THE MAPPING CLASS GROUP ACTS REDUCIBLY ON $\mathsf{SU}(n)$-CHARACTER VARIETIES

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ABSTRACT. When G is a connected compact Lie group, and π is a closed surface group, then $\mathsf{Hom}(\pi, G)/G$ contains an open dense $\mathsf{Out}(\pi)$-invariant subset which is a smooth symplectic manifold. This symplectic structure is $\mathsf{Out}(\pi)$-invariant and therefore defines an invariant measure μ, which has finite volume. The corresponding unitary representation of $\mathsf{Out}(\pi)$ on $L^2(\mathsf{Hom}(\pi, G)/G, \mu)$ contains no finite-dimensional subrepresentations besides the constants. This note gives a short proof that when $G = \mathsf{SU}(n)$, the representation $L^2(\mathsf{Hom}(\pi, G)/G, \mu)$ contains many other invariant subspaces.

Let $G = \mathsf{SU}(n)$ and π be the fundamental group of a closed oriented surface Σ. Let $\mathsf{Hom}(\pi, G)$ be the space of representations $\pi \longrightarrow G$. The group $\mathsf{Aut}(\pi) \times \mathsf{Aut}(G)$ acts on $\mathsf{Hom}(\pi, G)$. Let $\mathsf{Hom}(\pi, G)/G$ be the quotient of $\mathsf{Hom}(\pi, G)$ by $\{1\} \times \mathsf{Inn}(G)$. Then $\mathsf{Out}(\pi) := \mathsf{Aut}(\pi)/\mathsf{Inn}(\pi)$ acts on $\mathsf{Hom}(\pi, G)/G$. The $\mathsf{Out}(\pi)$-action preserves a symplectic structure on $\mathsf{Hom}(\pi, G)/G$ which determines a finite invariant smooth measure μ (on an invariant dense open subset which is a smooth manifold). When G is compact, the total measure is finite (Jeffrey-Weitsman [6, 7], Huebschmann [5]). There results a unitary representation of $\mathsf{Out}(\pi)$ on the Hilbert space

$$\mathcal{H} := L^2(\mathsf{Hom}(\pi, G)/G, \mu).$$

Let $C \subset \mathcal{H}$ denote the subspace corresponding to the constant functions.

The following theorem is proved in Goldman [3] for $n = 2$ and Pickrell-Xia [8, 9] in general:

Theorem. *The only finite dimensional* $\mathsf{Out}(\pi)$*-invariant subspace of* $\mathcal{H}$ *is* C.

According to [3], the only finite-dimensional invariant subspace of $\mathcal{H}$ consists of constants. Let $\mathcal{H}_0$ denote the orthocomplement of the constants in $\mathcal{H}$, that is, the set of $f \in \mathcal{H}$ such that $\int f d\mu = 0$.

The goal of this note is an elementary proof of the following:

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Theorem. *The representation of* $\mathsf{Out}(\pi)$ *on* $\mathcal{H}_0$ *is reducible.*

In general for a compact connected Lie group G, the components of $\mathsf{Hom}(\pi, G)$ and $\mathsf{Hom}(\pi, G)/G$ are indexed by $\pi_1(G')$ where G' is the commutator subgroup. In that case the ergodicity/mixing results of [3, 8, 9] imply that the only invariant finite dimensional subspaces are subspaces of the space of locally constant functions, a vector space of dimension $|vert\pi_1(G')|$. The method of proving reducibility requires a nontrivial element of the center of G. For simplicity, we only discuss the case $G = \mathsf{SU}(n)$ in this paper.

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1. The center of $\mathsf{SU}(n)$

Let $\mathfrak{Z} \cong \mathbb{Z}/n$ denote the center of G, the group consisting of all scalar matrices $\zeta\mathbb{I}$ where $\zeta^n = 1$. Then $\mathsf{Hom}(\pi, \mathfrak{Z}) \cong H^1(M; \mathbb{Z}/n)$ acts on $\mathsf{Hom}(\pi, G)/G$ by pointwise multiplication: If $\rho \in \mathsf{Hom}(\pi, G)$ and $u \in \mathsf{Hom}(\pi, \mathfrak{Z})$, then define the action $u \cdot \rho$ of u on ρ by:

$$\gamma \xmapsto{u\cdot\rho} \rho(\gamma)u(\gamma). \tag{1}$$

Recall the definition [2] of the symplectic structure on $\mathsf{Hom}(\pi, G)/G$. Suppose that $\rho \in \mathsf{Hom}(\pi, G)$ is an irreducible representation. By Weil [11] (compare also Raghunathan [10]), the Zariski tangent space to $\mathsf{Hom}(\pi, G)$ at ρ identifies with the space $Z^1(\pi, \mathfrak{g}_{\mathsf{Ad}\rho})$ of cocycles where $\mathfrak{g}_{\mathsf{Ad}\rho}$ is the π-module defined by the composition

