# Non-Linear Covers of Real Groups

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**Abstract**

Let $\mathbb{G}$ be a semisimple, simply connected, algebraic group defined over $\mathbb{R}$. The set of real points $G$ of $\mathbb{G}$ is not necessarily topologically simply connected, in which case $G$ admits a non–trivial covering group. We give simple uniform proofs of several basic properties of real non–linear groups, in particular a simple criterion for when such a cover exists. Some of these properties were previously known from a case–by–case check based on the classification of real groups and their covers.

## 1 Introduction

Let $\mathbb{G}$ be a semisimple, simply connected, algebraic group defined over $\mathbb{R}$. The set of real points $G$ of $\mathbb{G}$ is not necessarily topologically simply connected, in which case $G$ admits a non–trivial covering group. An example is the metaplectic group, the 2-fold cover of $Sp(2n,\mathbb{R})$. Such a group is not realizable as a linear (or algebraic, or matrix) group, and is said to be a *non–linear* cover of $G$.

Such groups are very common. In the p–adic case every isotropic group has a non–linear cover [8]. Over $\mathbb{R}$ this is false, for example if $G$ is a complex group or $Spin(n,1)$ with $n \geq 3$.

The main purpose of this paper is to give simple uniform proofs of several basic properties of real non–linear groups. Some of these were previously known from a case–by–case check based on the classification of real groups and their covers.

Fix a Cartan involution $\theta$ of $\mathbb{G}$ corresponding to $G$. Thus $K = G^\theta$ is a maximal compact subgroup of $G$. Let $\mathbb{T}$ be a $\theta$–stable Cartan subgroup of

$\mathbb{G}$. Then $\theta$ acts on the roots of $\mathbb{T}$ in $\mathbb{G}$. A root $\alpha$ is said to be imaginary (respectively real) if $\theta(\alpha) = \alpha$ (resp. $\theta(\alpha) = -\alpha$). If $\alpha$ is neither real nor imaginary then it is complex. If $\alpha$ is imaginary fix a root vector $X_\alpha$; then $\alpha$ is said to be compact (respectively non–compact) if $\theta(X_\alpha) = X_\alpha$ (resp. $\theta(X_\alpha) = -X_\alpha$). Let $\mathbb{T}$ be a fundamental (i.e. most compact) $\theta$–stable Cartan subgroup . For the basic definitions and properties of roots, Cartan involutions and Cartan subgroups see [11]. By convention all roots are long if the root system of $G$ is simply laced.

Note that $G$ has a non–trivial cover if and only if the fundamental group $\pi_1(G)$ is non–trivial.

**Theorem 1.1** $\pi_1(G) \neq 1$ *if and only if* $\mathbb{T}$ *has a long non–compact imaginary root.*

Alternatively we may state this in terms of long real roots of the maximally split Cartan subgroup (Proposition 4.3). Based on the classification of real groups and their covers, Gopal Prasad has made an observation closely related to this [8]. See Section 4.

If the root system of $\mathbb{G}$ is simply laced the statement is simpler, and a bit stronger. The proof is in Section 4.

**Corollary 1.2** *Suppose the root system of* $\mathbb{G}$ *is simply laced. Let* $\mathbb{T}$ *be any* $\theta$*–stable Cartan subgroup. Then* $\pi_1(G) \neq 1$ *if and only if* $\mathbb{T}$ *has a long real or long non-compact imaginary root.*

*Furthermore* $\pi_1(G) = 1$ *if and only if* $G$ *has precisely one conjugacy class of Cartan subgroups.*

The next corollary is an immediate consequence of the theory of nilpotent orbits. It suggests that the existence of a non–linear cover is closely related to the existence of small representations. Let $O_{min}$ be the minimal nilpotent orbit of the complexified Lie algebra of $G$.

**Corollary 1.3** $\pi_1(G) \neq 1$ *if and only if* $O_{min}$ *is defined over* $\mathbb{R}$*.*

Bertram Kostant pointed out, and gave a simple proof of, the implication ($\Leftarrow$). See Remark 3.13.

As usual we say $G$ is simple if its (real) Lie algebra is simple.

**Theorem 1.4** *Suppose* $G$ *is simple. Then* $\pi_1(G)$ *is trivial, or is isomorphic to* $\mathbb{Z}$ *or* $\mathbb{Z}/2\mathbb{Z}$*.*

The simple groups which do not have non–trivial covers are: compact groups, complex groups, $SL(n,\mathbb{H})$, $Spin(n,1)$ $(n \geq 3)$, $Sp(p,q)$, $E_6(F_4)$ and $F_4(B_4)$ (each real exceptional group is identified by its maximal compact subgroup). In Proposition 5.20 we give a method to determine $\pi_1(G)$ from the Kac diagram of $G$.

We continue to assume $G$ is simple, and fix a $\theta$–stable Cartan subgroup $\mathbb{T}$ and a real or non–compact imaginary root $\alpha$. Associated to $\alpha$ is the (real) root subgroup $M_\alpha \simeq SL(2,\mathbb{R})$. Inclusion of $M_\alpha$ in $G$ induces a map $\phi_\alpha : \pi_1(M_\alpha) \to \pi_1(G)$.

**Theorem 1.5** *If $\alpha$ is long then $\phi_\alpha$ is surjective.*

Thus a cover $\widetilde{G}$ of $G$ is determined by its restriction to $M_\alpha$ with $\alpha$ long: $\widetilde{G}$ is non-trivial if and only if the cover is non-trivial when restricted to $M_\alpha$.

If $\widetilde{G}$ is a non–trivial cover of $G$ we say a real or non–compact imaginary root $\alpha$ is *metaplectic* if $\phi_\alpha$ is non–trivial, or equivalently if $\widetilde{M_\alpha}$ is a non–trivial cover of $M_\alpha$.

**Theorem 1.6** *If $\mathbb{G} \neq \mathbb{G}_2$ then $\alpha$ is metaplectic if and only if it is long. If $G$ is the split real form of $G_2$ then every root is metaplectic.*

Theorems 1.4 and 1.5 have been observed by Gopal Prasad, based on the classification. We give uniform proofs in Section 5. The basic technique is to reduce the study of a non–linear group $\widetilde{G}$ to that of its maximal compact subgroup, which is an algebraic group.

Theorem 1.1 plays a key role in extending Vogan duality [12] to non–linear groups [1], which is a primary motivation for this result. This paper grew out of discussions with Peter Trapa on this topic. We originally emphasized Theorem 1.1, because of its application to Vogan duality. Gopal Prasad suggested proving Theorems 1.4 and 1.5 by these methods. The author thanks Gopal Prasad and Peter Trapa for their contributions to this work.

