Algorithms for Representation Theory of Real Reductive Groups

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Introduction

The irreducible admissible representations of a real reductive group such as $GL(n, \mathbb{R})$ have been classified by work of Langlands, Knapp, Zuckerman and Vogan. This classification is somewhat involved and requires a substantial number of prerequisites. See [12] for a reasonably accessible treatment. It is fair to say that it is difficult for a non-expert to understand any non-trivial case, not to mention a group such as E_8.

The purpose of these notes is to describe an algorithm to compute the irreducible admissible representations of a real reductive group. This algorithm has been implemented on a computer by the second author. An early version of the software (Version 0.3 as of April 2008), and other documentation and information, may be found on the web page of the Atlas of Lie Groups and Representations at `www.liegroups.org`.

Here is some more detail on what the algorithm and the software do:

(1) Allow the user to define

  (a) A complex reductive group G,

  (b) An inner class of real forms of G,

  (c) A particular real form $G(\mathbb{R})$ of G.

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(2) Enumerate the Cartan subgroups of $G(\mathbb{R})$, and describe them as real tori,

(3) For any Cartan subgroup $H(\mathbb{R})$ compute $W(G(\mathbb{R}), H(\mathbb{R}))$ (the "real" Weyl group),

(4) Compute a set $\mathcal{X}$ parametrizing the set $K\backslash G/B$ where B is a Borel subgroup and K is the complexification of a maximal compact subgroup of $G(\mathbb{R})$,

(5) Compute a set $\mathcal{Z}$ parametrizing the irreducible representations of $G(\mathbb{R})$ with regular integral infinitesimal character,

(6) Compute the cross action and Cayley transforms on $\mathcal{X}$ and $\mathcal{Z}$,

(7) Compute Kazhdan-Lusztig polynomials.

In fact the proper setting for all of the preceding computations is not a single real group $G(\mathbb{R})$, but an entire inner class of real forms, as described in Sections 2–4.

The approach used in these notes most closely follows [1]. This reference has the advantage over [3], which later supplanted it, in that it focuses on the case of regular integral infinitesimal character, and avoids some complications arising from the general case. There are a few changes in terminology from these references which are discussed in the remarks. We also make extensive use of [4]. This work has some overlap with, and was influenced by, that of Richardson and Springer [20].

A number or examples illustrating the algorithm and the software may be found in [2]. The reader is encouraged to use the software, and consult [2], while reading this paper.

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Fokko du Cloux died of ALS in November 2007. Fokko played a major role in development of the algorithm described in this paper, in addition to writing the atlas software. This work began in 2004 and was mostly complete by late 2006. Fokko was an active participant in the writing of this paper, which was originally submitted shortly after his death.

Index of Notation

$\mathrm{Aut}(G), \mathrm{Int}(G), \mathrm{Out}(G)$: the (holomorphic) automorphisms of G, the inner automorphisms, and the outer automorphisms $\mathrm{Aut}(G)/\mathrm{Int}(G)$, respectively (Section 2.1).

$\mathcal{B}$: set of Borel subgroups of G (Section 8), in bijection with G/B.

$(B, H, \{X_\alpha\})$: splitting or pinning data for G (Section 2.1). Defines a splitting $\mathrm{Out}(G) \to \mathrm{Aut}(G)$ (cf. (2.8)).

c^α, c_α: noncompact imaginary and real Cayley transforms, respectively (Definitions 14.1 and 14.8).

$D = (X, \Delta, X^\vee, \Delta^\vee)$: root datum defining a complex connected reductive group (2.1).

$D(G, H)$: root datum associated to a complex connected reductive group G and a Cartan subgroup H (2.5).

$D_b = (X, \Pi, X^\vee, \Pi^\vee), D_b(G, B, H)$: abstract *based* root datum, and the based root datum associated to G, H and a Borel subgroup B (2.6).

$D^\vee, D_b^\vee, G^\vee$: dual (based) root datum and dual group associated to D, D_b and G respectively (end of Section 2).

(G, γ): basic data consisting of a complex connected reductive group G and an involution $\gamma \in \mathrm{Out}(G)$ (Section 4).

G^Γ: given basic data (G, γ), the extended group containing G; $G^\Gamma = G \rtimes \Gamma$ where $\Gamma = \{1, \sigma\}$ and σ acts by a distinguished involution mapping to γ (Definition 5.1).

$(G^\vee, \gamma^\vee)$: basic data dual to (G, γ) (Section 4).

$H'_{-\tau}, H_{-\tau}, A_\tau, T_\tau$: various subgroups of the Cartan subgroup H, depending on a twisted involution τ ((11.1) and Remark 11.3). In particular $A_\tau \subset H_\tau \subset H'_\tau$, A_τ and T_τ are connected complex tori, and $H = T_\tau A_\tau$.

$\mathcal{I}(G, \gamma)$: set of strong involutions in G^Γ (Definition 5.5).

I: $\{\xi_i \,|\, i \in I\}$ is a set of representatives of the set of strong real forms, i.e. $\mathcal{I}(G, \gamma)/G$ (5.15)(a)

$\mathcal{I}_i$: for $i \in I$, strong involutions conjugate to ξ_i (5.15).

$\mathcal{I}_W$: twisted involutions in $W = \{\tau \in W^\Gamma \backslash W \,|\, \tau^2 = 1\}$ (9.11)(h) and (9.14).

$\mathcal{L}, \mathcal{L}_c$: set of equivalance classes of (complete) two-sided L-data for (G, γ) (7.14).

N, N^Γ : normalizer of H in G and G^Γ, respectively (9.11)(a).

$\tilde{p}, p$: maps from $\widetilde{\mathcal{X}}$ and $\mathcal{X}$ to $\mathcal{I}_W$, respectively (9.11)(i).

$P, P^\vee, P_{reg}, P^\vee_{reg}, P(G,H), P^\vee(G,H)$: weight and coweight lattices and their regular elements (Section 4).

$\mathcal{P} = \mathcal{P}(G,\gamma), \mathcal{P}_c = \mathcal{P}_c(G,\gamma)$: equivalence classes of (complete) one-sided L-data for (G,γ)) (7.9). Also $\mathcal{P}^\vee = \mathcal{P}(G^\vee,\gamma^\vee), \mathcal{P}^\vee_c = \mathcal{P}_c(G^\vee,\gamma^\vee)$.

$\mathcal{P}[\xi]$: set of equivalence classes of one-sided L-data (ξ', B) such that ξ is conjugate to ξ ((8.1)(b)). In bijection with $K_\xi\backslash G/B$ ((8.1)(e)).

$S = (\xi, B)$: one-sided L-datum for (G,γ), consisting of of a strong involution and a Borel subgroup of G (Definition 7.6). Also $S^\vee = (\eta, B_1^\vee)$ is a one-sided L-datum for $(G^\vee,\gamma^\vee)$.

$S_c^\vee = (\eta, B_1^\vee, \lambda)$: complete one-sided L-datum for $(G^\vee,\gamma^\vee)$, consisting of an L-datum $(\eta, B_1^\vee)$ and $\lambda \in \mathfrak{h}^\vee$ satisfying $\exp(2\pi i\lambda) = \eta^2$ (Definition 7.1).

$\mathbf{S} = (S, S^\vee)$: two-sided L-datum, where $S, S^\vee$ are L-data for (G,γ) and $(G^\vee,\gamma^\vee)$ respectively, satisfying an additional compatibility condition (Definition 7.10).

$\mathbf{S_c} = (S, S_c^\vee)$: complete two-sided L-datum, where S is an L-datum for (G,γ), $S_c^\vee$ is a complete L-datum for $(G^\vee,\gamma^\vee)$, satisfying an additional compatibility condition (Definition 7.10).

W, W^Γ: $W = N/H$ is the Weyl group, and $W = N^\Gamma/H$ is the extended Weyl group (9.11)(b).

$\widetilde{W}$: the Tits group; a subgroup of G mapping onto W (Definition 15.1).

$\mathcal{X} = \mathcal{X}(G,\gamma)$: the one-sided parameter space $= \widetilde{\mathcal{X}}/H = \{\xi \in \mathrm{Norm}_{G^\Gamma\backslash G}(H) \mid \xi^2 \in Z(G)\}/H$ (Definition 9.2).

$\mathcal{X}^r = \mathcal{X}^r(G,\gamma)$: the *reduced* parameter space; a finite subset of $\mathcal{X}$ (Definition 13.4).

