mathsf{Lie}(\xi_1, \xi_2)$. By (2.3.3),

$$\mathsf{det}\big(\mathsf{Lie}(\xi_1, \xi_2)\big) = 4 - (t_1^2 + t_2^2 + t_{12}^2 - t_1t_2t_{12}) = 2 - \kappa(t_1, t_2, t_{12}) \neq 0.$$

By (5.1.16), the polynomial

$$\mathcal{W} \xrightarrow{\mathsf{det}} \mathbb{C}$$

is nonconstant, and hence onto. Taking

$$\omega_1 \in (\mathsf{det}|_{\mathcal{W}}^{-1})(1),$$

the proof of Proposition 5.1.1 is complete assuming (5.1.16).

The case when $4 - t_1^2 - t_2^2 - t_{12}^2 + t_1t_2t_{12} = 0$ remains. Then

$$t_i = a_i + (a_i)^{-1}$$

for $i = 1, 2$, for some $a_1, a_2 \in \mathbb{C}^*$. Then either

$$t_{12} = a_1a_2 + (a_1a_2)^{-1} \tag{5.1.20}$$

or

$$t_{12} = a_1(a_2)^{-1} + (a_1)^{-1}a_2. \tag{5.1.21}$$

In the first case (5.1.20), set

$$\begin{aligned} \xi_1 &:= \begin{bmatrix} a_1 & t_{13} - a_1t_3 \\ 0 & (a_1)^{-1} \end{bmatrix}, \\ \xi_2 &:= \begin{bmatrix} a_2 & t_{23} - a_2t_3 \\ 0 & (a_2)^{-1} \end{bmatrix}, \\ \xi_3 &:= \begin{bmatrix} t_3 & -1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

and in the second case (5.1.21), set

$$\begin{aligned}
\xi_1 &:= \begin{bmatrix} (a_1)^{-1} & t_{13} - (a_1)^{-1} t_3 \\ 0 & a_1 \end{bmatrix}, \\
\xi_2 &:= \begin{bmatrix} a_2 & t_{23} - a_2 t_3 \\ 0 & (a_2)^{-1} \end{bmatrix}, \\
\xi_3 &:= \begin{bmatrix} t_3 & -1 \\ 1 & 0 \end{bmatrix}
\end{aligned}$$

obtaining $(\xi_1, \xi_2, \xi_3) \in \mathsf{SL}(2, \mathbb{C})^3$ explicitly solving (5.1.13) and (5.1.14). The proof of Proposition 5.1.1 is complete.

5.2 The four-holed sphere.

Let $\Sigma \approx \Sigma_{0,4}$ be the four-holed sphere, with boundary components A, B, C, D subject to the relation

$$ABCD = \mathbb{I}.$$

The fundamental group is freely generated by

$$A = X_1, B = X_2, C = X_3$$

which represent three of the boundary components. The fourth boundary component is represented by an element

$$D := (X_1 X_2 X_3)^{-1},$$

satisfying the relation

$$ABCD = \mathbb{I}.$$

The resulting redundant presentation of the free group is:

$$\pi = \langle A, B, C, D \mid ABCD = \mathbb{I} \rangle.$$

The elements

$$\begin{aligned}
X &:= X_1 X_2 \\
Y &:= X_2 X_3 \\
Z &:= X_1 X_3
\end{aligned}$$

correspond to simple loops on Σ separating Σ into two 3-holed spheres. (Compare Figure 8.) The (even more redundant) presentation

$$\begin{aligned}
\pi \;=\; \langle A, B, C, D, X, Y, Z \;|\; & ABCD = \mathbb{I}, \\
& X = AB,\; Y = BC,\; Z = CA \rangle
\end{aligned}$$

gives regular functions

$$\begin{aligned} a &= t_1 \\ b &= t_2 \\ c &= t_3 \\ x &= t_{12} \\ y &= t_{23} \\ z &= t_{13} \\ d &= t_{123}. \end{aligned}$$

generating the character ring. Using (5.1.1) to eliminate t_{132} as in (5.1.3), the product relation (5.1.2) implies:

$$\begin{aligned} x^2 + y^2 + z^2 + xyz = (ab + cd)x \\ + (ad + bc)y \\ + (ac + bd)z \\ + (4 - a^2 - b^2 - c^2 - d^2 - abcd). \end{aligned} \tag{5.2.1}$$

