Mutually algebraic structures and ‘automatic’ quantifier elimination

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Theorem (Zilber)

If T is strongly minimal, ω-categorical, and non-trivial, then T interprets an infinite group.

Strongly minimal: For every model M, there is a unique non-algebraic 1-type.
ω-categorical: Any two countable models are isomorphic.
non-trivial: For some $A \subseteq M$, $\mathrm{acl}(A) \neq \bigcup_{a \in A} \mathrm{acl}(\{a\})$.

Zilber's method led to 'Geometric Stability Theory' e.g., classifying the geometries of locally modular regular types.

Goncharov, Harizanov, Lempp, and McCoy: Under strong model theoretic hypotheses on $Th(M)$, can you bound the computational complexity of $ElDiag(M)$ in terms of $AtDiag(M)$?

e.g., If $AtDiag(M)$ is computable, must $ElDiag(M)$ be arithmetic?

Bounding the quantifier complexity is a sufficient condition.

Theorem (G-H-L-L-M)

If T is strongly minimal and trivial, then for any $M \models T$, the $L(M)$-theory $ElDiag(M)$ is model complete.

Proof: Messy induction on the complexity of $L(M)$-formulas φ showing that if $M \preceq N_1, N_2$ and $N_1 \subseteq N_2$, then φ is absolute between N_1 and N_2.

Thus, every $L(M)$-formula is equivalent to an existential formula, but we really can't see what they are...

Extend this?

Marker gave an example of a totally categorical, trivial theory of Morley rank 2 for which $ElDiag(M)$ is not model complete.

Theorem (Dolich-L.-Raichev)

If T is $\aleph_1$-categorical, trivial, of Morley rank 1, then for any $M \models T$, $ElDiag(M)$ is model complete.

Fix M any L-structure.

- A proper partition of variables $\overline{z} = \overline{x}\hat{}\overline{y}$ satisfies $lg(\overline{x}), \lg(\overline{y}) \geq 1$.
- An $L(M)$-formula $\varphi(\overline{z})$ is mutually algebraic if there is an integer K so that $M \models \forall\overline{x}\exists^{\leq K}\overline{y}\varphi(\overline{x}, \overline{y})$ for every proper partition $\overline{z} = \overline{x}\hat{}\overline{y}$.
- $MA(M)$ denotes all mutually algebraic $L(M)$-formulas.

Non-Examples

- The formula $x + y = z$ is not mutually algebraic;
- The graph of a pairing function $f : X \times Y \to Z$ is not mutually algebraic.

Membership in $MA(M)$ is fussy:

- Every definable subset of M^1 is mutually algebraic (no proper partitions);
- NOT closed under adjunction of dummy variables;
- Closed under conjunction only when free variables intersect;
- Closed under disjunction only when free variable sets are equal;
- Is closed under $\exists^{\geq m}\overline{y}\varphi(\overline{x},\overline{y})$ for all $m \geq 1$.

For M any L-structure, $MA^*(M)$ denotes the set of all Boolean combinations of mutually algebraic $L(M)$-formulas.

Proposition

For M any structure, $MA^*(M)$ is closed under projections.

Call a structure M mutually algebraic if every $L(M)$-definable set is in $MA^*(M)$.

Corollary

Every L-structure M has a maximal, mutually algebraic reduct.

If M is $(\mathbb{Q}, \leq)$, then the maximal, mutually algebraic reduct is just equality.

Challenge

What is the maximal, mutually algebraic reduct of $(\mathbb{C}, +, \cdot, 0, 1)$?

Note: Suppose M is a mutually algebraic L-structure and $A \subseteq M^n$ is a mutually algebraic subset. Let $L_P = L \cup \{P\}$, where P is n-ary. Then the L_P-structure (M, A) is mutually algebraic as well.

Proposition

Suppose M is a mutually algebraic structure. Then $ElDiag(M)$, hence $Th(M)$, has nfcp.

Corollary

If M is mutually algebraic, then (M, A) is mutually algebraic, hence has nfcp, for any unary expansion $A \subseteq M^1$.