$\widetilde{\mathcal{X}}$: strong involutions normalizing $H = \{\xi \in \mathrm{Norm}_{G^\Gamma\backslash G}(H) \mid \xi^2 \in Z(G)\}$ (9.11)(d).

$\mathcal{X}_\tau, \widetilde{\mathcal{X}}_\tau$: for τ a twisted involution, the fiber of the map $p : \mathcal{X} \to \mathcal{I}_W$ (resp. $\tilde{p} : \widetilde{\mathcal{X}} \to \mathcal{I}_W$) over τ (beginning of Section 11).

$\mathcal{X}[x]$: subset of $\mathcal{X}$ consisting of elements conjugate to x (9.7).

$\mathcal{X}(z)$: for $z \in Z(G)$ the elements of $\mathcal{X}$ satisfying $x^2 = z$ (9.13)(c).

$\mathcal{X}_\tau(z)$: for $z \in Z(G)$, the elements x of $\mathcal{X}_\tau$ satisfying $x^2 = z$ (beginning of Section 11).

$\mathcal{Z} = \mathcal{Z}(G, \gamma)$: the two-sided parameter space, contained in $\mathcal{X}(G, \gamma) \times \mathcal{X}(G^\vee, \gamma^\vee)$ (Definition 10.1).

δ: distinguished element of $\mathcal{X}$ or $\mathcal{I}_W$ (Definition 5.1 and before Lemma 9.12).

$\Delta_i, \Delta_r, \Delta_{cx}, W_i, W_{i,\tau}, W_r, W_{r,\tau}$, etc.: various root systems and Weyl groups (beginning of Section 12).

$\gamma^\vee$: element of $\mathrm{Out}(G^\vee)$ corresponding to $\gamma \in \mathrm{Out}(G)$ (Defintion 2.11).

θ_ξ, K_ξ: $\theta_\xi = \mathrm{int}(\xi)$, an involution of G, and $K_\xi = G^{\theta_\xi}$ (Definition 5.5). Thus θ_ξ is the Cartan involution of a real form of G, and K_ξ is the corresponding complexified maximal compact subgroup (Section 3).

θ_i, K_i: For $i \in I$, Cartan involution and complexified maximal compact subgroup associated to ξ_i (5.15).

ξ : strong involution (with respect to (G, γ)); $\xi \in G^\Gamma \backslash G$ satisfies $\xi^2 \in Z(G)$ (Definition 5.5). Then $\theta_\xi = \mathrm{int}(\xi)$ is an involution of G.

(ξ, π): Harish-Chandra module for a strong involution; a strong involution ξ and a $(\mathfrak{g}, K_\xi)$-module π. A *Harish-Chandra module for a strong real form* is a G-orbit of such pairs (Section 6).

$\Pi(G(\mathbb{R}), \lambda)$: set of equivalence classes of irreducible admissible representations of $G(\mathbb{R})$ with infinitesimal character λ (Definition 1.1).

$\Pi(G(\mathbb{R}), \Lambda)$: for $\Lambda \subset P_{reg}$, $\coprod_{\lambda \in \Lambda} \Pi(G(\mathbb{R}), \lambda)$ (before Theorem 7.17).

$\Pi(G, \xi)$: equivalence classes of $(\mathfrak{g}, K_\xi)$-modules with regular integral infinitesimal character (Definition 6.3).

$\Pi(G, \gamma)$: representations of strong real forms of G with regular integral infinitesimal character; $= \{(\xi, \pi) \,|\, \xi \in \mathcal{I}, \pi \in \Pi(G, \xi)\}/G$ (Definition 6.3).

$\Pi(G, \gamma, \Lambda)$: for $\Lambda \subset P_{reg}$, the subset of $\Pi(G, \gamma)$ consisting of representations with infinitesimal character contained in Λ.

$\Pi_\phi(G, \xi)$: L-packet of representations of strong real form ξ associated to the L-homomorphism ϕ (after 6.5).

$\Pi_\phi(G, \gamma)$: *large* L-packet; union of L-packets for all strong real forms (6.6).

$\Pi_{S_c^\vee}(G, \gamma)$: large L-packet $= \Pi_\phi(G, \gamma)$ where ϕ is defined by (7.3).

1 Overview

{s:overview}

The primary aim of this paper is to distill a well-known but difficult theory into an algorithm which can be implemented on a computer. While the resulting algorithm is self-contained and comparatively elementary, even understanding the algorithm itself requires a fairly deep knowledge of the mathematics. In this section we give a high level overview of the algorithm, before going into more detail in the remainder of the paper. An outline of the contents of the paper appears at the end of this Overview (Section 1.7).

We assume the reader is familiar with the theory of admissible representations of real reductive groups. A good introduction is Knapp's book [12].

In this section we write $G(\mathbb{C})$ for a complex reductive group, with real points $G(\mathbb{R})$. We will make various simplifying assumptions in the course of this overview; this is for ease of exposition, and the general statements will be found in the body of the paper. See the end of this section for a discussion of the issues involved.

Let $\mathfrak{g}$ be the Lie algebra of $G(\mathbb{C})$. Fix a Cartan involution θ of $G(\mathbb{C})$ corresponding to $G(\mathbb{R})$. Thus $K = G(\mathbb{R})^\theta$ is a maximal compact subgroup of $G(\mathbb{R})$. The basic goal is to parametrize the irreducible admissible representations of $G(\mathbb{R})$, or equivalently the irreducible admissible $(\mathfrak{g}, K)$-modules. This is an infinite (typically uncountable) set.

Suppose $H(\mathbb{C})$ is a Cartan subgroup of $G(\mathbb{C})$, and let $\mathfrak{h} = \mathrm{Lie}(H(\mathbb{C}))$. By the Harish-Chandra homomorphism, associated to $\lambda \in \mathfrak{h}^*$ is an *infinitesimal character* which we also denote λ. We say λ is *regular* (resp. *integral*) if $\langle \lambda, \alpha^\vee \rangle \neq 0$ (resp. $\in \mathbb{Z}$) for all roots α of $\mathfrak{h}$ in $\mathfrak{g}$.

Since we will be working with different Cartan subgroups simultaneously, it is convenient to fix one, denoted $H_a(\mathbb{C})$, the *abstract* Cartan subgroup. Then we fix once and for all $\lambda \in \mathfrak{h}_a^*$. The first basic reduction is to consider representations with infinitesimal character λ.

{d:infchar}

Definition 1.1 *Fix $\lambda \in \mathfrak{h}_a^*$, and let $\Pi(G(\mathbb{R}), \lambda)$ be the set of irreducible admissible representations of $G(\mathbb{R})$ with infinitesimal character λ.*

By a result of Harish-Chandra this is a finite set. The main result of this paper is an algorithm to compute $\Pi(G(\mathbb{R}), \lambda)$ when λ is regular and integral.

There are a number of approaches to classifying $\Pi(G(\mathbb{R}), \lambda)$. We focus on three of them, which are intertwined with each other, and each of which plays a role in the algorithm.

1.1 The Langlands classification

{s:langlands}

Many representations of reductive groups can be constructed using characters of Cartan subgroups. For example if G is a reductive group over a finite field these are the representations $R_T(\theta)$ of Deligne and Lusztig [8]. For generic θ $R_T(\theta)$ is irreducible.

For another example, suppose $\mathbb{F}$ is a finite or local field, $G = G(\mathbb{F})$ is split, and $B = HN$ is a Borel subgroup of G. If χ is a character of H then $\mathrm{Ind}_B^G(\chi \otimes 1)$ is a minimal principal series representation of G; again for generic χ this is irreducible.

For a final example assume $G(\mathbb{C})$ is semisimple and simply connected, and $G = G(\mathbb{R})$ is a real form of $G(\mathbb{C})$, containing a compact Cartan subgroup T. Suppose χ is a character of T such that $\langle d\chi, \alpha^\vee \rangle \neq 0$ for all roots α. Associated to χ is the discrete series representation of G with Harish-Chandra parameter $d\chi$, and every discrete series representation of G is of this form.

The Langlands classification for $G(\mathbb{R})$ is built out of the second two cases. For now we assume that $G(\mathbb{C})$ is acceptable, i.e. ρ (one half the sum of a set of positive roots) exponentiates to $H_a(\mathbb{C})$. This holds for example if $G(\mathbb{C})$ is semisimple and simply connected.

