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Invariant functions on Lie groups and Hamiltonian flows of surface group representations

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In [7] it was shown that if π is the fundamental group of a closed oriented surface S and G is Lie group satisfying very general conditions, then the space $\mathrm{Hom}(\pi, G)/G$ of conjugacy classes of representation $\pi \to G$ has a natural symplectic structure. This symplectic structure generalizes the Weil-Petersson Kähler form on Teichmüller space (taking $G = PSL(2, \mathbb{R})$), the cup-product linear symplectic structure on $H^1(S, \mathbb{R})$ (when $G = \mathbb{R}$), and the Kähler forms on the Jacobi variety of a Riemann surface M homeomorphic to S (when $G = U(1)$) and the Narasimhan-Seshadri moduli space of semistable vector bundles of rank n and degree 0 on M (when $G = U(n)$). The purpose of this paper is to investigate the geometry of this symplectic structure with the aid of a natural family of functions on $\mathrm{Hom}(\pi, G)/G$.

The inspiration for this paper is the recent work of Scott Wolpert on the Weil-Petersson symplectic geometry of Teichmüller space [18–20]. In particular he showed that the Fenchel-Nielsen "twist flows" on Teichmüller space are Hamiltonian flows (with respect to the Weil-Petersson Kähler form) whose associated potential functions are the geodesic length functions. Moreover he found striking formulas which underscore an intimate relationship between the symplectic geometry of Teichmüller space and the hyperbolic geometry (and hence the topology) of the surface. In particular the symplectic product of two twist vector fields (the Poisson bracket of two geodesic length functions) is interpreted in terms of the geometry of the surface.

We will reprove these formulas of Wolpert in our more general context. Accordingly our proofs are simpler and not restricted to Teichmüller space: while Wolpert's original proofs use much of the machinery of Teichmüller space theory, we give topological proofs which involve the multiplicative properties of homology with local coefficients and elementary properties of invariant functions.

Before stating the main results, it will be necessary to describe the ingredients of the symplectic geometry. Let G be a Lie group with Lie algebra $\mathfrak{g}$. The basic property we need concerning G is the existence of an orthogonal structure on G: an *orthogonal structure* on G is a nondegenerate symmetric bilinear form $\mathfrak{B}: \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$

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which is invariant under $\mathrm{Ad}\,G$. For example, if G is a reductive group of matrices, the trace form $\mathfrak{B}(X, Y) = \mathrm{tr}(XY)$ defines an orthogonal structure. (However, there are many "exotic" orthogonal structures on nonreductive groups.) Let $f: G \to \mathbb{R}$ be an *invariant function* (i.e. a function on G invariant under conjugation); its *variation function* (relative to $\mathfrak{B}$) is defined as the unique map $F: G \to \mathfrak{g}$ such that for all $X \in \mathfrak{g}$, $A \in G$

$$\mathfrak{B}(F(A), X) = \left.\frac{d}{dt}\right|_{t=0} f(A \exp tX).$$

It is easy to prove that $F(A)$ lies in the Lie algebra centralizer $\mathscr{Z}(A)$ of A.

The next ingredients are the spaces $\mathrm{Hom}(\pi, G)/G$ with their symplectic structure defined by $\mathfrak{B}$; for details we refer the reader to Goldman [7]. Recall that $\mathrm{Hom}(\pi, G)$ denotes the real analytic variety of all homomorphisms $\pi \to G$ and $\mathrm{Hom}(\pi, G)/G$ is its quotient by the action of G on $\mathrm{Hom}(\pi, G)$ by inner automorphisms. As shown in [7], the singular subset of $\mathrm{Hom}(\pi, G)$ consists of representations $\phi \in \mathrm{Hom}(\pi, G)$ such that the centralizer of $\phi(\pi)$ in $\mathrm{Ad}\,G$ has positive dimension; moreover G acts locally freely on the set of smooth points $\mathrm{Hom}(\pi, G)^{-}$. After removing possibly more G-invariant subsets of large codimension, one obtains a Zariski-open subset $\Omega \subset \mathrm{Hom}(\pi, G)$ such that Ω/G is a Hausdorff smooth manifold. (Alternatively one may consider the set of smooth points of the *character variety*, discussed in [15] for $G = \mathrm{SL}(2, \mathbb{C})$.) *In this paper, we shall always pretend that* $\mathrm{Hom}(\pi, G)/G$ *is a smooth manifold*. That is, we shall only really work on the smooth part Ω/G. All of our results extend to the singular part (suitably modified) but for simplicity we ignore the singularities of $\mathrm{Hom}(\pi, G)/G$, and relegate their discussion to [8].

The tangent space to $\mathrm{Hom}(\pi, G)/G$ at an equivalence class $[\phi]$, where $\phi \in \mathrm{Hom}(\pi, G)$, is the cohomology group $H^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$ (or $H^1(S; \mathfrak{g}_{\mathrm{Ad}\phi})$) where $\mathfrak{g}_{\mathrm{Ad}\phi}$ is the π-module (respectively, local coefficient system) with underlying space $\mathfrak{g}$ and action given by the composition $\pi \xrightarrow{\phi} G \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\mathfrak{g})$. Now $\mathfrak{B}$ defines a coefficient pairing $\mathfrak{g}_{\mathrm{Ad}\phi} \times \mathfrak{g}_{\mathrm{Ad}\phi} \to \mathbb{R}$ and cup product defines a pairing of tangent spaces

$$\omega(\phi): H^1(\pi, \mathfrak{g}_{\mathrm{Ad}\phi}) \times H^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi}) \to H^2(\pi; \mathbb{R}) \cong \mathbb{R}.$$

In [7], § 1, it is proved that ω is symplectic structure on $\mathrm{Hom}(\pi, G)/G$.

We shall investigate the symplectic geometry of $(\mathrm{Hom}(\pi, G)/G, \omega)$ in terms of certain "coordinate functions" on $\mathrm{Hom}(\pi, G)/G$. Namely, let $\alpha \in \pi$ and $f: G \to \mathbb{R}$ be an invariant function. Then there is a function $f_\alpha: \mathrm{Hom}(\pi, G)/G \to \mathbb{R}$ defined by $[\phi] \to f(\phi(\alpha))$. We are interested in the Hamiltonian flows associated to these functions.

Recall that if (X, ω) is a symplectic manifold and $\psi: \mathrm{X} \to \mathbb{R}$ is a smooth function, the Hamiltonian vector field associated to ψ (or with potential ψ) is the vector field $H\psi$ on X such that $\omega(H\psi, Y)$ equals the derivative $Y\psi$ for all vector fields Y. It is an easy consequence of the closedness of ω that the Hamiltonian flow generated by $H\psi$ leaves invariant ω as well as ψ.

To compute the Hamiltonian vector fields Hf_α of f_α, it is therefore necessary to differentiate f_α. Let $\alpha \in \pi$ and $f: G \to \mathbb{R}$ be an invariant function. Let $F: G \to \mathfrak{g}$ be the variation of f introduced above. Define a map $F_\alpha: \mathrm{Hom}(\pi, G) \to \mathfrak{g}$ by $F_\alpha(\phi) = F(\phi(\alpha))$. All of our general formulas will involve these functions F_α.

When α is simple, these flows have a simple description on $\mathrm{Hom}(\pi, G)$. If α is nonseparating, define a flow $\{\Xi_t\}_{t \in \mathbb{R}}$ on $\mathrm{Hom}(\pi, G)$ as follows: if $\gamma \in \pi$ is represented by a curve disjoint from α, let $(\Xi_t\phi)(\gamma) = \phi(\gamma)$ be constant; if $\beta \in \pi$ intersects α exactly once transversely "in the positive direction", let $(\Xi_t\phi)(\beta) = (\exp tF_\alpha(\phi))\phi(\beta)$. It can be shown that these conditions specify a unique flow Ξ on $\mathrm{Hom}(\pi, G)$. When α separates S into two components $S_1 \cup S_2$, define $(\Xi_t\phi)(\gamma) = \phi(\gamma)$ if γ is homotopic to a curve in S_1 and $(\Xi_t\phi)(\gamma) = \exp tF_\alpha(\phi)\phi(\alpha)\exp - tF_\alpha(\phi)$ if γ is homotopic to a carre in S_2. Then we have the following:

Duality Theorem. Let α be a simple loop on S and f_α, Ξ be as above. Then the flow Ξ on $\mathrm{Hom}(\pi, G)$ covers the Hamiltonian flow on $\mathrm{Hom}(\pi, G)/G$ associated to the function f_α on $\mathrm{Hom}(\pi, G)/G$.

As a corollary, we obtain Wolpert's duality theorem [19]: *the Fenchel-Nielsen twist flows on Teichmüller space are Hamiltonian (with respect to the Weil-Petersson Kähler form) with potential the geodesic length function l_α.*

When α is not simple, the Hamiltonian flow of f_α does not seem to admit a simple description as a "twist flow" as above, but the local flows nonetheless exist. (In general the local flows Hf_α are incomplete.) One can generally understand the behavior of these functions by forming derivatives $(Hf_\alpha)f_\alpha$ of f_β with respect to the Hamiltonian vector field Hf_α associated to f_α. It follows from the definition of Hf_α that $(Hf_\alpha)f_\beta$ equals the symplectic product $\omega(Hf_\alpha, Hf_\beta)$ which is by definition the Poisson bracket $\{f_\alpha, f_\beta\}$ of the two functions f_α and f_β. The following general result enables us to compute Poisson brackets:

Product formula. Let $\alpha, \beta \in \pi$ be represented by immersed curves in general position. Then the Poisson bracket of f_α and f_β is given by the function on $\mathrm{Hom}(\pi, G)/G$ defined by

$$\{f_\alpha, f_\beta\}([\phi]) = \sum_{p \in \alpha \# \beta} \varepsilon(p; \alpha, \beta)\mathfrak{B}(F_{\alpha_p}(\phi_p), F_{\beta_p}(\phi_p))$$

where $\varepsilon(p; \alpha, \beta) = \pm 1$ denotes the oriented intersection number of α and β at p, $\phi_p: \pi_1(S; p) \to G$ is a representative of $[\phi]$ and α_p, β_p are the elements of $\pi_1(S; p)$ corresponding to α and β.

This formula has the following meaning. The sum is taken over all of the intersection of α and β, which consists of transverse double points. Although $F_{\alpha_p}(\phi_p)$ depends on the choice of ϕ_p in an equivalence class $[\phi] \in \mathrm{Hom}(\pi, G)/G$, the pairing $\mathfrak{B}(F_{\alpha_p}(\phi_p), F_{\beta_p}(\phi_p))$ is independent of the choice of ϕ_p in its equivalence class. Finally we note that the functions f_α and f_β need not both be constructed from the same invariant function f.

As a corollary of the product formulas, we obtain the following basic fact due to Wolpert [18], [20]; (see also Kerckhoff [13]):

Cosine Formula. For any $\alpha \in \pi$ let $l_\alpha: \mathfrak{C}_S \to \mathbb{R}$ be the geodesic length function of α (i.e. the function which associates to a point $S \to M$ of Teichmüller space the length of the geodesic in M representing α). If $p \in \alpha \# \beta$ let θ_p denote the counterclockwise angle at p from the geodesic representing α to the geodesic representing β. Then the

Poisson bracket $\{l_\alpha, l_\beta\}$ is the function on Teichmüller space defined by

$$\{l_\alpha, l_\beta\} = \sum_{p \in \alpha \# \beta} \cos\theta_p .$$

(This formula has been generalized in several other directions; see Kerckhoff [13] for the case that α and β are measured geodesic laminations.)

In another direction, the product formula may be specialized as follows. Let G be a classical matrix group and let f be the character, and $\mathfrak{B}$ the trace form, of the standard embedding in $\mathrm{GL}(n,\mathbb{R})$. For example, $\mathrm{GL}(n,\mathbb{C})$ is embedded in $\mathrm{GL}(2n,\mathbb{R})$ as the subgroup centralizing the complex structure on $\mathbb{R}^{2n} \cong \mathbb{C}^n$, $f(A) = 2\,\mathrm{Re}\,\mathrm{tr}\,A$, if $A \in \mathrm{GL}(n,\mathbb{C})$ and if $X, Y \in \mathfrak{gl}(n,\mathbb{C})$, then $\mathfrak{B}(X,Y) = 2\,\mathrm{Re}\,\mathrm{tr}\,XY$. Also let $\tilde{f}(A) = 2\,\mathrm{Im}\,\mathrm{tr}\,A$. In the following theorem α, β represent immersed closed curves in general position (i.e. the immersion $\alpha \cup \beta$ has at worst transverse double points). If $p \in \alpha \# \beta$ is a double point intersection of α and β, let α_p (resp. β_p) denote the unique element of $\pi_1(S; p)$ corresponding to α (resp. β). We may use the group structure of $\pi_1(S; p)$ to form inverses and products such as $\alpha_p\beta_p$, $\alpha_p\beta_p^{-1}$ etc. The following result shows that the rational vector space spanned by the functions f_α, $\alpha \in \pi$, forms a Lie algebra under Poisson bracket:

Theorem. (i) *Let* $G = \mathrm{GL}(n,\mathbb{R})$, $\mathrm{GL}(n,\mathbb{C})$ *or* $\mathrm{GL}(n,\mathbb{H})$. *Then*

$$\{f_\alpha, f_\beta\} = \sum_{p \in \alpha \# \beta} \varepsilon(p; a, \beta)\, f_{\alpha_p\beta_p} .$$

(ii) *Let* $G = O(p,q)$, $O(n,\mathbb{C})$, $\mathrm{Sp}(n,\mathbb{H})$, $U(p,q)$, $\mathrm{Sp}(n,\mathbb{R})$, $\mathrm{Sp}(p,q)$ *or* $\mathrm{Sp}(n,\mathbb{C})$. *Then*

$$\{f_\alpha, f_\beta\} = \tfrac{1}{2} \sum_{p \in \alpha \# \beta} \varepsilon(p; \alpha, \beta)\,(f_{\alpha_p\beta_p} - f_{\alpha_p\beta_p^{-1}}) .$$

(iii) *Let* $G = \mathrm{SL}(n,\mathbb{R})$. *Then*

$$\{f_\alpha, f_\beta\} = \sum_{p \in \alpha \# \beta} \varepsilon(p; \alpha, \beta) \left(f_{\alpha_p\beta_p} - \frac{1}{n} f_\alpha f_\beta \right) .$$

(iv) *Let* $G = \mathrm{SL}(n,\mathbb{C})$ *or* $\mathrm{SL}(n,\mathbb{H})$. *Then*

$$\{f_\alpha, f_\beta\} = \sum_{p \in \alpha \# \beta} \varepsilon(p; \alpha, \beta) \left(f_{\alpha_p\beta_p} + \frac{1}{2n} (\tilde{f}_\alpha \tilde{f}_\beta - f_\alpha f_\beta) \right);$$

$$\{f_\alpha, \tilde{f}_\beta\} = \sum_{p \in \alpha \# \beta} \varepsilon(p; \alpha, \beta) \left(\tilde{f}_{\alpha_p\beta_p} - \frac{1}{2n} (f_\alpha \tilde{f}_\beta + \tilde{f}_\alpha f_\beta) \right);$$

$$\{\tilde{f}_\alpha, \tilde{f}_\beta\} = -\{f_\alpha, f_\beta\} .$$

(v) *Let* $G = \mathrm{SU}(p,q)$ *with* $p + q = n$. *Then*

$$\{f_\alpha, f_\beta\} = \tfrac{1}{2} \sum_{p \in \alpha \# \beta} \varepsilon(p; \alpha, \beta)\,(f_{\alpha_p\beta_p} - f_{\alpha_p\beta_p^{-1}} - (2/n)\tilde{f}_\alpha \tilde{f}_\beta);$$

$$\{f_\alpha, \tilde{f}_\beta\} = \tfrac{1}{2} \sum_{p \in \alpha \# \beta} \varepsilon(p; \alpha, \beta)\,(\tilde{f}_{\alpha_p\beta_p} - \tilde{f}_{\alpha_p\beta_p^{-1}});$$

$$\{\tilde{f}_\alpha, \tilde{f}_\beta\} = -\{f_\alpha, f_\beta\} .$$

(Here $\mathbb{H}$ denotes the noncommutative field of quaternions. In the notation of Helgason [10], $\mathrm{GL}(n, \mathbb{H})$, $\mathrm{Sp}(n, \mathbb{H})$, $\mathrm{SL}(n, \mathbb{H})$ are the groups $U^*(2n)$, $O^*(2n)$, $\mathrm{SU}^*(2n)$.)

The preceding theorem suggests that there is a formally defined Lie algebra structure on certain spaces based on the set $\hat{\pi}$ of homotopy classes of closed oriented curves on S. The simplest of these is defined as follows. Let $\mathbb{Z}\hat{\pi}$ be the free $\mathbb{Z}$-module with basis $\hat{\pi}$. If α, β are immersed oriented closed curves which intersect transversely in double points, then define the formal sum

$$[\alpha, \beta] = \sum_{p \in \alpha \# \beta} \varepsilon(p; \alpha, \beta) |\alpha_p \beta_p| \in \mathbb{Z}\hat{\pi}$$

where $\varepsilon(p; \alpha, \beta)$, α_p, β_p are as in the product formula, $\alpha_p \beta_p$ denotes the product in $\pi_1(S; p)$ and $|\alpha_p \beta_p| \in \hat{\pi}$ denotes the free homotopy class of $\alpha_p \beta_p \in \pi_1(S; p)$. The operation $[\alpha, \beta]$ depends only on the free homotopy classes of α and β and thus extends by linearity to a bilinear map $\mathbb{Z}\hat{\pi} \times \mathbb{Z}\hat{\pi} \to \mathbb{Z}\hat{\pi}$.

Proposition. (a) *Under this operation, $\mathbb{Z}\hat{\pi}$ is a Lie algebra over $\mathbb{Z}$.*

(b) *Let $G = \mathrm{GL}(n, \mathbb{R}), \mathrm{GL}(n, \mathbb{C})$ or $\mathrm{GL}(n, \mathbb{H})$. Let $\varrho: \mathbb{Z}\hat{\pi} \mapsto C^\infty$ $(\mathrm{Hom}(\pi, G)/G$ be the linear map defined by $\varrho: \alpha \mapsto f_\alpha$. Then ϱ is a homomorphism of the Lie algebra $\mathbb{Z}\hat{\pi}$ into the Lie algebra of functions on $\mathrm{Hom}(\pi, G)/G$ under Poisson bracket.*

There is a similar construction for a Lie algebra based on unoriented curves. The map $\hat{\pi} \to \hat{\pi}$, $\alpha \mapsto \alpha^{-1}$ which reverses the orientations of oriented loops extends to an automorphism of $\mathbb{Z}\hat{\pi}$ of order two. Its stationary set is additively a free module $\mathbb{Z}\bar{\pi}$ based on the set $\bar{\pi}$ of homotopy classes of unoriented closed curves on S; as a subalgebra $\mathbb{Z}\bar{\pi}$ has bracket relations defined by (ii).

Proposition. *Let $G = O(p, q)$, $O(n, \mathbb{C})$, $O(n, \mathbb{H})$, $U(p, q)$, $\mathrm{Sp}(p, q)$, $\mathrm{Sp}(n, \mathbb{R})$ or $\mathrm{Sp}(n, \mathbb{C})$. Let $\varrho: \mathbb{Z}\bar{\pi} \to C^\infty(\mathrm{Hom}(\pi, G)/G$ be the linear map defined by $\alpha \mapsto f_\alpha$. Then ϱ is a homomorphism of the Lie algebra $\mathbb{Z}\bar{\pi}$ into the Lie algebra of functions on $\mathrm{Hom}(\pi, G)/G$ under Poisson bracket.*

(The existence of the Lie algebra $\mathbb{Z}\bar{\pi}$ was hinted by Wolpert [20] who considered its homomorphic image of vector fields on Teichmüller space, which he called the "twist lattice". That the Lie algebra $\mathbb{Z}\bar{\pi}$ could be embedded in a Lie algebra $\mathbb{Z}\hat{\pi}$ based on oriented curves was first observed by Dennis Johnson.)

There is a dual formulation of the above propositions, which is very suggestive. For concreteness let us consider $\mathbb{Z}\hat{\pi}$, but everything we say will hold for $\mathbb{Z}\bar{\pi}$ and some of the other Lie algebras we consider, with minor modifications. Namely, let $\mathbb{Z}\hat{\pi}^*$ denote the real vector space dual to $\mathbb{Z}\hat{\pi}$. Then $\mathbb{Z}\hat{\pi}$ acts by derivations on the $\mathbb{Z}\pi^*$, which we may say is the "coadjoint module" of $\mathbb{Z}\hat{\pi}$. Indeed, $\mathbb{Z}\hat{\pi}^*$ is just the space of all real-valued class functions on π. Then to each point $[\phi] \in \mathrm{Hom}(\pi, G)/G$ is associated the function $\chi([\phi]): \mathbb{Z}\hat{\pi} \to \mathbb{R}$ defined by $\chi([\phi])(\alpha) = \varrho(\phi(\alpha))$. The resulting map $\chi: \mathrm{Hom}(\pi, G)/G \to \mathbb{Z}\hat{\pi}^*$ is the *moment map* for the Poîsson action of $\mathbb{Z}\hat{\pi}$ on $\mathrm{Hom}(\pi, G)/G$. (See the references [1], [2], and [17] for the definition of this concept.) In more familiar terms, $\chi(\phi)$ is just the character of the representation $\phi: \pi \to G \hookrightarrow \mathrm{GL}(n, \mathbb{R})$. The image of the moment map is just the character "variety" which consists of all characters of representations in $\mathrm{Hom}(\pi, G)$.