This leads to a presentation of the character ring as a quotient of the polynomial ring $\mathbb{C}[a, b, c, d, x, y, z]$ by the principal ideal (Φ) generated by

$$\begin{aligned} \Phi(a, b, c, d; x, y, z) = x^2 + y^2 + z^2 + xyz \\ - (ab + cd)x - (ad + bc)y - (ac + bd)z \\ + a^2 + b^2 + c^2 + d^2 + abcd - 4. \end{aligned} \tag{5.2.2}$$

Thus the $\mathsf{SL}(2, \mathbb{C})$-*character variety* is a quartic hypersurface in $\mathbb{C}^7$, and for fixed boundary traces $(a, b, c, d) \in \mathbb{C}^4$, the *relative* $\mathsf{SL}(2, \mathbb{C})$-*character variety* is the cubic surface in $\mathbb{C}^3$ defined by (5.2.1), as was known to Fricke and Vogt. (Compare Benedetto-Goldman [2], Goldman [28], Goldman-Neumann [33], Cantat-Loray [11] and Cantat [10], Iwasaki [40].)

The Fricke space of $\mathbf{\Sigma_{0,4}}$. We identify the Fricke space $\mathfrak{F}(\Sigma_{0,4})$ in terms of trace coordinates.

Theorem 5.2.1. *The Fricke space of a four-holed sphere with boundary traces $a, b, c, d > 2$ is defined by the inequalities in $\mathbb{R}^4 \times \mathbb{R}^3$ for $(a, b, c, d) \in \mathbb{R}^4$ and $(x, y, z) \in \mathbb{R}^3$:*

$$\begin{cases} a, b, c, d & \geq 2, \quad x < -2, \\ F^-, F^+ & > 0 \\ F^- F^+ & = \frac{(x^2 + a^2 + b^2 - abx - 4)(x^2 + c^2 + d^2 - cdx - 4)}{x^2 - 4} \end{cases}$$

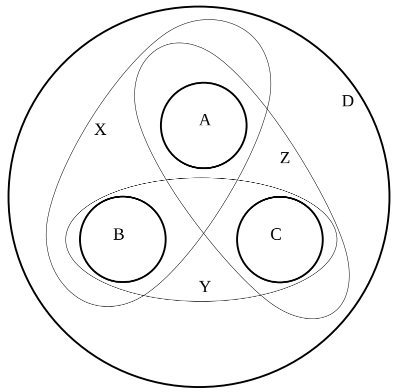

Figure 8. Seven simple curves on $\Sigma_{0,4}$.

where

$$F^- = \sqrt{2-x}\left(y - z - \frac{(a-b)(d-c)}{2-x}\right) - \sqrt{-2-x}\left(y + z - \frac{(a+b)(d+c)}{-2-x}\right)$$
$$F^+ = \sqrt{2-x}\left(y - z - \frac{(a-b)(d-c)}{2-x}\right) + \sqrt{-2-x}\left(y + z - \frac{(a+b)(d+c)}{-2-x}\right).$$

Proof. For a given hyperbolic structure on Σ, the holonomy generators $\rho(A)$, $\rho(B)$, $\rho(C)$, $\rho(D) \in \mathsf{PSL}(2,\mathbb{R})$ are hyperbolic or parabolic. Choose lifts

$$\tilde\rho(A), \tilde\rho(B), \tilde\rho(C), \tilde\rho(D) \in \mathsf{SL}(2,\mathbb{R})$$

which have positive trace. Since ρ is a representation $\pi_1(\Sigma) \longrightarrow \mathsf{PSL}(2,\mathbb{R})$,

$$\tilde\rho(A)\tilde\rho(B)\tilde\rho(C)\tilde\rho(D) = \pm\mathbb{I}.$$

We claim that $\tilde\rho(A)\tilde\rho(B)\tilde\rho(C)\tilde\rho(D) = \mathbb{I}$. Since

$$\begin{aligned}\mathsf{tr}\big(\tilde\rho(A)\big) &\geq 2\\ \mathsf{tr}\big(\tilde\rho(B)\big) &\geq 2\\ \mathsf{tr}\big(\tilde\rho(C)\big) &\geq 2\\ \mathsf{tr}\big(\tilde\rho(D)\big) &\geq 2,\end{aligned}$$