Consider the set of pairs $(H(\mathbb{R}), \chi)$ where $H(\mathbb{R})$ is a Cartan subgroup of $G(\mathbb{R})$, χ is a character of $H(\mathbb{R})$, and $d\chi$ is $G(\mathbb{C})$-conjugate to λ. The group $G(\mathbb{R})$ acts on these pairs by conjugation. We define *character data* as follows:

$$\mathcal{C}(G(\mathbb{R}), \lambda) = \{(H(\mathbb{R}), \chi) \mid d\chi \text{ is } G(\mathbb{C})\text{-conjugate to } \lambda\}/G(\mathbb{R}). \tag{1.2}$$

{p:LC}

Proposition 1.3 (Langlands Classification) *Assume $G(\mathbb{C})$ is acceptable and λ is regular. There is a natural bijection*

$$\Pi(G(\mathbb{R}), \lambda) \overset{1-1}{\longleftrightarrow} \mathcal{C}(G(\mathbb{R}), \lambda). \tag{1.4}$$

There are many versions of the Langlands classification, for example see [12] or [25]. This restatement of (a special case of) the classification is taken from [4, Theorem 8.29]. Since $G(\mathbb{C})$ is acceptable we do not need the ρ-cover of H, and since λ is regular we can take Ψ to be the set of positive, real integral roots defined by λ (notation as in [4]).

Thus one version of our algorithm would be to replace the right hand side of (1.4) by a computable combinatorial object. Here is an idea of what this involves.

First we fix a set $H_1(\mathbb{R}), \dots, H_n(\mathbb{R})$ of representatives of the conjugacy classes of Cartan subgroups. An algorithm for computing this set is given by Kostant[13], [14], and independently by Borel (unpublished) and Sugiura [23].

Now fix i and let $H(\mathbb{R}) = H_i(\mathbb{R})$. Let $W(G(\mathbb{R}), H(\mathbb{R}))$ be the "real" Weyl group $\mathrm{Norm}_{G(\mathbb{R})}(H(\mathbb{R}))/H(\mathbb{R})$. This is a subgroup of the Weyl group $W(G(\mathbb{C}), H(\mathbb{C})) = W(\mathfrak{g}, \mathfrak{h})$. Unlike $W(\mathfrak{g}, \mathfrak{h})$, $W(G(\mathbb{R}), H(\mathbb{R}))$ depends on i, is not necessarily the Weyl group of a root system, and can be somewhat difficult to compute. An algorithm is given in [11]; also see [26, Proposition 4.16].

{l:number}

Lemma 1.5 ([25], Theorem 2.2.4) *In the setting of Proposition 1.3 there is a bijection*

$$(1.6) \qquad \Pi(G(\mathbb{R}), \lambda) \overset{1-1}{\longleftrightarrow} \coprod_{i}^{n} (W/W(G(\mathbb{R}), H_i(\mathbb{R})) \times [H_i(\mathbb{R})/H_i(\mathbb{R})^0]\widehat{\ }.$$

In particular

$$(1.7) \qquad |\Pi(G(\mathbb{R}), \lambda)| = \sum_{i=1}^{n} |W/W(G(\mathbb{R}), H_i(\mathbb{R}))||H_i(\mathbb{R})/H_i(\mathbb{R})^0|.$$

See [25, Theorem 2.2.4].

The bijection (1.6) depends on a number of choices. In addition to the choice of the $H_i(\mathbb{R})$, for each i we have to choose a Borel subgroup containing $H_i(\mathbb{C})$, and an embedding of $H_i(\mathbb{R})/H_i(\mathbb{R})^0$ in $H_i(\mathbb{R})$. Nevertheless this result gives a good idea of what we need to compute: {en:need}

(1) the Cartan subgroups $H_1(\mathbb{R}), \dots, H_n(\mathbb{R})$; and for each i,

(2) $W(G(\mathbb{R}), H_i(\mathbb{R}))$,

(3) $H_i(\mathbb{R})/H_i(\mathbb{R})^0$.

In any event it is clear that any parametrization of $\Pi(G(\mathbb{R}), \lambda)$ must (if only implicitly) include a description of items (1-3).

1.2 $\mathcal{D}$-modules

{s:dmodules}

We now describe the classification of $\Pi(G(\mathbb{R}),\lambda)$ in terms of $\mathcal{D}$-modules. The basic reference is [5], or see [17] for a good introduction to the subject.

Recall θ is a Cartan involution of $G(\mathbb{C})$ corresponding to $G(\mathbb{R})$. Let $K(\mathbb{C}) = G(\mathbb{C})^\theta$. Then $K(\mathbb{C})$ is a reductive group (possibly disconnected), with real points $K = G(\mathbb{R})^\theta$. By the relationship between finite dimensional representations of K and $K(\mathbb{C})$, we may view admissible $(\mathfrak{g}, K)$-modules as $(\mathfrak{g}, K(\mathbb{C}))$-modules.

Let $\mathcal{B}$ be the variety of Borel subgroups of $G(\mathbb{C})$, so $\mathcal{B} = G(\mathbb{C})/B(\mathbb{C})$ where $B(\mathbb{C})$ is a fixed Borel subgroup. Then $K(\mathbb{C})$ acts on $\mathcal{B}$ with finitely many orbits. Let $\mathcal{D}_\lambda$ be the sheaf of twisted differential operators on $\mathcal{B}$ corresponding to λ. We consider *$\mathcal{D}$-module data*:

$$\mathcal{D}(G(\mathbb{R}),\lambda) = \{(\mathcal{O},\tau)\}/K(\mathbb{C}). \tag{1.8}$$

where $\mathcal{O}$ is a $K(\mathbb{C})$-orbit on $\mathcal{B}$, and τ is an irreducible $K(\mathbb{C})$-equivariant $\mathcal{D}_\lambda$-module.

Here is how to make τ more concrete. Fix $x \in \mathcal{O}$ and let $K_x(\mathbb{C}) = \mathrm{Stab}_{K(\mathbb{C})}(x)$. Let $B(\mathbb{C})$ be the Borel subgroup of $G(\mathbb{C})$ corresponding to x, and let $H(\mathbb{C})$ be a θ-stable Cartan subgroup contained in $B(\mathbb{C})$. Then τ can be viewed as a character of $H(\mathbb{R})$ whose differential $d\chi$ is $G(\mathbb{C})$-conjugate to λ.

{t:caD}

Proposition 1.9 ([5]; [17], Theorem 3.9) *There is a natural bijection*

$$\Pi(G(\mathbb{R}),\lambda) \xleftrightarrow{1-1} \mathcal{D}(G(\mathbb{R}),\lambda). \tag{1.10}$$

It is reasonable to look for an algorithm to compute the right hand side.

Fix an orbit $\mathcal{O}$ of $K(\mathbb{C})$ on $\mathcal{B}$. By the Proposition, for every $\mathcal{D}_\lambda$-module τ on $\mathcal{O}$ we obtain an irreducible representation. Varying τ we obtain a finite set of representations associated to $\mathcal{O}$. For (not very good) reasons which will become clear in the next section (also see [26, Section 8]) we refer to this set as an *R-packet*, and denote it $\Pi_R(G(\mathbb{R}),\mathcal{O},\lambda)$. Thus $\Pi(G(\mathbb{R}),\lambda)$ is a disjoint union of R-packets:

$$\Pi(G(\mathbb{R}),\lambda) = \coprod_{K(\mathbb{C})\backslash\mathcal{B}} \Pi_R(G(\mathbb{R}),\mathcal{O},\lambda). \tag{1.11}$$

1.3 The Langlands classification using L-groups

{s:langlands2}

We now consider a version of the Langlands classification in terms of L-groups. Given our group $G(\mathbb{C})$, with real points $G(\mathbb{R})$, let $G^{\vee\Gamma}$ be the L-group of $G(\mathbb{R})$ [6]. Thus $G^{\vee\Gamma} = G^\vee(\mathbb{C}) \rtimes \Gamma$, where $G^\vee(\mathbb{C})$ is the dual group of $G(\mathbb{C})$, $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$, and $G^{\vee\Gamma}$ is a certain semidirect product. For example if $G(\mathbb{R})$ is split this is a direct product.