# 2 The Fundamental Group

In this section $\mathbb{G}$ is a connected, semisimple, simply connected group defined over $\mathbb{R}$, with real points $G$. Fix a Cartan involution $\theta$ corresponding to $G$ and a $\theta$–stable fundamental Cartan subgroup $\mathbb{T}$.

Let $X^*(\mathbb{T})$ and $X_*(\mathbb{T})$ be the character and co–character lattices of $\mathbb{T}$ respectively. Let $\Delta = \Delta(\mathbb{G},\mathbb{T}) \subset X^*(\mathbb{T})$ be the set of roots of $\mathbb{T}$ in $\mathbb{G}$, and

$\Delta^\vee = \Delta^\vee(\mathbb{G},\mathbb{T}) \subset X_*(\mathbb{T})$ the co–roots. Let $R(\mathbb{G},\mathbb{T}) = \mathbb{Z}\langle\Delta(\mathbb{G},\mathbb{T})\rangle$ be the root lattice, and $R^\vee(\mathbb{G},\mathbb{T}) = \mathbb{Z}\langle\Delta^\vee(\mathbb{G},\mathbb{T})\rangle$ the co–root lattice. We write $\langle,\rangle$ for the pairing of roots and co-roots. We use the subscripts $r, i, cx, c, nc$ on $\Delta$ to denote the real, imaginary, complex, compact (imaginary) and non–compact (imaginary) roots, respectively.

Let $\mathbb{T}_K = \mathbb{T}^\theta$; this is a Cartan subgroup of $\mathbb{K} = \mathbb{G}^\theta$. Let $\Delta(\mathbb{K},\mathbb{T}_K)$ be the roots of $\mathbb{T}_K$ in $\mathbb{K}$, with co-roots $\Delta^\vee(\mathbb{K},\mathbb{T}_K)$.

We have

$$\pi_1(\mathbb{G}) \simeq X_*(\mathbb{T})/R^\vee(\mathbb{G},\mathbb{T}) \tag{2.1}$$

and since $\mathbb{G}$ is simply connected $X_*(T) = R^\vee(\mathbb{G},\mathbb{T})$. On the other hand (for example see [7, page 198])

$$\pi_1(G) \simeq \pi_1(K) \simeq \pi_1(\mathbb{K}) \simeq X_*(\mathbb{T}_K)/R^\vee(\mathbb{K},\mathbb{T}_K). \tag{2.2}$$

The embedding $\mathbb{T}_K \subset \mathbb{T}$ induces an embedding $X_*(\mathbb{T}_K) \subset X_*(\mathbb{T})$ and

$$\begin{aligned} X_*(\mathbb{T}_K) &= X_*(\mathbb{T})^\theta \\ &= R^\vee(\mathbb{G},\mathbb{T})^\theta \end{aligned} \tag{2.3}$$

and therefore

$$\pi_1(G) \simeq R^\vee(\mathbb{G},\mathbb{T})^\theta/R^\vee(\mathbb{G},\mathbb{T}). \tag{2.4}$$

We need an intermediate result to compute $R^\vee(\mathbb{G},\mathbb{T})^\theta$.

Let $\Delta(\mathbb{G},\mathbb{T}_K)$ be the roots of $\mathbb{T}_K$ in $\mathbb{G}$; this is a (possibly non reduced) root system. For $\beta \in \Delta(\mathbb{G},\mathbb{T}_K)$ we may identify $\beta^\vee$ with an element of $\mathfrak{t}_K$ (the complex Lie algebra of $\mathbb{T}_K$, thought of as a complex group), and hence with a one–parameter subgroup of $\mathbb{T}_K$.

**Lemma 2.5** *Suppose $\alpha \in \Delta_{cx}(\mathbb{G},\mathbb{T})$. Then one of the following conditions hold:*

*(1) $\langle\alpha,\theta\alpha^\vee\rangle = 0$, and $\alpha^\vee + \theta\alpha^\vee = \beta^\vee$ where $\beta = \alpha|_{\mathbb{T}_K} \in \Delta(\mathbb{K},\mathbb{T}_K)$,*

*(2) $\langle\alpha,\theta\alpha^\vee\rangle = -1$, and $\alpha^\vee + \theta\alpha^\vee = \gamma^\vee$ where $\gamma = \alpha + \theta\alpha \in \Delta_{nc}$.*

In (1) we have used the inclusions $\beta^\vee \in \Delta^\vee(\mathbb{K}, \mathbb{T}_K) \subset X_*(\mathbb{T}_K) \subset X_*(\mathbb{T}) = R^\vee(\mathbb{G}, \mathbb{T})$.

**Proof.** Let $\beta = \alpha|_{\mathbb{T}_K} \in \Delta(\mathbb{G}, \mathbb{T}_K)$. It is easy to see that

$$\beta^\vee = c(\alpha^\vee + \theta\alpha^\vee) \tag{2.6}$$

for some rational number $c$; each side corresponds to the same one-parameter subgroup of $\mathbb{T}_K$. (To be precise we should clear denominators and have a relation with integral coefficients; it will turn out that $c \in \{1, 2\}$). To compute $c$ we consider (for $z \in \mathbb{C}^\times$)

$$\beta(\beta^\vee(z)) = \beta(c(\alpha^\vee + \theta\alpha^\vee)(z)).$$

The left–hand side is equal to $z^2$. Since $\alpha^\vee + \theta\alpha^\vee \in X_*(\mathbb{T}_K)$ the right hand side is equal to $\alpha(c(\alpha^\vee + \theta\alpha^\vee)(z)) = z^{c(2+\langle\alpha,\theta\alpha^\vee\rangle)}$. Setting these equal we see

$$c = \frac{2}{2 + \langle\alpha, \theta\alpha^\vee\rangle}. \tag{2.7}$$

Since $\alpha$ and $\theta\alpha$ have the same length it is a standard fact that $\langle\alpha, \theta\alpha^\vee\rangle \in \{0, \pm 1\}$ [2, Chapter VI, §1.3], and $\langle\alpha, \theta\alpha^\vee\rangle = \pm 1$ if and only if $\alpha \mp \theta\alpha$ is a root. If $\alpha - \theta\alpha$ is a root then it is real; since $\mathbb{T}$ has no real roots this is impossible, so $\langle\alpha, \theta\alpha^\vee\rangle \in \{0, -1\}$.