each of $\tilde\rho(A), \tilde\rho(B), \tilde\rho(C), \tilde\rho(D)$ lies in a unique one-parameter subgroup of $\mathsf{SL}(2,\mathbb{R})$. The corresponding embeddings define trivializations of the corresponding flat $\mathsf{PSL}(2,\mathbb{R})$-bundle over each component of $\partial\Sigma$ as in Goldman [22, 26]. Namely, since each component $\partial_i(\Sigma)$ is a closed 1-manifold, lifting a homeomorphism $\mathbb{R}/\mathbb{Z} \longrightarrow \partial_i(\Sigma)$ to

$$\mathbb{R} \longrightarrow \widetilde{\partial_i(\Sigma)}$$

the flat bundle over $\partial_i(\Sigma)$ with holonomy γ_i lifts to the quotient of the trivial principal $\mathsf{PSL}(2,\mathbb{R})$-bundle $\mathbb{R}\times\mathsf{PSL}(2,\mathbb{R})$ by the $\mathbb{Z}$-action generated by

$$(t,g)\longmapsto(t+1,\gamma_i g).$$

The corresponding trivialization is covered by the $\mathbb{Z}$-equivariant isomorphism

$$(t,g)\longmapsto\Big(t,\exp\big(-t\log(\gamma_i)g\big)\Big)$$

where

$$\Big\{\exp\big(t\log(\gamma_i)\big)\Big\}_{t\in\mathbb{R}}$$

is the unique one-parameter subgroup of $\mathsf{PSL}(2,\mathbb{R})$ containing γ_i as above.

Since $\chi(\Sigma)=-2$, the Euler class of the representation ρ equals -2 and is even. The obstruction to lifting a representation to the double covering space

$$\mathsf{SL}(2,\mathbb{R})\longrightarrow\mathsf{PSL}(2,\mathbb{R})$$

is the second Stiefel-Whitney class, which is the reduction of the Euler class modulo 2. Therefore $\tilde\rho$ defines a representation and $\tilde\rho(A)\tilde\rho(B)\tilde\rho(C)\tilde\rho(D)=\mathbb{I}$ as claimed.

Furthermore, if ρ is a Fuchsian representation, then X is represented by a unique closed geodesic on Σ and

$$\rho(X)=\rho(A)\rho(B)$$

is hyperbolic. The relative Euler classes of the restriction of ρ to the subsurfaces complementary to X sum to ±2. Since they are constrained to equal $-1,0,+1$, they both must be equal to $+1$ or both equal to -1. (Compare [22, 26].) It follows that the trace $x=\mathsf{tr}(X)<-2$.

We study (5.2.1) using the following identity:

$$\begin{aligned}
4(4-x^2)\Big\{&x^2+y^2+z^2+xyz\\
&-\big((ab+cd)x+(ad+bc)y+(ac+bd)z\big)\\
&+\big(a^2+b^2+c^2+d^2+abcd-4\big)\Big\}\\
=(2+x)&\Big\{(y-z)(2-x)+(a-b)(c-d)\Big\}^2\\
&+(2-x)\Big\{(y+z)(2+x)-(a+b)(c+d)\Big\}^2\\
&-4\kappa_{a,b}(x)\kappa_{c,d}(x).
\end{aligned}\tag{5.2.3}$$

where

$$\kappa_{p,q}(x) := x^2 + p^2 + q^2 - pqx - 4. \tag{5.2.4}$$

This function equals $\kappa(p, q, x) - 2$, where κ is the commutator trace function defined in (2.2.8).

When $x \neq \pm 2$, rewrite (5.2.1) using (5.2.3) as follows:

$$\begin{aligned} \frac{2+x}{4}\left((y+z) - \frac{(a+b)(d+c)}{2+x}\right)^2 & \\ + \frac{2-x}{4}\left((y-z) - \frac{(a-b)(d-c)}{2-x}\right)^2 &. \\ = \frac{\kappa_{a,b}(x)\kappa_{c,d}(x)}{4-x^2} & \end{aligned} \tag{5.2.5}$$

(Compare (3-3) of Benedetto-Goldman [2].)