Let $W_\mathbb{R}$ be the Weil group of $\mathbb{R}$, i.e. $W_\mathbb{R} = \langle \mathbb{C}^\times, j\rangle$ where $jzj^{-1} = \overline{z}$ and $j^2 = -1$. We consider the space $\mathrm{Hom}_{\mathrm{adm}}(W_\mathbb{R}, G^{\vee\Gamma})$ of *admissible homomorphisms* of $W_\mathbb{R}$ into $G^{\vee\Gamma}$. These are the continuous homomorphisms such that $\phi(\mathbb{C}^\times)$ consists of semisimple elements of $G^\vee(\mathbb{C})$, and $\phi(j) \notin G^\vee(\mathbb{C})$.

The version of the Langlands classification in terms of L-groups says that associated to an admissible homomorphism ϕ is a finite set of irreducible admissible representations $G(\mathbb{R})$, called an *L-packet*. The L-packet of ϕ only depends on the $G^\vee(\mathbb{C})$-orbit of ϕ, and we write $\Pi_L(G(\mathbb{R}), \mathcal{O}^\vee)$ for the L-packet associated to such an orbit $\mathcal{O}^\vee$. An L-packet may be empty (if $G(\mathbb{R})$ is not quasi-split); non-empty L-packets associated to two orbits are equal if only if the orbits are equal. Finally the admissible dual of $G(\mathbb{R})$ is the disjoint union of L-packets.

The representations in $\Pi_L(G(\mathbb{R}), \mathcal{O}^\vee)$ all have the same infinitesimal character, which is determined by $\mathcal{O}^\vee$ (see Section 7). Let $\mathrm{Hom}_{\mathrm{adm}}(W_\mathbb{R}, G^{\vee\Gamma}, \lambda)$ be the admissible homomorphisms for which $\Pi_L(G(\mathbb{R}), \phi)$ has infinitesimal character λ. Thus (compare (1.11)) $\Pi(G(\mathbb{R}), \lambda)$ is a disjoint union of L-packets:

$$\Pi(G(\mathbb{R}), \lambda) = \coprod_{\mathcal{O}^\vee} \Pi_L(G(\mathbb{R}), \mathcal{O}^\vee) \tag{1.12}$$

where the union runs over $G^\vee(\mathbb{C})$ orbits on $\mathrm{Hom}_{\mathrm{adm}}(W_\mathbb{R}, G^{\vee\Gamma}, \lambda)$.

Suppose an L-packet $\Pi_L(G(\mathbb{R}), \mathcal{O}^\vee)$ is non-empty. Some additional data is required to specify a particular representation in $\Pi_L(G(\mathbb{R}), \mathcal{O}^\vee)$. Choose $\phi \in \mathcal{O}^\vee$, let S_ϕ be the centralizer in $G^\vee(\mathbb{C})$ of the image of ϕ, and let $\mathbb{S}_\phi = S_\phi/S_\phi^0$. Roughly speaking $\Pi_L(G(\mathbb{R}), \mathcal{O}^\vee)$ should be parametrized by characters of $\mathbb{S}_\phi$. This leads us to define the set of *Langlands data*

$$\mathcal{L}(G^{\vee\Gamma}, \lambda) = \{(\phi, \chi) \mid \phi \in \mathrm{Hom}_{\mathrm{adm}}(W_\mathbb{R}, G^{\vee\Gamma}, \lambda), \chi \in \widehat{\mathbb{S}_\phi}\}/G^\vee(\mathbb{C}). \tag{1.13}$$

Suppose $G_1(\mathbb{R}), G_2(\mathbb{R})$ are two real forms of $G(\mathbb{C})$. It may be that the associated L-groups are isomorphic (we say the real forms are inner to each other if this holds). For example this is the case for all real forms of $G(\mathbb{C})$ if $G(\mathbb{C})$ is semisimple and its Dynkin diagram admits no non-trivial automorphisms. Unlike $\mathcal{C}(G(\mathbb{R}),\lambda)$ and $\mathcal{D}(G(\mathbb{R}),\lambda)$, $\mathcal{L}(G^{\vee\Gamma},\lambda)$ should be related to representations of all these real forms.

To make this precise it is convenient to assume that $G(\mathbb{C})$ is adjoint. Let $G_1(\mathbb{R}),\dots,G_n(\mathbb{R})$ be the inequivalent real forms of $G(\mathbb{C})$ with L-group $G^{\vee\Gamma}$.

{t:caL}

Proposition 1.14 ([1], Theorem 3-2) *Assume $Z(G(\mathbb{C}))=1$. There is a natural bijection*

$$\coprod_{i=1}^{n}\Pi(G_i(\mathbb{R}),\lambda)\overset{1-1}{\longleftrightarrow}\mathcal{L}(G^{\vee\Gamma},\lambda). \tag{1.15}$$

Again it is reasonable to look for an algorithm to compute the right hand side.

1.4 L-packets and R-packets

{s:packets}

Fix a real form $G(\mathbb{R})$ of $G(\mathbb{C})$. Consider for the moment the problem of explicitly parametrizing $\Pi(G(\mathbb{R}),\lambda)$ using L-homomorphisms.

Recall (1.12) $\Pi(G(\mathbb{R}),\lambda)$ is the disjoint union of L-packets $\Pi_L(G(\mathbb{R}),\mathcal{O}^\vee)$ (some of these may be empty). Specifying a single representation in an L-packet $\Pi_L(G(\mathbb{R}),\mathcal{O}^\vee)$ amounts to specifying a character of the two-group $\mathbb{S}_\phi$ ($\phi\in\mathcal{O}^\vee$), a problem we prefer to avoid.

On the other hand $\Pi(G(\mathbb{R}),\lambda)$ is also the disjoint union of R-packets $\Pi_R(G(\mathbb{R}),\mathcal{O},\lambda)$ (see (1.11)). Again specifying a single representation in an R-packet requires specifying a character of a two-group, in this case the component group of a torus.

The key to our parametrization is that the intersection of an L-packet and an R-packet is at most one element. We thereby obtain a classification in terms of pairs of packets, i.e. pairs of orbits. Here is a weak version of this result, which does not require any further assumptions on $G(\mathbb{C})$:

{l:Ophi}

Lemma 1.16 ([26], Proposition 8.3) *Suppose $\Pi_R(G(\mathbb{R}),\mathcal{O},\lambda)$ is an R-packet, and $\Pi_L(G(\mathbb{R}),\mathcal{O}^\vee)$ is an L-packet. Then $\Pi_L(G(\mathbb{R}),\mathcal{O}^\vee)\cap\Pi_R(G(\mathbb{R}),\mathcal{O},\lambda)$ has at most one element.*

This reduces the problem of parametrizing $\Pi(G(\mathbb{R}),\lambda)$ to the following problems. Let $K(\mathbb{C})$ be the complexification of a maximal compact subgroup of $G(\mathbb{R})$.

(1) Parametrize $K(\mathbb{C})$-orbits on $\mathcal{B}$,

(2) Parametrize $G^\vee(\mathbb{C})$-orbits on $\mathrm{Hom}_{\mathrm{adm}}(W_\mathbb{R}, G^{\vee\Gamma}, \lambda)$.

It turns out that (2) is equivalent to

(2′) Parametrize $K^\vee(\mathbb{C})$-orbits on $\mathcal{B}^\vee$.

Here $\mathcal{B}^\vee$ is the variety of Borel subgroups of $G^\vee(\mathbb{C})$ and $K^\vee(\mathbb{C})$ is the fixed points of an involution of $G^\vee(\mathbb{C})$, i.e. the analogue of (1) on the dual side. We also need to determine when the intersection of an L-packet and an R-packet is non-empty.

We therefore turn to the problem of computing the space of $K(\mathbb{C})$ orbits on $\mathcal{B}$, before returning the parametrization of $\Pi(G(\mathbb{R}), \lambda)$ in Section 1.6.

1.5 The Parameter Space $\mathcal{X}$

{s:X}

Fix $G(\mathbb{C})$. As in Section 1.2 we are interested in computing the space of $K(\mathbb{C})$-orbits on $\mathcal{B}$, where $K(\mathbb{C})$ is the fixed points of a Cartan involution of $G(\mathbb{C})$. It turns out it is easier to treat all Cartan involutions simultaneously. For the purposes of this Overview we make two simplifying assumptions:

(1) the center of $G(\mathbb{C})$ is trivial,

(2) every automorphism of $G(\mathbb{C})$ is inner.

Assuming (1), condition (2) is equivalent to:

(2′) the Dynkin diagram of $G(\mathbb{C})$ has no nontrivial automorphisms.