If $\langle\alpha, \theta\alpha^\vee\rangle = 0$ then by (2.6) and (2.7) $\alpha^\vee + \theta\alpha^\vee = \beta^\vee \in \Delta^\vee(\mathbb{K}, \mathbb{T}_K)$. On the other hand suppose $\langle\alpha, \theta\alpha^\vee\rangle = -1$, so $\gamma = \alpha + \theta\alpha$ is an imaginary root. Then similar considerations give $\alpha^\vee + \theta\alpha^\vee = \gamma^\vee$. Note that $\gamma|_{\mathbb{T}_K} = 2\beta$. If $\gamma$ is compact then both $\beta$ and $2\beta$ are contained in $\Delta(\mathbb{K}, \mathbb{T}_K)$, contradicting the fact the $\Delta(\mathbb{K}, \mathbb{T}_K)$ is a reduced root system. Therefore $\gamma \in \Delta_{\mathrm{nc}}$. This completes the proof. ■

**Proposition 2.8**

$$R^\vee(\mathbb{G}, \mathbb{T})^\theta = R^\vee(\mathbb{G}, \mathbb{T}_K).$$

**Proof.** Choose a $\theta$–stable set of simple roots of $\Delta(\mathbb{G}, \mathbb{T})$; such a set always exists (for example see [5, Section VI.8]). Then the left hand side is spanned by the co–roots $\alpha^\vee$ with $\alpha$ imaginary, together with the elements $\alpha^\vee + \theta\alpha^\vee$ where $\alpha$ is complex, i.e. the following set:

$$\Delta_{\mathrm{i}}^\vee \cup \{\alpha^\vee + \theta\alpha^\vee \mid \alpha \in \Delta_{\mathrm{cx}}^\vee\}.$$

where $\Delta_{\mathrm{i}}^\vee = \{\alpha^\vee \,|\, \alpha \in \Delta_{\mathrm{i}}\}$, and similarly $\Delta_{cx}^\vee$.

The roots of $\mathbb{T}_K$ in $\mathbb{G}$ are restrictions of the roots of $\mathbb{T}$ in $\mathbb{G}$ to $\mathbb{T}_K$. If $\alpha \in \Delta(\mathbb{G},\mathbb{T})$ is imaginary and $\beta = \alpha|_{\mathbb{T}_K}$ then $\alpha^\vee = \beta^\vee$. Therefore the first term is contained in $R^\vee(\mathbb{G},\mathbb{T}_K)$. By Lemma 2.5 the second term is also, so $R^\vee(\mathbb{G},\mathbb{T})^\theta \subset R^\vee(\mathbb{G},\mathbb{T}_K)$.

For the reverse inclusion, suppose $\beta \in \Delta(\mathbb{G},\mathbb{T}_K)$ and choose $\alpha \in \Delta(\mathbb{G},\mathbb{T})$ which restricts to $\beta$. If $\alpha$ is imaginary then as before $\beta^\vee = \alpha^\vee \in \Delta^\vee(\mathbb{G},\mathbb{T})$. If $\alpha$ is complex then we are in the setting of Lemma 2.5. In case (1) $\beta^\vee = \alpha^\vee + \theta\alpha^\vee$; in case (2) $\beta = \frac{1}{2}\gamma$ and $\beta^\vee = 2\gamma^\vee = 2(\alpha^\vee + \theta\alpha^\vee)$. ■

**Remark 2.9** The involution $\theta$ acts on $\Delta = \Delta(\mathbb{G},\mathbb{T})$, and the quotient is naturally a (possibly non–reduced) root system [9], which we denote $\Delta_\theta$. Restriction from $\mathbb{T}$ to $\mathbb{T}^\theta$ defines an isomorphism $\Delta_\theta \simeq \Delta(\mathbb{G},\mathbb{T}_K)$. Write $R^\vee(\Delta) = \mathbb{Z}\langle\Delta^\vee\rangle$ for the co–root lattice of $\Delta$, and $R^\vee(\Delta_\theta)$ similarly. $\theta$ acts on $R^\vee(\Delta)$ and the Proposition may be stated:

$$R^\vee(\Delta)^\theta = R^\vee(\Delta_\theta).$$

This has been proved by Thomas Haines and Bao Chau Ngo using the affine Weyl group [3].

Proposition 2.8 and (2.4) imply

**Proposition 2.10**

$$\pi_1(G) \simeq R^\vee(\mathbb{G},\mathbb{T}_K)/R^\vee(\mathbb{K},\mathbb{T}_K).$$

It is convenient to express this in a different form. By Lemma 2.5 we may write

$$R^\vee(\mathbb{G},\mathbb{T}_K) = R^\vee(\mathbb{K},\mathbb{T}_K) + \mathbb{Z}\langle\Delta_{\mathrm{nc}}^\vee\rangle. \tag{2.11}$$

Therefore

$$\pi_1(G) \simeq \mathbb{Z}\langle\Delta_{\mathrm{nc}}^\vee\rangle/R^\vee(\mathbb{K},\mathbb{T}_K) \cap \mathbb{Z}\langle\Delta_{\mathrm{nc}}^\vee\rangle. \tag{2.12}$$

and in particular

$$\pi_1(G) = 1 \Leftrightarrow \Delta_{\mathrm{nc}}^\vee \subset R^\vee(\mathbb{K},\mathbb{T}_K). \tag{2.13}$$

## 3 Proof of Theorem 1.1

We continue with the notation of Section 2. In particular we have a fundamental Cartan subgroup $\mathbb{T}$ containing $\mathbb{T}_K = \mathbb{T}^\theta$.

We have to show:

$$\pi_1(G) = 1 \Leftrightarrow \Delta_{\mathrm{nc,long}} = \emptyset.$$

By (2.13) it is enough to show

$$\Delta_{\mathrm{nc}}^\vee \subset R^\vee(\mathbb{K}, \mathbb{T}_K) \Leftrightarrow \Delta_{\mathrm{nc,long}} = \emptyset. \tag{3.1}$$

**Example 3.2** If $G$ is a complex group then every root is complex; if $G$ is compact then every root is imaginary and compact. In each case $\Delta_{nc} = \emptyset$ and $\pi_1(G) = 1$.

**Remark 3.3** If $\mathbb{T}_K = \mathbb{T}$ then $R^\vee(\mathbb{K}, \mathbb{T}_K) = \mathbb{Z}\langle\Delta_c^\vee\rangle$ and we have to show

$$\Delta_{nc}^\vee \subset \mathbb{Z}\langle\Delta_c^\vee\rangle \Leftrightarrow \Delta_{nc,long} = \emptyset.$$

Suppose furthermore that $\mathbb{G}$ is simply laced. We have to show

$$\Delta_{\mathrm{nc}}^\vee \subset \mathbb{Z}\langle\Delta_{\mathrm{c}}^\vee\rangle \Leftrightarrow \Delta_{\mathrm{nc}} = \emptyset.$$

The implication ($\Leftarrow$) is immediate. On the other hand suppose $\alpha \in \Delta_{\mathrm{nc}}$ and $\alpha^\vee \in \mathbb{Z}\langle\Delta_c^\vee\rangle$, i.e.

$$\alpha^\vee = \sum_{\beta\in\Delta_c} m_\beta \beta^\vee \quad (m_\beta \in \mathbb{Z}).$$

In the simply laced case this implies $\alpha = \sum_\beta m_\beta \beta$, contradicting the fact that the classification of imaginary roots as compact or non–compact is a grading [12, Definition 3.13 and Proposition 4.14], i.e. if a sum of compact roots is a root then it is compact. This proves the reverse implication.