We fix $a, b, c, d \geq 2$. As x varies, (5.2.5) defines a family of conics parametrized by x. For $x < -2$, this conic is a hyperbola, denoted $H_{a,b,c,d;x}$. The solutions of (5.2.5) for $a, b, c, d \geq 2$ and $x < -2$ fall into two connected components corresponding to the two components of the hyperbolas.

We explicitly describe these components. First observe that if $a, b \geq 2$ and $x < -2$, then

$$\kappa_{a,b}(x) > 16, \qquad \kappa_{c,d}(x) > 16, \qquad 4 - x^2 < 0$$

so the left-hand side (5.2.5) is negative. For notational simplicity denote its opposite by $k = \kappa_{a,b,c,d;x}$:

$$k = \kappa_{a,b,c,d;x} := \frac{\kappa_{a,b}(x)\kappa_{c,d}(x)}{x^2 - 4} \; > \; 0.$$

Rewrite (5.2.5) as:

$$\begin{aligned} \frac{2-x}{4}\left((y-z) - \frac{(a-b)(d-c)}{2-x}\right)^2 & \\ - \frac{-2-x}{4}\left((y+z) - \frac{(a+b)(d+c)}{2+x}\right)^2 &= k \end{aligned}$$

Factoring the left-hand side of this equation, rewrite (5.2.5) as:

$$F^+(y,z)\, F^-(y,z) \; = \; k \tag{5.2.6}$$

where the functions $F^{\pm}(y,z)$ are defined as:

$$F^-(y,z) = F^-_{a,b,c,d;x}(y,z) := \sqrt{2-x}\left(y - z - \frac{(a-b)(d-c)}{2-x}\right) - \sqrt{-2-x}\left(y + z - \frac{(a+b)(d+c)}{-2-x}\right). \quad (5.2.7)$$

$$F^+(y,z) = F^+_{a,b,c,d;x}(y,z) := \sqrt{2-x}\left(y - z - \frac{(a-b)(d-c)}{2-x}\right) + \sqrt{-2-x}\left(y + z - \frac{(a+b)(d+c)}{-2-x}\right) \quad (5.2.8)$$

For fixed $a,b,c,d \geq 2$ and $x < -2$, the functions $F^{\pm}(y,z)$ are affine functions of y,z.

We identify each of the two components of the hyperbola $H_{a,b,c,d;x}$ defined by (5.2.6). One component, denoted $H^+_{a,b,c,d;x}$, is the intersection of $H_{a,b,c,d;x}$ with the open half plane

$$F^-_{a,b,c,d;x}(y,z) > 0.$$

Equivalently, $H^+_{a,b,c,d;x}$ is the intersection of $H_{a,b,c,d;x}$ with the open half-plane

$$F^+_{a,b,c,d;x}(y,z) > 0.$$

Similarly the other component $H^-_{a,b,c,d;x}$ is the intersection of $H_{a,b,c,d;x}$ with the open half-plane

$$F^-_{a,b,c,d;x}(y,z) < 0,$$

or, equivalently,

$$F^+_{a,b,c,d;x}(y,z) < 0.$$

The union of these hyperbola components correspond to values of the relative Euler class (compare [26]) as follows. Either

$$H^+ := \bigcup_{a,b,c,d \geq 2, x < -2} H^+_{a,b,c,d;x}$$

or

$$H^- := \bigcup_{a,b,c,d \geq 2, x < -2} H^-_{a,b,c,d;x}$$

corresponds to characters of representations with relative Euler class 0. The other component corresponds to representations with relative Euler class ± 1. There is no way to distinguish between relative Euler class $+1$ and -1 since the characters are equivalence classes under the group $\mathsf{PGL}(2,\mathbb{R})$, which does not preserve orientation. To determine which one is which, it suffices to check one single example and use continuity of the integer-valued relative Euler class.