For example this holds if $G(\mathbb{C})$ is a simple adjoint group of type B_n,C_n,E_7, E_8,F_4 or G_2.

Under assumption (2) every involutive automorphism of $G(\mathbb{C})$ is of the form $\mathrm{int}(x)$ for some involution $x \in G(\mathbb{C})$ (where $\mathrm{int}(x)$ is conjugation by x). It follows that the (equivalence classes of) real forms of $G(\mathbb{C})$ are in bijection with conjugacy classes of involutions in $G(\mathbb{C})$. If x is such an involution then $\theta_x = \mathrm{int}(x)$ is an involution of $G(\mathbb{C})$, and is the Cartan involution of a real form. Conversely the Cartan involution of every real form is of the form θ_x. See Section 3.

Let $\{x_1, \dots, x_n\}$ be a set of representatives of the conjugacy classes of involutions of $G(\mathbb{C})$. For $1 \le i \le n$ let $\theta_i = \mathrm{int}(x_i), K_i(\mathbb{C}) = G(\mathbb{C})^{\theta_i}$ and write $G_i(\mathbb{R})$ for the corresponding real form of $G(\mathbb{C})$. Fix a Cartan subgroup $H(\mathbb{C})$.

{d:overview:X}

Definition 1.17 (see Definition 9.2)

$$\mathcal{X}(G(\mathbb{C})) = \{x \in Norm_{G(\mathbb{C})}(H(\mathbb{C})) \mid x^2 = 1\}/H(\mathbb{C}). \tag{1.18}$$

(The quotient is by conjugation by $H(\mathbb{C})$.) This is a finite set.

{p:overview:XGC}

Proposition 1.19 *Assume (1) and (2). There is a natural bijection*

$$\mathcal{X}(G(\mathbb{C})) \xleftrightarrow{1-1} \coprod_{i=1}^{n} K_i(\mathbb{C})\backslash\mathcal{B}. \tag{1.20}$$

Sketch of proof. Let

$$\mathcal{P} = \{(x, B(\mathbb{C})) \mid x \in G(\mathbb{C}), x^2 = 1, B(\mathbb{C}) \text{ a Borel subgroup}\}/G(\mathbb{C}). \tag{1.21}$$

Every element of $\mathcal{P}$ is conjugate to one of the form $(x_i, B(\mathbb{C}))$; the set of conjugacy classes of pairs $(x_i, B(\mathbb{C}))$ is isomorphic to $K_i(\mathbb{C})\backslash\mathcal{B}$. This gives a bijection $\mathcal{P} \xleftrightarrow{1-1} \coprod_{i=1}^{n} K_i(\mathbb{C})\backslash\mathcal{B}$.

On the other hand fix a Borel subgroup $B_0(\mathbb{C})$ containing $H(\mathbb{C})$. Every element of $\mathcal{P}$ conjugate to one of the form $(x, B_0(\mathbb{C}))$. Furthermore by conjugating by $B_0(\mathbb{C})$ we may assume $x \in \mathrm{Norm}_{G(\mathbb{C})}(H(\mathbb{C}))$, which gives a bijection $\mathcal{P} \xleftrightarrow{1-1} \mathcal{X}(G(\mathbb{C})))$. See Section 8. □

We turn next to the computation of $\mathrm{Hom}_{\mathrm{adm}}(W_{\mathbb{R}}, G^{\vee\Gamma}, \lambda)/G^\vee(\mathbb{C})$. A remarkable fact, mentioned at the end of the previous section, is that the space $\mathcal{X}$ (applied on the dual side) provides this parametrization. To make this precise it is convenient to assume that the center of $G^\vee(\mathbb{C})$ is trivial.

Fix a Cartan subgroup $H^\vee(\mathbb{C})$ of $G^\vee(\mathbb{C})$. Applying Definition 1.17 to $G^{\vee\Gamma}$ we have

$$\mathcal{X}(G^\vee(\mathbb{C})) = \{x \in \mathrm{Norm}_{G^\vee(\mathbb{C})}(H^\vee(\mathbb{C})) \mid x^2 = 1\}/H^\vee(\mathbb{C}). \tag{1.22}$$

{p:overview:XchGC}

Proposition 1.23 *Assume $Z(G^\vee(\mathbb{C})) = 1$. There is a natural bijection*

$$\mathcal{X}(G^\vee(\mathbb{C})) \xleftrightarrow{1-1} Hom_{adm}(W_{\mathbb{R}}, G^{\vee\Gamma}, \lambda)/G^\vee(\mathbb{C}). \tag{1.24}$$

Sketch of proof. We may view λ as an element of the Lie algebra of $H^\vee(\mathbb{C})$. If $x \in \mathrm{Norm}_{G^\vee(\mathbb{C})}(H^\vee(\mathbb{C}))$, $x^2 = 1$ define

$$\begin{aligned} \phi(z) &= z^{\lambda}\overline{z}^{\mathrm{Ad}(x)\lambda} \\ \phi(j) &= e^{-\pi i\lambda}x. \end{aligned} \tag{1.25}$$

Then $\phi \in \mathrm{Hom}_{\mathrm{adm}}(W_\mathbb{R}, G^{\vee\Gamma}, \lambda)$, and every element of $\mathrm{Hom}_{\mathrm{adm}}(W_\mathbb{R}, G^{\vee\Gamma}, \lambda)$ is conjugate to one of this form. See Section 7. □

Now assume $Z(G(\mathbb{C}))$ and $Z(G^\vee(\mathbb{C}))$ are trivial, and $\mathrm{Out}(G(\mathbb{C})) = 1$. Then Proposition 1.19 holds here as well, so there are bijections

$$\mathrm{Hom}_{\mathrm{adm}}(W_\mathbb{R}, G^{\vee\Gamma}, \lambda) \overset{1-1}{\longleftrightarrow} \mathcal{X}(G^\vee(\mathbb{C})) \overset{1-1}{\longleftrightarrow} \coprod_{i=1}^{m} K_i^\vee(\mathbb{C})\backslash\mathcal{B}^\vee. \tag{1.26}$$

Here $\mathcal{B}^\vee$ is the space of Borel subgroups of $G^\vee(\mathbb{C})$, $y_1, \dots, y_m$ are representatives of the conjugacy classes of involutions in $G^\vee(\mathbb{C})$, and for each i $K_i^\vee(\mathbb{C}) = \mathrm{Cent}_{G^\vee(\mathbb{C})}(y_i)$.

1.6 The parameter space $\mathcal{Z}$

{s:overview:Z}

We now combine the results of the previous section with Lemma 1.16 to define the parameter space $\mathcal{Z}$ for $\Pi(G(\mathbb{R}), \lambda)$.

To avoid some technical issues in the previous section we assumed the center of $G(\mathbb{C})$ is trivial and $G(\mathbb{C})$ has no outer automorphisms. For consideration of $\mathcal{L}(G^{\vee\Gamma}, \lambda)$ we assumed the corresponding facts for $G^\vee(\mathbb{C})$. Therefore we assume $G(\mathbb{C})$ is both simply connected and adjoint. We may as well also assume $G(\mathbb{C})$ is simple, i.e. of type G_2, F_4 or E_8.

As in the previous section let $x_1, \dots, x_n$ be representaitves of the conjugacy classes of involutions of $G(\mathbb{C})$, and let $K_i(\mathbb{C}) = \mathrm{Cent}_{G(\mathbb{C})}(x_i)$. Dually let $y_1, \dots, y_m$ be representatives of the conjugacy classes of involutions of $G^\vee(\mathbb{C})$, and let $K_j^\vee(\mathbb{C}) = \mathrm{Cent}_{G^\vee(\mathbb{C})}(y_j)$.