It is a well-known fact that the characters of representations locally separate points on the highest-dimensional strata of $\mathrm{Hom}(\pi, G)/G$. Using this fact, it follows that the Hamiltonian vector fields of trace functions generate a transitive action on the stratum $\mathrm{Hom}(\pi, G)^-/G$. Since the moment map (or character map) $\chi: \mathrm{Hom}(\pi, G)/G \to \mathbb{Z}\hat{\pi}^*$ is equivariant with respect to the action of the Lie algebra $\mathbb{Z}\hat{\pi}$, it follows that the open strata of the character variety are represented as *coadjoint orbits* of the Lie algebra $\mathbb{Z}\hat{\pi}$. This gives a new perspective on the symplectic structure of $\mathrm{Hom}(\pi, G)/G$ and suggests that the piecewise-linear symplectic structure on Thurston's space of measured geodesic laminations can be understood as a coadjoint orbit in $\mathbb{Z}\hat{\pi}^*$ also. (Compare Papadopolous [22] and Fathi [26].)

This paper is organized as follows. Section 1 is a preliminary discussion of invariant functions on Lie groups. In particular variation maps are computed for the standard choice of f above as well as the hyperbolic displacement length function on $PSL(2, \mathbb{R})$. Section 2 reviews homology with local coefficients, Poincaré duality and intersection theory.

In Sect. 3 the product formula is proved. The cosine formula is deduced as a corollary. The main step in the proof of the product formula is Proposition 3.7 which gives a geometric cycle representing the Hamiltonian vector field Hf_α. The remainder of § 3 computes Poisson brackets on $\mathrm{Hom}(\pi, G)/G$ for all classical groups G.

In Sect. 4 we assume that α is a *simple* closed curve. After a brief discussion of Fenchel-Nielsen twist flows on Teichmüller space to motivate the definition of "generalized twist flows", we prove the duality formula. The proof is by direct calculation, and uses Proposition 3.7. As a corollary we prove Wolpert's length-twist duality.

Section 5 is devoted to the abstract theory of Lie algebras based on closed curves. The above two propositions are proved recasting the generalizations of Kerckhoff's and Wolpert's cosine formula in a more algebraic setting. Several elementary algebraic properties of these Lie algebras are proved: for example several constructions of nontrivial ideals and subalgebras are presented. We describe the modifications needed to construct Lie algebras based on curves which act on $\mathrm{Hom}(\pi, G)/G$ for the other classical Lie groups G. Finally we interpret (using the moment-map construction) the character variety of representations of a surface group into a classical Lie group as a coadjoint orbit in a Lie algebra based on curves. These facts are used to characterize those closed curves in $\mathbb{Z}\hat{\pi}$ and $\mathbb{Z}\bar{\pi}$ which commute with a fixed simple closed curve α as those which are homotopic to curves disjoint from α.

§ 1. Invariant functions on Lie groups

1.1. Let G be a real Lie group with finitely many components. A C^1 function $f: G \to \mathbb{R}$ is said to be *invariant* if it is invariant under inner automorphisms: $f(PAP^{-1}) = f(A)$ for all A, $P \in G$. The prototypical example of an invariant function is the character of a real linear representation $\varrho: G \to \mathrm{GL}(n, \mathbb{R})$, i.e. $f(A) = \mathrm{tr}\,\varrho(A)$ for $A \in G$. In this section we show how, in many cases, an invariant function f gives rise to an equivariant map $F: G \to \mathfrak{g}$, which we call its *variation*, and compute various examples of invariant functions and their variations.

1.2. Let $f: G \to \mathbb{R}$ be a C^1 invariant function. Its differential df is a 1-form on G. We identify the Lie algebra $\mathfrak{g}$ of G with the space of all left-invariant vector fields on G; its dual vector space $\mathfrak{g}^*$ is identified with the space of left-invariant 1-forms on G. For $A \in G$, $df(A) \in T_A^* G$ is a covector on G and extends uniquely to a left-invariant 1-form $\hat{F}(A) \in \mathfrak{g}^*$. Invariance of f implies that $\hat{F}: G \to \mathfrak{g}^*$ is G-equivariant:

$$\hat{F}(PAP^{-1}) = \mathrm{Ad}^* P(\hat{F}(A))$$

whenever $P, A \in G$. Alternatively, $\hat{F}(A)$ is the linear functional on $\mathfrak{g}$ defined by

$$\hat{F}(A): X \to \left.\frac{d}{dt}\right|_{t=0} f(A \exp tX)$$

for $X \in \mathfrak{g}$.

1.3. An *orthogonal structure* on G is a nondegenerate symmetric bilinear form $\mathfrak{B}: \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ which is invariant under the adjoint representation. It is well known that on a semisimple Lie group, the Killing form defines an orthogonal structure; more generally the trace form of a reductive matrix group (see below) defines an orthogonal structure. However, there are non-reductive groups which admit orthogonal structures and it does not seem to be known how to characterize which Lie groups possess such structures. See Medina [21] for more information.

Suppose G is a Lie group with an orthogonal structure $\mathfrak{B}$. Let $f: G \to \mathbb{R}$ be an invariant function and $\hat{F}: G \to \mathfrak{g}^*$ its associated equivariant map. Since $\mathfrak{B}$ is nondegenerate, it defines an isomorphism $\tilde{\mathfrak{B}}: \mathfrak{g} \to \mathfrak{g}^*$ of G-modules. Define the *variation* of f (with respect to $\mathfrak{B}$) to be the composition $F = \tilde{\mathfrak{B}}^{-1} \circ \hat{F}: G \to \mathfrak{g}^* \to \mathfrak{g}$. Then $F: G \to \mathfrak{g}$ is G-equivariant:

$$F(PAP^{-1}) = \mathrm{Ad}\, P(F(A))$$

for $P, A \in G$. Moreover $F(A)$ may be alternatively defined by

$$\mathfrak{B}(F(A), X) = \left.\frac{d}{dt}\right|_{t=0} f(A \exp tX)$$

where X ranges over all of $\mathfrak{g}$.

If $A \in G$, let $\mathscr{Z}(A)$ denote the Lie algebra centralizer of A, i.e. the subalgebra of $\mathfrak{g}$ fixed by $\mathrm{Ad}\, A$. Equivariance implies that $F(A)$ lies in the intersection of all $\mathscr{Z}(P)$ where $PAP^{-1} = A$. For example, F maps the center of G to the center of $\mathfrak{g}$. *Taking* $P = A$, *we see that* $F(A) \in \mathscr{Z}(A)$ *for all* $A \in G$.

1.4. The most common invariant functions and orthogonal structures arise from linear representations of reductive Lie groups. Namely, if $\varrho: G \to \mathrm{GL}(n, \mathbb{R})$ is a homomorphism, then its character $A \mapsto \mathrm{tr}\,\varrho(A)$ is an invariant function on G and its *trace form* $\mathfrak{B}(X, Y) = \mathrm{tr}\,\varrho(X)\varrho(Y)$ is a symmetric $\mathrm{Ad}\, G$-invariant bilinear form $\mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. (Here we have used $\varrho: \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{R})$ for the associated representation of Lie algebras.) If ϱ is a local isomorphism of G onto a reductive subgroup of

$\mathrm{GL}(n, \mathbb{R})$ (i.e. one for which the representation ϱ and all of its tensor powers are completely reducible), then $\mathfrak{B}$ is nondegenerate and defines an orthogonal structure on G.

Proposition. *Let $\varrho : G \to \mathrm{GL}(n, \mathbb{R})$ be a local isomorphism onto a reductive subgroup of $\mathrm{GL}(n, \mathbb{R})$ and let f and $\mathfrak{B}$ denote the character and trace form of ϱ respectively. Then the variation $F : G \to \mathfrak{g}$ of f with respect to $\mathfrak{B}$ can be expressed as the composition*

$$G \xrightarrow{\varrho} \mathrm{GL}(n, \mathbb{R}) \xrightarrow{\mathrm{i}} \mathfrak{gl}(n, \mathbb{R}) \xrightarrow{\mathrm{pr}} \mathfrak{g}$$

where $\mathrm{i} : \mathrm{GL}(n, \mathbb{R}) \to \mathfrak{gl}(n, \mathbb{R})$ is the natural inclusion of invertible $n \times n$ matrices in all $n \times n$ matrices and $\mathrm{pr} : \mathfrak{gl}(n, \mathbb{R}) \to \mathfrak{g}$ is the $\mathfrak{B}$-orthogonal projection onto $\varrho(\mathfrak{g}) \subset \mathfrak{gl}(n, \mathbb{R})$ followed by the inverse of the isomorphism $\varrho : \mathfrak{g} \to \varrho(\mathfrak{g})$.

Proof. Clearly we may assume ϱ is the identity. Then, for $A \in G$, $X \in \mathfrak{g}$,

$$\begin{aligned} \mathfrak{B}(F(A), X) &= \left.\frac{d}{dt}\right|_{t=0} f(A \exp tX) \\ &= \left.\frac{d}{dt}\right|_{t=0} \mathrm{tr}(A(I + tX + O(t^2))) \\ &= \mathrm{tr}\, AX = \mathrm{tr}\, \mathrm{pr}(\mathrm{i}(A))X = \mathfrak{B}(\mathrm{pr}(\mathrm{i}(A)), X). \end{aligned}$$

Since $\mathfrak{B}$ is nondegenerate, $F = \mathrm{pr} \circ \mathrm{i}$ as claimed. Q.E.D.

1.5. Corollary. *Let $G = \mathrm{GL}(n, \mathbb{R})$ and let f and $\mathfrak{B}$ be the character and trace form, respectively of the identity representation. Then the variation $F : G \to \mathfrak{g}$ is the inclusion i of invertible $n \times n$ matrices in all $n \times n$ matrices.*

1.6. There are analogous results for the general linear groups over the complex field $\mathbb{C}$ and the field $\mathbb{H}$ of quaternions. Let J be the $2n \times 2n$ matrix $\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ where I_n denotes the $n \times n$ identity matrix. Then $\mathrm{GL}(n, \mathbb{C})$ may be identified with the subgroup of $\mathrm{GL}(2n, \mathbb{R})$ consisting of matrices A satisfying $JA = AJ$, i.e. $A = -JAJ$. Similarly $\mathfrak{gl}(n, \mathbb{C}) \subset \mathfrak{gl}(2n, \mathbb{R})$ is defined by the same equation. The projection map $\mathfrak{gl}(2n, \mathbb{R}) \to \mathfrak{gl}(n, \mathbb{C})$ is $\mathrm{pr}(X) = \frac{1}{2}(X - JXJ)$. After a simple application of Proposition 1.4, we obtain:

Corollary. *Let $G = \mathrm{GL}(n, \mathbb{C})$ and let f and $\mathfrak{B}$ be the character and the trace form of the representation $\mathrm{GL}(n, \mathbb{C}) \subset \mathrm{GL}(2n, \mathbb{R})$. Then the variation $F : G \to \mathfrak{g}$ of f with respect to $\mathfrak{B}$ is the inclusion of $n \times n$ invertible complex matrices in all $n \times n$ complex matrices.*

In complex coordinates, $f(A) = 2\,\mathrm{Re}\,\mathrm{tr}\, A$ and $\mathfrak{B}(X, Y) = 2\,\mathrm{Re}\,\mathrm{tr}\, XY$. (We would obtain the same variation function F if we had used $f(A) = 2\,\mathrm{Re}\,\mathrm{tr}\, A$ and $\mathfrak{B}(X, Y) = \mathrm{Re}\,\mathrm{tr}\, XY$ instead.) If $\varrho : G \to \mathrm{GL}(n, \mathbb{C})$ is a local isomorphism onto a reductive subgroup of $\mathrm{GL}(n, \mathbb{C})$, then the real parts of the character and trace form of ϱ define an invariant function $f : G \to \mathbb{R}$ and an orthogonal structure $\mathfrak{B}$. Then the conclusion of Proposition 1.4 will hold with $\mathrm{GL}(n,\mathbb{R})$ and $\mathfrak{gl}(n,\mathbb{R})$ replaced by $\mathrm{GL}(n,\mathbb{C})$ and $\mathfrak{gl}(n,\mathbb{C})$.

As an example, let $G=\mathrm{GL}(n,\mathbb{H})$ be the subgroup of $\mathrm{GL}(2n,\mathbb{C})$ consisting of matrices A satisfying $JA=AJ$. (In the notation of Helgason [10], $G=U^*(2n)$.) The Lie algebra $\mathfrak{g}$ consists of all $X\in\mathfrak{gl}(2n,\mathbb{C})$ satisfying the same equation and thus we obtain the following analogue of Corollary 1.6:

1.7. Corollary. *Let $G=\mathrm{GL}(n,\mathbb{H})$ f and $\mathfrak{B}$ as above. Then the variation $F: G\to\mathfrak{g}$ is the inclusion of invertible quaternionic matrices in all quaternionic matrices.*

As another direct application of 1.4, we obtain:

1.8. Corollar.y *Let $G=\mathrm{SL}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{C})$, or $\mathrm{SL}(n,\mathbb{H})$. If $A\in G$ define $f(A)=\mathrm{Re}\,\mathrm{tr}\,A$: for $X,Y\in\mathfrak{g}$, let $\mathfrak{B}(X,Y)=\mathrm{Re}\,\mathrm{tr}\,XY$. Then $F: G\to\mathfrak{g}$ is given by $F(A)=A-(\mathrm{tr}\,A/n)I$, for $A\in G$.*

Corollary 1.8 has a variant which applies the subgroups $\mathrm{UL}(n,\mathbb{C})$, $\mathrm{UL}(n,\mathbb{H})$, of $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{GL}(n,\mathbb{H})$ defined by $|\det A|=1$. For these groups the Lie algebra is defined by $\mathrm{Re}\,\mathrm{tr}\,A=0$. If $f(A)=\mathrm{Re}\,\mathrm{tr}\,A$ and $\mathfrak{B}(X,Y)=\mathrm{Re}\,\mathrm{tr}\,XY$ then 1.4 implies that $F: G\to\mathfrak{g}$ is given by $F(A)=A-(\mathrm{Re}\,\mathrm{tr}\,A)/nI$.

When Corollary 1.8 is applied to $n=2$ one gets an alternative formula for F. The characteristic equation for $A\in\mathrm{SL}(2,\mathbb{R})$ gives $A+A^{-1}=(\mathrm{tr}\,A)I$. Then 1.8 implies

$$F(A)=A-(\mathrm{tr}\,A/2)I=\tfrac{1}{2}(A-A^{-1}).$$

We shall presently see that this formula is ubiquitous:

1.9. Corollary. *Let G be one of the groups $O(p,q)$, $O(n,\mathbb{C})$, $U(p,q)$, $\mathrm{Sp}(n,\mathbb{R})$, $\mathrm{Sp}(n,\mathbb{C})$, $\mathrm{Sp}(p,q)$, $\mathrm{Sp}(n,\mathbb{H})$. Let f and $\mathfrak{B}$ be the real parts of the character and the trace form respectively of the standard representations (in $\mathrm{GL}(n,\mathbb{C})$ for $O(p,q)$, $O(n,\mathbb{C})$ and $U(p,q)$ $(n=p+q)$, in $\mathrm{GL}(2n,\mathbb{C})$ for $\mathrm{Sp}(n,\mathbb{R})$, $\mathrm{Sp}(n,\mathbb{C})$, $\mathrm{Sp}(p,q)$, $(2n=p+q)$ and $\mathrm{Sp}(n,\mathbb{H})$). Then the variation of f with respect to $\mathfrak{B}$ is given by $F(A)=\frac{1}{2}(A-A^{-1})$. (In terminology of [10], $\mathrm{Sp}(n,\mathbb{H})=O^*(2n)=O(2n,\mathbb{C})\cap\mathrm{GL}(n,\mathbb{H})$.)*

Proof.

Orthogonal groups $O(p,q)$ and $O(n,\mathbb{C})$. Let $I_{p,q}=\begin{pmatrix} I_p & O \\ O & -I_q \end{pmatrix}$. Then $A\in G\Leftrightarrow AI_{p,q}{}^tA=I_{p,q}$ and $X\in\mathfrak{g}\Leftrightarrow XI_{p,q}+I_{p,q}{}^tX=0\Leftrightarrow X=-I_{p,q}XI_{p,q}$. Orthogonal projection $\mathfrak{gl}(n)\to\mathfrak{g}$ is given by $X\mapsto\frac{1}{2}(X-I_{p,q}{}^tXI_{p,q})$. Thus if $A\in G$, Proposition 1.4 imples that $F(A)=\frac{1}{2}(A-I_{p,q}{}^tAI_{p,q})=\frac{1}{2}(A-A^{-1})$.

Unitary groups $U(p,q)$. $A\in G\Leftrightarrow AI_{p,q}{}^t\bar{A}=I_{p,q}$; $X\in\mathfrak{g}\Leftrightarrow X=-I_{p,q}{}^t\bar{X}I_{p,q}$. Orthogonal projection $\mathfrak{gl}(n,\mathbb{C})\to\mathfrak{g}$ is given by $X\to\frac{1}{2}(X-I_{p,q}{}^t\bar{X}I_{p,q})$. Thus if $A\in G$, 1.4 imples $F(A)=\frac{1}{2}(A-I_{p,q}{}^t\bar{A}I_{p,q})=\frac{1}{2}(A-A^{-1})$.

Symplectic groups $\mathrm{Sp}(n,\mathbb{R})$ and $\mathrm{Sp}(n,\mathbb{C})$. $A\in G\Leftrightarrow AJ^tA=J$; $X\in\mathfrak{g}\Leftrightarrow XJ+J^tX=0\Leftrightarrow X=J^tXJ$. Orthogonal projection $\mathfrak{gl}(2n)\to\mathfrak{g}$ is $X\to\frac{1}{2}(X+J^tXJ)$. Thus if $A\in G$, $F(A)=\frac{1}{2}(A+J^tAJ)=\frac{1}{2}(A-A^{-1})$.

Symplectic groups $\mathrm{Sp}(p,q)$. $A\in G\Leftrightarrow AJ^tA=J,\ AI_{p,q}{}^t\bar{A}=I_{p,q};\ X\in\mathfrak{g}\Leftrightarrow XJ+J^tX=XI_{p,q}+I_{p,q}{}^t\bar{X}=0\Leftrightarrow X=J^tXJ=-I_{p,q}{}^t\bar{X}I_{p,q}$, Orthogonal projection $\mathfrak{gl}(2n,\mathbb{C})\to\mathfrak{g}$ is given by $X\mapsto\frac{1}{4}(X-J^tXJ-I_{p,q}{}^t\bar{X}I_{p,q}-JI_{p,q}{}^t\bar{X}I_{p,q}J)$. If $A\in G$, then $F(A)=\frac{1}{4}(A+J^tAJ-I_{p,q}{}^t\bar{A}I_{p,q}-JI_{p,q}{}^t\bar{A}I_{p,q}J)=\frac{1}{4}(A-A^{-1}-A^{-1}+A)=\frac{1}{2}(A-A^{-1})$.

Quaternionic symplectic group $\mathrm{Sp}(n,\mathbb{H})$. $A\in G\Leftrightarrow JAJ=-\bar{A}$ and $A^tA=I$; $X\in\mathfrak{g}\Leftrightarrow X=-J\bar{X}J=-{}^tX$. Orthogonal projection $\mathfrak{gl}(2n,\mathbb{C})\to\mathfrak{g}$ is $X\mapsto\frac{1}{4}(X+J\bar{X}J-{}^tX-J^t\bar{X}J)$ so if $A\in G$, $F(A)=\frac{1}{4}(A+J\bar{A}J-{}^tA-J^t\bar{A}J)=\frac{1}{4}(A+A-A^{-1}-A^{-1})=\frac{1}{2}(A-A^{-1})$. Q.E.D.

1.10. Corollary. *Let $G=\mathrm{SU}(p,q)$ and f, $\mathfrak{B}$ be the real parts of the character and trace form of the standard representation in $\mathrm{GL}(n,\mathbb{C})$, $n=p+q$. Then the variation of f is given by $F(A)=\frac{1}{2}(A-A^{-1})-(i/n)\mathrm{Im}\,\mathrm{tr}\,A$.*

Proof. $A\in G\Leftrightarrow AI_{p,q}{}^t\bar{A}=I_{p,q}$, $\det A=1$; $X\in\mathfrak{g}\Leftrightarrow XI_{p,q}+I_{p,q}{}^t\bar{X}=0$, $\mathrm{tr}\,A=0$. Orthogonal projection $\mathfrak{gl}(n,\mathbb{C})\to\mathfrak{g}$ is given by $X\mapsto\frac{1}{2}(X-I_{p,q}{}^t\bar{X}I_{p,q})-(i/n)\mathrm{Im}\,\mathrm{tr}\,X$. Thus, if $A\in G$, $F(A)=\frac{1}{2}(A-A^{-1})-(i/n)\mathrm{Im}\,\mathrm{tr}\,A$. Q.E.D.