Here is an example whose relative Euler class is zero. Choose

$$\rho(A) = \rho(D)^{-1}, \qquad \rho(B) = \rho(C)^{-1}$$

so that the relation

$$\rho(A)\rho(B)\rho(C)\rho(D) = \mathbb{I}$$

is trivially satisfied. Clearly such a representation depends only on the pair $\rho(A), \rho(B)$ which is arbitrary. Furthermore we restrict the boundary traces to satisfy:

$$a = \mathsf{tr}\big(\rho(A)\big) \geq 2, \qquad b = \mathsf{tr}\big(\rho(B)\big) \geq 2.$$

This space is connected and contains the character of the trivial representation, whose relative Euler class is zero. Now consider the specific example:

$$\rho(A) := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \rho(B) := \begin{bmatrix} 1 & 0 \\ x-2 & 1 \end{bmatrix}$$

where $x < -2$ is arbitrary. Then $a = b = c = d = 2$ and

$$y = 2, \qquad z = 4 - x.$$

In particular $y - z < 0$ and $y + z > 0$. and therefore (5.2.7) implies

$$F^{\pm}_{a,b,c,d;x} < 0$$

proving that this representation has a character in H^-, proving Theorem 5.2.1. □

5.3 The two-holed torus.

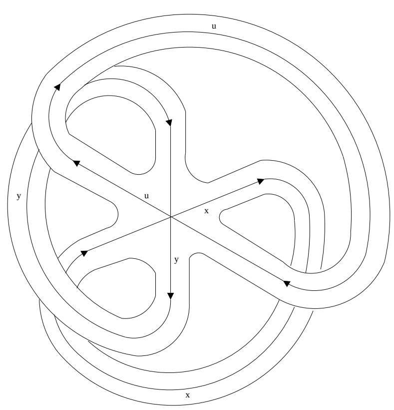

Figure 9. A ribbon graph representing a 2-holed torus

The two-holed torus admits a redundant geometric presentation corresponding to the ribbon graph depicted in Figure 9:

$$\langle A, B, U, X, Y \mid A = UXY, \quad B = UYX \rangle.$$

The 1-handles correspond to free generators U, X, Y and the boundary components correspond to the triple products

$$A = UXY,$$
$$B = UYX.$$

Since the curves corresponding to X, Y, U in Figure 9 intersect transversely at the basepoint, the double products

$$V = UX,$$
$$W = UY,$$
$$Z = XY$$

are represented by simple loops as well. The Sum Relation (5.1.1) and the Product Relation (5.1.2) imply that the relative character variety for $\Sigma_{1,2}$ is defined by

$$\begin{aligned} a + b &= yv + xw + zu - uxy \\ ab &= x^2 + y^2 + u^2 + v^2 + w^2 + z^2 - xyz - yuw - uxv + vwz - 4. \end{aligned} \tag{5.3.1}$$

Button [9] gives defining inequalities for the Fricke space, where the boundary components are mapped to parabolics, as follows. First consider the $\mathbb{R}$-locus of the character variety, defined as the set of all $(a, b; u, x, y, z, v, w) \in \mathbb{R}^8$ satisfying (5.3.1) above, and

$$a = b = 2.$$

The regular neighborhood of the union of the loops corresponding to a pair of X, Y, U is an embedded one-holed torus. For example corresponding to the pair X, Y is a one-holed torus whose boundary corresponds to the commutator $[X, Y]$, and cuts Σ into the one-holed torus and a three-holed sphere. This commutator has trace $\kappa(x, y, z)$. Similarly the pair Y, U determines a separating curve whose corresponding trace function is $\kappa(y, u, w)$ and the pair U, X determines a separating curve whose corresponding trace function is $\kappa(u, x, v)$. The preceding discussions of the Fricke spaces of the one-holed torus and the three-holed sphere imply:

$$\kappa(x, y, z) \ < \ -2, \quad \kappa(y, u, w) \ < \ -2, \quad \kappa(u, x, v) \ < \ -2.$$

Button [9] shows that these necessary conditions are sufficient, thus obtaining an explicit description of the Fricke-Teichmüller space of $\Sigma_{1,2}$ in terms of traces. (The reader should draw these curves on the ribbon graph depicted in Figure 9.)

5.4 Orientable double covering spaces.

Let Σ be a nonorientable surface of $\chi(\Sigma) = -1$ and $\hat{\Sigma} \xrightarrow{\hat{\Pi}} \Sigma$ be its orientable covering space. There are two cases:

- $\Sigma \approx C_{0,2}$ and $\hat{\Sigma} \approx \Sigma_{0,4}$;
- $\Sigma \approx C_{1,1}$ and $\hat{\Sigma} \approx \Sigma_{1,2}$.