{d:overview:Z}

Definition 1.27 *Assume $G(\mathbb{C})$ is simple, simply connected and adjoint. Fix Cartan subgroups $H(\mathbb{C})$ of $G(\mathbb{C})$ and $H^\vee(\mathbb{C})$ of $G^\vee(\mathbb{C})$. Let $\mathfrak{h}, \mathfrak{h}^\vee$ be the Lie algebras of $H(\mathbb{C})$ and $H^\vee(\mathbb{C})$, respectively. Let*

$$\begin{aligned} \mathcal{X} = \mathcal{X}(G(\mathbb{C})) &= \{x \in \mathit{Norm}_{G(\mathbb{C})}(H(\mathbb{C}))\,|\,x^2 = 1\}/H(\mathbb{C}) \\ \mathcal{X}^\vee = \mathcal{X}(G^\vee(\mathbb{C})) &= \{y \in \mathit{Norm}_{G^\vee(\mathbb{C})}(H^\vee(\mathbb{C}))\,|\,y^2 = 1\}/H^\vee(\mathbb{C}). \end{aligned} \tag{1.28)(a}$$

There is a natural adjoint map $End(\mathfrak{h}) \ni X \mapsto X^t \in End(\mathfrak{h}^\vee)$. Let

$$\mathcal{Z} = \{(x,y) \in \mathcal{X} \times \mathcal{X}^\vee \mid (Ad(x)|_{\mathfrak{h}})^t = -Ad(y)|_{\mathfrak{h}^\vee}\}. \tag{1.29}$$

By (1.20) and (1.26) $\mathcal{Z}$ may be viewed as a subset of

$$\coprod_{i=1}^{n} K_i(\mathbb{C})\backslash\mathcal{B} \times \coprod_{i=1}^{m} K_i^\vee(\mathbb{C})\backslash\mathcal{B}^\vee. \tag{1.30}$$

See (10.7). Here is a special case of the main result:

{t:overviewmain}

Theorem 1.31 (see Theorem 7.15) *Assume $G(\mathbb{C})$ is simple, simply connected and adjoint. Write $G_1(\mathbb{R}),\dots,G_n(\mathbb{R})$ for the equivalence classes of real forms of $G(\mathbb{C})$.*

There is a natural bijection

$$\mathcal{Z} \overset{1-1}{\longleftrightarrow} \coprod_{i=1}^{n} \Pi(G_i(\mathbb{R}),\lambda). \tag{1.32}$$

This is a refinement of Lemma 1.16, and is a restatement of [1, Theorem 2-12].

Sketch of proof.

Here are three ways to think of the proof of this result.

Fix $1 \le i \le n$ and $1 \le j \le m$. Also fix an orbit $\mathcal{O}$ of $K_i(\mathbb{C})$ on $\mathcal{B}$, and an orbit $\mathcal{O}^\vee$ of $K_j^\vee(\mathbb{C})$ on $\mathcal{B}^\vee$. By Lemma 1.16 $\Pi_L(G_i(\mathbb{R}),\mathcal{O}^\vee) \cap \Pi_R(G_i(\mathbb{R}),\mathcal{O},\lambda)$ consists of at most element. Condition (1.29) makes this intersection nonempty, and we obtain every representation exactly once this way.

Alternatively, fix $1 \le i \le m$ and an orbit $\mathcal{O}^\vee$ of $K_i^\vee(\mathbb{C})$ on $\mathcal{B}^\vee$. Fix $1 \le j \le n$, and let $\Pi_L(G_j(\mathbb{R}),\mathcal{O}^\vee)$ be the corresponding L-packet. The choice of an orbit of $K_j(\mathbb{C})$ on $\mathcal{B}$ defines a character χ of $\mathbb{S}_\phi$, and by Proposition 1.14 this defines a representation of $G_j(\mathbb{R})$. This is the proof in Section 7 (see [1]).

Finally, fix $1 \le i \le n$, an orbit $\mathcal{O}$ of $K_i(\mathbb{C})$ on $\mathcal{B}$, and let $\Pi_R(G_i(\mathbb{R}),\mathcal{O},\lambda)$ be the corresponding R-packet. The data needed to specify a $\mathcal{D}_\lambda$ module supported on $\mathcal{O}$ is precisely an orbit of $K_j^\vee(\mathbb{C})$ on $\mathcal{B}^\vee$ for some $1 \le j \le m$. By Proposition 1.9 this defines an irreducible representation of $G_i(\mathbb{R})$. $\square$

As is clear from the statement, to explicitly parametrize $\Pi(G(\mathbb{R}),\lambda)$ the main issue is to understand the spaces $\mathcal{X}$ and $\mathcal{X}^\vee$. It is not immediately obvious how to explicitly compute these, but this can be done using the Tits group. This is discussed in Section 15. In any event all of the structural data discussed in Section 1 can be read off from the space $\mathcal{X}$.

Proposition 1.33 (see Proposition 12.19) *Use the notation of Theorem 1.31.* {l:or}

(1) The real forms of G are parametrized by $\mathcal{X}\cap H(\mathbb{C})/W$. Write $x_1,\dots,x_n$ for representatives of this set.

(2) $\mathcal{X}\overset{1-1}{\longleftrightarrow}\coprod_{i=1}^n K_{x_i}(\mathbb{C})\backslash\mathcal{B}$.

(3) Associated to each $x\in\mathcal{X}$ is a pair $(G_x(\mathbb{R}),H_x(\mathbb{R}))$ consisting of a real form of $G(\mathbb{C})$ and a Cartan subgroup of $G_x(\mathbb{R})$. This induces a bijection between $\mathcal{X}/W$ and the union, over real forms $G(\mathbb{R})$ of $G(\mathbb{C})$, of the conjugacy classes of Cartan subgroups of $G(\mathbb{R})$.

(4) For $x\in\mathcal{X}$ we have

$$W(G_x(\mathbb{R}),H_x(\mathbb{R}))\simeq Stab_W(x). \tag{1.34}$$

See Section 12.1.

It is clear from the discussion that the setting is entirely symmetric in $G(\mathbb{C})$ and $G^\vee(\mathbb{C})$. This is a manifestation of *Vogan duality* [26], which is an essential guiding principal in the definitions. By symmetry we have, in the setting of (1.32),

$$\coprod_{i=1}^n \Pi(G_i(\mathbb{R}),\lambda)\overset{1-1}{\longleftrightarrow}\mathcal{Z}\overset{1-1}{\longleftrightarrow}\coprod_{j=1}^m \Pi(G_j^\vee(\mathbb{R}),\lambda^\vee) \tag{1.35}$$

(here $\lambda^\vee$ is a regular integral infinitesimal character for $G^\vee(\mathbb{C})$). See Corollary 10.8 for the general statement, without the restrictions on $G(\mathbb{C})$. In fact this is a refinement of Vogan duality of [26], which considers only a single real form at a time. See [3, Theorems 1.24 and 15.12].

This explains the nature of R-packets: it is clear from the discussion that the Vogan dual of an R-packet for $G(\mathbb{R})$ is an L-packet for some real form of $G^\vee(\mathbb{C})$.

For ease of exposition in this Overview we have made various assumptions. Removing these constraints involves a number of closely related technical issues:

(1) If $G(\mathbb{C})$ is not adjoint we weaken the assumption $x^2 = 1$ to $x^2 \in Z(G(\mathbb{C}))$. This introduces the notion of *strong real form* (Section 5).

(2) If (the derived group of) $G(\mathbb{C})$ is not simply connected there are a finite number of different infinitesimal characters at which the representation theory looks different. Dually, since $G^\vee(\mathbb{C})$ is not necessarily adjoint, we allow $y^2 \in Z(G^\vee(\mathbb{C}))$.

(3) If ρ does not exponentiate to $G(\mathbb{C})$ we allow characters of the ρ-cover of $H(\mathbb{R})$ [4, Definition 8.11].

(4) If $G(\mathbb{C})$ admits non-trivial outer automorphisms, we specify an involution γ in $\mathrm{Out}(G(\mathbb{C}))$. Then all of the objects discussed above become "twisted" by γ.

Here is an example which takes some of these issues into account. We change notation to be more in agreement with the rest of the paper.

Let $G = Sp(2n,\mathbb{C})$ and let $G^\vee$ be the dual group, i.e. $SO(2n+1,\mathbb{C})$. Fix Cartan subgroups H of G and $H^\vee$ of $G^\vee$, with Lie algebras $\mathfrak{h}$ and $\mathfrak{h}^\vee$.

Theorem 1.36 The irreducible representations of $Sp(2n,\mathbb{R})$ with the same infinitesimal character as the trivial representation are parametrized by:

$$\{(x,y)\}/H \times H^\vee$$

where

$$\begin{aligned} &x \in \mathrm{Norm}_G(H), \quad x^2 = -I \\ &y \in \mathrm{Norm}_{G^\vee}(H^\vee) \quad y^2 = I \end{aligned} \tag{1.37}$$

and

$$(\mathrm{Ad}(x)|_{\mathfrak{h}})^t = -Ad(y)|_{\mathfrak{h}^\vee}. \tag{1.38}$$

The quotient is by the conjugation action of $H \times H^\vee$. This is a finite set. The number of elements is given by the following table:

n	1	2	3	4	5	6	7	8	9
	4	18	88	460	2,544	1,4776	89,632	565,392	3,695,680

This set also parametrizes the irreducible representations of real forms of $SO(p,q)$ $(p+q=2n+1)$ with trivial infinitesimal character. This is an example of Vogan duality (see (1.35)).