1.11. Finally we close this discussion with a trivial observation which will be useful in § 3.

Lemma. *Let G be a complex Lie group with an orthogonal structure $\mathfrak{B}:\mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ which is invariant under the complex structure on $\mathfrak{g}$ (i.e. $\mathfrak{B}(iX,Y)=\mathfrak{B}(X,iY)$ where i is the complex structure on $\mathfrak{g}$). Let $f:G\to\mathbb{R}$ be an invariant function which is the real part of a holomorphic complex invariant function $Z:G\to\mathbb{C}$, and let $F:G\to\mathfrak{g}$ be the variation of f with respect to $\mathfrak{B}$. If $\tilde{f}:G\to\mathbb{R}$ denotes the invariant function which is the imaginary part of Z, then its variation is given by $\tilde{F}(A)=-iF(A)$.*

Proof. For any $X\in\mathfrak{g}$, $A\in G$, we have

$$\begin{aligned}\mathfrak{B}(-iF(A),X)=\mathfrak{B}(F(A),-iX)&=\left.\frac{d}{dt}\right|_{t=0} f(A\exp(-itX))\\ &=\mathrm{Re}\left.\frac{d}{dt}\right|_{t=0} z(A\exp(-itX))=\mathrm{Re}-i\left.\frac{d}{dt}\right|_{t=0} z(A\exp(tX))\\ &=\mathrm{Im}\left.\frac{d}{dt}\right|_{t=0} z(A\exp(tX))=\mathfrak{B}(\tilde{F}(A),X).\quad \text{Q.E.D.}\end{aligned}$$

(The hypothesis that $\mathfrak{B}$ is invariant under the complex structure is equivalent to the existence of a complex orthogonal structure $\mathfrak{B}_{\mathbb{C}}:\mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ having $\mathfrak{B}$ as its real part.)

1.12. Example. The displacement length function for hyperbolic elements of $\mathrm{SL}(2,\mathbb{R})$. Invariant functions need not be defined on all of G; we may just as well consider invariant functions defined on invariant open subsets Ω of G. Then everything we have said goes through as before, except of course that the functions $\hat{F}$, F are only defined on Ω.

A particularly important example occurs for $G=\mathrm{SL}(2,\mathbb{R})$, which acts isometrically on the Poincaré upper half-plane H^2. For $A\in G$, the *displacement length* of A is defined as $l(A)=\inf\{\mathrm{dist}(x,Ax):x\in H^2\}$. The subset of G for which $l(A)>0$

equals the subset of hyperbolic elements $\mathrm{Hyp}=\{A\in G: |\mathrm{trace}\,A|>2\}$. For $A\in\mathrm{Hyp}$, the displacement length l is related to the trace by the formula

$$|\mathrm{trace}\,A| = 2\cosh(l(A)/2)\,. \tag{2.10}$$

Let $L:\mathrm{Hyp}\to\mathfrak{g}$ be the variation function of l and let $\mathfrak{B}$ be the trace form for $\mathrm{SL}(2,\mathbb{R})$. It is easy to see that every $A\in\mathrm{Hyp}$ is conjugate to a matrix of the form $\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}$ for $\lambda>0$. Thus in view of (1.3) it suffices to compute L for diagonal matrices. Let $X\in\mathfrak{g}$ be the matrix $\begin{pmatrix}a & b\\ c & -a\end{pmatrix}$ and $A=\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}$. Then

$$\begin{aligned}\mathfrak{B}(L(A),X) &= \frac{d}{dt}\bigg|_{t=0} l((\exp tX)A)\\ &= \frac{d}{dt}\bigg|_{t=0} 2\cosh^{-1}\tfrac{1}{2}|\mathrm{trace}((\exp tX)A)|\\ &= \frac{d}{dt}\bigg|_{t=0} 2\cosh^{-1}\left|\tfrac{1}{2}\mathrm{trace}\left(\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}+t\begin{pmatrix}a & b\\ c & -a\end{pmatrix}\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}+O(t^2)\right)\right|\\ &= \frac{d}{dt}\bigg|_{t=0} 2\cosh^{-1}|((\lambda+\lambda^{-1})/2+ta(\lambda-\lambda^{-1})/2)| = 2a\,.\end{aligned} \tag{1.10}$$

It follows that $L(A)=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$.

Since $l(-A)=l(A)$, it follows easily that $L(-A)=L(A)$. Thus (replacing A by $-A$ if necessary) we may assume that A has positive eigenvalues. Note also that L is constant along one-parameter subgroups. As one-parameter subgroups in Hyp correspond to geodesics in H^2, we see that for a hyperbolic element A with positive eigenvalues, $L(A)$ may be characterized as the unique element of g satisfying:

(i) $$\mathfrak{B}(L(A),L(A))=2$$

(ii) $$A=\exp tL(A)\quad\text{for some}\quad t>0\,.$$

(In fact (i) and (ii) together imply $t=\frac{1}{2}l(A)$ is the unique real number satisfying (ii).)

§ 2. Homology with local coefficients

In this section we briefly summarize algebraic topological results we need which concern homology and cohomology in a flat vector bundle (including multiplicative structure), Poincaré duality and intersection theory. For more details, we refer to Steenrod [16], § 31, for basic definitions, Brown [3] for the multiplicative structure, and Dold [5] for the relationship between Poincaré duality, cup products, and the intersection pairing on homology (with ordinary coefficients). Compare also Johnson-Millson [12].

2.1. Let M^n be a closed oriented connected smooth manifold. Let ξ be a *flat vector bundle* over M; recall that this means there is a coordinate covering for ξ such that the coordinate changes are locally constant maps into the general linear group of the fibre. Thus if s is a section of ξ defined over an open subset of M it makes sense to ask whether in these local charts s is the graph of a constant map into the fibre. Such a section will be called *flat* (or "parallel", "covariant constant", etc.).

Let $\mathfrak{C}_*(M)$ denote the complex of smooth singular chains on M. A basis of $\mathfrak{C}_k(M)$ consists of all smooth maps $\sigma: \Delta^k \to M$ where Δ^k is the standard k-simplex. The boundary $\partial\sigma$ of σ is the $(k-1)$-chain $\sum_{i=0}^{k}(-1)^i\partial_i\sigma$ where $\partial_i\sigma$ is the i-th face of σ. Let $\mathfrak{C}_*(M;\xi)$ denote the complex of smooth singular chains on M with values in ξ: a basis for $\mathfrak{C}_k(M;\xi)$ consists of smooth maps $\sigma: \Delta^k \to \mathrm{M}$ together with a flat section s of $\sigma^*\xi$ over Δ^k. Abusing notation, we denote such a ξ-valued k-simplex by $\sigma\otimes s$. The boundary of such a simplex is the $(k-1)$-chain $\partial(\sigma\otimes s) = \sum_{i=0}^{k}(-1)^i\partial_i\sigma\otimes s_i$ where s_i is the restriction of s to the i-th face $\partial_i\Delta^k$ of Δ^k. It is easy to show that $\mathfrak{C}_*(M;\xi)$ is a chain complex, and its homology is denoted $H_*(M;\xi)$.

Here is a particularly simple construction of cycles in $\mathfrak{C}_k(M;\xi)$. Let V^k be a closed oriented k-manifold, $f: V^k \to M$ a map, and s a flat section of $f^*\xi$ over V. Then there is a ξ-valued k-cycle on M, denoted $f\otimes s$ for brevity, which is given by $\sum_{i=1}^{m}(f\circ\sigma_i)\otimes s_i$ where $\{\sigma_i\}_{i=1,\dots,m}$ is a triangulation of V and s_i is the restriction of s to σ_i. All of the cycles we use in this paper are of this form.

In a similar way cohomology with coefficients in a flat vector bundle is defined. If ξ is a flat vector bundle over M then a *ξ-valued k-cochain* on M is a function which assigns to each singular k-simplex $\sigma: \Delta^k \to M$ a flat section of $\sigma^*\xi$ over Δ^k. The collection of all such cochains forms a complex $\mathfrak{C}^*(M;\xi)$ from which we obtain the *cohomology of M with coefficients in ξ, $H^*(M;\xi)$*.

It is well known that every flat vector bundle is associated to a linear representation of the fundamental group in the following way. Let $h: \pi_1(M) \to \mathrm{GL}(V)$ be a representation of $\pi_1(M)$ on a vector space V and let $\tilde{M} \to M$ be a universal covering space of M. Then $\pi_1(M)$ acts on $\tilde{M}\times V$ diagonally, by deck transformations of $\tilde{M}$ and linearly by h on V. The quotient of $\tilde{M}\times V$ by this action is the total space of a flat vector bundle ξ over M. Furthermore two flat vector bundles arising from $h_1, h_2 \in \mathrm{Hom}(\pi_1(M), \mathrm{GL}(V))$ are equivalent if and only if h_1 and h_2 differ by an inner automorphism of G. A representation determining ξ is called a holonomy representation for ξ. It is easy to see that flat sections of ξ are in bijective correspondence with vectors in V stationary under h. If M is an Eilenberg-MacLane space of type $K(\pi,1)$, then there is a canonical isomorphism of $H_k(M;\xi)$ and $H^k(M;\xi)$ with the Eilenberg-MacLane group homology and cohomology $H_k(\pi;V_h)$ and $H^k(\pi;V_h)$ respectively, where V_h is the π-module determined by $h: \pi \to \mathrm{GL}(V)$.

2.2. Products and Poincaré duality. Let $\mathfrak{B}: \xi_1 \times \xi_2 \to \xi_3$ be a bilinear pairing of flat vector bundles over M. The usual cup- and cap-product operations define pairings of complexes

$$\mathscr{B}_*(\cup): \mathfrak{C}^k(M;\xi_1)\times\mathfrak{C}^l(M;\xi_2)\to\mathfrak{C}^{k+l}(M;\xi_3)$$

$$\mathscr{B}_*(\cap): \mathfrak{C}^k(M;\xi_1)\times\mathfrak{C}_l(M;\xi_2)\to\mathfrak{C}_{l-k}(M;\xi_3)$$

which induce homology pairings

$$\mathscr{B}_*(\cup): H^k(M;\xi_1)\times H^l(M;\xi_2)\to H^{k+l}(M;\xi_3)$$
$$\mathscr{B}_*(\cap): H^k(M;\xi_1)\times H_l(M;\xi_2)\to H_{l-k}(M;\xi_3)$$

respectively.

The usual Poincaré duality isomorphism is given by cap product $\cap[M]: H^k(M)\to H_{n-k}(M)$ with the fundamental homology class $[M]\in H_n(M)$. On the chain level this isomorphism may be described geometrically as follows. Let A be a smooth $(n-k)$-cycle in M. Then $(\cap[M])^{-1}[A]$ is represented by the k-cocycle which assigns to every k-chain B which intersects A transversely the intersection number $\sum_{p\in A\# B}\varepsilon(p;A,B)$. Here $\varepsilon(p;A,B)=\pm 1$ is the oriented intersection number at p and the sum is over the set $A\# B$ of transverse intersections of A with B. (Of course these intersections are counted with multiplicity. That is, if $A=\sum_{i=1}^{l}\alpha_i$, $B=\sum_{j=1}^{m}\beta_j$, where $\alpha_i:\Delta^{n-k}\to M$ and $\beta_j:\Delta^k\to$ M are singular simplices, then $A\# B$ really consists of pairs $(p,q)\in\Delta^k\times\Delta^{n-k}$ with $\alpha_i(p)=\beta_j(q)$ for some i,j. We will usually avoid this technicality by assuming that the intersections $A\# B$ are transverse double points: then (p,q) is uniquely determined by $\alpha_i(p)=\beta_j(q)$.) Moreover, although this formula is only defined on transverse k-chains B, $(\cap[M])^{-1}[A]$ is uniquely determined. Since it is a cocycle, its value on a singular simplex $\beta:\Delta^k\to M$ is the same as on any β' homotopic to β rel $\partial\beta$, and thus we may replace each k-simplex by one which is transverse to A. For further detail, see Dold [5].

This construction works equally well with coefficients in a flat vector bundle; for more details consult Cohen [4]. The map $\cap[M]: H^k(M;\xi)\to H_{n-k}(M;\xi)$ is an isomorphism. If $A=\sum_{i=1}^{m}\sigma_i\otimes a_i$ is a ξ-valued $(n-k)$-cycle, then $(\cap[M])^{-1}[A]$ is the ξ-valued k-cocycle which assigns to each k-simplex $\tau:\Delta^k\to M$ which is transverse to the σ_i the flat section of $\tau^*\xi$ given by

$$\sum_{i=1}^{m}\sum_{p\in\sigma_i\#\tau}\varepsilon(p;\sigma_i,\tau)\tau^*a_i.$$

In a similar vein, cup product has a geometric interpretation in terms of intersection. If A is an k-cycle and B is an $(n-k)$-cycle transverse to A then the Poincaré dual of the cup product of the cocycles Poincaré dual to $[A]$ and $[B]$ is given by the intersection pairing $A\cdot B$ of A and N, i.e.

$$((\cap[M])^{-1}[A]\cup(\cap[M])^{-1}[B])\cap[M]=A\cdot B=\sum_{p\in A\# B}\varepsilon(p;A,B)\in H_0(M;\mathbb{Z})\cong\mathbb{Z}$$

A similar formula holds for cycles with coefficients in flat vector bundles. Namely, let ξ_1, ξ_2, ξ_3 be flat vector bundles over M and $\mathfrak{B}: \xi_1 \times \xi_2 \to \xi_3$ a pairing. Let $A = \sum_{i=1}^{l} \sigma_i \otimes a_i$ be a ξ_1-valued k-cycle and $B = \sum_{j=1}^{m} \tau_j \otimes b_j$ a ξ_2-valued $(n-k)$-cycle. Then the cup-product of the Poincaré duals of $[A]$ and $[B]$ is Poincaré dual to the intersection

$$A \cdot B = \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{p \in \sigma_i \# \tau_j} \varepsilon(p; \sigma_i, \tau_j) \mathfrak{B}(a_i(p), b_j(p)) \in H_0(M; \xi_3).$$

In the cases of interest here, ξ_3 is the trivial $\mathbb{R}$-bundle $M \times \mathbb{R}$ so $A \cdot B$ lies in $H_0(M; \mathbb{R}) = \mathbb{R}$.

2.3. In this paper we will be exclusively interested in the case when S is a surface $(n=2)$. Let ξ be a flat vector bundle over S. Suppose $\alpha: S^1 \to S$ is an oriented closed smooth curve in S. Generically α is an immersion whose only self-intersections are transverse double points. For every *simple point* p of α (i.e. $p \in \alpha(S^1)$ and $\alpha^{-1}(p)$ is a single point) there is an element α_p of the fundamental group $\pi_1(S; p)$. (We will often confuse a parametrized curve α with its image $\alpha(S^1)$ in S, and often write $p \in \alpha$; the meaning shall be clear from the context.)

Let $p \in \alpha$ be a simple point and let $\varrho: \pi_1(S; p) \to \mathrm{GL}(V)$ be a holonomy representation (where V denotes the fibre of ξ above p). A flat section over α then corresponds to a vector $v \in V$ which is fixed under $\varrho(\alpha_p)$. In particular we obtain a cycle on S with coefficients in ξ, which we denote by $\alpha \otimes v$. (Here we have identified fixed vectors of $h(\alpha_p)$ in V with flat sections of $\alpha^* \xi$.)

Now we pair two such cycles. Suppose that ξ, η are two flat vector bundles over S and $\mathfrak{B}$ is a pairing of vector bundles $\xi \times \eta \to \mathbb{R}$. Let α, β be two oriented smooth closed curves. We say that α and β *intersect transversely in double points* if the maps $S^1 \to S$ representing α and β are transverse and for each $p \in S$, $\alpha^{-1}(p) \cap \beta^{-1}(p)$ has at most two elements. In that case each $p \in \alpha \# \beta$ is a simple point of α and of β so we obtain well-defined elements α_p, $\beta_p \in \pi_1(S; p)$. Let $\xi(p)$ and $\eta(p)$ denote the fibres over p of ξ and η respectively, and choose holonomy representations

$$h_\xi(p): \pi_1(S; p) \to \mathrm{GL}(\xi(p)), \qquad h_\eta(p): \pi_1(S; p) \to \mathrm{GL}(\eta(p))$$

for ξ and η respectively. For any flat section s (resp. s_η) of $\alpha^* \xi$ (resp. $\beta^* \eta$), let $v_\xi(p) \in \xi(p)$ (resp. $v_\eta(p) \in \eta(p)$) be the corresponding fixed vectors. Then the intersection product of the two cycles $\alpha \otimes s_\xi$, $\beta \otimes s_\eta$ with respect to the coefficient pairing $\mathfrak{B}$ is given by

$$\mathfrak{B}_*((\alpha \otimes s_\xi) \cdot (\beta \otimes s_\eta)) = \sum_{p \in \alpha \# \beta} \varepsilon(p; \alpha, \beta) \mathfrak{B}(v_\xi(p), v_\eta(p)).$$

Finally we mention that cycles of the form $\alpha \otimes v$ have a simple description in group homology. Recall that in group homology the group of 1-chains $C_1(\pi; V)$ of a group π with values in a π-module V is $\mathbb{Z}\pi \otimes_{\mathbb{Z}} V$ where $\mathbb{Z}\pi$ is the integral group ring of π. The 0-chain group $C_0(\pi; V)$ equals V. The boundary operator $\partial: C_1(\pi; V) \to C_0(\pi, V)$ associates to a "V-valued 1-simplex" $\alpha \otimes v$ (where $\alpha \in \pi$, $v \in V$) the element $v - \alpha v$ of $V = C_0(\pi, V)$. Clearly $\alpha \otimes v$ is itself a cycle if and only if

$\alpha v = v$, and this 1-cycle in $Z_1(\pi; V)$ corresponds to the geometrically defined class in $H^1(S; \xi)$ where $\pi = \pi_1(S)$ and ξ is the flat vector bundle associated to the π-module V. For a discussion of higher-dimensional cycles of this type, with applications to deformations of hyperbolic structures, see Johnson-Millson [12].

§ 3. The product formula

3.1. For the remainder of this paper we fix the following notation. S is a closed oriented surface of genus $g > 1$ and $\tilde{S} \to S$ is a universal covering space. Let π denote the group of deck transformations of $\tilde{S}$. If α is an oriented closed curve on S and $p \in S$ is a simple point of α, let α_p denote the unique element of $\pi_1(S; p)$ determined by α. Let G denote a Lie group with Lie algebra $\mathfrak{g}$ and we fix a orthogonal structure $\mathfrak{B} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. If $\phi \in \mathrm{Hom}(\pi, G)$ we denote its equivalence class in $\mathrm{Hom}(\pi, G)/G$ by $[\phi]$. Let $\mathfrak{g}_{\mathrm{Ad}\phi}$ denote the π-module $\mathfrak{g}$ when π-action is given by the composition $\pi \xrightarrow{\phi} G \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\mathfrak{g})$. We also denote by $\mathfrak{g}_{\mathrm{Ad}\phi}$ the corresponding flat vector bundle over S. We denote the dual of $\mathfrak{g}_{\mathrm{Ad}\phi}$ by $\mathfrak{g}^*_{\mathrm{Ad}\phi}$. Note that $\mathfrak{B}$ determines an isomorphism $\tilde{\mathfrak{B}} : \mathfrak{g}_{\mathrm{Ad}\phi} \to \mathfrak{g}^*_{\mathrm{Ad}\phi}$ of π-modules. Finally, if $[\phi] \in \mathrm{Hom}(\pi, G)/G$ and $p \in S$, we will systematically abuse notation and write $\phi : \pi_1(S; p) \to G$ although there will usually be no canonical choice of isomorphism of $\pi_1(S; p)$ with π.

3.2. A *symplectic structure* on a manifold is a closed nondegenerate exterior 2-form. In [7] it is shown that an orthogonal structure $\mathfrak{B}$ on G determines a natural symplectic structure on $\mathrm{Hom}(\pi, G)/G$. (Although $\mathrm{Hom}(\pi, G)/G$ rarely is a manifold, it is real analytically stratified by symplectic manifolds; see [7] and [8] for further details.)

We briefly recall the construction from [7]. The Zariski tangent space to $\mathrm{Hom}(\pi, G)$ at ϕ is the space $Z^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$ of 1-cocycles of π with values in $\mathfrak{g}_{\mathrm{Ad}\phi}$. To see this we consider a real analytic path ϕ_t in $\mathrm{Hom}(\pi, G)$ starting at $\phi_0 = \phi$ and write $\phi_t(x) = \exp(tu(x) + 0(t^2))\phi(x)$. The defining equations for $\mathrm{Hom}(\pi, G)$ in the form $\phi_t(xy) = \phi_t(x)\phi_t(y)$ imply that the "first variation" $u : \pi \to \mathfrak{g}$ is a 1-cocycle, i.e. $u(xy) = u(x) + \mathrm{Ad}\,\phi(x)u(y)$. Moreover the tangent space to the G-orbit of ϕ is the space of 1-coboundaries: let $g_t = \exp(tu_0 + O(t^2))$ be a path in G starting at 1; then the first variation of the path ϕ_t defined by $\phi_t(x) = g_t\phi(x)g_t^{-1}$ equals $u(x) = u_0 - \mathrm{Ad}\,\phi(x)u_0$ which, as a 1-cochain $\pi \to \mathfrak{g}$ is the coboundary of the 0-cochain $u_0 \in \mathfrak{g}$. Thus $T_{[\phi]}\mathrm{Hom}(\pi, G)/G = H^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$, which we identify by a canonical isomorphism, with $H^1(S; \mathfrak{g}_{\mathrm{Ad}\phi})$.

The cup-product in S and the coefficient pairing $\mathfrak{B} : \mathfrak{g}_{\mathrm{Ad}\phi} \times \mathfrak{g}_{\mathrm{Ad}\phi} \to \mathbb{R}$ combine to produce a pairing $\omega(\phi) : H^1(S; \mathfrak{g}_{\mathrm{Ad}\phi}) \times H^1(S; \mathfrak{g}_{\mathrm{Ad}\phi}) \to H^2(S; \mathbb{R}) \cong \mathbb{R}$. Explicitly if $a, b \in H^1(S; \mathfrak{g}_{\mathrm{Ad}\phi})$, then we write $\omega_\phi(a, b) = \mathfrak{B}_*(a \cup b) \cap [S] \in \mathbb{R}$. In [7] it is proved that $\omega(\phi)$ is a nondegenerate skew-symmetric bilinear form on $H^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$. Furthermore it is proved there that ω defines a closed exterior 2-form, hence a symplectic structure, on $\mathrm{Hom}(\pi, G)/G$.