Then $\pi_1(\Sigma) \cong \mathbb{F}_2$ and $\pi_1(\hat{\Sigma}) \cong \mathbb{F}_3$. Denote a set of free generators of $\pi_1(\Sigma)$ by X_1, X_2 which correspond to orientation-reversing loops on Σ. The image of

$$\pi_1(\hat{\Sigma}) \overset{\hat{\Pi}_*}{\hookrightarrow} \pi_1(\Sigma) = \langle X_1, X_2 \rangle$$

equals the kernel of the homomorphism

$$\begin{aligned} \pi_1(\Sigma) &\longrightarrow \{\pm 1\} \\ X_1 &\longmapsto -1 \\ X_2 &\longmapsto -1, \end{aligned}$$

which is freely generated by, for example,

$$\begin{aligned} Y_1 &= X_1^2 \\ Y_2 &= X_1^{-1} X_2^{-1} \\ Y_3 &= X_2^2. \end{aligned}$$

The deck transformation of $\hat{\Sigma} \xrightarrow{\hat{\Pi}} \Sigma$ is induced by the restriction of the inner automorphism $\mathsf{Inn}(X_1)$ to $(\hat{\Pi})_*\big(\pi_1(\Sigma)\big)$:

$$\begin{aligned} Y_1 &\longmapsto Y_1 \\ Y_2 &\longmapsto Y_3^{-1} Y_2^{-1} Y_1^{-1} \\ Y_3 &\longmapsto Y_1 Y_2 Y_3 Y_2^{-1} Y_1^{-1}. \end{aligned}$$

The character ring of $\pi_1(\Sigma) \cong \mathbb{F}_2$ is the polynomial ring $\mathbb{C}[x_1, x_2, x_{12}]$. The character ring of $\pi_1(\hat{\Sigma}) \cong \mathbb{F}_3$ is the quotient

$$\mathbb{C}[y_1, y_2, y_3, y_{123}, y_{12}, y_{23}, y_{13}]/(\mathfrak{I})$$

where $(\mathfrak{I})$ is the principal ideal generated by

$$\begin{aligned} &\Phi(y_1, y_2, y_3, y_{123}, y_{12}, y_{23}, y_{13}) \\ &\quad = y_{12}^2 + y_{13}^2 + y_{23}^2 + y_{12} y_{13} y_{23} \\ &\qquad - (y_1 y_2 + y_3 y_{123}) y_{12} - (y_1 y_{123} + y_2 y_3) y_{23} \\ &\qquad - (y_1 y_3 + y_2 y_{123}) y_{13} + y_1^2 + y_2^2 + y_3^2 + y_{123}^2 + y_1 y_2 y_3 y_{123} - 4 \end{aligned} \tag{5.4.1}$$

where Φ is the polynomial (5.2.2). The automorphism

$$\mathsf{Inn}(X_1)|_{(\hat{\Pi})_*\big(\pi_1(\Sigma)\big)}$$

corresponding to the deck transformation induces the involution of character rings:

$$\begin{aligned}
\mathfrak{R}_3 &\leftrightarrow \mathfrak{R}_3 \\
y_1 &\leftrightarrow y_1 \\
y_2 &\leftrightarrow y_{123} \\
y_3 &\leftrightarrow y_3 \\
y_{12} &\leftrightarrow y_{23} \\
y_{13} &\leftrightarrow y_1 y_3 - y_{13} - y_{12} y_{23} + y_{123} y_2 \\
y_{23} &\leftrightarrow y_{12} \\
y_{123} &\leftrightarrow y_2
\end{aligned}$$

The covering space $\hat{\Pi}$ induces the embedding of character rings

$$\begin{aligned}
\mathfrak{R}_2 &\hookrightarrow \mathfrak{R}_3 \\
y_1 &\longmapsto x_1^2 - 2 \\
y_2 &\longmapsto x_{12} \\
y_3 &\longmapsto x_2^2 - 2 \\
y_{12} &\longmapsto x_1 x_2 - x_{12} \\
y_{13} &\longmapsto x_1 x_2 x_{12} - x_1^2 - x_2^2 + 2 \\
y_{23} &\longmapsto x_1 x_2 - x_{12} \\
y_{123} &\longmapsto x_{12}
\end{aligned}$$

The two topological types for Σ differ by their choice of *peripheral structure:*