$$\pi \xrightarrow{\rho} G \xrightarrow{\mathsf{Ad}} \mathsf{Aut}(\mathfrak{g})$$

and $\mathfrak{g}$ is the Lie algebra of G. Since ρ is irreducible, G acts locally freely and $\mathsf{Hom}(\pi, G)/G$ has a smooth structure in a neighborhood of $[\rho]$. The tangent space to the orbit $G\cdot\rho$ equals the space of coboundaries $B^1(\pi, \mathfrak{g}_{\mathsf{Ad}\rho})$. The tangent space to $\mathsf{Hom}(\pi, G)/G$ at $[\rho]$ identifies with the cohomology group $H^1(\pi, \mathfrak{g}_{\mathsf{Ad}\rho})$.

Let $u \in \mathsf{Hom}(\pi, \mathfrak{Z})$. Since $\mathsf{Ad}(\mathfrak{Z})$ is trivial, the action of u induces an identification of π-modules $\mathfrak{g}_{\mathsf{Ad}\rho} \longrightarrow \mathfrak{g}_{\mathsf{Ad}(u\cdot\rho)}$ and hence of tangent spaces

$$\begin{aligned} T_{[\rho]}\mathsf{Hom}(\pi, G)/G = H^1(\pi, \mathfrak{g}_{\mathsf{Ad}\rho}) \\ \longrightarrow \quad T_{[u\cdot\rho]}\mathsf{Hom}(\pi, G)/G = H^1(\pi, \mathfrak{g}_{\mathsf{Ad}(u\cdot\rho)}). \end{aligned}$$

The symplectic form ω_ρ at $[\rho]$ is defined by the cup product

$$H^1(\pi, \mathfrak{g}_{\mathsf{Ad}\rho}) \times H^1(\pi, \mathfrak{g}_{\mathsf{Ad}\rho}) \longrightarrow H^2(\pi; \mathbb{R})$$

using the pairing of π-modules induced by an Ad-invariant nondegenerate symmetric bilinear form on $\mathfrak{g}$ as a coeefficient pairing. Evidently the symplectic form ω, and therefore the corresponding measure μ, are $\mathsf{Hom}(\pi, \mathfrak{Z})$-invariant.

2. Trace functions

Suppose that $\gamma \in \pi$. The function

$$\begin{aligned} \mathsf{Hom}(\pi, G) &\longrightarrow \mathbb{C} \\ \rho &\longmapsto \mathsf{tr}(\rho(\gamma)) \end{aligned}$$

is $\mathsf{Inn}(G)$-invariant and defines a function

$$\mathsf{Hom}(\pi, G)/G \xrightarrow{t_\gamma} \mathbb{C}.$$

We extend this definition to products of more than one element of π: Let $\gamma = (\gamma_1, \dots, \gamma_s) \in \pi^s$. Define the *trace function* of γ as the product:

$$t_\gamma([\rho]) := t_{\gamma_1}([\rho]) \dots t_{\gamma_s}([\rho])$$

Using the definition (1) of the action of $\mathsf{Hom}(\pi, \mathfrak{Z})$ on $\mathsf{Hom}(\pi, G)/G$, the action of $u \in \mathsf{Hom}(\pi, \mathfrak{Z})$ on the trace function t_γ is given by $u \cdot t_\gamma$, defined by:

$$\rho \xmapsto{u \cdot t_\gamma} u(\gamma)^{-1} t_\gamma\big(\rho\big) \tag{2}$$

since

$$\begin{aligned} \big(u \cdot t_\gamma\big)\big(\rho\big) &:= t_\gamma(u^{-1} \cdot \rho) \\ &= \mathsf{tr}\big((u^{-1} \cdot \rho)(\gamma)\big) \\ &= \mathsf{tr}\big(u(\gamma)^{-1} \rho(\gamma)\big) \\ &= u(\gamma)^{-1} \mathsf{tr}(\rho(\gamma)) \\ &= u(\gamma)^{-1} t_\gamma\big(\rho\big) \end{aligned}$$

Since $\mathfrak{Z} \cong \mathbb{Z}/n$, the evaluation $u(\gamma)$ depends only on the *total homology class*

$$[\gamma] := [\gamma_1] + \dots + [\gamma_s] \in H_1(\Sigma; \mathbb{Z}/n),$$

defined as the sum of the $\mathbb{Z}/n$-homology classes of the γ_i. The evaluation of $u \in \mathsf{Hom}(\pi, \mathfrak{Z})$ on γ is just the natural pairing

$$\begin{aligned} \mathsf{Hom}(\pi, \mathfrak{Z}) \times H_1(\Sigma; \mathbb{Z}/n) &\longrightarrow \mathbb{Z}/n \cong Z \\ (u, [\gamma]) &\longmapsto u \cdot [\gamma] \end{aligned}$$

where $[\gamma]$ is the total homology class of γ.