If we instead require $x^2=I$ we will parametrize representations of the groups $Sp(p,q)$. Dually we will obtain representations of $SO(p,q)$ with infinitesimal character 2ρ.

We conclude this section with a comment about the requirements that the infinitesimal character λ be regular and integral. It is straightforward to remove both of these conditions, but introduces some extra complications.

First of all if λ is regular but not integral we replace $G^\vee(\mathbb{C})$ with a subgroup whose root system is dual to the integral root system defined by λ. This follows the program of [26], and with this change many of the preceding constructions hold.

Secondly if λ is singular (and possibly non-integral) let λ' be a regular element such that $\lambda-\lambda'$ is a sum of roots. By Zuckerman's translation principle [27] $\Pi(G(\mathbb{R}),\lambda)$ is a an explicitly computable subset of $\Pi(G(\mathbb{R}),\lambda')$, thereby reducing this case to the one of regular infinitesimal character.

1.7 Outline

{s:outline}

Here is an outline of the contents of the paper.

Section 2 defines the basic objects of study, i.e. reductive groups and root data. In Section 3 we discuss real forms of a complex group.

We put the information from Sections 2 and 3 together to define *basic data* in Section 4. This consists of either a pair (G,γ) consisting of a complex reductive group and an involution in $\mathrm{Out}(G)$, or a pair (D_b,γ) consisting of a based root datum and an involution of it.

Given basic data we define the *extended group* $G^\Gamma=G\rtimes\Gamma$ in Section 5. We also define the notions of *strong involution* and *strong real form.* Harish-Chandra modules for strong real forms are discussed in Section 6.

The first main step in constructing the parameter space is L-data (Section 7). The relation with K-orbits on G/B is the subject of Section 8.

The primary combinatorial construction is the *one-sided parameter space* $\mathcal{X}$ of Section 9. Once we have $\mathcal{X}$ it is straightforward to define the parameter space $\mathcal{Z}\subset\mathcal{X}\times\mathcal{X}^\vee$. The main result is Theorem 10.3.

After stating the main result, we go back down into some of the details of the space $\mathcal{X}$ in Sections 11-14 and relate this space to structure theory of G. A summary of the relationship between $\mathcal{X}$ and structure theory of G is found in Section 12.1. We discuss the *reduced parameter space* $\mathcal{X}^r$ in Section 13. In Section 15 we use the Tits group to explicitly compute the space $\mathcal{X}$.

Some examples are discussed in the body of the paper. In particular the very informative cases of $SL(2)$ and $PSL(2)$ are discussed in examples 3.4, 5.6, 12.20, and 12.25. More examples may be found in [2].

2 Reductive Groups and Root Data

{s:groups}

For many purposes one may identify a connected reductive algebraic group with its group of complex points. For the discussion of real forms (Section 3), and to keep the exposition as elementary as possible, we choose to work with complex groups. Experts, and those with an interest in other fields, may wish to convert to the language of algebraic groups where appropriate.

We now describe the parameters for a connected reductive complex group. These are provided by *root data* and *based root data*. Good references are the books by Humphreys [10] and Springer [22].

We begin with a pair $X, X^\vee$ of free abelian groups of finite rank, together with a perfect pairing $\langle\,,\,\rangle : X \times X^\vee \to \mathbb{Z}$. Suppose $\Delta \subset X, \Delta^\vee \subset X^\vee$ are finite sets, equipped with a bijection $\alpha \to \alpha^\vee$. For $\alpha \in \Delta$ define the reflection $s_\alpha \in \mathrm{Hom}(X, X)$:

$$s_\alpha(x) = x - \langle x, \alpha^\vee\rangle\alpha \quad (x \in X)$$

and define $s_{\alpha^\vee} \in \mathrm{Hom}(X^\vee, X^\vee)$ similarly.

A *root datum* is a quadruple

$$D = (X, \Delta, X^\vee, \Delta^\vee) \tag{2.1}$$

where $X, X^\vee$ are free abelian groups of finite rank, in duality via a perfect pairing $\langle\,,\,\rangle$, and $\Delta, \Delta^\vee$ are finite subsets of $X, X^\vee$, respectively. We assume there is a bijection $\Delta \ni \alpha \mapsto \alpha^\vee \in \Delta^\vee$ such that for all $\alpha \in \Delta$,

$$\langle \alpha, \alpha^\vee\rangle = 2,\ s_\alpha(\Delta) = \Delta,\ s_{\alpha^\vee}(\Delta^\vee) = \Delta^\vee. \tag{2.2}$$

Suppose we are given $X, X^\vee$ and finite subsets $\Delta \subset X$ and $\Delta^\vee \subset X^\vee$. By [7, Lemma VI.1.1] applied to $\mathbb{Q}\langle\Delta\rangle$ and $\mathbb{Q}\langle\Delta^\vee\rangle$ there is at most one bijection $\alpha \mapsto \alpha^\vee$ satisfying (2.2). Alternatively suppose we are given only a finite subset Δ of X, satisfying $X \subset \mathbb{Q}\langle\Delta\rangle$. By (loc. cit.) there is at most one subset $\Delta^\vee$, and bijection $\alpha \mapsto \alpha^\vee$, satisfying (2.2). The condition $X \subset \mathbb{Q}\langle\Delta\rangle$ holds if and only if the corresponding group is semisimple.

Suppose $D_i = (X_i, \Delta_i, X_i^\vee, \Delta_i^\vee)$ $(i = 1, 2)$ are root data. We say they are isomorphic if there is an isomorphism $\phi \in \mathrm{Hom}(X_1, X_2)$ satisfying $\phi(\Delta_1) = \Delta_2$ and $\phi^t(\Delta_2^\vee) = \Delta_1^\vee$. Here $\phi^t \in \mathrm{Hom}(X_2^\vee, X_1^\vee)$ is defined by

$$\langle \phi(x_1), x_2^\vee \rangle_2 = \langle x_1, \phi^t(x_2^\vee) \rangle_1 \quad (x_1 \in X_1, x_2^\vee \in X_2^\vee). \tag{2.3}$$

Let G be a connected reductive complex group and choose a Cartan subgroup H of G. Let $X^*(H), X_*(H)$ be the character and co-character lattices of H respectively. We have canonical isomorphisms

$$\mathfrak{h} \simeq X_*(H) \otimes_{\mathbb{Z}} \mathbb{C}, \quad \mathfrak{h}^* \simeq X^*(H) \otimes_{\mathbb{Z}} \mathbb{C} \tag{2.4}$$

where $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{h}^* = \mathrm{Hom}(\mathfrak{h}, \mathbb{C})$. (If G has rank n then $H \simeq (\mathbb{C}^\times)^n$, $X_*(H) \simeq \mathbb{Z}^n$ and $\mathfrak{h} \simeq \mathbb{C}^n$.) Let $\Delta = \Delta(G, H)$ be the set of roots of H in G, and $\Delta^\vee = \Delta^\vee(G, H)$ the corrresponding co-roots. Associated to (G, H) is the root datum

$$D(G, H) = (X^*(H), \Delta, X_*(H), \Delta^\vee). \tag{2.5}$$

If H' is another Cartan subgroup then there is an element $g \in G$ so that $gHg^{-1} = H'$. Let $D' = (X^*(H'), \Delta(G, H'), X_*(H'), \Delta^\vee(G, H'))$ be the corresponding root datum. The inverse transpose action on characters induces an isomorphism

$$(Ad(g)^t)^{-1} : X^*(H) \rightarrow X^*(H')$$

which gives an isomorphism $D(G, H) \simeq D(G, H')$.

Now suppose in addition to H we have chosen a Borel subgroup B containing H. Let Δ^+ be the corresponding set of positive roots of Δ, with simple roots Π. Then $\Pi^\vee = \{\alpha^\vee \mid \alpha \in \Pi\}$ is a set of simple roots of $\Delta^\vee$, and

$$D_b(G, B, H) = (X, \Pi, X^\vee, \Pi^\vee) \tag{2.6}$$

is a *based root datum.* If $H' \subset B'$ are another Cartan and Borel subgroup then there is a *canonical* isomorphism $D_b(G, B, H) \simeq D_b(G, B', H')$.