- $\Sigma \approx C_{0,2}$ has two boundary components corresponding to:

$$\delta_1 := Y_2 = X_1^{-1} X_2^{-1},\ \delta_2 = Y_1 Y_2 = X_1 X_2^{-1}.$$

- $\Sigma \approx C_{1,1}$ has one boundary component corresponding to

$$\delta := Y_1 Y_3 = X_1^2 X_2^2.$$

5.5 The two-holed cross-surface.

The fundamental group of the two-holed cross-surface $C_{0,2}$ is free of rank two, with presentation

$$\pi_1(C_{0,2}) \ := \ \langle U, V, W, W' \ \mid \ W = UV,\, W' = V^{-1}U \rangle \cong \mathbb{F}_2.$$

The free generators U, V correspond to orientation-reversing simple curves on $C_{0,2}$ and W, W' correspond to the components of $\partial C_{0,2}$.

The orientable double covering-space $\widehat{C_{0,2}} \longrightarrow C_{0,2}$ is connected, has four boundary components (since $C_{0,2}$ has two boundary components, each of which is orientable) and has Euler characteristic $-2 = 2\chi(C_{0,2})$. Therefore $\widehat{C_{0,2}} \approx \Sigma_{0,4}$, the four-holed sphere, and has presentation

$$\pi = \langle A, B, C, D \mid ABCD = \mathbb{I} \rangle.$$

The corresponding monomorphism of fundamental groups is:

$$\begin{aligned} \pi_1(\Sigma_{0,4}) &\hookrightarrow \pi_1(C_{0,2}) \\ A &\longmapsto W = UV \\ B &\longmapsto W' = V^{-1}U \\ C &\longmapsto \mathsf{Inn}(U^{-1})(W')^{-1} = U^{-2}VU \\ D &\longmapsto \mathsf{Inn}(U^{-1})(W)^{-1} = U^{-1}V^{-1}. \end{aligned}$$

The character ring of $\pi_1(C_{0,2})$ is the polynomial ring $\mathfrak{R}_2 \cong \mathbb{C}[u, v, w]$ and the character ring of $\pi_1(\Sigma_{0,4})$ is the quotient of $\mathbb{C}[a, b, c, d, x, y, z]$ by the relation defined by (5.2.1). The induced homomorphism of character rings is:

$$\begin{aligned} \mathfrak{R}_3 &\longrightarrow \mathfrak{R}_2 = \mathbb{C}[u, v, w] \\ a &\longmapsto w \\ b &\longmapsto uv - w = w' \\ c &\longmapsto uv - w = w' \\ d &\longmapsto w \\ x &\longmapsto u^2 - 2 \\ y &\longmapsto u^2 + v^2 + w^2 - uvw - 2 \\ z &\longmapsto v^2 - u^2\big(u^2 + v^2 + w^2 - uvw - 2\big) - 2, \end{aligned}$$

evidently satisfying the defining equation (5.2.1) for the character variety of $\Sigma_{0,4}$.

5.6 The one-holed Klein bottle.

The fundamental group of the one-holed Klein bottle $C_{1,1}$ is free of rank two, with presentation

$$\pi_1(C_{1,1}) \;:=\; \langle P, Q, R, D \;\mid\; PQR = P^2Q^2D = \mathbb{I} \rangle \cong \mathbb{F}_2.$$

The free generators P, Q correspond to orientation-reversing simple curves on $C_{1,1}$ and D corresponds to $\partial C_{1,1}$.

The orientable double covering-space $\widehat{C_{1,1}} \longrightarrow C_{1,1}$ is connected, has two boundary components (since $\partial C_{1,1}$) is connected and orientable) and has Euler characteristic $-2 = 2\chi(C_{1,1})$. Therefore $\widehat{C_{0,2}} \approx \Sigma_{1,2}$, the two-holed torus, and its fundamental group has presentation