Perhaps the only surprise is that in the case of two root lengths we may have $\emptyset \neq \Delta_{\mathrm{nc}}^\vee \subset \mathbb{Z}\langle\Delta_{\mathrm{c}}^\vee\rangle$.

**Example 3.4** Let $G = Sp(p, q)$ with $pq \neq 0$. This group contains a compact Cartan, all roots are imaginary, and in the usual coordinates the roots are $\pm e_i \pm e_j, \pm 2e_i$ $1 \leq i, j \leq p + q$. The roots $\pm e_i \pm e_j$ with $i \leq p < j$ are non–compact, and all other roots are compact. Note that $\Delta_{nc,long} = \emptyset$. Let

$\alpha = e_i \pm e_j$ with $i \leq p < j$, $\beta = 2e_i$, and $\gamma = 2e_j$. Note that $\alpha \notin \mathbb{Z}\langle\Delta_c\rangle$, but $\alpha^\vee = \beta^\vee \pm \gamma^\vee \in \mathbb{Z}\langle\Delta_c^\vee\rangle$. Therefore $\Delta_{nc}^\vee \subset \mathbb{Z}\langle\Delta_c^\vee\rangle$, confirming (3.1) in this case. This group does not have a non–trivial cover: $K \simeq Sp(p) \times Sp(q)$, which is simply connected.

We will show:

$$\alpha \in \Delta_{\text{nc,long}} \Rightarrow \alpha^\vee \notin R^\vee(\mathbb{K}, \mathbb{T}_K) \tag{3.5}$$

and

$$\Delta_{\text{nc,long}} = \emptyset \Rightarrow \Delta^\vee_{\text{nc, short}} \subset R^\vee(\mathbb{K}, \mathbb{T}_K). \tag{3.6}$$

These are enough to show (3.1) and complete the proof of Theorem 1.1.

Fix a Weyl group invariant inner product $(,)$ on $R(\mathbb{G}, \mathbb{T})$ satisfying $(\alpha, \alpha) = 2$ if $\alpha$ is long. Let

$$c = \max_{\alpha,\beta} \frac{(\alpha, \alpha)}{(\beta, \beta)} \in \{1, 2, 3\} \tag{3.7}$$

[2, Chapter VI, §1.3].

Identifying $\alpha^\vee$ with $\frac{2\alpha}{(\alpha,\alpha)}$ we may write

$$\alpha^\vee = \begin{cases} \alpha & \alpha \text{ long} \\ c\alpha & \alpha \text{ short.} \end{cases} \tag{3.8}$$

Suppose $\alpha \in \Delta_{\text{nc,long}}$ and $\alpha^\vee \in R^\vee(\mathbb{K}, \mathbb{T}_K)$. Then

$$\alpha^\vee = \sum_{\beta \in \Delta_{c,short}} m_\beta \beta^\vee + \sum_{\gamma \in \Delta_{c,long}} m_\gamma \gamma^\vee \tag{3.9}$$

with $m_\beta, m_\gamma \in \mathbb{Z}$. Inserting (3.8) gives

$$\alpha = \sum_{\beta \in \Delta_{c,short}} m_\beta \beta + c \sum_{\gamma \in \Delta_{c,long}} m_\gamma \gamma. \tag{3.10}$$

As in remark 3.3 this is a contradiction proving (3.5).

Now consider (3.6), so assume $\Delta_{\text{nc,long}} = \emptyset$ and $\alpha \in \Delta_{\text{nc, short}}$, so we are in the non–simply laced case. In this case $T$ is is compact and every root is imaginary $\tau = 1$ in the notation of Section 5). It is enough to show that we can write

$$\alpha^\vee = \beta^\vee + \gamma^\vee \tag{3.11}$$

with $\beta, \gamma \in \Delta_c$. For this we reduce to the rank 2 case. Since the split group $G_2$ has long non–compact roots we may assume $G \neq G_2$.

The long roots span $R$ (since their span is a Weyl group invariant sub–root system). Therefore we may choose $\beta \in \Delta_{\text{long}}$ such that $\langle \beta, \alpha^\vee \rangle \neq 0$. Then $\langle \alpha, \beta^\vee \rangle = \pm 1$ and we may assume $\langle \alpha, \beta^\vee \rangle = 1$. The root system spanned by $\alpha, \beta$ is of type $B_2 \simeq C_2$. Letting $\gamma = -s_\alpha(\beta) = -\beta + 2\alpha$ we have

$$2\alpha = \beta + \gamma$$

Here $\alpha$ is short and $\beta, \gamma$ are long. It follows that

$$\alpha^\vee = \beta^\vee + \gamma^\vee$$

and since $\Delta_{\text{nc,long}} = \emptyset$, $\beta, \gamma \in \Delta_c$. This completes the proof of Theorem 1.1.

**Remark 3.12** The rank 2 group at the end of the preceding proof is isomorphic to $Sp(1,1) \simeq Spin(4,1)$. See Example 3.4.

If we do not assume $\Delta_{\text{nc,long}} = \emptyset$ this rank 2 group may be $Sp(4, \mathbb{R})$. In this case $\alpha^\vee \notin R^\vee(\mathbb{K}, \mathbb{T}_K)$, i.e. (3.6) would fail without the first assumption. Note that $2\alpha^\vee \in X_*(\mathbb{T}_K)$ in this case.

**Remark 3.13** Kostant has given the following short proof of (3.5). Since $\alpha$ is long then $\langle \beta, \alpha^\vee \rangle \in \{-1, 0, 1\}$ for all $\beta \neq \alpha$; in particular this holds for all $\beta \in \Delta(\mathbb{K}, \mathbb{T}_K)$. Choosing an appropriate set of positive roots we see $\alpha^\vee$ is in the fundamental chamber for the affine Weyl group of $K$. It is a standard fact that the only element of the co-root lattice in the fundamental chamber is 0. Therefore $\alpha^\vee \notin R^\vee(\mathbb{K}, \mathbb{T}_K)$.

In our context the following Lemma is sufficient to give a proof of (3.5) along these lines.