3.3. We briefly review the formalism of Hamiltonian mechanics and symplectic geometry. For further information, the reader is referred to the books of Abraham-Marsden [1], Arnold [2], and Weinstein [17].

Let (X, ω) be a symplectic manifold, i.e. X is a smooth manifold and ω is a symplectic structure on X. Let $\tilde{\omega}: TX \to T^*X$ be the isomorphism of the tangent and cotangent bundles determined by ω. If $\psi \in C^\infty(X)$ is a smooth function, its exterior derivative $d\psi$ is a 1-form and $\tilde{\omega}^{-1} \circ d\psi$ is a vector field we denote by $H\psi$. The vector field $H\psi$ is the symplectic analogue of the gradient of ψ with respect to a Riemannian metric. We call $H\psi$ a *Hamiltonian vector field*, the flow it generates a *Hamiltonian flow* and ψ the *potential*.

The following elementary facts are well known:

Proposition. (i) *The Lie derivative of ω with respect to a Hamiltonian vector field is zero, i.e. a Hamiltonian flow preserves ω.*

(ii) *If $\psi_1, \psi_2 \in C^\infty(X)$, then the symplectic product $\omega(H\psi_1, H\psi_2)$ equals either of the directional derivatives $(H\psi_1)\psi_2 = -(H\psi_2)\psi_1$. (This function is the Poisson bracket of ψ_1, ψ_2 and is denoted $\{\psi_1, \psi_2\}$.)*

(iii) *Under Poisson bracket, the space $C^\infty(M)$ of smooth functions on M becomes a Lie algebra $C^\infty(M; \omega)$. Moreover H is a Lie algebra homomorphism from $C^\infty(M; \omega)$ into the Lie algebra of vector fields on M with Lie bracket. Finally, the adjoint representation* $\mathrm{ad}: C^\infty(M; \omega) \to \mathrm{Der}\, C^\infty(M)$, *defined by* $\mathrm{ad}\,\psi: f \mapsto \{\psi, f\}$ *is a Lie algebra homomorphism from $C^\infty(M; \omega)$ to the Lie algebra of derivations of the commutative ring $C^\infty(M)$ under multiplication, i.e. $\{\psi_1, \psi_2\psi_3\} = \{\psi_1, \psi_2\}\psi_3 + \{\psi_1, \psi_3\}\psi_2$ for ψ_1, ψ_2, $\psi_3 \in C^\infty(M; \omega)$.*

3.4. We are interested in a certain family of functions on the symplectic "manifold" $(\mathrm{Hom}(\pi, G)/G, \omega)$. Namely let $\alpha \in \pi$ and $f: G \to \mathbb{R}$ an invariant function on G. Consider the function $f_\alpha: \mathrm{Hom}(\pi, G) \to \mathbb{R}$ defined by $\phi \to f(\phi(\alpha))$. Since f is invariant, f_α is G-invariant and defines a function $\mathrm{Hom}(\pi, G)/G \to \mathbb{R}$, also denoted f_α. A further consequence of invariance of f is that f_α depends only on the conjugacy class of α in π. Thus we may take any oriented closed curve α in S and define f_α since the set $\hat{\pi}$ of conjugacy classes in π equals the set of free homotopy classes of oriented closed curves in S. Let $f': G \to \mathbb{R}$ be another invariant function and let $F, F': G \to \mathfrak{g}$ be the variation functions constructed in § 1.

3.5 Theorem. *Let α, β be two oriented closed curves which are immersions with transverse double points. Then the Poisson bracket $\{f_\alpha, f'_\alpha\}$ equals the function $\mathrm{Hom}(\pi, G)/G \to \mathbb{R}$ defined by*

$$[\phi] \mapsto \sum_{p \in \alpha \# \beta} \varepsilon(p; \alpha, \beta)\mathfrak{B}(F(\phi(\alpha_p)), F'(\phi(\beta_p)))\,.$$

Note that in this expression we have to choose, for each $p \in \alpha \# \beta$, a representative $\phi: \pi_1(S; p) \to G$ for $[\phi]$. However, since F and F' are Ad-equivariant and $\mathfrak{B}$ is Ad-invariant, each summand is independent of these particular choices.

3.6 Corollary. *If α and β are disjoint then f_α and f'_β Poisson-commute.*

The proof of 3.5 uses an explicit cycle Poincaré dual to the Hamiltonian of f_α. The Hamiltonian vector field Hf_α assigns to $[\phi]$ an element of the tangent space $T_{[\phi]}\mathrm{Hom}(\pi, G)/G = H^1(\pi, \mathfrak{g}_{\mathrm{Ad}\,\phi})$. We denote this element of $H^1(\pi; \mathfrak{g}_{\mathrm{Ad}\,\phi}) = H^1(S; \mathfrak{g}_{\mathrm{Ad}\,\phi})$ by $Hf_\alpha(\phi)$. (We shall often identify $H^*(\pi; \mathfrak{g}_{\mathrm{Ad}\,\phi})$ and $H^*(S; \mathfrak{g}_{\mathrm{Ad}\,\phi})$, etc. without explicit mention of the canonical isomorphisms.) Although it seems complicated to express $Hf_\alpha(\phi)$ by an explicit cocycle in $Z^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$ it is much simpler to find a cycle representing the Poincaré dual homology class $Hf_\alpha(\phi) \cap[S] \in H_1(S; \mathfrak{g}_{\mathrm{Ad}\phi})$.

3.7 Proposition. *Choose representatives $\phi: \pi_1(S; p) \to G$ for $[\phi]$ and $\alpha_p \in \pi_1(S; p)$ for α. Then the Poincaré duality isomorphism $\cap[S]: H^1(S; \mathfrak{g}_{\mathrm{Ad}\phi}) \to H_1(S; \mathfrak{g}_{\mathrm{Ad}\phi})$ carries $Hf_\alpha(\phi) \in H^1(S; \mathfrak{g}_{\mathrm{Ad}\phi})$ to the homology class of the cycle $\alpha \otimes F(\phi(\alpha_p))$ on S with coefficient in $\mathfrak{g}_{\mathrm{Ad}\phi}$.*

Note that since $\mathrm{Ad}\phi(\alpha_p)F(\phi(\alpha_p)) = F(\phi(\alpha_p))$, the chain $\alpha \otimes F(\phi(\alpha_p))$ is a cycle as claimed. We emphasize, however, that in order to write down the cycle, we have had to choose representatives ϕ and α_p.

Proof of 3.5 assuming 3.7. By 3.7 $Hf_\alpha(\phi) \cap [S] = [\alpha \otimes F(\phi(\alpha_p))]$ and $Hf'_\beta(\phi) \cap [S] = [\beta \otimes F'(\phi(\beta_p))]$ for each $p \in \alpha \# \beta$. By the duality between cup-product and intersection pairing, as in 2.2,

$$\begin{aligned}\{f_\alpha, f'_\beta\} &= \omega(Hf_\alpha, Hf'_\alpha) = \mathfrak{B}_*(Hf_\alpha(\phi) \cup Hf'_\beta(\phi)) \cap [S] \\ &= \mathfrak{B}_*((Hf_\alpha(\phi) \cap [S]) \cdot (Hf_\beta(\phi) \cap [S]) \\ &= \mathfrak{B}_*([\alpha \otimes F(\phi(\alpha_p))] \cdot [\beta \otimes F'(\phi(\beta_p))]) .\end{aligned}$$

By (2.3) this last quantity equals

$$\sum_{p \in \alpha \# \beta} \varepsilon(p; \alpha, \beta) \mathfrak{B}(F(\phi(\alpha_p)), F'(\phi(\beta_p))) ,$$

proving 3.5. Q.E.D.

Proof of 3.7. We first compute the differential of $f_\alpha: \mathrm{Hom}(\pi, G)/G \to \mathbb{R}$. For $[\phi] \in \mathrm{Hom}(\pi, G)/G$, $df_\alpha(\phi) \in T^*_{[\phi]} \mathrm{Hom}(\pi, G)/G = H^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})^*$. Thus suppose that $u \in Z^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$ is a cocycle representing $[u] \in H^1(\pi; \mathfrak{g}_{\mathrm{Ad}\phi})$; we compute the effect of $df_\alpha(\phi)$ on $[u]$. Now u is tangent to a path ϕ_t in $\mathrm{Hom}(\pi, G)$ in the sense that $\phi_t(x) = \exp(tu(x) + O(t^2))\phi(x)$. Now

$$\begin{aligned} df_\alpha(\phi): [u] &\mapsto \left.\frac{d}{dt}\right|_{t=0} f(\phi_t(\alpha_p)) \\ &= \left.\frac{d}{dt}\right|_{t=0} f((\exp tu(\alpha_p) + O(t^2))\phi(\alpha_p)) = \hat{F}(\phi(\alpha_p))\,(u(\alpha_p)) \\ &= \mathfrak{B}(F(\phi(\alpha_p)), u(\alpha_p)) .\end{aligned}$$

Let ${}^t\tilde{\mathfrak{B}}: H^1(\pi; \mathfrak{g}^*)^* \to H^1(\pi; \mathfrak{g})^*$ be the transpose of the isomorphism $H^1(\pi; \mathfrak{g}) \to H^1(\pi; \mathfrak{g}^*)$ effected by the coefficient isomorphism $\tilde{\mathfrak{B}}: \mathfrak{g} \to \mathfrak{g}^*$. (We shall henceforth drop the subscript $\mathrm{Ad}\phi$ from $\mathfrak{g}_{\mathrm{Ad}\phi}$ and $\mathfrak{g}^*_{\mathrm{Ad}\phi}$ as the context will be clear.) Let $\eta: H_1(\pi; \mathfrak{g}) \to H^1(\pi; \mathfrak{g}^*)^*$ be the map arising from the cap product pairing $H^1(\pi; \mathfrak{g}^*) \times H_1(\pi; \mathfrak{g}) \to H^0(\pi; \mathbb{R}) = \mathbb{R}$. (On the chain level, this is the canonical pairing of $\mathfrak{g}$-valued chains with $\mathfrak{g}^*$-valued cochains.) Since $df_\alpha(\phi)$ takes a cohomology class $[u] \in H^1(\pi; \mathfrak{g}^*)$ to $\mathfrak{B}(F(\phi(\alpha_p)), u(\alpha_p))$, we have $({}^t\tilde{\mathfrak{B}})^{-1}(df_\alpha(\phi))$

$=\eta(\alpha_p \otimes F(\phi(\alpha_p)))$. Let $\theta: H^1(\pi; \mathfrak{g}) \to H^1(\pi; \mathfrak{g}^*)^*$ be the map arising from the pairing

$$H^1(\pi; \mathfrak{g}) \times H^1(\pi; \mathfrak{g}^*) \xrightarrow{\cup} H^2(\pi; \mathbb{R}) \xrightarrow{\cap[\pi]} H_0(\pi; \mathbb{R}) = \mathbb{R}$$

where $[\pi]$ denotes the fundamental class in $H^2(\pi; \mathbb{R})$.

3.8. Lemma. *The diagram*

$$\begin{array}{ccc} H^1(\pi; \mathfrak{g}) & \xrightarrow{\cap[\pi]} & H_1(\pi; \mathfrak{g}) \\ {\scriptstyle \tilde{\omega}} \downarrow & \searrow^{\theta} & \downarrow {\scriptstyle \eta} \\ H^1(\pi; \mathfrak{g})^* & \xleftarrow[{}^t\tilde{\mathfrak{B}}]{} & H^1(\pi; \mathfrak{g}^*)^* \end{array}$$

is commutative.

Proof. The lower left triangle is commutative by the definition of $\tilde{\omega}$. The commutativity of the upper right triangle arises from the "associative law" relating cup and cap products. Namely, let $X \in H^1(\pi; \mathfrak{g}^*)$, $Y \in H^1(\pi; \mathfrak{g})$. Then $\theta(Y): X \mapsto (X \cup Y) \cap [\pi]$ and $\eta(Y \cap [\pi]): X \mapsto X \cap (Y \cap [\pi])$. Since $(X \cup Y) \cap [\pi] = X \cap (Y \cap [\pi])$, the diagram commutes. Q.E.D.

Now we finish the proof of 3.7. For

$$\begin{aligned} \alpha \otimes F(\phi(\alpha_p)) &= \eta^{-1} \circ ({}^t\tilde{\mathfrak{B}})^{-1}(df_\alpha(\phi)) \\ &= (\tilde{\omega}^{-1}(df_\alpha(\phi))) \cap [\pi] = Hf_\alpha(\phi) \cap [\pi]. \end{aligned}$$

The proof of 3.7 is now complete.

3.9. The Cosine formula

As our first application of Theorem 3.5 we prove the following basic formula due to Wolpert [18], [20].

Theorem. *Let $\mathfrak{T}_S$ denote the Teichmüller space of S. For each closed curve α in S, let $l_\alpha: \mathfrak{T}_S \to \mathbb{R}$ be the function which assigns to $m \in \mathfrak{T}_S$ the length of the unique geodesic $\alpha(m)$ homotopic to α on the hyperbolic surface S_m corresponding to m. Then, in the Weil-Petersson symplectic structure,*

$$\{l_\alpha, l_\beta\} = \sum_{p \in \alpha(m) \# \beta(m)} \cos \theta_p$$

where θ_p is the counterclockwise angle from $\alpha(m)$ to $\beta(m)$ at p. (Since $\alpha(m)$ and $\beta(m)$ are uniquely determined by α, β, and m, the right-hand side is a well-defined function on $\mathfrak{T}_S$.)

Proof. Let $G = \mathrm{SL}(2, \mathbb{R})$. Then $\mathfrak{T}_S$ may be defined as one of the connected components of $\mathrm{Hom}(\pi, G)/G$ consisting of $[\phi]$ for which $\phi(\alpha)$ is hyperbolic for all $\alpha \neq 1$ in π. Then the geodesic length function l_α is a smooth function on $\{[\phi] \in \mathrm{Hom}(\pi, G)/G : \phi(\alpha) \in \mathrm{Hyp}\}$ (and hence on $\mathfrak{T}_S$) which is constructed from the invariant function $l: \mathrm{Hyp} \to \mathbb{R}$ discussed in 1.13. The Weil-Petersson Kähler form on $\mathfrak{C}_S$ is the restriction of the symplectic structure on $\mathrm{Hom}(\pi, G)/G$ determined by $(-1/8)$ times the Killing form on $\mathfrak{g}=\mathfrak{gl}(2,\mathbb{R})$, i.e. $(-1/2)$ times the trace form $\mathfrak{B}$ of the standard representation on $\mathbb{R}^2$. Thus 3.5 applies and we have

$$\{l_\alpha, l_\beta\} = -1/2 \sum_{p\in\alpha\#\beta} \varepsilon(p;\alpha,\beta)\mathfrak{B}(L(\phi(\alpha_p)), L(\phi(\beta_p)))\,.$$

Thus it remains to identify $-1/2\varepsilon(p;\alpha,\beta)\mathfrak{B}(L(\phi(\alpha_p)), L(\phi(\beta_p)))$ as $\cos\theta_p$.

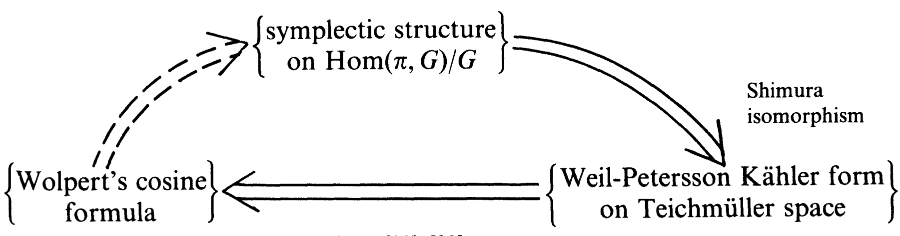
Let $A, B\in \mathrm{Hyp}$ be hyperbolic elements of $G=\mathrm{SL}(2,\mathbb{R})$. Then by 1.13 $L(A)$ is the unique element of $\mathfrak{g}$ such that $A=\exp tL(A)$ for $t>0$ and $\mathfrak{B}(L(A), L(A))=\frac{1}{2}$; similarly for $L(B)$. Let a (resp. b) be the unique oriented geodesic invariant under A (resp. B) so that A (resp. B) moves points in the forward direction. Suppose that a and b intersect at p; let θ be the counterclockwise angle from a to b at p. It suffices to prove:

$$\cos\theta = -1/2\varepsilon(p;a,b)\mathfrak{B}(L(A), L(B))\,. \tag{3.10}$$

First we note that by changing A to A^{-1} the orientation of a changes and both $\varepsilon(p;\alpha,\beta)$ and $L(A)$ are replaced by their negatives; similarly for B. Thus both sides of (3.10) depend only on the unoriented geodesics a and b. Now $L(A)$ is conjugate to the matrix $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and thus $L(A)$ has determinant -1. Writing $L(A) = \begin{pmatrix}p & q\\ r & s\end{pmatrix}$, we note that it acts on the upper half plane as an orientation-reversing isometry $\bar{z}\to(p\bar{z}+q)/(r\bar{z}+s)$. Furthermore since $L(A)$ has trace zero, it acts by reflection about a. Similarly $L(B)$ acts by reflection in b. Their product $(L(A))\,(L(B))$ acts by rotation about p through angle 2θ. Since $\det(L(A))(L(B))=1$, $\mathrm{tr}(L(A))\,(L(B))=\pm 2\cos\theta$. Hence $\mathfrak{B}(L(A), L(B))=\mathrm{tr}\,L(A)\,L(B) = \pm 2\cos\theta$.

It remains to check the sign. Both sides of (3.10) are continuous functions in the open subset W of $\mathrm{Hyp}\times\mathrm{Hyp}$ consisting of (A,B) such that their invariant axes cross. Thus it suffices to check (3.10) for a single example in each component of W. Now W has two components, detected by $\varepsilon(p;a,b)$; changing (A,B) to (A^{-1},B) interchanges the two components. Thus it suffices to check the sign in (3.10) for a single $(A,B)\in W$; this calculation is straightforward, uninspiring and omitted. Q.E.D.

3.11. Remark. By reversing the proofs of 3.7 and 3.9 one sees that the cosine formula is actually equivalent to the description (originally due to Shimura) of the Weil-Petersson Kähler form on Teichmüller space in terms of the cup-product, fundamental cycle, and Killing form, as in Goldman [7], §2. Thus the following circle of ideas is now complete:



3.12. Lie algebras of trace functions

Although the cosine formula 3.9 expresses a beautiful relationship between the Riemannian geometry of a hyperbolic surface and the symplectic geometry of Teichmüller space, its usefulness for the more general space $\mathrm{Hom}(\pi, G)/G$ is limited since the geodesic length functions, etc. are only defined on Teichmüller space. Hence we are led to consider variants of these functions which come from globally defined invariant functions on G and hence are globally defined on $\mathrm{Hom}(\pi, G)/G$.

We shall compute Poisson brackets of certain functions f_α for some of the most natural invariant functions f on all of the classical Lie groups. Namely we shall consider the "standard" representation $G \to \mathrm{GL}(n, \mathbb{R})$ (i.e. the representation by which G is defined) and take for f and $\mathfrak{B}$ the character and the trace form of this representation. When G is defined as a subgroup of $\mathrm{GL}(n, \mathbb{C})$, then we compose the standard representation of G in $\mathrm{GL}(n, \mathbb{C})$ with the standard representation $\mathrm{GL}(n, \mathbb{C}) \subset \mathrm{GL}(2n, \mathbb{R})$; alternatively we may work in $\mathrm{GL}(n, \mathbb{C})$ and take f (resp. $\mathfrak{B}$) to be twice the real part of the character (resp. the trace form) of the standard representation $G \subset \mathrm{GL}(n, \mathbb{C})$, i.e. $f(A) = 2 \operatorname{Re} \operatorname{tr} A$ for $A \in G \subset \mathrm{GL}(n, \mathbb{C})$, and $\mathfrak{B}(X, Y) = 2 \operatorname{Re} \operatorname{tr} XY$ for $X, Y \in \mathfrak{g} \subset \mathfrak{g}(n, \mathbb{C})$. For the standard representations of the classical groups and their characters and trace forms, we refer to 1.6–1.12.

In what follows α and β are closed curves on S which are in general position, i.e. α and β are immersions and which intersect each other in transverse double points (we do not require that they intersect minimally. Thus each $p \in \alpha \# \beta$ is a simple point of α and of β; thus α and β determine well-defined elements α_p and β_p, respectively, of $\pi_1(S; p)$.

3.13. Theorem. *Let $G = \mathrm{GL}(n, \mathbb{R})$, $\mathrm{GL}(n, \mathbb{C})$ or $\mathrm{GL}(n, \mathbb{H})$. Let f, $\mathfrak{B}$, α, β be as in 3.12. Then the Poisson bracket of f_α and f_β is given by the formula*

$$\{f_\alpha, f_\beta\} = \sum_{p \in \alpha \# \beta} \varepsilon(p; \alpha, \beta) f_{\alpha_p \beta_p}$$

where $\alpha_p \beta_p$ denotes the product in $\pi_1(S; p)$ of the elements α_p, $\beta_p \in \pi_1(S; p)$.