If M is a mutually algebraic structure, what can $Th(M)$ be?

Proposition

If M is mutually algebraic, then $Th(M)$ is weakly minimal and trivial.

T is weakly minimal if, for any $M \preceq N$, every non-algebraic $p \in S_1(M)$ has a unique non-algebraic extension $q \in S_1(N)$. Trivial is with respect to algebraic closure, $\text{acl}(A) = \bigcup_{a \in A} \text{acl}(\{a\})$.

Back to quantifier elimination:

Theorem

If T is weakly minimal and trivial, then for every $M \models T$

1. *Every quantifier-free $L(M)$-formula is equivalent to a Boolean combination of quantifier-free mutually algebraic formulas;*
2. *Every $L(M)$-formula $\varphi(\overline{x})$ is equivalent to a Boolean combination of (mutually algebraic) formulas of the form $\exists \overline{y} R(\overline{x}, \overline{y})$, where $R(\overline{x}, \overline{y})$ is quantifier-free mutually algebraic;*
3. *$ElDiag(M)$ is near model complete and M is a mutually algebraic structure.*

Put the last few slides together:

Theorem

TFAE for a consistent theory T:

- *Every model of T is a mutually algebraic structure;*
- *For every $M \models T$, every unary expansion (M, A) has nfcp;*
- *Every completion of T is weakly minimal and trivial.*

For any model M of such a T, every $L(M)$-formula $\varphi(\overline{x})$ is $ElDiag(M)$-equivalent to a boolean combination of formulas $\exists \overline{y} R(\overline{x}, \overline{y})$, where $R(\overline{x} \hat{\ } \overline{y})$ is mutually algebraic and quantifier free.

Suppose T is strongly minimal and trivial and $M \models T$.

On one hand, $ElDiag(M)$ is model complete, hence every $L(M)$-formula is equivalent to an existential formula.

On the other hand, every $L(M)$-formula is equivalent to a Boolean combination of mutually algebraic formulas of a specific form.

Can we combine these?

- $S(\overline{w})$ is a partial equality diagram of $\overline{w}$ if it is a boolean combination of $w = w'$ for various $w, w' \in \overline{w}$.
- Suppose $\overline{x}, \overline{y}, \overline{z}$ are disjoint with $\lg(\overline{x}) \geq 1$. A preferred formula $\theta(\overline{x}, \overline{z})$ has the form

$$\exists \overline{y}(R(\overline{x}, \overline{y}) \wedge S(\overline{x}, \overline{y}, \overline{z}))$$

where $R(\overline{x}\hat{\ }\overline{y})$ is q.f., mutually algebraic, and S is a partial equality diagram of $\overline{x}\hat{\ }\overline{y}\hat{\ }\overline{z}$.

- $\mathscr{P} = \{$all formulas equivalent to a positive boolean combination of preferred formulas$\}$.

Question

Is $\neg R(\overline{z}) \in \mathscr{P}$ for every q.f., mutually algebraic formula $R(\overline{z})$?

Answer: Yes, if you can count.

Theorem

The following are equivalent for a mutually algebraic M:

1. $\exists^{=m}\overline{y}R(x,\overline{y}) \in \mathscr{P}$ *for all q.f., mutually algebraic* $R(x,\overline{y})$ *with* $\lg(x) = 1$ *and all* $m \in \omega$*;*
2. $\mathscr{P}$ *is closed under negation;*
3. $L(M) = \mathscr{P}$*;*
4. $ElDiag(M)$ *is model complete.*

Corollary

If T is strongly minimal and trivial, then these conditions hold.

Can we go beyond rank one?

Conjecture (Dolich)

If T is $\aleph_1$-categorical and trivial, then the quantifier complexity of $ElDiag(M)$ is bounded by the Morley rank of T.

Marker's method of 'fuzzifying' gives, for each integer n, a totally categorical, trivial theory T_n of Morley rank n where $ElDiag(M_n)$ admits quantifiers down to Σ_n, but no lower.

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