**Lemma 3.14** *Suppose $\Delta$ is a root system and $\gamma$ is a miniscule weight, i.e. $\langle \gamma, \alpha^\vee \rangle \in \{-1, 0, 1\}$ for all $\alpha \in \Delta$. Then $\gamma$ is not in the root lattice.*

**Proof.** Suppose $\gamma = \sum_S \alpha$ where $S$ is a set of roots with multiplicity. We may assume $\alpha \in S \Rightarrow -\alpha \notin S$. We proceed by induction on the order of $S$. If $S = \{\alpha\}$ then $\langle \alpha, \alpha^\vee \rangle = 2$, contradicting the assumption. Assume $|S| > 1$, and choose $\alpha \in S$. Then $\langle \gamma, \alpha^\vee \rangle = 2 + \sum_{\beta \in S - \alpha} \langle \beta, \alpha^\vee \rangle$, so $\langle \beta, \alpha^\vee \rangle < 0$ for some $\beta \in S$. But this implies $\alpha + \beta$ is a root, so we may write $\gamma$ as a sum of $|S| - 1$ roots, contradicting the inductive hypothesis. ■

# 4 Relation with the condition of Prasad

Gopal Prasad [8] has observed Theorem 1.1 in a somewhat different form. We explain the relation between the two statements and show they are equivalent. The main point is that Prasad is working with the maximally split Cartan subgroup, whereas our proof of Theorem 1.1 uses the maximally compact Cartan subgroup.

We continue in the setting of Section 2. We will assume some familiarity with the theory of Cayley transforms [11, Definition 8.3.4], [12].

Let $\mathbb{A}$ be a maximal isotropic torus of $\mathbb{G}$, with real points $A$. Choose a $\theta$–stable Cartan subgroup $\mathbb{T}_s$ containing $\mathbb{A}$; this is a maximally split Cartan subgroup of $\mathbb{G}$.

Let $\Delta(\mathbb{G}, \mathbb{A})$ be the set of roots of $\mathbb{A}$ in $\mathbb{G}$. These roots are sometimes referred to as real roots, but as this terminology conflicts with that of the preceding sections we do not use it. This root system is possibly non–reduced. Define the multiplicity $m(\alpha)$ of a root $\alpha \in \Delta(\mathbb{G}, \mathbb{A})$ to be the dimension of the corresponding root space, i.e.

$$m(\alpha) = |\{\beta \in \Delta(\mathbb{G}, \mathbb{T}) \mid \beta|_{\mathbb{T}_s} = \alpha\}|.$$

**Proposition 4.1 ([8])** *$\pi_1(G) \neq 1$ if and only if $m(\alpha) = 1$ for all $\alpha \in \Delta_{long}(\mathbb{G}, \mathbb{A})$.*

The proof in [8] is case–by-case, based on the classification of real simple groups and their covers, cf. [10] or [7]. (The table [7, page 319] incorrectly states that $Spin(n, 1)$ is not simply connected.) We prove this directly by showing this is equivalent to the condition of Theorem 1.1.

We first state a simple result which follows from the theory of Cayley transforms. This applies to our situation, and also proves Corollary 1.2.

Let $\mathbb{T}_f$ be a fundamental Cartan subgroup of $\mathbb{G}$.

**Lemma 4.2** *(1)* $\Delta_{nc,long}(\mathbb{G},\mathbb{T}_f)\neq\emptyset \Leftrightarrow \Delta_{r,long}(\mathbb{G},\mathbb{T}_s)\neq\emptyset$.

*(2) If $\mathbb{G}$ is simply laced and $\mathbb{T}$ is any Cartan subgroup then*

$$\Delta_{nc}(\mathbb{G},\mathbb{T}_f)\neq\emptyset \Leftrightarrow \Delta_{nc}(\mathbb{G},\mathbb{T})\cup\Delta_r(\mathbb{G},\mathbb{T})\neq\emptyset.$$

**Proof.** Suppose $\alpha\in\Delta_{r,long}(\mathbb{G},\mathbb{T}_s)$. We claim there is a subset $S=\{\alpha_1=\alpha,\alpha_2,\dots,\alpha_n\}$ of $\Delta_r(\mathbb{G},\mathbb{T}_s)$ consisting of strongly orthogonal roots, so that the corresponding Cayley transform $c_S$ takes $\mathbb{T}_s$ to $\mathbb{T}_f$ (after possibly replacing $\mathbb{T}_f$ by a $G$–conjugate).

To see this, first pass to the the Cayley transform $\mathbb{T}'=c_\alpha(\mathbb{T}_s)$ of $\mathbb{T}_s$ by $\alpha$, and then choose a set $\{\beta_1,\dots,\beta_{n-1}\}$ of strongly orthogonal real roots for $\mathbb{T}'$ taking $\mathbb{T}'$ to $\mathbb{T}_f$. Since $\alpha$ is long, $\alpha,\beta$ are strongly orthogonal if and only if they are orthogonal. It follows that $\{\alpha,c_\alpha(\beta_1),\dots,c_\alpha(\beta_n)\}$ satisfies the condition.

It follows that $c_S(\alpha)\in\Delta_{nc,long}(\mathbb{G},\mathbb{T}_f)$.

For the converse it is only necessary to choose a subset $S=\{\alpha_1=\alpha,\alpha_2,\dots,\alpha_n\}$ of $\Delta_{nc}(\mathbb{G},\mathbb{T}_f)$ so that $c_S$ is defined and takes $\mathbb{T}_f$ to (a $G$–conjugate of) $\mathbb{T}_s$.

The proof of (2) is similar. ■

**Proof of Corollary 1.2.** The first statement follows immediately from Lemma 4.2(2). The second is an immediate consequence of the theory of Cayley transforms. ■

By Lemma 4.2 we convert Theorem 1.1 to a statement about $\mathbb{T}_s$.

**Proposition 4.3** $\pi_1(G)\neq 1$ *if and only if* $\mathbb{T}_s$ *has a long real root.*

**Proof of Proposition 4.1.** By Proposition 4.3 it is enough to show:

$$\Delta_{r,long}(\mathbb{G},\mathbb{T}_s)\neq\emptyset \Leftrightarrow m(\beta)=1 \text{ for all } \beta\in\Delta_{\mathrm{long}}(\mathbb{G},\mathbb{A}). \tag{4.4}$$

The main point is that if $\alpha\in\Delta_{\mathrm{cx}}(\mathbb{G},\mathbb{T}_s)$ then $\alpha|_{\mathbb{A}}$ has multiplicity greater than 1 since $\alpha|_{\mathbb{A}}=-\theta\alpha|_{\mathbb{A}}$. Furthermore complex roots become shorter on restriction to $\mathbb{A}$, whereas real roots stay the same length.

Fix an inner product $(,)$ on $X^*(\mathbb{T}_s)$ which is invariant by the Weyl group and $\theta$, and use the same notation for $(,)$ restricted to $\mathbb{A}$.