Proof. To avoid repetition we consider all three cases $G = \mathrm{GL}(n, \mathbb{R})$, $\mathrm{GL}(n, \mathbb{C})$, $\mathrm{GL}(n, \mathbb{H})$ simultaneously. Write $k = 1$ in case $G = \mathrm{GL}(n, \mathbb{R})$; otherwise let $k = 2$. Then in the standard representations by complex matrices (see 1.6–1.7) we have $f(A) = k \operatorname{Re} \operatorname{tr} A$ and $\mathfrak{B}(X, Y) = k \operatorname{Re} \operatorname{tr} XY$. By 1.5–1.7 the variation map $F: G \to \mathfrak{g}$ of f with respect to $\mathfrak{B}$ is the natural inclusion of invertible $n \times n$ real (resp. complex, quaternionic) matrices. The product formula 3.5 imples that the Poisson bracket $\{f_\alpha, f_\beta\}$ is the function on $\mathrm{Hom}(\pi, G)/G$ given by

$$\begin{aligned}
[\phi] \mapsto & \sum_{p \in \alpha \# \beta} \varepsilon(p; \alpha, \beta) \mathfrak{B}(F(\phi(\alpha_p)), F(\phi(\beta_p))) \\
= & \sum_{p \in \alpha \# \beta} \varepsilon(p; \alpha, \beta) k \operatorname{Re} \operatorname{tr} \phi(\alpha_p) \phi(\beta_p) \\
= & \sum_{p \in \alpha \# \beta} \varepsilon(p; \alpha, \beta) f_{\alpha_p \beta_p}(\phi)
\end{aligned}$$

as claimed. Q.E.D.

3.14. Theorem. *Let G be one of the groups $O(p,q)$, $O(n,\mathbb{C})$, $O(n,\mathbb{H})$, $U(p,q)$, $\mathrm{Sp}(n,\mathbb{R})$, $\mathrm{Sp}(p,q)$ and let f, $\mathfrak{B}$, α, β be as in 3.12. Then*

$$\{f_\alpha, f_\beta\} = \frac{1}{2} \sum_{p \in \alpha \# \beta} \varepsilon(p;\alpha,\beta)\,(f_{\alpha_p\beta_p} - f_{\alpha_p\beta_p^{-1}}).$$

Remark. This formula is due to Wolpert [20] for the case of Teichmüller space $\mathfrak{C}_S \subset \mathrm{Hom}(\pi, G)/G$, $G = \mathrm{SL}(2,\mathbb{R})$.

Proof. Write $k=1$ for $G=O(p,q)$ and $\mathrm{Sp}(n,\mathbb{R})$; $k=2$ for all the other groups. We claim that if $A \in G$, then $f(A)=f(A^{-1})$. If G is an orthogonal or a symplectic group, this follows from the fact that A and A^{-1} have the same eigenvalues. While this is no longer true for $G=U(p,q)$, the eigenvalues of A^{-1} are the conjugates of those of A. Since $f(A)=2\,\mathrm{Re}\,\mathrm{tr}\,A$, the claim follows for $G=U(p,q)$ as well.

By 1.9, the variation map $F: G \to \mathfrak{g}$ is given by $F(A)=\frac{1}{2}(A-A^{-1})$. If A, $B \in G$, then

$$\begin{aligned}\mathfrak{B}(F(A),F(B)) &= (k/4)\,\mathrm{Re}\,\mathrm{tr}(A-A^{-1})(B-B^{-1})\\ &= (k/4)\,\mathrm{Re}\,\mathrm{tr}(AB+A^{-1}B^{-1}-A^{-1}B-AB^{-1})\\ &= (f(AB)+f(A^{-1}B^{-1})-f(A^{-1}B)-f(AB^{-1}))/4\\ &= \tfrac{1}{2}(f(AB)-f(AB^{-1})).\end{aligned}$$

Now substitute $A=\phi(\alpha_p)$, $B=\phi(\beta_p)$ and apply 3.5. Q.E.D.

3.15. Theorem. *Let $G=\mathrm{SL}(n,\mathbb{R})$, f, $\mathfrak{B}$, α, β as above. Then*

$$\{f_\alpha, f_\beta\} = \sum_{p \in \alpha \# \beta} \varepsilon(p;\alpha,\beta)\left(f_{\alpha_p\beta_p} - \frac{1}{n} f_\alpha f_\beta\right).$$

Proof. By 1.8, the variation map $F: G \to \mathfrak{g}$ is $F(A)=A-(\mathrm{tr}\,A)/nI$. If A, $B \in G$, then

$$\begin{aligned}\mathfrak{B}(F(A),F(B)) &= \mathrm{tr}((A-(\mathrm{tr}\,A)/nI)(B-(\mathrm{tr}\,B)/nI))\\ &= \mathrm{tr}(AB-(\mathrm{tr}\,A)/nB-(\mathrm{tr}\,B)/nA+(\mathrm{tr}\,A)(\mathrm{tr}\,B)/n^2I)\\ &= \mathrm{tr}\,AB-(\mathrm{tr}\,A\,\mathrm{tr}\,B)/n = f(AB)-f(A)f(B)/n.\end{aligned}$$

Now substitute $A=\phi(\alpha_p)$, $B=\phi(\beta_p)$ and apply 3.5. Q.E.D.

Remarks. (1) The same formula holds for

$$G=\mathrm{UL}(n,\mathbb{C})=\{A \in \mathrm{GL}(n,\mathbb{C}): |\det A|=1\}$$

and

$$G=\mathrm{UL}(n,\mathbb{H})=\mathrm{UL}(2n,\mathbb{C}) \cap \mathrm{GL}(n,\mathbb{H}).$$

The proof is identical.

(2) The formula in 3.15 differs from the formulas in 3.13 and 3.14 in an important respect. 3.13 and 3.14 express a Poisson bracket of trace functions f_α as a linear combination of trace functions. On the other hand, 3.15 expresses the Poisson bracket of trace functions as a nonlinear polynomial in trace functions.

When $n=2$, trace identities in $\mathrm{SL}(2,\mathbb{R})$ reduce 3.15 to 3.14 (e.g., when $G=\mathrm{Sp}(1,\mathbb{R})$); for $n>2$ it seems impossible to write 3.15 as a linear combination of traces. This complication can be traced to the fact that for $n>2$, the determinant map $\det:\mathrm{GL}(n,\mathbb{R})\to\mathbb{R}^*$ has degree >2. (All of the groups in 3.13 are defined by linear equations in some $\mathrm{GL}(m,\mathbb{R})$ and all of the groups in 3.14 are defined by quadratic equations.) This leads to mild complications in the abstract Lie algebra theory developed in § 4.

3.16. Now we discuss the groups $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{SL}(n,\mathbb{H})$ and $\mathrm{SU}(p,q)$. Taking $f(A)=2\,\mathrm{Re\,tr}\,A$, $\mathfrak{B}(X,Y)=2\,\mathrm{Re\,tr}\,XY$, as usual, we find, unfortunately, that the Poisson bracket of trace functions f_α cannot be expressed solely in terms of f_α's alone. Rather we must introduce auxiliary functions $\tilde{f}_\alpha$ derived from the invariant function $\tilde{f}:G\to\mathbb{R}$ defined by $\tilde{f}(A)=2\,\mathrm{Im\,tr}\,A$.

Theorem. *Let $G=\mathrm{SL}(n,\mathbb{C})$ or $\mathrm{SL}(n,\mathbb{H})$, f, $\tilde{f}$, $\mathfrak{B}$, α, β as above. Then*

$$\{f_\alpha, f_\beta\}=\sum_{p\in\alpha\#\beta}\varepsilon(p;a,\beta)\left(f_{\alpha_p\beta_p}+\frac{1}{2n}(\tilde{f}_\alpha\tilde{f}_\beta-f_\alpha f_\beta)\right)$$

$$\{f_\alpha, \tilde{f}_\beta\}=\sum_{p\in\alpha\#\beta}\varepsilon(p;\alpha,\beta)\left(\tilde{f}_{\alpha_p\beta_p}-\frac{1}{2n}(f_\alpha\tilde{f}_\beta+\tilde{f}_\alpha f_\beta)\right)$$

$$\{\tilde{f}_\alpha, \tilde{f}_\beta\}=\sum_{p\in\alpha\#\beta}\varepsilon(p;\alpha,\beta)\left(-f_{\alpha_p\beta_p}+\frac{1}{2n}(f_\alpha f_\beta-\tilde{f}_\alpha\tilde{f}_\beta)\right).$$

Proof. By 1.11, the variation $\tilde{F}$ of $\tilde{f}$ satisfies $\tilde{F}(A)=-iF(A)$, where $F(A)=A-(\mathrm{tr}\,A)/nI$ by 1.8. If $A,B\in G$, then

$$\begin{aligned}\mathfrak{B}(F(A),F(B))&=2\,\mathrm{Re\,tr}(A-(\mathrm{tr}\,A)/nI)\,(B-(\mathrm{tr}\,B)/nI)\\&=f(AB)-2\,\mathrm{Re}(\mathrm{tr}\,A\,\mathrm{tr}\,B)/n\\&=f(AB)-(f(A)f(B)-\tilde{f}(A)\tilde{f}(B))/2n.\end{aligned}$$

Similarly

$$\mathfrak{B}(F(A),\tilde{F}(B))=\tilde{f}(AB)-(f(A)\tilde{f}(B)+\tilde{f}(A)f(B))/2n$$

and

$$\mathfrak{B}(\tilde{F}(A),\tilde{F}(B))=-f(AB)+(f(A)f(B)-\tilde{f}(A)\tilde{f}(B))/2n.$$

Substitute $A=\phi(\alpha_p)$ and $B-\phi(\beta_p)$ and apply 3.5. Q.E.D.

3.17. Theorem. *Let $G=\mathrm{SU}(p,q)$ with $p+q=n$, and let $f,\tilde{f},\mathfrak{B},\alpha,\beta$ be as in 3.16. Then*

$$\{f_\alpha, f_\beta\}=\tfrac{1}{2}\sum_{p\in\alpha\#\beta}\varepsilon(p;\alpha,\beta)\,(f_{\alpha_p\beta_p}-f_{\alpha_p\beta_p^{-1}}+(1/2n)\tilde{f}_\alpha\tilde{f}_\beta)$$

$$\{f_\alpha, \tilde{f}_\beta\}=\tfrac{1}{2}\sum_{p\in\alpha\#\beta}\varepsilon(p;\alpha,\beta)\,(\tilde{f}_{\alpha_p\beta_p}-\tilde{f}_{\alpha_p\beta_p^{-1}})$$

$$\{\tilde{f}_\alpha, \tilde{f}_\beta\}=\tfrac{1}{2}\sum_{p\in\alpha\#\beta}\varepsilon(p;\alpha,\beta)\,(f_{\alpha_p\beta_p^{-1}}-f_{\alpha_p\beta_p}-(1/2n)\tilde{f}_\alpha\tilde{f}_\beta).$$

Proof. By 1.10, $F(A)=(A-A^{-1})/2-((i/n)\operatorname{Im}\operatorname{tr}A)I$. Thus

$$F(A)F(B)=(A-A^{-1})(B-B^{-1})/4-(i/2n)((\operatorname{Im}\operatorname{tr}A)\cdot(B-B^{-1})+(\operatorname{Im}\operatorname{tr}B)(A-A^{-1}))-(1/n^2)\cdot(\operatorname{Im}\operatorname{tr}A(\operatorname{Im}\operatorname{tr}B)I).$$

Applying the identity $-i\operatorname{tr}(A-A^{-1})=2\operatorname{Im}\operatorname{tr}A$ we compute

$$\operatorname{tr}F(A)F(B)=\operatorname{tr}(A-A^{-1})(B-B^{-1})/4+(2/n)(\operatorname{Im}\operatorname{tr}A)(\operatorname{Im}\operatorname{tr}B)-(1/n)(\operatorname{Im}\operatorname{tr}A)(\operatorname{Im}\operatorname{tr}B).$$

Thus

$$\mathfrak{B}(F(A),F(B))=(f(AB)-f(AB^{-1}))/2+\tilde{f}(A)\tilde{f}(B)/2n;$$
$$\mathfrak{B}(F(A),\tilde{F}(B))=(\tilde{f}(AB)-\tilde{f}(AB^{-1}))/2;$$
$$\mathfrak{B}(\tilde{F}(A),\tilde{F}(B))=-\mathfrak{B}(F(A),F(B)).$$

Now substitute and apply 3.5. Q.E.D.

§ 4. Hamiltonian twist flows

4.1. When α is a *simple* closed curve, then the Hamiltonian flows generated by a function $f_\alpha:\operatorname{Hom}(\pi,G)/G\to\mathbb{R}$ associated to an invariant function $f:G\to\mathbb{R}$ has a very neat explicit description. In particular there is a natural family of vector fields $\tilde{H}f_\alpha$ on $\operatorname{Hom}(\pi,G)$ covering the Hamiltonian vector field Hf_α on $\operatorname{Hom}(\pi,G)/G$. The description of these vector fields and the flows they generate generalize the Fenchel-Nielsen twist flows on Teichmüller space.

To motivate the definition of these "generalized twist flows" we briefly recall the Fenchel-Nielsen flows on Teichmüller space, referring to Wolpert [18] and Kerckhoff [13] for further details. The Teichmüller space $\mathfrak{T}_S$ of S is defined as the set of equivalence classes of pairs (M,f) where M is a hyperbolic surface and $f:S\to M$ is a homotopy equivalence; (M,f) and (M',f') are equivalent if there is an isometry $h:M\to M'$ such that $h\circ f\simeq f'$. For every nontrivial homotopy class α of simple closed geodesics on S and for every point m in $\mathfrak{T}_S$, there is a unique simple closed geodesic $\alpha(m)$ on M_m in the homotopy class $(f_m)_*\alpha$.

Definition. Let S by a surface and $\alpha\subset S$ a simple closed curve. S *split along* α is the surface-with-boundary $S|\alpha$ for which there exists a quotient map $j:S|\alpha\to S$ which maps $\operatorname{int}(S|\alpha)$ homeomorphically onto $S-\alpha$ and identifies the two components α_+, α_- of $\partial(S|\alpha)$ to α.

The *Fenchel-Nielsen twist flow about* α is defined as follows. For any point $m\in\mathfrak{T}_S$ we will describe its image $m(t)$ under the time t map of the flow. Let (M_m,f_m) represent $m\in\mathfrak{T}_S$ and let $(M|\alpha)_m$ denote the split surface $M_m|\alpha(m)$. There is a canonical isometry $i_\alpha:\alpha_+(m)\to\alpha_-(m)$ between the two components of $\partial(M|\alpha)_m$. Since $\alpha_+(m)$ and $\alpha_-(m)$ are geodesics, there is a whole one-parameter family of isometries $\theta_t^\pm:\alpha_\pm(m)\to\alpha_\mp(m)$, $t\in\mathbb{R}$, $i_\alpha(0)=id$, which is uniquely determined by the requirement that each $\theta_t^{\pm} x$ is a positively oriented unit speed path on $\alpha_{\pm}(m)$. Let $M_{m(t)}$ denote the hyperbolic surface which is $(M|\alpha)_m$ with $\alpha_+(m)$ and $\alpha_-(m)$ identified via $i_\alpha \circ \theta_t^+$. To define the flow line $m(t)$ in $\mathfrak{C}_S$, it suffices to define homotopy-equivalences $S \to M_{m(t)}$. This can be accomplished as follows. For each oriented curve γ in M_m which is transverse to $\alpha(m)$, there is a split curve $\gamma|\alpha$ in $(M|\alpha)_m$ which projects to γ under the quotient map. $\gamma|\alpha$ is a disjoint union of oriented intervals whose endpoints lie on α_+ and α_-. We shall modify $\gamma|\alpha$ by inserting arcs to obtain a union of intervals $(\gamma|\alpha)_t$ on $(M|\alpha)_m$ which projects to a closed curve on $M_{m(t)}$. For each component of $\gamma|\alpha$ whose terminal endpoint is $x \in \alpha_{\pm}(m)$, insert the arc $s \mapsto \theta^{\pm}_{\pm st}(x)$, $0 \leqq s \leqq 1$. The resulting curve on $(M|\alpha)_m$ now covers a closed curve γ_t on $M_{m(t)}$. Moreover the map $\gamma \mapsto \gamma_t$ induces an isomorphism $\pi_1(M_m) \to \pi_1(M_{m(t)})$, hence a homotopy-equivalence $h_t: M_m \to M_{m(t)}$. Thus we obtain a well-defined point $m(t)$ in $\mathfrak{C}_S$ represented by $(M_m, h_t \circ f)$.

There is a canonical map $\mathfrak{C}_S \to \mathrm{Hom}(\pi, G)/G$, $G = PSL(2, \mathbb{R})$, defined as follows. Suppose $m \in \mathfrak{C}_S$ is represented by (M, f), then f_* maps $\pi = \pi_1(S)$ isomorphically onto $\pi_1(M)$ which is a discrete subgroup of G. Changing representatives (M, f) only changes the inclusion $\pi_1(M) \subset G$ by an inner automorphism of G. It is known (see e.g. [7], [15]) that the resulting map $\mathfrak{C}_S \to \mathrm{Hom}(\pi, G)/G$ is a homeomorphism of $\mathfrak{C}_S$ onto a connected component of $\mathrm{Hom}(\pi, G)/G$.

The deformation $m(t)$ is "concentrated" at α; since $(M|\alpha)_m \approx (M|\alpha)_{m(t)}$, the only objects in M which geometrically change in $M_{m(t)}$ are those which intersect α. Let C be a component of $S|\alpha$; since α is nontrivial in π, the natural map $\pi_1(C) \to \pi_1(S)$ is injective. Since the split surfaces $(M|\alpha)_{m(t)}$ remain isometric the restriction to $\pi_1(C)$ of the corresponding path in $\mathrm{Hom}(\pi, G)/G$ is constant. This motivates the following general definition:

4.2. Definition. A *generalized twist flow* about α is a flow $\{\eta_t\}_{t \in \mathbb{R}}$ on $\mathrm{Hom}(\pi, G)$ such that for each $\phi \in \mathrm{Hom}(\pi, G)$, the deformation $\eta_t \phi$ restricts to a *trivial deformation* $\pi_1(C) \to G$ for each component C of $S|\alpha$, i.e. there exists a path $g_t = g_t(C)$ in G such that $\eta_t \phi(\gamma) = g_t \phi(\gamma) g_t^{-1}$ for $\gamma \in \pi_1(C) \subset \pi$. If $\{\eta_t\}_{t \in \mathbb{R}}$ is a generalized twist flow on Hom (π, G), then the tangent vector field $\left.\frac{d}{dt}\right|_{t=0} \eta_t$ will be called a *generalized twist field* (about α) on $\mathrm{Hom}(\pi, G)$. It is easy to see that a vector field ξ on $\mathrm{Hom}(\pi, G)$ is a generalized twist field about α if and only if for each component C of $S|\alpha$, the restriction of $\xi(\phi) \in T_\phi \mathrm{Hom}(\pi, G) = Z^1(\pi; \mathfrak{g}_{\mathrm{Ad}\,\phi})$ to $\pi_1(C)$ is a coboundary, i.e. for each C, there exists $\lambda_C \in \mathfrak{g}$ such that $\xi(\phi)(\gamma) = \lambda_C - \mathrm{Ad}\,\phi(\gamma)\lambda_C$ for $\gamma \in \pi_1(C)$.

4.3. Theorem. *Let $\alpha \subset S$ be a simple loop and $f: G \to \mathbb{R}$ an invariant function. There exists a generalized twist flow $\{\Xi_t\}_{t \in \mathbb{R}}$ about α on* $\mathrm{Hom}(\pi, G)$ *which covers the Hamiltonian flow on* $\mathrm{Hom}(\pi, G)/G$ *associated to f_α:* $\mathrm{Hom}(\pi, G)/G \to \mathbb{R}$.

It will be convenient to normalize these flows as follows. If α is *nonseparating*, ($S|\alpha$ is connected) then by applying an inner automorphism, we may assume that a generalized twist flow about α is constant on the subgroup $\pi_1(S|\alpha) \subset \pi_1(S)$, i.e. $\Xi_t \phi(\gamma) = \phi(\gamma)$ for all $\gamma \in \pi_1(S|\alpha)$. In case α is separating ($S|\alpha$ has two components) then we normalize $\Xi_t \phi$ by requiring it be constant on the fundamental group of one component of $S|\alpha$.

Theorem 4.3 will be proved constructively: we shall give an explicit twist field covering Hf_α. Thus we shall prove a statement which is stronger than 4.3. Hence we will split the discussion into two cases, depending on whether α is separating or not.

4.4. The twist flow associated to a separating loop

Let $\alpha \subset S$ be a separating simple loop. Then $S|\alpha$ is a disjoint union $S_1 \cup S_2$ where $\text{genus}(S_1)+\text{genus}(S_2)=\text{genus}(S)$. Accordingly the fundamental group π of S is a free product of its subgroups $\pi_1(S_1)$ and $\pi_1(S_2)$ amalgamated over the cyclic subgroup generated by α.

Suppose that $\{\Xi_t\}_{t\in\mathbb{R}}$ is any generalized twist flow about α on $\text{Hom}(\pi, G)$, normalized so that $\Xi_t\phi(\gamma)=\phi(\gamma)$ for $\gamma\in\pi_1(S_1)$. Then since $\Xi_t\phi$ is a trivial deformation when restricted to $\pi_1(S_2)$, there exists a path $\{\zeta_t(\phi)\}_{t\in\mathbb{R}}$ in G such that $\Xi_t\phi(\gamma)=\zeta_t(\phi)\phi(\gamma)\zeta_t(\phi)^{-1}$ for $\gamma\in\pi_1(S_2)$. Since $\alpha\in\pi_1 S_1\cap\pi_1 S_2$, $\phi(\alpha)=\Xi_t\phi(\alpha)=\zeta_t(\phi)\phi(\alpha)\zeta_t(\phi)^{-1}$ whence $\zeta_t(\phi)$ is a path in the centralizer $Z(\phi(\alpha))$ of $\phi(\alpha)$ in G. Conversely, given any such path $\zeta_t(\phi)$, the conditions

$$\Xi_t\phi:\gamma\mapsto\begin{cases}\phi(\gamma) & \text{if } \gamma\in\pi_1(S_1)\\ \zeta_t(\phi)\phi(\gamma)\zeta_t(\phi)^{-1} & \text{if } \gamma\in\pi_1(S_2)\end{cases} \tag{4.4}$$

uniquely determine a family of homomorphism $\Xi_t\phi\in\text{Hom}(\pi, G)$.