This covering-space $\Sigma_{1,2} \longrightarrow C_{1,1}$ induces the monomorphism

$$\begin{aligned} \pi_1(\Sigma_{1,2}) &\hookrightarrow \pi_1(C_{1,1}) \\ U &\longmapsto PQ \\ X &\longmapsto QP^{-1} \\ Y &\longmapsto P^2 \\ A &\longmapsto PQ^2P \sim P^2Q^2 \\ B &\longmapsto PQP^2QP^{-1} \sim P^2Q^2 \end{aligned}$$

where $\pi_1(\Sigma_{1,2})$ is presented as:

$$\langle A, B, U, X, Y \mid A = UXY, \quad B = UYX \rangle.$$

The character ring of $\pi_1(C_{1,1})$ is the polynomial ring $\mathfrak{R}_2 \cong \mathbb{C}[p, q, r]$ and the character ring of $\pi_1(\Sigma_{1,2})$ is the quotient of $\mathbb{C}[a, b, x, y, z, u, v, w]$ by the relations defined by (5.3.1). The covering space $\Sigma_{1,2} \longrightarrow C_{1,1}$ induces the homomorphism of character rings:

$$\begin{aligned} \mathfrak{R}_3 &\longrightarrow \mathfrak{R}_2 = \mathbb{C}[p, q, r] \\ u &\longmapsto r \\ x &\longmapsto pq - r \\ y &\longmapsto p^2 - 2 \\ v &\longmapsto q^2 - 2 \\ w &\longmapsto r \\ z &\longmapsto p(pr - q) - r \\ a &\longmapsto 2 - p^2 - q^2 + pr \\ b &\longmapsto 2 - p^2 - q^2 + pr \end{aligned}$$

which evidently satisfies the relations of (5.3.1).

We briefly give a geometric description of the deck transformation of the double covering of the $\mathsf{SL}(2, \mathbb{C})$-character variety V_3 of the rank three free group $\mathbb{F}_3$.

Consider the elliptic involution ι of the torus $\Sigma_{1,0}$. Writing $\Sigma_{1,0}$ as the quotient $\mathbb{R}^2/\mathbb{Z}^2$, this involution is induced by the map

$$\begin{aligned} \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ u &\longmapsto -u. \end{aligned}$$

This involution has four fixed points, and its quotient orbifold is S^2 with four branch points of order two. Choose a small disc $D \subset \Sigma_{1.0}$ such that D and its image $\iota(D)$ are disjoint. Then ι induces an involution on the complement

$$\Sigma_{1,0} \setminus (D \cup \iota(D)) \;\approx\; \Sigma_{1,2}.$$

This involution of the two-holed torus $\Sigma_{1,2}$ induces the involution of $\pi_1(\Sigma_{1,2}) = \langle U, X, Y\rangle$:

$$\begin{aligned} U &\longmapsto U^{-1} \\ X &\longmapsto X^{-1} \\ Y &\longmapsto Y^{-1}. \end{aligned}$$

The quotient orbifold is a disc with four branch points of order two. The corresponding involution of character varieties is the branched double covering (5.1.4) of the character variety V_3 over $\mathbb{C}^6$ described in Proposition 5.1.1.

5.7 Free groups of rank ≥ 3.

The basic trace identity (2.2.5), the Sum Relation (5.1.1) and the Product Relation (5.1.2) imply that the trace polynomial f_w of any word $w(X_1, X_2, \dots, X_n)$ can be written in terms of trace polyomials of words

$$X_{i_1} X_{i_2} \dots X_{i_1} X_{i_r}$$

where $1 \leq i_1 < i_2 < \cdots < i_r \leq n$. The following identity, which may be found in Vogt [74], implies that it suffices to choose $r \leq 3$:

$$\begin{aligned} 2\, t_{1234} \;=\; & t_1 t_2 t_3 t_4 + t_1 t_{234} + t_2 t_{341} + t_3 t_{412} + t_4 t_{123} \\ & + t_{12} t_{34} + t_{41} t_{23} - t_{13} t_2 t_{24} - t_1 t_2 t_{34} - t_{12} t_3 t_4 - t_4 t_1 t_{23} - t_{41} t_2 t_3 \end{aligned}$$

The $\mathsf{SL}(2,\mathbb{C})$-character variety of a rank n free group has dimension $3n - 3$, as it corresponds to the quotient of the $3n$-dimensional complex manifold $\mathsf{SL}(2,\mathbb{C})^n$ by the generically free action of the 3-dimensional group $\mathsf{PGL}(2,\mathbb{C})$. Thus the transcendence degree of the field of fractions of the *character ring* equals $3n - 3$. In contrast, the above discussion implies that this ring has

$$n + \binom{n}{2} + \binom{n}{3} = \frac{n(5 + n^2)}{6}$$

generators, considerably larger than the dimension $3n - 3$ of the character variety.

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