**Lemma 4.5** *Choose $\alpha \in \Delta(\mathbb{G}, \mathbb{T}_s)$, and let $\beta = \alpha|_{\mathbb{A}} \in \Delta(\mathbb{G}, \mathbb{A})$. Then*

$$(\beta, \beta) = \begin{cases} (\alpha, \alpha) & \alpha \text{ real}, \\ \frac{1}{2}(\alpha, \alpha) & \alpha \text{ complex}, \langle \alpha, \theta\alpha^\vee \rangle = 0, \\ \frac{1}{4}(\alpha, \alpha) & \alpha \text{ complex}, \langle \alpha, \theta\alpha^\vee \rangle = 1. \end{cases}$$

**Proof.** Let $\mathbb{T}_c = \mathbb{T}_s^\theta$, so $\mathbb{T}_s = \mathbb{T}_c\mathbb{A}$ (not necessarily a direct product). Note that $(X^*(\mathbb{T}_c), X^*(\mathbb{A})) = 0$.

If $\alpha$ is a real root then $(\alpha, \alpha) = (\alpha|_{\mathbb{A}}, \alpha|_{\mathbb{A}})$. Suppose $\alpha$ a is complex root and let $\beta = \alpha|_{\mathbb{A}}$. Then $(\alpha - \theta\alpha)|_{\mathbb{A}} = 2\beta$ and $(\alpha - \theta\alpha)|_{\mathbb{T}_c} = 1$. Therefore $(\beta, \beta) = \frac{1}{4}(\alpha - \theta\alpha, \alpha - \theta\alpha) = \frac{1}{2}(\alpha, \alpha) - \frac{1}{2}(\alpha, \theta\alpha)$. Write $(\alpha, \theta\alpha) = \frac{(\alpha,\alpha)}{2}\frac{2(\alpha,\theta\alpha)}{(\alpha,\alpha)} = \frac{1}{2}(\alpha, \alpha)\langle \alpha, \theta\alpha^\vee \rangle$ to obtain

$$(\beta, \beta) = \frac{1}{2}(\alpha, \alpha)(1 - \frac{1}{2}\langle \alpha, \theta\alpha^\vee \rangle).$$

with $\langle \alpha, \theta\alpha^\vee \rangle \in \{0, \pm 1\}$. If $\langle \alpha, \theta(\alpha^\vee) \rangle = -1$ then (compare Section 2) $\beta = \alpha + \theta(\alpha)$ is an imaginary root. By a calculation in $SU(2,1)$ $\beta$ is non–compact, and the corresponding Cayley transform then gives a more split Cartan subgroup, contradicting the fact that $\mathbb{T}_s$ is maximally split. Therefore $\langle \alpha, \theta(\alpha^\vee) \rangle \in \{1, 0\}$. ■

The implication ($\Rightarrow$) in (4.4) is now clear: if $\Delta(\mathbb{G}, \mathbb{T}_s)$ contains a long real root, then all long roots of $\Delta(\mathbb{G}, \mathbb{A})$ are real, and have multiplicity one.

For the reverse implication suppose $\beta \in \Delta_{\text{long}}(\mathbb{G}, \mathbb{A})$. Since $m(\beta) = 1$ we have $\beta = \alpha|_{\mathbb{A}}$ where $\alpha$ is a real root. We claim $\Delta_{\text{r,long}}(\mathbb{G}, \mathbb{T}_s) \neq \emptyset$.

If the root system of $\mathbb{G}$ is simply laced this is immediate: $\alpha$ is a long real root. However, for example in $Spin(4,1)$, a root $\beta \in \Delta_{\text{long}}(\mathbb{G}, \mathbb{A})$ may be the restriction of a root $\alpha \in \Delta_{\text{r,short}}(\mathbb{G}, \mathbb{T}_s)$. In this example $\beta = \delta|_{\mathbb{A}} = \theta\delta|_{\mathbb{A}}$ with $\delta \in \Delta_{\text{cx}}(\mathbb{G}, \mathbb{T}_s)$, so $m(\beta) > 1$, contradicting the assumption.

The proof is completed by reducing the general case to this one, as at the end of Section 3. That is, suppose $\alpha \in \Delta_{\text{r, short}}(\mathbb{G}, \mathbb{T}_s)$ and $\Delta_{\text{r, long}}(\mathbb{G}, \mathbb{T}_s) = \emptyset$. Then there exists $\delta \in \Delta_{\text{cx, long}}(\mathbb{G}, \mathbb{T}_s)$ such that $\delta - s_\alpha\delta = 2\alpha$. Since $\theta\alpha = -\alpha$ it follows that $s_\alpha(\delta) = \theta(\delta)$ and $\delta - \theta\delta = \alpha$. Consequently $\beta = \alpha|_{\mathbb{A}} = \delta|_{\mathbb{A}} = \theta\delta|_{\mathbb{A}}$, contradicting the assumption that $m(\beta) = 1$. ■

# 5 Proofs of Theorems 1.4-1.6

In this section we let $\mathbb{G}$ be a simply connected group defined over $\mathbb{R}$, with real points $G$. We assume $G$ is simple. Thus either $\mathbb{G}$ is simple or $G$ is complex, in which case $\mathbb{G} = \mathbb{G}_1 \times \mathbb{G}_1$ where $\mathbb{G}_1$ is simple, and $G = \mathbb{G}_1(\mathbb{C})$ (viewed as a real group).

We need a few preliminaries about the parametrization of real groups via Kac diagrams. We will summarize the facts which we need. We parametrize real forms by their Cartan involutions. See [4, Chapter X, §5] or [7, Chapter 5, §1] for details.

Fix an element $\tau$ of the outer automorphism group $\mathrm{Out}(\mathbb{G})$ of $\mathbb{G}$ of order $k \leq 2$. Associated to $(\mathbb{G}, \tau)$ is a Dynkin diagram, which we now describe.

Fix a Cartan subgroup $\mathbb{B}$ of $\mathbb{G}$ and a Cartan subgroup $\mathbb{T} \subset \mathbb{B}$. Then we may choose a splitting $\iota$ of the exact sequence $1 \to \mathrm{Int}(\mathbb{G}) \to \mathrm{Aut}(\mathbb{G}) \to \mathrm{Out}(\mathbb{G}) \to 1$ so that, identifying $\tau$ with its image in $\mathrm{Aut}(\mathbb{G})$, $\mathbb{B}$ and $\mathbb{T}$ are $\tau$–stable. Let $\mathbb{T}^\tau$ be the fixed points of $\tau$ acting on $\mathbb{T}$.