Lemma 1. *If* $[\gamma] \neq 0 \in H_1(\Sigma; \mathbb{Z}/n)$, *then*

$$\int t_\gamma d\mu = 0.$$

Proof. Since the homology class of γ in $H_1(\Sigma, \mathbb{Z}/n)$ is nonzero, $u(\gamma) \neq 1$ for some $u \in \mathsf{Hom}(\pi, \mathfrak{Z})$. Since $(u^{-1})_* \mu = \mu$,

$$\int t_\gamma \, d\mu \ = \ \int (t_\gamma \circ u^{-1}) d(u^{-1})_* \mu = u(\gamma)^{-1} \int t_\gamma d\mu$$

by (2). Since $u(\gamma)^{-1} \neq 1$, the integral is zero as claimed. □

An element $\gamma \in \pi$ is *nonseparating simple* if it contains a nonseparating simple loop which is homotopically nontrivial and not homotopic to $\partial\Sigma$. Since such an element has nontrivial homology class $[\gamma] \in H_1(\Sigma, \mathbb{Z}/n)$, Lemma 1 implies:

Corollary 2. *Suppose* γ *is nonseparating simple. Then* $t_\gamma \in \mathcal{H}_0$.

For $n = 2$, Frohman and Kania-Bartoszynska [1] have calculated $\int t_\gamma d\mu$ for separating simple loops γ in terms of Bernoulli numbers and $6j$-symbols.

3. Invariant subspaces

Let $\mathcal{W} \subset \mathcal{H}_0$ denote the closure of the span of all t_α, where α is nonseparating simple loops. (Alternatively, let α run over all elements of π with nonzero homology class modulo n.) Clearly $\mathcal{W}$ is $\mathsf{Out}(\pi)$-invariant.

Proposition 3. $\mathcal{W}$ *is a proper subspace of* $\mathcal{H}_0$.

Proof. Choose $\gamma \in \pi^s$ such that:

- t_γ is not constant;
- γ has trivial homology class in $H_1(\Sigma, \mathbb{Z}/n)$.

Such elements are easy to find: for example if γ consists of one nontrivial separating simple loop, or if γ consists of one n-th power, or if $\gamma = (\gamma_1, \ldots, \gamma_1)$ where s is a positive multiple of n and γ_1 is nontrivial.

Normalize t_γ so that it lies in $\mathcal{H}_0$:

$$\hat{t}_\gamma := t_\gamma - \int t_\gamma d\mu.$$

We claim that $\hat{t}_\gamma$ is a nonzero vector in $\mathcal{H}_0$ orthogonal to $\mathcal{W}$. It is nontrivial since t_γ is nonconstant. Furthermore if $\alpha \in \pi$ has nonzero homology class $[\alpha] \in H_1(\Sigma, \mathbb{Z}/n)$ (for example if it nonseparating simple), then t_α is orthogonal to $\hat{t}_\gamma$.

Since $\rho(\alpha)$ is unitary, $\overline{t_\alpha} = t_{\alpha^{-1}}$. Thus

$$\begin{aligned}\langle \hat{t}_\gamma, t_\alpha \rangle &= \int \hat{t}_\gamma \overline{t_\alpha} d\mu = \int \hat{t}_\gamma t_{\alpha^{-1}} d\mu \\ &= \int t_\gamma t_{\alpha^{-1}} d\mu \text{ (by Corollary 2)} \\ &= \int t_\eta d\mu\end{aligned}$$

where $\eta = (\gamma_1, \ldots, \gamma_s, \alpha^{-1})$. Since

$$[\eta] = [\gamma] - [\alpha] \neq 0,$$

Lemma 1 implies that $\langle \hat{t}_\gamma, t_\alpha \rangle = 0$, as desired. □

When π is a free group and $n = 2$, then ergodicity and weak-mixing of the action of $\mathsf{Out}(\pi)$ on $\mathsf{Hom}(\pi, G)/G$ is proved in Goldman [4]. Exactly the same argument as above (with $\mathcal{W}$ replaced by the closure of the span of t_γ where γ is an element of a free generating set of π) shows that the unitary representation of $\mathsf{Out}(\pi)$ on $L^2(\mathsf{Hom}(\pi, G)/G, \mu)$ contains invariant (necessarily infinite-dimensional) subspaces other than the constants.

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