4.5 Theorem. *Let α be a separating simple loop and let S_1, S_2 be the components of $S|\alpha$. For each $\phi\in\text{Hom}(\pi, G)$, $t\in\mathbb{R}$, let $\zeta_t(\phi)=\exp tF(\phi(\alpha))$ where $F: G\to\mathfrak{g}$ is the variation function discussed in* § 1. *Then the flow $\{\Xi_t\}_{t\in\mathbb{R}}$ defined by* (4.4) *is a flow on* $\text{Hom}(\pi, G)$ *which covers the Hamiltonian flow on* $\text{Hom}(\pi, G)/G$ *associated to f_α.*

4.6. The twist flow associated to a nonseparating loop

Let $\alpha\subset S$ be a nonseparating simple loop. Then there exists another simple loop β which intersects α one transversely with positive intersection number. Then $\pi_1(S)$ is generated by the subgroup $\pi_1(S|\alpha)$ and β with the relation $\beta\alpha_+\beta^{-1}\alpha_-=1$, i.e. $\pi_1(S)$ is an *HNN* extension of $\pi_1(S|\alpha)$ where the elements α_+, $(\alpha_-)^{-1}$ of $\pi_1(S|\alpha)$ are made conjugate. We may assume that the image of α_+ under $\pi_1(S|\alpha)\to\pi_1(S)$ is α.

Suppose that $\{\Xi_t\}_{t\in\mathbb{R}}$ is any generalized twist flow on $\text{Hom}(\pi, G)$ about α, normalized so that $\Xi_t\phi(\gamma)=\phi(\gamma)$ for $\gamma\in\pi_1(S|\alpha)$. Writing $\Xi_t\phi(\beta)=\phi(\beta)\zeta_t(\phi)$, we find that

$$\Xi_t\phi(\beta)\phi(\alpha)\,(\Xi_t\phi(\beta))^{-1}=\Xi_t\phi(\beta\alpha\beta^{-1})=\phi(\beta\alpha\beta^{-1})=\phi(\beta)\phi(\alpha)\phi(\beta)^{-1}$$

whence $\zeta_t(\phi)$ is a path in $Z(\phi(\alpha))$. Conversely if $\zeta_t(\phi)$ is such a path,

$$\Xi_t\phi:\begin{cases}\gamma\mapsto\phi(\gamma) & \text{if } \gamma\in\pi_1(S|\alpha)\\ \beta\mapsto\phi(\beta)\zeta_t(\phi)\end{cases} \tag{4.6}$$

defines a family of homomorphisms $\pi\to G$.

4.7. Theorem. *Let α be a nonseparating simple loop on S and β as above. For each $\phi \in \operatorname{Hom}(\pi, G)$ let $\zeta_t(\phi) = \exp tF(\phi(\alpha))$. Then the flow defined by* (4.6) *is a generalized twist flow on* $\operatorname{Hom}(\pi, G)$ *which covers the Hamiltonian flow on* $\operatorname{Hom}(\pi, G)/G$ *associated to f_α.*

Proof of 4.5. Let $F_\alpha(\phi) = F(\phi(\alpha))$. Let ξ denote the generalized twist field tangent to $\{\Xi_t\}_{t \in \mathbb{R}}$; for $\phi \in \operatorname{Hom}(\pi, G)$, the cocycle $\xi\phi \in Z^1(\pi; \mathfrak{g}_{\operatorname{Ad}\phi})$ is defined by

$$\xi\phi(\gamma) = \begin{cases} 0 & \text{if } \gamma \in \pi_1(S_1) \\ F_\alpha(\phi) - \operatorname{Ad}\phi(\gamma) F_\alpha(\phi) & \text{if } \gamma \in \pi_1(S_2). \end{cases}$$

We shall prove $[\xi\phi] = Hf_\alpha$ by showing they have the some Poincaré dual. Thus by 2.2 and 3.7 it suffices to show that for any cycle $z \in Z_1(S; \mathfrak{g}^*_{\operatorname{Ad}\phi})$, the cap-product $[\xi\phi] \cap [z]$ equals the intersection product $(\alpha \otimes F_\alpha(\phi)) \cdot z$. Since $\xi\phi$ is a cocycle we may replace z by a homologous cycle which is better adapted to α. For example we may assume z is a sum of 1-simplices $\gamma \otimes c$ where each $\gamma: [0,1] \to S$ has $\gamma(0) = \gamma(1) = p_0 \in \alpha$ and γ descends to an immersion $\tilde{\gamma}: S^1 = [0,1]/0 \sim 1 \to S$ which intersects α transversely in double points. Order the double points as they are encountered along $\gamma: \alpha \# \gamma = \{p_1, \ldots, p_{2k+1}\}$ where $p_1 = \gamma(0)$, $\gamma^{-1}(p_i) < \gamma^{-1}(p_{i+1})$, $p_{2k+1} = \gamma(1) = p_1$, etc. Since γ is transverse to α, it is split into segments $\bar{\gamma}_1, \ldots, \bar{\gamma}_{2k}$ satisfying $\partial \bar{\gamma}_1 = \{p_i, p_{i+1}\}$, $\bar{\gamma}_{2i+1} \subset S_1$, $\bar{\gamma}_{2i} \subset S_2$, for example. Furthermore the intersection numbers $\varepsilon_i = \varepsilon(p_i; \gamma, \alpha)$ alternate ± 1, i.e. $\varepsilon_i = (-1)^{i+1} \varepsilon_1$.

Now choose subarcs α_i of α which run from p_1 to p_i; for $i = 1, 2k+1$, let α_i be the constant path p_1. Let γ_i be the loop $\alpha_i * \bar{\gamma}_i * \alpha_{i+1}^{-1}$ based at p_1. Then

$$\begin{aligned} \gamma = \bar{\gamma}_1 * \ldots * \bar{\gamma}_n &\cong \alpha_1 * \bar{\gamma}_1 * \alpha_2^{-1} * \alpha_2 * \bar{\gamma}_2 * \ldots * \bar{\gamma}_{2k} * \alpha_{2k+1}^{-1} \\ &= \gamma_1 \ldots \gamma_{2k}. \end{aligned}$$

Here $\gamma_{2i+1} \in \pi_1(S_1)$ and $\gamma_{2i} \in \pi_1(S_2)$. Accordingly we can decompose the flat section over γ which equals c at p_1 as (with our customary abuse of notation)

$$\gamma \otimes c = \sum_{i=1}^{2k} \gamma_i \otimes c_i$$

with $c_i = \operatorname{Ad}\phi((\gamma_1 \ldots \gamma_i)^{-1}) c$ the parallel transport of c over the arc of γ from p_1 to p_i. Then evaluating the 1-cocycle $\xi\phi$ on the 1-chain $\gamma \otimes c$ we obtain

$$\begin{aligned} \xi\phi \cap (\gamma \otimes c) &= \langle \xi\phi(\tilde{\gamma}), c \rangle = \langle \xi\phi(\gamma_1 \ldots \gamma_{2k}), c \rangle \\ &= \sum_{i=1}^{k} \langle \operatorname{Ad}\phi(\gamma_1 \ldots \gamma_{2i-2}) (I - \operatorname{Ad}\phi(\gamma_{2i-1})) F_\alpha(\phi), c \rangle \\ &= \sum_{i=1}^{2k} (-1)^i \langle \operatorname{Ad}\phi(\gamma_1 \ldots \gamma_{i-1}) F_\alpha(\phi), c \rangle \\ &= \sum_{p_i \in \alpha \# \gamma} \varepsilon_i \langle F_\alpha(\phi), c_i \rangle = (\alpha \otimes F_\alpha(\phi)) \cdot (\gamma \otimes c). \quad \text{Q.E.D.} \end{aligned}$$

Proof of 4.7.

The idea of the proof of 4.7 is identical to that of 4.5; therefore we give only the necessary modifications for that proof. Again we consider a chain $\gamma \otimes c$ where $\gamma : [0,1] \to S$ descends to an immersed closed curve $\tilde{\gamma}$ which intersects α transversely, in double points. Again we order $\alpha \# \gamma$ as $\{p_1, \ldots, p_{k+1}\}$ where $p_1 = \gamma(0), \ldots, p_{k+1} = \gamma(1) = p_1$; similarly we choose arcs α_i on α from p_1 to p_i. Again let $\varepsilon_i = \varepsilon(p_i; \alpha, \gamma)$.

The split curve $\gamma|\alpha$ on $S|\alpha$ is a disjoint union of intervals $\bar{\gamma}_i$ whose endpoints lie on α_+ or α_-. Using the obvious notation, we see that $\bar{\gamma}_i$ has initial endpoint $p_i^{\varepsilon_1 \ldots \varepsilon_i} \in \alpha_{\varepsilon_1 \ldots \varepsilon_i}$ and terminal endpoint $p_{i+1}^{-\varepsilon_1 \ldots \varepsilon_{i+1}} \in \alpha_{-\varepsilon_1 \ldots \varepsilon_{i+1}}$. Let $\bar{\beta}$ be the arc $\beta|\alpha$ which runs from p_1^+ to p_1^-. Let γ_i be the loop

$$\bar{\beta}^{(1-\varepsilon_1 \ldots \varepsilon_i)/2} * \alpha_i^{\varepsilon_1 \ldots \varepsilon_i} * \bar{\gamma}_i * \alpha_i^{-\varepsilon_1 \ldots \varepsilon_i - 1} * (\bar{\beta}^{-1})^{(1+\varepsilon_1 \ldots \varepsilon_{i+1})/2}$$

in $S|\alpha$ based at p_1. In terms of the generating set $\pi_1(S|\alpha) \cup \{\beta\}$ for π, we see that $\gamma \simeq \beta^{\varepsilon_1}\gamma_1 \ldots \beta^{\varepsilon_k}\gamma_k$. Accordingly we shall decompose the flat section $\gamma \otimes c = \sum \gamma_i \otimes c_i$. Observe that for each p_i with $\varepsilon_i = -1$, the loop γ_i has travelled an extra β_i to get from p_1^+ to p_1^+ via $p_{i+1}^{-\varepsilon_1 \ldots \varepsilon_{i+1}}$. Thus parallel transporting c along $\gamma_1 \ldots \gamma_i$ gives $c_j = \operatorname{Ad}\phi((\beta^{\varepsilon_1}\gamma_1 \ldots \beta^{\varepsilon_{i-1}}\gamma_{i-1}\beta^{(\varepsilon_i - 1)/2})^{-1})c$. Now we compute, just as for 4.5:

$$\begin{aligned}
\xi\phi \frown (\gamma \otimes c) &= \langle \xi\phi(\tilde{\gamma}), c \rangle = \langle \xi\phi(\beta^{\varepsilon_1}\gamma_1 \ldots \beta^{\varepsilon_k}\gamma_k), c \rangle \\
&= \sum_{i=1}^{k} \varepsilon_i \langle \operatorname{Ad}\phi(\beta^{\varepsilon_1}\gamma_1 \ldots \beta^{\varepsilon_{i-1}}\gamma_{i-1}\beta^{(\varepsilon_i - 1)/2}) F_\alpha(\phi), c \rangle \\
&= \sum_{i=1}^{k} \varepsilon_i \langle F_\alpha(\phi), \operatorname{Ad}\phi((\beta^{\varepsilon_1} \ldots \gamma_{i-1}\beta^{(\varepsilon_i - 1)/2})^{-1})\, c \rangle \\
&= \sum_{p_i \in \gamma \# \alpha} \varepsilon_i \langle F_\alpha(\phi), c_i \rangle = (\alpha \otimes F_\alpha(\phi)) \cdot (\gamma \otimes c). \quad \text{Q.E.D.}
\end{aligned}$$

4.11. Wolpert's duality formula

As an application of 4.3 we prove the following theorem, due to Wolpert [19]:

Theorem. *Let $\mathfrak{T}_S$ be the Teichmüller space of S with its Weil-Petersson symplectic structure. Let α be a simple loop and $l_\alpha : \mathfrak{T}_S \to \mathbb{R}$ the geodesic length function associated to α. Then the Hamiltonian flow associated to l_α is the Fenchel-Nielsen twist flow about α.*

Proof. It is easy to see from the discussion in 4.1 that the Fenchel-Nielsen twist flow about α is covered by a generalized twist flow $\{\Xi_t\}_{t \in \mathbb{R}}$ on the subset of $\operatorname{Hom}(\pi, G)$ above $\mathfrak{T}_S$. Since the boundary components α_+ and α_- of $M|\alpha$ are identified by the isometries $i_\alpha \circ \theta_t : \alpha_+ \to \alpha_-$ which moves points forward at unit speed along α_-, the corresponding representations ϕ are deformed via a path $\xi_t(\phi)$ in G which translates points on the $\phi(\alpha)$-invariant geodesic with unit speed in the same direction that $\phi(\alpha)$ moves them. In other words, $\{\Xi_t(\phi)\}_{t \in \mathbb{R}}$ is the unique one-parameter subgroup of G which contains $\phi(\alpha)$ with $\xi_t(\phi) = \phi(\alpha)$ for exactly $t = \frac{1}{2} l_\alpha(\phi)$.

Let $\mathfrak{B}$, l, and L be as in 1.13. It is proved in 1.13 that $\xi_t(\phi)$, as described above, equals $\exp tL(\phi(\alpha))$. Thus 4.5 and 4.7 imply that the generalized twist flow $\{\Xi_t\}_{t \in \mathbb{R}}$ covering the Fenchel-Nielsen flow covers the Hamiltonian flow associated to l_α.

§5. Lie algebras of curves on a surface

5.1. The product formula 3.5 led us to relate Poisson brackets of trace functions f_α, f_β to sums of the form $\sum_{p\in\alpha\#\beta} \varepsilon(p;\alpha,\beta)u(\alpha_p,\beta_p)$. The purpose of this section is to define abstract Lie algebra structures on spaces based on closed curves. We shall obtain Lie algebras which map homomorphically into the Lie algebra of regular functions on $\mathrm{Hom}(\pi, G)/G$ under Poisson bracket, for various G.

Let S be any oriented surface (not necessarily compact) and let $\pi=\pi_1(S)$. Let $\hat{\pi}$ denote the set of conjugacy classes in π, i.e. the set of homotopy classes of oriented closed curves in S. If $\alpha\in\pi$, denote its conjugacy class by $|\alpha|$. Let $\mathbb{Z}\hat{\pi}$ denote the free abelian group with basis $\hat{\pi}$; then the map $\pi\to\hat{\pi}$ given by $\alpha\mapsto|\alpha|$ extends by linearity to a linear map $\mathbb{Z}\pi\to\mathbb{Z}\hat{\pi}$ also denoted by $|\ |$. If α is a closed curve and $p\in\alpha$ is a simple point (i.e. if in a parametrization $f: S^1\to S$ of α, the inverse image $f^{-1}(p)$ is connected) we denote by α_p the loop α based at p, as well as its homotopy class in $\pi_1(S;p)$.

Suppose that α, β are immersed loops in S which are *generic* (i.e. the map $\alpha\cup\beta: S^1\cup S^1\to S$ is an immersion with at worst transverse double points). Then their bracket is defined as the following element of $\mathbb{Z}\hat{\pi}$:

$$[\alpha,\beta]=\sum_{p\in\alpha\#\beta}\varepsilon(p;\alpha,\beta)\,|\alpha_p\beta_p|\,.$$

We shall first prove that $[\alpha,\beta]$ depends only on the homotopy classes of α and β:

5.2. Theorem. *Let α, β be a generic pair of immersed loops and let α', β' be another such pair such that α' is freely homotopic to α and β' is freely homotopic to β. Then $[\alpha,\beta]=[\alpha',\beta']$ in $\mathbb{Z}\hat{\pi}$.*

It follows that the bracket gives a well-defined map $\hat{\pi}\times\hat{\pi}\to\mathbb{Z}\hat{\pi}$, since every loop in S is homotopic to an immersion and every pair of loops in S is homotopic to a generic pair of immersions. We extend the bracket by linearity to a bilinear map $[\ ,\ ]:\mathbb{Z}\hat{\pi}\times\mathbb{Z}\hat{\pi}\to\mathbb{Z}\hat{\pi}$.

5.3 Theorem. *$\mathbb{Z}\hat{\pi}$ is a Lie algebra under* $[\ ,\ ]$.

Theorem 3.13 says that on $\mathrm{Hom}(\pi, G)/G$, where $G=\mathrm{GL}(n,\mathbb{R})$, $\mathrm{GL}(n,\mathbb{C})$ or $\mathrm{GL}(n,\mathbb{H})$ the trace functions f_α satisfy the same commutation relations as do curves α in $\mathbb{Z}\hat{\pi}$. We restate this formally using the following definition: Let (X,ω) be a symplectic manifold and $C^\infty(X,\omega)$ the Lie algebra of smooth functions on X under Poisson bracket. By a Poisson action *of a Lie algebra* $\mathfrak{g}$ *on* (X,ω) we shall mean a Lie algebra homomorphism $\varrho:\mathfrak{g}\to C^\infty(X,\omega)$.

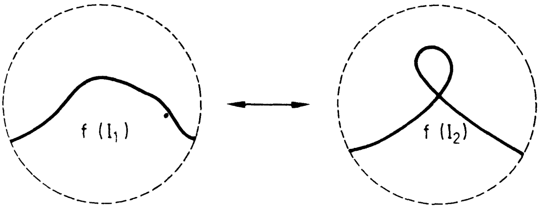
5.4. Theorem. *Let* $G=\mathrm{GL}(n,\mathbb{R})$, $\mathrm{GL}(n,\mathbb{C})$, *or* $\mathrm{GL}(n,\mathbb{H})$. *Then $\alpha\mapsto f_\alpha$ defines a Poisson action of $\mathbb{Z}\hat{\pi}$ on* $\mathrm{Hom}(\pi, G)/G$.

The first part of this section is devoted to the proofs of 5.2 and 5.3. Then we discuss the most elementary properties of the homomorphism of 5.4. In particular this homomorphism seems never to be injective: linear relations between traces of words in G gives rise to elements of the kernel. For $G=\mathrm{GL}(1,\mathbb{R})$ or $\mathrm{GL}(1,\mathbb{C})$, the representation of $\mathbb{Z}\hat{\pi}$ on $\mathrm{Hom}(\pi, G)/G$ (which is just $H^1(S:\mathbb{R}^*)$ or $H^1(S:\mathbb{C}^*)$) is completely analyzed. We show that its image is a Lie algebra $\mathbb{Z}H$ which is additively a free $\mathbb{Z}$-module with basis the integral homology $H = H_1(S;\mathbb{Z})$. The map $\hat{\pi} \to H$ obtained by taking homology class defines a Lie algebra homomorphism $\mathbb{Z}\hat{\pi} \to \mathbb{Z}H$ through which the representation of 5.4 factors, when G is abelian.

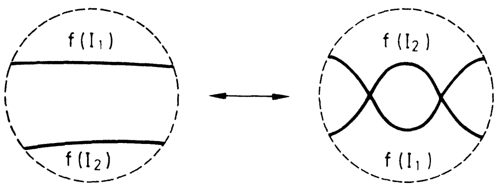
5.5. Now we turn to the proofs of 5.2 The idea is that the bracket $[\alpha, \beta]$ is unchanged as α and β undergo a generic homotopy of closed curves. Let M denote a closed 1-manifold and recall that a smooth map $f: M \to S$ is *generic* if it is an immersion whose only self-intersections are transverse double points. Generic immersions thus form a dense open subset $\mathrm{Imm}(M, S)_0$ of both the space of all immersions $\mathrm{Imm}(M, S)$ and the space of all smooth maps $C^\infty(M, S)$ (with the Fréchet topology).

5.6. Lemma. *Suppose f, $g \in \mathrm{Imm}(M, S)_0$ are generic immersions which are homotopic in $C^\infty(M, S)$ (i.e. are homotopic maps $M \to S$). Then there exists a sequence $f_1 = f, f_2, f_3, \ldots, f_{k-1}, f_k = g$ of generic immersions such that f_{i+1} is related to f_i be one of the following standard moves:*

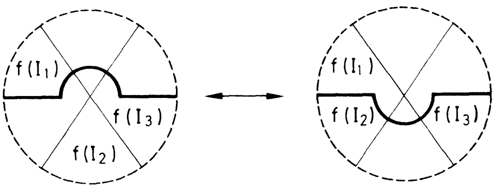
(ω1) *birth-death of monogons*



(ω2) *birth-death of bigons*



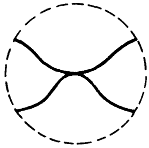
(ω3) *jumping over a double point*



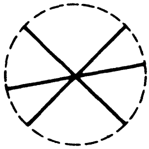
To apply a move (ωj) to a generic map $f \in \mathrm{Imm}(M, S)_0$, we find j intervals $I_1, \ldots, I_j (1 \leqq j \leqq 3)$ for which $f|_I$ (where $I = I_1 \cup \ldots \cup I_j$) is one of the cases pictured above for (ωj), and replace $f|_I$ with the other picture, leaving $f|_{M-I}$ fixed. (We emphasize, however, that these pictures are only "immersed over" $f(M-I)$ and are not embeddings. That is, we cannot necessarily assume there is an open disc containing the double points that is unchanged under the move and is furthermore disjoint from $f(M-I)$. Questions concerning the existence of embedded bigons are quite subtle; see Hass-Scott [9] for more information.)