Let $\Delta = \Delta(\mathbb{G}, \mathbb{T}^\tau)$. This is a (possibly non–reduced) root system. Then $\mathbb{B}$ defines a set of positive roots of $\Delta$, let $\Pi$ be the corresponding set of simple roots, and $D$ the corresponding Dynkin diagram.

Assume $\Delta$ is not reduced and $G$ is not complex. Let

$$\beta = \begin{cases} \text{highest root of } \Delta & k = 1 \\ \text{highest short root of } \Delta & k = 2 \end{cases} \tag{5.6}$$

If $\Delta$ is not reduced or $G$ is complex let $\beta$ be the highest root of $\Delta$. Let $\overline{\Pi} = \Pi \cup \{-\beta\}$, and let $\overline{D}$ be the Dynkin diagram of $\overline{\Pi}$.

**Remark 5.7** If $k = 1$ $\beta$ is the highest root of $\Delta$. If $k = 2$ $\beta$ is the highest root of $\mathfrak{t}^\tau$ in $\mathfrak{p}$, where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of $Lie_{\mathbb{C}}(G)$ with respect to $\tau$ and $\mathfrak{t}^\tau = Lie_{\mathbb{C}}(T^\tau)$.

**Remark 5.8** The roots of $\overline{\Pi}$ are the linear parts of a set simple roots of an affine root system $S$ in the sense of [6, §1.2], and $\overline{D}$ is the corresponding affine Dynkin diagram. The underlying finite root system of $S$ is $\Delta$, and the root $\beta$ is special [loc. cit.]. By the classification of affine root systems [loc. cit.] $\overline{D}$ is either the completed Dynkin diagram $\widehat{E}$ of a finite Dynkin diagram $E$ or its dual $(\widehat{E})^\vee$. In the first case in (5.6), and also if $\Delta$ is not reduced or $G$ is complex, $\overline{D} = \widehat{D}$. In the second $\overline{D} = (\widehat{E})^\vee$ with $E = D^\vee$.

**Remark 5.9** If there is a vertex of the Dynkin diagram of $\Delta(\mathbb{G},\mathbb{T})$ fixed by $\tau$ then $\Delta$ is reduced; this only fails in the case $\mathbb{G} = SL(2n+1), k = 2$. In this case $\Delta$ is of type $BC_n$ and $\overline{D}$ is self–dual.

Define positive integers $n(\alpha)$ by the conditions that they are relatively prime and

$$\sum_{\alpha\in\overline{\Pi}} n(\alpha)\alpha = 0. \tag{5.10}$$

Fix $m \le 2$ and let $S$ be a subset of $\overline{D}$ satisfying

$$k\sum_{\alpha\in S} n(\alpha) = m. \tag{5.11}$$

Note that $|S| = 1$ or 2. Associated to $S$ is an involution $\theta$ of $\mathbb{G}$ of order $m$ such that $\mathbb{T}$ is $\theta$ invariant and

$$\overline{\Pi} - S \text{ is a set of simple roots of } \Delta(\mathbb{G}^\theta, \mathbb{T}^\tau). \tag{5.12}$$

Every involution of $\mathbb{G}$ is conjugate to one obtained this way. We refer to the pair $(\overline{D}, S)$ as the Kac diagram of $G$.

**Remark 5.13** If $k = m = 1$ then $G$ is compact, and if $k = m = 2$ then $G$ is complex. In each case $S$ consists of a single special root, which up to automorphism of $\overline{D}$ may be taken to be $\beta$. In the literature these cases are usually disregarded in the discussion of Kac diagrams.

The dual Dynkin diagram $\overline{D}^\vee$ is the Dynkin diagram of $\overline{\Pi}^\vee = \{\alpha^\vee \mid \alpha \in \overline{\Pi}\}$ and integers $n(\alpha^\vee)$ are defined by (5.10) applied to $\overline{\Pi}^\vee$.

**Lemma 5.14**

$$kn(\alpha) \le 2 \Rightarrow n(\alpha^\vee) \le 2. \tag{5.15}$$

This can be checked easily from the classification of affine root systems, cf. [7, Table 6] or [6, Section 1.3]. For example if $\Delta$ is classical then $n(\alpha^\vee) \le 2$ for all $\alpha$. We give a case–free proof below.

Using the Lemma we complete the proof of Theorem 1.4.

**Proof of Theorem 1.4.** Let $(\overline{D}, S)$ be the Kac diagram of $G$. Recall Proposition 2.10:

$$\pi_1(G) \simeq R^\vee(\mathbb{G}, \mathbb{T}_K)/R^\vee(\mathbb{K}, \mathbb{T}_K). \tag{5.16}$$

The numerator is

$$\mathbb{Z}\langle\{\alpha^\vee \,|\, \alpha \in \Pi\}\rangle$$

and by (5.12) the denominator is

$$\mathbb{Z}\langle\{\alpha^\vee \,|\, \alpha \in \widehat{\Pi} - S\}\rangle.$$

Therefore, via the isomorphism (5.16), $\pi_1(G)$ is generated by $\{\alpha^\vee \,|\, \alpha \in S\}$.

Case 1: $S = \{\alpha\}$

It follows immediately that $\pi_1(G)$ is cyclic and generated by $\alpha^\vee$. By (5.10) applied to $\overline{\Pi}^\vee$

$$n(\alpha^\vee)\alpha^\vee = -\sum_{\beta \notin S} n(\beta^\vee)\beta^\vee \in R^\vee(\mathbb{K}, \mathbb{T}_K).$$

By (5.11) $kn(\alpha) \leq 2$, so by Lemma 5.14 $n(\alpha^\vee) = 1$ or 2, and $|\pi_1(G)| = 1$ or 2 accordingly.

Case 2: $S = \{\alpha_1, \alpha_2\}$

In this case $k = 1$, $n(\alpha_1) = n(\alpha_2) = 1$, and the center of $K$ is one-dimensional, i.e. $G$ has Hermitian symmetric domain. Since $k = 1$, $\mathbb{T} = \mathbb{T}_K$ and

$$\pi_1(G) = R^\vee(\mathbb{G}, \mathbb{T})/R^\vee(\mathbb{K}, \mathbb{T}).$$

It is well known that there is a set of simple roots for $\Delta(\mathbb{G}, \mathbb{K})$ for which precisely one root $\beta$ is non–compact. This corresponds to the fact that (up to automorphism of $\overline{D}$) we may take $S = \{\beta, \alpha\}$ and then $\Delta(\mathbb{K}, \mathbb{T}) = \Delta - \{\alpha\}$. It follows that $\pi_1(G)$ is cyclic, with generator $\alpha^\vee$. Since $K$ has a one-dimensional center the fundamental group is infinite. In our terms the rank of $R^\vee(\mathbb{K}, \mathbb{T})$ is one less than the rank of $R^\vee(\mathbb{G}, \mathbb{T})$, with quotient generated by $\alpha^\vee$. ∎

**Remark 5.17** Theorem 1.1 also follows from these considerations.

**Proof of Lemma 5.14.** As in Section 3 subsitute $\alpha = \frac{(\alpha,\alpha)}{2}\alpha^\vee$ in (5.10) to give

$$\sum_{\alpha\in\overline{\Pi}} \frac{(\alpha,\alpha)}{2} n(\alpha)\alpha^\vee = 0.$$

On the other hand $\sum_{\alpha\in\overline{\Pi}} n(\alpha^\vee)\alpha^\vee = 0$. We conclude there is a constant $d$ so that

$$dn(\alpha^\vee) = \frac{(\alpha,\alpha)}{2} n(\alpha)$$

for all $\alpha \in \overline{\Pi}$. Taking $\alpha = \beta$ (5.6) gives $d = \frac{(\beta,\beta)n(\beta)}{2n(\beta^\vee)}$, and therefore

$$n(\alpha^\vee) = \frac{(\alpha,\alpha)}{(\beta,\beta)} \frac{n(\beta^\vee)}{n(\beta)} n(\alpha). \tag{5.18}$$

Finally using $n(\alpha) \leq \frac{2}{k}$ by assumption we have to show

$$\frac{(\alpha,\alpha)}{(\beta,\beta)} \frac{n(\beta^\vee)}{n(\beta)} \leq k. \tag{5.19}$$

Recall (Remark 5.8) $\beta$ is special for the affine root system $S$ defined by $\overline{D}$ and $\Delta$ is the underlying finite root system of $S$ [6]. It is straightforward to see if $\Delta$ is reduced then a root $\alpha$ of $S$ is special if and only if $\alpha^\vee$ is a special root of $S^\vee$. (This amounts to the fact that if $\Phi$ is a set of simple roots for a finite reduced root system $\Delta$, then $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$ is a set of simple roots for $\Delta^\vee$. This is false if $\Delta$ is of type $BC_n$.)

Therefore if $\Delta$ is reduced then $\beta^\vee$ is special, and $n(\beta^\vee) = 1$. In this case if $\beta$ is long (5.19) is clear; if $\beta$ is short then $k = 2$ and it holds as well.

If $\Delta$ is not reduced (i.e. of type $BC_n$) then $n(\beta^\vee) = k = 2$ and we have to show $\frac{(\alpha,\alpha)}{(\beta,\beta)} \leq 1$, which is true since $\beta$ is long. ∎

From this discussion, in particular (5.18), we may read off $\pi_1(G)$ from the Kac diagram as follows.

**Proposition 5.20** *Let $(\overline{D}, S)$ be the Kac diagram of $G$.*

*(1) If $|S| = 2$ then $\pi_1(G) \simeq \mathbb{Z}$.*

*(2) Suppose $S=\{\alpha\}$. Then*

$$\pi_1(G)=\begin{cases}1 & n(\alpha^\vee)=1\\ \mathbb{Z}/2\mathbb{Z} & n(\alpha^\vee)=2.\end{cases}$$

*If $G$ is not compact or complex then the condition may be written*

$$\pi_1(G)=\begin{cases}1 & \alpha \textit{ short}\\ \mathbb{Z}/2\mathbb{Z} & \alpha \textit{ long},\end{cases}$$

*and $G$ has a non–trivial cover unless $S=\{\alpha\}$ where $\alpha$ is short.*

Theorem 1.5 now also follows easily.

**Proof of Theorem 1.5.** If $\alpha$ is real replace it with its Cayley transform $c_\alpha(\alpha)\in\Delta_{nc,long}(\mathbb{G},\mathbb{T}_\alpha)$ where $\mathbb{T}_\alpha$ is the compact (mod center) Cartan subgroup of $M_\alpha$. Without loss of generality we may assume $\mathbb{T}_\alpha\cap\mathbb{K}\subset\mathbb{T}_K$, so $\alpha^\vee\in X_*(\mathbb{T}\cap\mathbb{K})\subset X_*(\mathbb{T}_f)$ where $\mathbb{T}_f$ is a fundamental Cartan subgroup.

By Proposition 2.10 the fundamental group of $M_\alpha$ is

$$\pi_1(M_\alpha)\simeq\mathbb{Z}\langle\alpha^\vee\rangle/R^\vee(\mathbb{K},\mathbb{T}_K)\cap\mathbb{Z}\langle\alpha^\vee\rangle$$

so $\phi_\alpha$ is identified with the natural map

$$\phi_\alpha:\mathbb{Z}\langle\alpha^\vee\rangle/R^\vee(\mathbb{K},\mathbb{T}_K)\cap\mathbb{Z}\langle\alpha^\vee\rangle\to R^\vee(\mathbb{G},\mathbb{T}_K)/R^\vee(\mathbb{K},\mathbb{T}_K).$$

As in the proof of Lemma 4.2 $\alpha^\vee=\beta^\vee$ for some $\beta\in\Delta_{nc,long}(\mathbb{G},\mathbb{T}_f)$. By (3.5) $\phi_\alpha$ is non–trivial, so if $\pi_1(G)=\mathbb{Z}/2\mathbb{Z}$ we are done. Assume $\pi_1(G)=\mathbb{Z}$. We have already seen that in this case $T$ is compact and $\pi_1(G)$ is generated by $\beta^\vee$ where $\beta$ is a long non–compact root, and the result follows. ■

**Proof of Theorem 1.6.** The inverse image $\widetilde{K}$ of $K$ in $\widetilde{G}$ is the real points of an algebraic group $\widetilde{\mathbb{K}}$, which contains a Cartan subgroup $\widetilde{\mathbb{T}}_K$ covering $\mathbb{T}_K$. We have the containments

$$X_*(\mathbb{T}_K)\supset X_*(\widetilde{\mathbb{T}_K})\supset R^\vee(\mathbb{K},\mathbb{T}_K).$$

The first containment is of index 2, the second is equality unless $G$ has Hermitian symmetric domain, and the quotient of the first term by the last is isomorphic to $\pi_1(G)$.

The statement is then equivalent to

$$\alpha \in \Delta_{\mathrm{nc,long}} \Leftrightarrow \alpha^\vee \notin X_*(\widetilde{\mathbb{T}}_K). \tag{5.21}$$

Compare (3.5) and (3.6). The implication ($\Rightarrow$) follows immediately from Theorem 1.5. The reverse implication follows by a reduction to rank 2, by Remark 3.12. (The case of the split group $G_2$ is easy, since $X_*(\widetilde{\mathbb{T}}) = R^\vee(\mathbb{K}, \mathbb{T})$ and the compact roots are of type $A_1 \times A_1$.) ■

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