Proof of 5.6. Standard results in transversality (as in Hirsch [11]) imply that the subset $\mathrm{Imm}(M,S)_1$ consisting of immersions $f: M \to S$ with finitely many transverse double points and exactly one of either of the following singularities:

(tangential double point)



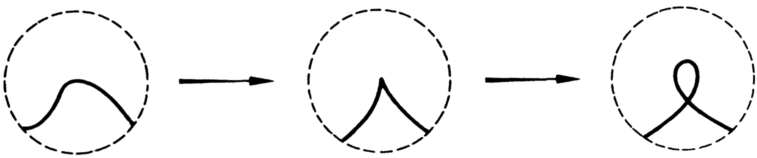
(transverse triple point)



has codimension one in $\mathrm{Imm}(M,S)$. Let $\mathfrak{C}$ be the subset of $C^\infty(M,S)$ consisting of maps $f: M \to S$ such that there exists a unique $x \in M$ such that $f|_{M-\{x\}} \in \mathrm{Imm}(M-\{x\},\ S)_0$, $df(x)=0$ but $d^2f(x) \neq 0$. Then $\mathfrak{C} \cup \mathrm{Imm}(M,S)_1 = C^\infty(M,S)_1$ has codimension one in $C^\infty(M,S)$.

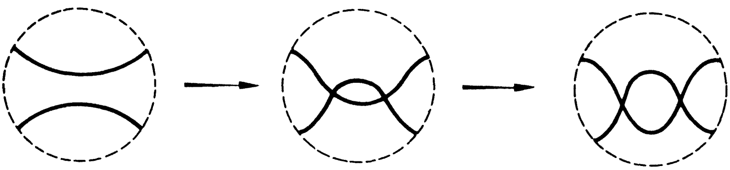
Since $f, g \in C^\infty(M,S)_0$ are homotopic, by transversality they are homotopic via a generic path in $C^\infty(M,S)$, i.e. one which completely misses the codimension $\geqq 2$ subset $C^\infty(M,S) - (\mathrm{Imm}(M,S)_0 \cup C^\infty(M,S)_1)$ and meets $C^\infty(M,S)_1$ transversely. Thus there is a homotopy $F_t: M \to S$, $F_0 = f$, $F_1 = g$, such that $F_t \in \mathrm{Imm}(M,S)_0 \cup C^\infty(M,S)_1$ for all $0 \leqq t \leqq 1$, and for only finitely many values of $t (t = t_1, \ldots, t_k)$ does $F_t \in C^\infty(M,S)_1$. Over each open interval (t_i, t_{i+1}), this homotopy is carried by an isotopy of S. When $F_t \in C^\infty(M,S)_1$, there are three possibilities:

(i) $F_t \in \mathfrak{C}$. In that case the homotopy in a neighborhood of the singular point $x \in M$ is a homotopy between maps pictured as

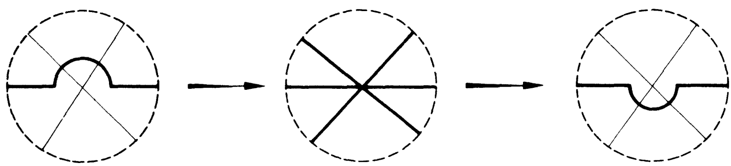


and passing the critical point corresponds to move $(\omega 1)$.

(ii) $F_t \in \mathrm{Imm}(M,S)_1$ *and has one tangency*. In that case the homotopy in a neighborhood of the two preimages of the tangential double point looks like



(iii) $F_t \in \mathrm{Imm}(M,S)_1$ *and has one triple point*. In that case the homotopy looks like

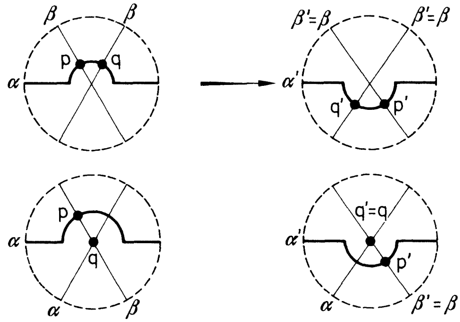


and passing the triple point corresponds to move ($\omega 3$).

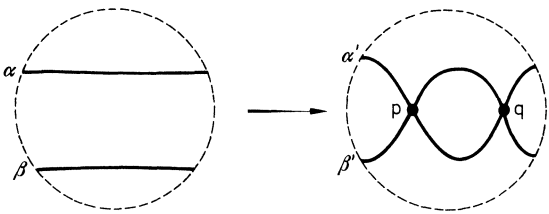
To find the sequence $f_1, \dots, f_k$ take $f_1 = f$, $f_k = g$ and $F_{i+1} = F_{t_i+\varepsilon}$ where $0 < \varepsilon < \min_{i=1,\dots,k-1} (t_{i+1} - t_i)$. The proof of 5.8 is now complete. Q.E.D.

Proof of 5.2. Applying 5.8 to the case M is a disjoint union of two circles and $f = \alpha \cup \beta \in \mathrm{Imm}(M,S)$ and $g = \alpha' \cup \beta' \in \mathrm{Imm}(M,S)$, we readily reduce 5.2 to showing $[\alpha, \beta] = [\alpha', \beta']$ when $\alpha \cup \beta$ and $\alpha' \cup \beta'$ are related by one of three moves ($\omega 1$), ($\omega 2$), or ($\omega 3$). Since ($\omega 1$) does not affect the intersection $\alpha \# \beta$ and $\alpha \cong \alpha'$, $\beta \cong \beta'$, the brackets $[\alpha, \beta]$ and $[\alpha', \beta']$ are equal.

Next suppose that $\alpha \cup \beta$ and $\alpha' \cup \beta'$ are related by move ($\omega 3$). We may assume $\alpha \# \beta \neq \alpha' \# \beta'$ since otherwise the assertion is obvious, as above. Then there exists a pair $\{p, q\}$ of double points in $\alpha \# \beta$ which are replaced by a pair $\{p', q'\}$ of double points in $\alpha' \# \beta'$ and $\alpha \# \beta - \{p, q\} = \alpha' \# \beta' - \{p', q'\}$. We may assume there is a disc $D \subset S$ so that on $(\alpha \cup \beta)^{-1}(S - D)$, both $\alpha \cup \beta$ and $\alpha' \cup \beta'$ agree. Inside D we find an arc from p to p' (resp. q to q') which induce isomorphism $\pi_1(S; p) \cong \pi_1(S; p')$ (resp. $\pi_1(S; q) \cong \pi_1(S; q')$) which take α_p, β_p to $\alpha_{p'}$, $\beta_{p'}$ (resp. α_q, β_q to $\alpha_{q'}$, $\beta_{q'}$) correspond. In particular $|\alpha_p \beta_p| = |\alpha'_{p'} \beta'_{p'}|$ and $|\alpha_q \beta_q| = |\alpha'_{q'} \beta'_{q'}|$. Furthermore $\varepsilon(p; \alpha, \beta) = \varepsilon(p'; \alpha', \beta')$ and $\varepsilon(q; \alpha, \beta) = \varepsilon(q'; \alpha', \beta')$ whence $[\alpha, \beta] = [\alpha', \beta']$. Two typical cases are illustrated below:



Finally suppose that $\alpha \cup \beta$ and $\alpha' \cup \beta'$ are related by ($\omega 2$). If $\alpha \# \beta = \alpha' \# \beta'$ again there is nothing to prove. If $\alpha \# \beta \neq \alpha' \# \beta'$, then we may assume $\alpha' \# \beta' = \alpha \# \beta \cup \{p, q\}$ where p and q are the vertices of a bigon whose sides are arcs in α' and β':



Now $\alpha'_p\beta'_p$ and $\alpha'_q\beta'_q$ are homotopic and $\varepsilon(p;\alpha',\beta')=-\varepsilon(q;\alpha',\beta')$. Thus $[\alpha,\beta]=[\alpha',\beta']$, and the proof of 5.2 is complete.

Proof of 5.3. We must verify that $[,]$ is alternating and satisfies the Jacobi identity.

Alternating. If $p\in\alpha\#\beta$, then $\alpha_p\beta_p$ and $\beta_p\alpha_p$ are conjugate in $\pi_1(S;p)$, i.e. $|\alpha_p\beta_p|=|\beta_p\alpha_p|$. Now $\varepsilon(p;\alpha,\beta)=-\varepsilon(p;\beta,\alpha)$, so

$$[\alpha,\beta]=\sum_{p\in\alpha\#\beta}\varepsilon(p;\alpha,\beta)\,|\alpha_p\beta_p|=-\sum_{p\in\alpha\#\beta}\varepsilon(p;\alpha,\beta)\,|\beta_p\alpha_p|=-[\beta,\alpha].$$

Jacobi identity. By 5.2 we may assume $\alpha\cup\beta\cup\gamma$ is a generic immersion of a disjoint union of 3 circles into S. The bracket $[[\alpha,\beta],\gamma]$ is a sum of terms, one for each $(p,q)\in(\alpha\#\beta)\times((\alpha\cup\beta)\#\gamma)$. The contribution from (p,q) is

$$\varepsilon(p;\alpha,\beta)\varepsilon(q;\alpha_p\beta_p,\gamma)\,|(\alpha_p\beta_p)_q\gamma_q|.$$

There are two cases, depending on whether $q\in\alpha$ or $q\in\beta$. Suppose $q\in\alpha$. Then in $[[\gamma,\alpha],\beta]$ there is a contribution

$$\varepsilon(q;\gamma,\alpha)\varepsilon(p;\gamma_q\alpha_q,\beta)\,|(\gamma_q\alpha_q)_p\beta_p|$$

coming from (q,p). These two terms cancel: it is easy to see that $|(\alpha_p\beta_p)_q\gamma_q|=|(\gamma_q\alpha_q)_p\beta_p|$; moreover

$$\begin{aligned}\varepsilon(p;\alpha,\beta)\varepsilon(q;\alpha_p\beta_p,\gamma)&=\varepsilon(p;\alpha,\beta)\varepsilon(q;\alpha,\gamma)\\&=-\varepsilon(q;\gamma,\alpha)\varepsilon(p;\alpha,\beta)=-\varepsilon(q;\gamma,\alpha)\varepsilon(p;\gamma_q\alpha_q,\beta).\end{aligned}$$

The case $q\in\beta$ is handled similarly as are the other terms in $[[\beta,\gamma],\alpha]+[[\gamma,\alpha],\beta]$ coming from $(\beta\#\alpha)\times(\alpha\#\gamma)$. Thus all the terms in $[[\alpha,\beta],\gamma]+[[\beta,\gamma],\alpha]+[[\gamma,\alpha],\beta]$ cancel. The proof of 5.3 is now complete. Q.E.D.

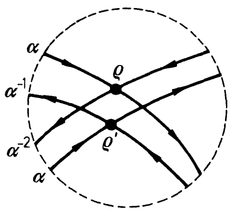
Algebraic properties of $\mathbb{Z}\hat{\pi}$

5.9. Now that we have established that $\mathbb{Z}\hat{\pi}$ is a Lie algebra, we briefly mention a few of its algebraic properties. First of all, if $N\lhd\pi$ is a normal subgroup, then the submodule of $\mathbb{Z}\hat{\pi}$ generated by the image $\hat{N}$ of N under $|\,|:\pi\to\hat{\pi}$ is a subalgebra. In particular, taking $N=\{1\}$ the identity subgroup we find a one-dimensional subalgebra $\mathbb{Z}1\subset\mathbb{Z}\hat{\pi}$. Since 1 is represented by the trivial loop, the subalgebra $\mathbb{Z}1$ is central (probably it is the whole center of $\mathbb{Z}\hat{\pi}$). It is interesting to note that this central ideal is a direct summand of $\mathbb{Z}\hat{\pi}$.

Proposition. *Let $\hat{\pi}'=\hat{\pi}-\{1\}$ and let $\mathbb{Z}\hat{\pi}'$ be the free submodule of $\mathbb{Z}\hat{\pi}$ generated by $\hat{\pi}'$. Then $\mathbb{Z}\hat{\pi}'$ is an ideal in $\mathbb{Z}\hat{\pi}$ and there is a direct-sum decomposition $\mathbb{Z}\hat{\pi}=\mathbb{Z}1\oplus\mathbb{Z}\hat{\pi}'$.*

Proof. It suffices to prove that $\mathbb{Z}\hat{\pi}'$ is a subalgebra, since $\mathbb{Z}\hat{\pi}=\mathbb{Z}1\oplus\mathbb{Z}\hat{\pi}'$ as abelian groups and $\mathbb{Z}1$ is central. Thus we must show that if $\alpha,\beta\in\hat{\pi}'$, then $[\alpha,\beta]\in\mathbb{Z}\hat{\pi}'$. If not, there exists $p\in\alpha\#\beta$ with $|\alpha_p\beta_p|=1$. Thus α is freely homotopic to β^{-1} and we may assume, by 5.2 that $\beta=\alpha^{-1}$. It thus suffices to prove $[\alpha,\alpha^{-1}]=0$.

Represent α by a generic immersion and α^{-1} be a generic immersion such that $\alpha \cup \alpha^{-1}$ cobounds a narrow annulus. Then the double points in $\alpha \# \alpha^{-1}$ occur in pairs p, p', one pair for each self-intersection of α:



Now $|\alpha_p \alpha_p^{-1}| = |\alpha_{p'} \alpha_{p'}^{-1}|$ and $\varepsilon(p; \alpha, \alpha^{-1}) = -\varepsilon(p'; \alpha, \alpha^{-1})$ as is clear from the picture. It follows that $[\alpha, \alpha^{-1}] = 0$. Q.E.D.

Remark. This proposition has the following consequence for the Poisson actions of 5.4. If (X, ω) is any symplectic manifold then the exact sequence

$$0 \to \mathbb{R} \to C^\infty(X, \omega) \to \mathrm{Ham}(X, \omega) \to 0$$

(where $\mathrm{Ham}(X, \omega)$ is the Lie algebra of Hamiltonian vector fields) is a central extension of Lie algebras. Hence for any Poisson action $\varrho: \mathfrak{g} \to C^\infty(X, \omega)$ the preimage $\varrho^{-1}(\mathbb{R})$ of the constants $\mathbb{R} \subset C^\infty(X, \omega)$ is a central subgroup and there is an associated central extension $0 \to \varrho^{-1}(\mathbb{R}) \to \mathfrak{g} \to \mathfrak{g}/\varrho^{-1}(\mathbb{R}) \to 0$. For the Poisson actions of 5.4, $\mathbb{Z}1$ maps into constants and the central extensions is $0 \to \mathbb{Z}1 \to \mathbb{Z}\hat{\pi} \to \mathbb{Z}\hat{\pi}' \to 0$. Proposition 5.9 thus shows that this central extension is split, i.e. for every Poisson action on $\mathrm{Hom}(\pi, G)/G$ generated by trace functions f_α, the constant functions split off as a direct summand.

5.10. Now we discuss the Poisson action of $\mathbb{Z}\hat{\pi}$ on $\mathrm{Hom}(\pi, G)/G$ when $G = \mathrm{GL}(1, \mathbb{R})$. In that case $\mathrm{Hom}(\pi, G)/G = \mathrm{Hom}(\pi, G)$ is the abelian group $H^1(S, \mathbb{Z}/2) \oplus H^1(S; \mathbb{R})$. The identity component $\mathrm{Hom}(\pi, \mathrm{GL}_+(\mathbb{R}))$ is the symplectic vector space $H^1(S; \mathbb{R})$. In this case we may explicitly describe the Poisson action *in global coordinates*.

Let $X_1, \ldots, X_g$, $Y_1, \ldots, Y_g$ be the standard generators for π, i.e. π has the presentation with $X_1 Y_1 X_1^{-1} Y_1^{-1} \ldots X_g Y_g X_g^{-1} Y_g^{-1} = I$. (Here g denotes the genus of S.) Let $\xi_1, \ldots, \xi_g, \eta_1, \ldots, \eta_g$ be the images of $X_1, \ldots, X_g$, $Y_1, \ldots, Y_g$ in $H = H_1(S; \mathbb{Z})$. Let $(x_1, y_1, \ldots, x_g, y_g)$ be coordinates on $H^1(S; \mathbb{R})$ dual to $\xi_1, \eta_1, \ldots, \xi_2, \eta_g$; then the symplectic structure on $H^1(S; \mathbb{R})$ equals $\omega = \sum_{i=1}^{g} dx_i \wedge dy_i$.

There is an explicit diffeomorphism

$$\mathbb{R}^{2g} = H^1(S, \mathbb{R}) \to \mathrm{Hom}(\pi, G)/G(G = \mathrm{GL}_+(\mathbb{R}))$$

which associates to a point $(x_1, y_1, \ldots, x_g, y_g)$ the homomorphism which takes X_i to e^{x_i} and Y_i to e^{y_i}. If $\alpha \in \pi$ has homology class $\sum_{i=1}^{g} m_i \xi_i + n_i \xi_i$, then the trace function $f_\alpha: \mathrm{Hom}(\pi, G)/G \to \mathbb{R}$ is given by

$$f_\alpha(x_1, \ldots, y_g) = \exp\left(\sum_{i=1}^{g} (m_i x_i + n_i y_i) \right).$$

(Since G is abelian, f_α depends only on the homology class $[\alpha]\in H$.) It is easy to check the $\mathbb{Z}$-linear span of these functions form a Lie algebra of functions on $\mathbb{R}^{2g}$ which is a homomorphic image of $\mathbb{Z}\hat{\pi}$.

This Lie algebra can be described abstractly as follows. Let $\mathbb{Z}H$ be the free $\mathbb{Z}$-module on $H=H_1(S;\mathbb{Z})$. If $\alpha,\beta\in H$, define their bracket by $[\alpha,\beta]=(\alpha\cdot\beta)[\alpha+\beta]$ where $\alpha\cdot\beta$ denotes intersection pairing and $[\alpha+\beta]$ is the element of H (basis element of $\mathbb{Z}H$) corresponding to the sum of α and β in $H_1(S;\mathbb{Z})$. The natural map $[\,]:\mathbb{Z}\hat{\pi}\to\mathbb{Z}H$ defined by taking homology class, is a Lie algebra homomorphism through which the Poisson action of $\mathbb{Z}\hat{\pi}$ on $\mathrm{Hom}(\pi,\mathrm{GL}_+(1,\mathbb{R}))$ factors. Its kernel is the ideal of $\mathbb{Z}\hat{\pi}$ additively generated by 1 and formal sums $\alpha+\beta-\gamma$, where $[\alpha]+[\beta]=[\gamma]$.

5.11. Even if G is nonabelian, the representation $\varrho:\mathbb{Z}\hat{\pi}\to C^\infty(\mathrm{Hom}(\pi,G)/G)$ need not be faithful (we conjecture that it *never* is). Consider the case $G=\mathrm{GL}(2,\mathbb{R})$ or $\mathrm{GL}(2,\mathbb{C})$. Then if α lies in the commutator subgroup of π, then for all $\phi\in\mathrm{Hom}(\pi,G)$, $\mathrm{tr}\,\phi(\alpha)=\mathrm{tr}\,\phi(\alpha^{-1})$ although α is never conjugate to α^{-1} (unless $\alpha=1$). Thus, for example, the element $|\alpha|-|\alpha^{-1}|$ lies in the kernel of ϱ, if $\alpha\in[\pi,\pi]$. However there are much more complicated ways for two elements $\alpha,\beta\in\pi$ to satisfy $\mathrm{tr}\,\phi(\alpha)=\mathrm{tr}\,\phi(\beta)$ for all $\phi\in\mathrm{Hom}(\pi,G)$. The prototypical construction is the following. Let $w(x,y)$ be a word in x and y. Then $w(x,y)$ and $w(x^{-1},y^{-1})^{-1}$ (i.e. w "read backwards") have the same trace, whenever $x,y\in\mathrm{GL}(2,\mathbb{C})$. Such a pair of elements need not be conjugate or conjugate to each other's inverse. For a simple example, take $\alpha=xyx^{-1}y^2$ and $\beta=x^{-1}yxy^2$: $\mathrm{tr}\,\phi(\alpha)=\mathrm{tr}\,\phi(\beta)$ for all $\phi\in\mathrm{Hom}(\pi,G)$, $x,y\in\pi$, but α is not conjugate to β or β^{-1}. In particular we obtain many elements $|\alpha|-|\beta|$ which lie in $\mathrm{Ker}\,\varrho$. For more information, see Magnus [14] and the references quoted there

Lie algebras based on unoriented curves

5.12. Let $\iota:\hat{\pi}\to\hat{\pi}$ be the involution which reverses the orientation on an oriented curve, i.e. $\iota:|\alpha|\to|\alpha^{-1}|$. Extend ι linearly to $\mathbb{Z}$-linear involution $\iota:\mathbb{Z}\hat{\pi}\to\mathbb{Z}\hat{\pi}$.

Proposition. *ι is a Lie algebra automorphism of $\mathbb{Z}\hat{\pi}$.*

Proof. Let $\alpha,\beta\in\pi$. Then

$$[\iota(\alpha),\iota(\beta)]=[\alpha^{-1},\beta^{-1}]=\sum_{p\in\alpha\#\beta}\varepsilon(p;\alpha^{-1},\beta^{-1})\,|\alpha_p^{-1}\beta_p^{-1}|$$

$$=\sum_{p\in\alpha\#\beta}\varepsilon(p;\alpha,\beta)\,|(\alpha_p\beta_p)^{-1}|=\iota([\alpha,\beta])\,.\quad \text{Q.E.D.}$$

The stationary set of ι is a subalgebra $\mathbb{Z}\bar{\pi}$ of $\mathbb{Z}\hat{\pi}$. If $\alpha\in\hat{\pi}$, let $\bar{\alpha}=\alpha+\iota(\alpha)$. Then $\mathbb{Z}\bar{\pi}$ is freely generated (additively) by the set $\bar{\pi}$ of all $\bar{\alpha}$, for $\alpha\in\hat{\pi}$. If $\alpha,\beta\in\hat{\pi}$ we compute the bracket of $\bar{\alpha}$ and $\bar{\beta}$ as follows:

$$\begin{aligned}[\bar{\alpha},\bar{\beta}]&=[\alpha+\iota(\alpha),\beta+\iota(\beta)]=([\alpha,\beta]+[\iota(\alpha),\iota(\beta)])\\&\quad+([\iota(\alpha),\beta]+[\alpha,\iota(\beta)])=\overline{[\alpha,\beta]}+\overline{[\alpha,\iota(\beta)]}\\&=\sum_{p\in\alpha\#\beta}\varepsilon(p;\alpha,\beta)\left(\overline{|\alpha_p\beta_p|}-\overline{|\alpha_p\beta_p^{-1}|}\right).\end{aligned}$$

This is the same commutation relation (up to a factor of 2) that the trace functions f_α satisfy on $\mathrm{Hom}(\pi, G)/G$ when G is an orthogonal, unitary, or symplectic group by Theorem 3.14. In other words:

5.13. Theorem. *Let $\varrho: \mathbb{Z}\bar{\pi} \to C^\infty(\mathrm{Hom}(\pi, G)/G)$ be the linear map defined by $\varrho(\bar{\alpha}) = 2f_\alpha$ where G, f, etc. are all as in* 3.14. *Then ϱ is a Poisson action of $\mathbb{Z}\bar{\pi}$.*

When G is abelian (i.e. $O(2)$, $O(1,1)$, or $U(1)$), then the image of ϱ is a Lie algebra $\mathbb{Z}\bar{H}$ which is a free $\mathbb{Z}$-module on $\bar{H} = H_1(S;\mathbb{Z})/\{\pm 1\}$, and may be obtained from $\mathbb{Z}H$ as the stationary set of the canonical involution.

5.14. Furthermore there is a homomorphism $\mathbb{Z}\bar{\pi} \to \mathbb{Z}H$ such that the diagram

$$\begin{array}{ccc} \mathbb{Z}\bar{\pi} & \to & \mathbb{Z}\bar{H} \\ \downarrow & & \downarrow \\ \mathbb{Z}\hat{\pi} & \to & \mathbb{Z}H \end{array}$$

commutes. This homomorphism is defined analogously to the homomorphism $\mathbb{Z}\tilde{\pi} \to \mathbb{Z}H$ of 5.10.

There seems to be one marked difference between the Lie algebras $\mathbb{Z}H$ and $\mathbb{Z}\bar{H}$ (and thus between the algebras $\mathbb{Z}\hat{\pi}$ and $\mathbb{Z}\bar{\pi}$). While $\mathbb{Z}H$ admits no homomorphism to an abelian Lie algebra over $\mathbb{R}$, its subalgebra $\mathbb{Z}\bar{H}$ admits a nontrivial homomorphism to an abelian Lie algebra (over $\mathbb{R}$) of dimension 4^g. This homomorphism is defined as follows. Let H_2 denote the set $H_1(S;\mathbb{Z}/2)$ and as usual $\mathbb{Z}H_2$ the free abelian group with basis H_2, with the trivial Lie algebra structure. Then the canonical map $H \to H_2$ given by reduction mod 2 evidently factors through $\bar{H}$ and extends to a homomorphism $\mathbb{Z}H \to \mathbb{Z}H_2$ which factors through $\mathbb{Z}\bar{H}$. Indeed, if $[\alpha]$ and $[\beta]$ are homology classes in H, we have:

$$[[\bar{\alpha}], [\bar{\beta}]] = [\alpha] \cdot [\beta] \, ([\overline{(\alpha+\beta)}] - [\overline{(\alpha-\beta)}]) \, .$$

Since $\alpha+\beta$ and $\alpha-\beta$ are mod 2 homologous, it follows that $\mathbb{Z}\bar{H} \to \mathbb{Z}H_2$ is a Lie algebra homomorphism.

Spaces of characters, the moment map, and coadjoint orbits in $\mathbb{Z}\pi$

5.15. Dual to the Lie algebra homomorphism $\varrho: \mathbb{Z}\hat{\pi} \to C^\infty(\mathrm{Hom}(\pi, G)/G, \omega)$ is the so-called moment map which maps the symplectic manifold $\mathrm{Hom}(\pi, G)/G$ into the real linear dual $\mathbb{Z}\hat{\pi}^*$ of the Lie algebra $\mathbb{Z}\hat{\pi}$. This dualistic approach suggests another way to think of the spaces $\mathrm{Hom}(\pi, G)/G$, in terms of coadjoint orbits associated to the Lie algebras based on curves.

The formal construction is as follows. Consider a Lie algebra $\mathfrak{G}$. The real vector space $\mathfrak{G}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{G}, \mathbb{R})$ dual to $\mathfrak{G}$ is known as the coadjoint module. The Lie algebra $\mathfrak{G}$ acts on $\mathfrak{G}^*$ via the coadjoint action, which is the transpose of the adjoint action of $\mathfrak{G}$ on itself, $\mathrm{ad}(x)(y) = [x, y]$. If G is a Lie group having $\mathfrak{G}$ as its Lie algebra, a coadjoint orbit of G is defined to be an orbit of G on $\mathfrak{G}^*$ under the coadjoint action. In the absence of a definite Lie group G, we may define a coadjoint orbit to be an integral submanifold in $\mathfrak{G}^*$ of the integrable plane field P on $\mathfrak{G}^*$ defined by $P(x) = \mathrm{ad}^*(\mathfrak{G})(x)$. It is a well-known fact (due to Kirillov and Kostant in the 1960's and apparently also Sophus Lie in the 1880's (compare Weinstein [24])) that coadjoint orbits carry invariant symplectic structures. Indeed, every simply connected symplectic manifold X upon which $\mathfrak{G}$ acts by symplectic vector fields covers a coadjoint orbit.

Mapping the symplectic manifold M into the coadjoint module is effected by the famous "moment-map". Assuming that the action of $\mathfrak{G}$ on M is Hamiltonian (i.e. the homomorphism $\mathfrak{G} \to \mathrm{Vect}(M, \omega)$ lifts to a Lie algebra homomorphism $\varrho: \mathfrak{G} \to C^\infty(M, \omega)$), the moment map $\chi: M \to \mathfrak{G}^*$ is defined by duality: $\chi(x)$ is the linear functional on $\mathfrak{G}$ which sends $\gamma \in \mathfrak{G}$ to the real number $\varrho(\gamma)(x)$. Under the above hypothesis that ϱ is a Lie algebra homomorphism, it follows that χ is equivariant with respect to the given action on M and the coadjoint action on $\mathfrak{G}^*$. In particular, if $\mathfrak{G}$ acts transitively on M (i.e. at every $x \in M$, the tangent vectors $\gamma(x)$, $\gamma \in \mathfrak{G}$, span $T_x M$) then it follows that the moment map is a covering map of M onto a coadjoint orbit in $\mathfrak{G}^*$.

The coadjoint module of $\mathbb{Z}\hat{\pi}$ is the space of real class functions on π, i.e. invariant functions $\pi \to \mathbb{R}$. As the Lie algebra homomorphism $\varrho: \mathbb{Z}\hat{\pi} \to C^\infty(\mathrm{Hom}(\pi, G)/G, \omega)$ is given by the trace function $\varrho(\alpha)([\phi]): \alpha \mapsto \mathrm{tr}(\phi(\alpha))$, the dual moment map is the map $\chi: \mathrm{Hom}(\pi, G)/G \to \mathbb{Z}\hat{\pi}^*$ associating to each $[\phi] \in \mathrm{Hom}(\pi, G)/G$ the character of the representation ϕ. The image of the moment map is the space of all characters of representations $\pi \to G$.

5.16. Theorem. *Let $G = \mathrm{GL}(n; \mathbb{R})$ and let $\mathrm{Hom}(\pi, G)^-$ be the subset of $\mathrm{Hom}(\pi, G)$ consisting of irreducible representations $\pi \to G$. Then the character map $\chi: \mathrm{Hom}(\pi, G)^-/G \to \mathbb{Z}\hat{\pi}^*$ is a covering map onto its image, and its image, the space of characters of irreducible representations $\pi \to G$, is a coadjoint orbit in $\mathbb{Z}\pi^*$.*

Remark. The same techniques show that other moduli spaces (e.g. Teichmüller space, Jacobi varieties, $H^1(S; \mathbb{R})$, moduli of stable vector bundles, etc.) can be similarly represented as coverings of coadjoint orbits in Lie algebras based on curves. The proofs for the analogous statements for the other classical Lie groups G are identical, and therefore omitted.

Proof. Since the moment map $\chi: \mathrm{Hom}(\pi, G)^-/G \to \mathbb{Z}\hat{\pi}^*$ is $\mathbb{Z}\hat{\pi}$-equivariant, all the assertions of this theorem follow once we know that the action of $\mathbb{Z}\hat{\pi}$ is locally transitive. That is, we must show that the Hamiltonian vector fields Ht_α, $\alpha \in \hat{\pi}$, span the tangent space $T_{[\phi]}\mathrm{Hom}(\pi, G)/G$ at ϕ. Since the vector fields Ht_α are symplectically dual to the differentials dt_α, it suffices to show that the 1-forms dt_α, $\alpha \in \hat{\pi}$, span the cotangent space $T^*_{[\phi]}\mathrm{Hom}(\pi, G)/G$ at $[\phi]$.

To this end we use the theorem of Procesi [23] that the functions t_α, $\alpha \in \hat{\pi}$, generate the coordinate ring of the algebro-geometric quotient $\mathrm{Hom}(\pi, \mathrm{GL}(n; \mathbb{C}))/\mathrm{GL}(n; \mathbb{C})$ when π is a finitely generated free group, and follows from this result for a surface group as well. In particular, the differentials of the restrictions of t_α to $\mathrm{Hom}(\pi, G)/G$ span the cotangent space at α, as desired. Q.E.D.

5.17. Commuting curves in $\mathbb{Z}\hat{\pi}$ and $\mathbb{Z}\bar{\pi}$

Theorem. *Let $\alpha, \beta \in \hat{\pi}$, where α is represented by a simple closed curve. Then:*
(i) *$[\alpha, \beta]=0$ in $\mathbb{Z}\hat{\pi}$ if and only if α and β are freely homotopic to disjoint curves.*
(ii) *$[\bar{\alpha}, \bar{\beta}]=0$ in $\mathbb{Z}\bar{\pi}$ if and only if α and β are freely homotopic to disjoint curves.*

(I am grateful to S. Wolpert for suggesting the proof of part (ii).)

Proof. The "if" implications follow immediately from 5.2 and the definitions of the bracket in these Lie algebras. By 5.2, the bracket may be computed using a disjoint pair of representative curves, in which case the bracket is given by an empty sum.

For the other implications, we use the Poisson actions of $\mathbb{Z}\bar{\pi}$ on Teichmuller space $\mathfrak{T}_S$ of S, as well as the Poisson actions of $\mathbb{Z}\hat{\pi}$ on $\mathrm{Hom}(\pi, \mathrm{GL}(n, \mathbb{R})/\mathrm{GL}(n, \mathbb{R})$, for $n=1, 2$. We begin with the proof of (2).

Suppose α is a simple closed curve and β is a closed curve which is not homotopic to a curve disjoint from α (i.e. the geometric intersection number $i(\alpha, \beta)>0$). Then the following inequality concerning Poisson brackets of geodesic length functions on Teichmuller space is proved in Wolpert [20], Theorem 3.4 (and the first paragraph of p. 224):

$$(*) \qquad \{l_\alpha, \{l_\alpha, l_\beta\}\}>0$$

(This is the basic convexity result of geodesic length functions along earthquake paths, first noticed by Kerckhoff in his solution of the Nielsen realization problem [13].)

To apply inequality (*) to the Lie algebra $\mathbb{Z}\bar{\pi}$ we fix a component X of

$$\mathrm{Hom}(\pi, SL(2, \mathbb{R}))/SL(2, \mathbb{R})$$

which maps to the Teichmuller space

$$\mathfrak{T}_S \subset \mathrm{Hom}(\pi, PSL(2, \mathbb{R}))/PSL(2, \mathbb{R})$$

under the map

$$\mathrm{Hom}(\pi, SL(2, \mathbb{R}))/SL(2, \mathbb{R}) \rightarrow \mathrm{Hom}(\pi, PSL(2, \mathbb{R}))/PSL(2, \mathbb{R})$$

induced by the epimorphism $SL(2, \mathbb{R}) \rightarrow PSL(2, \mathbb{R})$. Let t and l be the invariant functions introduced in §2. To each $\alpha \in \hat{\pi}$, we may write

$$t_\alpha=2\varepsilon_\alpha \cosh(l_\alpha/2)$$

where the sign ε_α of t_α is a constant function (± 1) on X (which may depend on the choice of the component X). The Poisson bracket $\{f, h \circ g\}$ of a function f with the composition of another function g with a real function $h: \mathbb{R} \rightarrow \mathbb{R}$ obeys the following chain rule:

$$\{f, h \circ g\}=h' \circ g\{f, g\}$$

By repeated applications of this chain rule we obtain:

$$(**)\quad \varepsilon_\beta\{t_\alpha, \{t_\alpha, t_\beta\}\}$$
$$=\varepsilon_\beta\{2\varepsilon_\alpha \cosh(l_\alpha/2\}, \{2\varepsilon_\alpha \cosh(l_\alpha/2), 2\varepsilon_\beta \cosh(l_\beta/2)\}\}$$
$$=\sinh^2(l_\alpha/2)(\cosh(l_\beta/2)\{l_\alpha, l_\beta\}^2 + 2\sinh(l_\beta/2)\{l_\alpha, \{l_\alpha, l_\beta\}\} > 0$$

by (*).

Thus $\{t_\alpha, \{t_\alpha, t_\beta\}\}$ is nonzero, so that $\{t_\alpha, t_\beta\}$ is also nonzero. Since $\bar{\alpha} \mapsto t_\alpha$ is a Lie algebra homomorphism $\mathbb{Z}\hat{\pi} \to C^\infty(X)$, it follows that $[\bar{\alpha}, \bar{\beta}] \neq 0$. This proves (ii).

Now we prove (i). Suppose for the sake of contradiction that α and $\beta \in \hat{\pi}$, α represented by a simple curve, $[\alpha, \beta]=0$ but β is not homotopic to a curve disjoint from α. First observe that $[\alpha, \beta]=0$ implies that the algebraic intersection number $\alpha \cdot \beta = 0$: the Lie algebra homomorphism $\mathbb{Z}\pi \to \mathbb{Z}H$ takes $[\alpha, \beta]$ to the element $\alpha \cdot \beta[\alpha+\beta]$ of $\mathbb{Z}H$ (where $[\alpha+\beta]$ is the basis element of H corresponding to homology class $[\alpha]+[\beta]$). Thus $\alpha \cdot \beta = 0$.

Let $G = \mathrm{GL}_+(2, \mathbb{R})$ be the subgroup of orientation-preserving linear automorphisms of $\mathbb{R}^2$ and consider the following two invariant functions $G \to \mathbb{R}$.

$$\delta(A) = \det(A)^{1/2}.$$
$$\tilde{t}(A) = \mathrm{tr}(A).$$

Then $A \mapsto (\delta(A)^{-1}A, \delta(A))$ defines an isomorphism $G \to SL(2, \mathbb{R}) \times \mathbb{R}_+$. We extend the trace function $t: SL(2, \mathbb{R}) \to \mathbb{R}$ to G by the formula

$$t(A) = \delta(A)^{-1}\tilde{t}(A).$$

It is easy to check that the variation function $\Delta: G \to \mathfrak{G}$ of δ is given by $\Delta(A) = \delta(A)I$.

The isomorphism $G \to SL(2, \mathbb{R}) \times \mathbb{R}_+$ determines a symplectic isomorphism

$$\mathrm{Hom}(\pi, G)/G \to \mathrm{Hom}(\pi, SL(2, \mathbb{R}))/SL(2, \mathbb{R}) \times \mathrm{Hom}(\pi, \mathbb{R}_+).$$

Let X' be the component of Hom(π, G)/G which corresponds to $X \times \mathrm{Hom}(\pi, \mathbb{R}_+)$ (where X is as above). Then $\alpha \mapsto \tilde{t}_\alpha$ defines a Lie algebra homomorphism $\mathbb{Z}\hat{\pi} \to C^\infty(X')$.

Now

$$\{\tilde{t}_\alpha, \tilde{t}_\beta\} = \{\delta_\alpha t_\alpha, \delta_\alpha t_\beta\} = \{t_\alpha, t_\beta\} + \{\delta_\alpha, t_\beta\} + \{t_\alpha, \delta_\beta\} + \{\delta_\alpha, \delta_\beta\}$$

and we claim that all but the first summand must vanish. By 3.5, $\{\delta_\alpha, t_\beta\}$ is a sum over $\alpha \# \beta$ of expressions of the form

$$\mathrm{tr}(\delta(A)(B - (1/2)\,\mathrm{tr}(B)I))$$

which is evidently zero. Similarly $\{t_\alpha, \delta_\beta\} = 0$. Finally $\{\delta_\alpha, \delta_\beta\} = (\alpha \cdot \beta)\delta_{\alpha+\beta} = 0$.

Thus $\{\tilde{t}_\alpha, \tilde{t}_\beta\} = \{t_\alpha, t_\beta\}$ which is nonzero by (**). This contradiction proves (ii). Q.E.D.

Remark. It would be interesting to have a purely topological proof of this topological theorem. The condition that one of the curves be simple, however, is crucial, as was recently pointed out by Peter Scott. If α is not representable as a power of a simple loop, then α is not homotopic to a curve disjoint from itself, even though $[\alpha, \alpha]=0$ in $\mathbb{Z}\hat{\pi}$ (and similarly in $\mathbb{Z}\bar{\pi}$). Thus although two curves may commute in these formal Lie algebras, they need not be representable as geometrically disjoint curves unless one of them is simple.

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References

1. Abraham, R., Marsden, J.: Foundations of Mechanics. Second Edition. Reading, Massachusetts: Benjamin/Cummings 1978
2. Arnold, V.I.: Mathematical methods of classical mechanics. Graduate Texts in Mathematics 60, Berlin-Heidelberg-New York: Springer 1978
3. Brown, K.S.: Cohomology of groups. Graduate Texts in Mathematics 87. Berlin-Heidelberg-New York: Springer 1982
4. Cohen, J.M.: Poincaré 2-Complexes II. Chin. J. Math. **6**, 25–44 (1978)
5. Dold, A.: Lectures on Algebraic Topology, Berlin-Heidelberg-New York: Springer 1972
6. Goldman, W.: Discontinuous groups and the Euler class. Doctoral disseration. University of California 1980
7. Goldman, W.: The symplectic nature of fundamental groups of surfaces. Adv. Math. **54**, 200–225 (1984)
8. Goldman, W.: Representations of fundamental groups of surfaces. Proceedings of Special Year in Topology, Maryland 1983–1984, Lect. Notes Math. (to appear)
9. Hass, J., Scott, G.P.: Intersections of curves on surfaces. Isr. J. Math. **51**, 90–120 (1985)
10. Helgason, S.: Differential geometry, Lie groups, and symmetric spaces. New York: Academic Press 1978
11. Hirsch, M.W.: Differential Topology, Graduate texts in mathematics 33, Berlin-Heidelberg-New York: Springer 1976
12. Johnson, D., Millson, J.: Deformation spaces of compact hyperbolic manifolds. In: Discrete groups in geometry and analysis. Proceedings of a Conference Held at Yale University in Honor of G. D. Mostow on his Sixtieth Birthday (to appear)
13. Kerckhoff, S.: The Nielsen realization problem. Ann. Math. **117**, 235–265 (1983)
14. Magnus, W.: The uses of 2 by 2 matrices in combinatorial group theory. A survey, Result. Math. **4**, 171–192 (1981)
15. Morgan, J.W., Shalen, P.B.: Valuations, trees, and degeneration of hyperbolic structures, I. Ann. Math. **120**, 401–476 (1984)
16. Steenrod, N.E.: The Topology of Fibre Bundles, Princeton NJ: Princeton University Press 1951
17. Weinstein, A.: Lectures on symplectic manifolds. C.B.M.S. 29, Am. Math. Soc., Providence R.I., 1977
18. Wolpert, S.: An elementary formula for thc Fenchel-Nielsen twist. Comm. Math. Helv. **56**, 132–135 (1981)

19. Wolpert, S.: The Fenchel-Nielsen deformation, Ann. Math. **115**, 501–528 (1982)
20. Wolpert, S.: On the symplectic geometry of deformations of a hyperbolic surface. Ann. Math. **117**, 207–234 (1983)
21. Medina-Perea, A.: Groupes de Lie munis de pseudo-métriques de Riemann bi-invariantes. Séminaire de Géométrie Différentielle 1981–1982, Exposé 4, Institut de Mathématiques, Université des Sciences et Techniques du Languedoc, Montpellier
22. Papadopoulos, A.: Geometric intersection functions and Hamiltonian flows on the space of measured foliations on a surface. I.A.S. (preprint, 1984)
23. Procesi, C.: The invariant theory of $n \times n$ matrices. Adv. Math. **19**, 306–381 (1976)
24. Weinstein, A. The local structure of Poisson manifolds. J. Diff. Geom. **18**, 523–557 (1983)
25. Weinstein, A.: Poisson structures and Lie algebras, Proceedings of a conference on The Mathematical Heritage of Elie Cartan, Lyon, June 1984 (to appear in Asterisque)
26. Fathi, A.: The Poisson bracket on the space of measured foliations on a surface (preprint)

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