Each root datum is the root datum of a reductive algebraic group, which is determined uniquely up to isomorphism, and the same holds for based root data.

Note that a connected reductive complex group G of rank n is determined by a small finite set of data: two sets (of order the semisimple rank of G) of integral n-vectors, subject only to condition (2.2), which may be expressed by requiring that the matrix of dot products is a Cartan matrix.

Example 2.7 Suppose G is of rank 2 and semisimple rank 1. Then a root datum is given by an ordered pair (v, w) with $v, w \in \mathbb{Z}^2$, satisfying $v \cdot w = 2$. Equivalence is given by the action of $GL(2, \mathbb{Z})$, where $g \cdot (v, w) = (gv, {}^t g^{-1} w)$. It is an interesting exercise to see that there are precisely three such pairs, up to equivalence: $((2,0),(1,0)), ((1,0),(2,0))$ and $((1,1),(1,1))$, corresponding to $SL(2,\mathbb{C}) \times \mathbb{C}^\times, PGL(2,\mathbb{C}) \times \mathbb{C}^\times$ and $GL(2,\mathbb{C})$, respectively.

If $D = (X, \Delta, X^\vee, \Delta^\vee)$ is a root datum then the dual root datum is defined to be $D^\vee = (X^\vee, \Delta^\vee, X, \Delta)$. Given G with root datum $D = (X, \Delta, X^\vee, \Delta^\vee)$ the *dual group* is the group $G^\vee$ defined by $D^\vee$. We define duality of based root data similarly.

2.1 Automorphisms

{s:automorphisms}

There is an exact sequence

$$1 \to \mathrm{Int}(G) \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1 \tag{2.8}$$

where $\mathrm{Int}(G)$ is the group of inner automorphisms of G, $\mathrm{Aut}(G)$ is the group of (holomorphic) automorphisms of G, and $\mathrm{Out}(G) \simeq \mathrm{Aut}(G)/\mathrm{Int}(G)$ is the group of outer automorphisms.

A *splitting datum* or *pinning* for G is a set $(B, H, \{X_\alpha\})$ where B is a Borel subgroup, H is a Cartan subgroup contained in B and $\{X_\alpha\}$ is a set of root vectors for the simple roots of H in B.

{d:distinguished}

Definition 2.9 *An involution of G is said to be distinguished if it preserves a splitting datum.*

For example an inner involution is distinguished if and only if it is the identity; this is the Cartan involution of the compact real form. More generally, among all of the involutions mapping to a fixed involution in $\mathrm{Out}(G)$, a distinguished involution makes as many roots as possible imaginary and compact (cf. Section 12); it is the Cartan involution of a "maximally compact" real form of G in the given inner class (cf. Definition 3.6).

The group $\mathrm{Int}(G)$ acts simply transitively on the set of splitting data. Given a splitting datum $(B, H, \{X_\alpha\})$ this gives an isomorphism

$$\text{(2.10)(a)} \qquad \phi_S : \mathrm{Out}(G) \simeq \mathrm{Stab}_{\mathrm{Aut}(G)}(S) \subset \mathrm{Aut}(G)$$

and this is a splitting of the exact sequence (2.8). We call a splitting *distinguished* if it fixes a splitting datum. We obtain isomorphisms

$$\text{(2.10)(b)} \qquad \mathrm{Out}(G) \simeq \mathrm{Aut}(D_b) \simeq \mathrm{Aut}(D)/W.$$

If G is semisimple then there is an injection of $\mathrm{Out}(G)$ into the automorphism group of the Dynkin diagram of G; if G is semisimple and simply connected or adjoint then this is an isomorphism.

If $\tau \in \mathrm{Aut}(D)$ then $-\tau^t \in \mathrm{Aut}(D^\vee)$ (cf. 2.3). Now suppose $\tau \in \mathrm{Aut}(D_b)$. While $-\tau^t$ is probably not in $\mathrm{Aut}(D_b^\vee)$, if we let w_0 be the long element of the Weyl group we have $-w_0\tau^t \in \mathrm{Aut}(D_b^\vee)$.

{d:chtau}

Definition 2.11 *Suppose $\tau \in Aut(D_b)$. Let $\tau^\vee = -w_0\tau^t \in Aut(D_b^\vee)$. This defines a bijection $Aut(D_b) \overset{1-1}{\longleftrightarrow} Aut(D_b^\vee)$. By* (2.10)(b) *we obtain a bijection $Out(G) \leftrightarrow Out(G^\vee)$ by composition:*

$$\text{(2.12)} \qquad Out(G) \leftrightarrow Aut(D_b) \leftrightarrow Aut(D_b^\vee) \leftrightarrow Out(G^\vee).$$

For $\gamma \in Out(G)$ we write $\gamma^\vee$ for the corresponding element of $Out(G^\vee)$. The map $\gamma \mapsto \gamma^\vee$ is a bijection of sets.

Remark 2.13 This is not necessarily an isomorphism of groups. For example the identity goes to the image of $-w_0$ in $\mathrm{Out}(G)$, which is the identity if and only G is semisimple and $-1 \in W$.

Example 2.14 Let $G = PGL(n)$ $(n \geq 3)$. Then $G^\vee = SL(n)$ and $\mathrm{Out}(G) \simeq \mathrm{Out}(G^\vee) \simeq \mathbb{Z}/2\mathbb{Z}$. If $\gamma = 1 \in \mathrm{Out}(G)$ then $\gamma^\vee$ is the non-trivial element of $\mathrm{Out}(G^\vee)$. It is represented by the automorphism $\tau^\vee(g) = {}^t g^{-1}$ of $G^\vee$.

3 Involutions of Reductive Groups

{s:involutions}

Fix a connected reductive complex group G. A real form of G is a subgroup $G(\mathbb{R})$ which is the fixed points of an *antiholomorphic* involution of G. Two such real forms are said to be equivalent if they are conjugate by G. The Cartan involution provides a description of real forms in terms of *holomorphic* involutions which is better suited to our purposes. We illustrate this with an example.

Example 3.1 Let $G = GL(n,\mathbb{C})$. The group $GL(n,\mathbb{R})$ is a real form of $GL(n,\mathbb{C})$; it is the fixed points of the antiholomorphic involution $\sigma(g) = \overline{g}$. The orthogonal group $O(n)$ is a maximal compact subgroup of $GL(n,\mathbb{R})$; it is the fixed points of the involution $\theta(g) = {}^tg^{-1}$ of $GL(n,\mathbb{R})$. This involution extends to a holomorphic involution of $GL(n,\mathbb{C})$, with fixed points $O(n,\mathbb{C})$.

Suppose σ is an antiholomorphic involution of G, with fixed points $G(\mathbb{R})$. Then there is a holomorphic involution θ of G such that $G(\mathbb{R})^\theta$ is a maximal compact subgroup of $G(\mathbb{R})$. It turns out this gives a bijection on the level of G-conjugacy classes. See [18, Theorem 5.1.4], [4] and [9, Chapter X, Section 1].

{t:realforms}

Theorem 3.2 *The map $\sigma \mapsto \theta$ gives a bijection between G-conjugacy classes of antiholomorphic involutions of G and G-conjugay classes of holomorphic involutions of G.*

Using this result we classify real forms in terms of *holomorphic* involutions. We also prefer to incorporate the notion of equivalence in the definition of real forms.

{d:involution}

Definition 3.3 *Let G be a connected reductive complex group. An involution of G is an involution in Aut(G), i.e. a holomorphic automorphism θ of G satisfying $\theta^2 = 1$. A real form of G is a G-conjugacy class of involutions.*

Our definition of equivalence of real forms differs from the usual one in one subtle way: it only allows conjugacy by G, rather than all of $\mathrm{Aut}(G)$. The theorem also holds with G-conjugacy replaced by conjugacy by $\mathrm{Aut}(G)$, and this is how it is usually stated (for example [18, Theorem 5.1.4]).

{ex:sl2C}

Example 3.4 We consider the case of $SL(2,\mathbb{C})$. Up to conjugacy there are two antiholomorphic involutions of $SL(2,\mathbb{C})$: $\sigma^s(g) = \overline{g}$ and $\sigma^c(g) = {}^t\overline{g}^{-1}$.

These are the two real forms $G(\mathbb{R}) = SL(2,\mathbb{R})$ (split) and $G(\mathbb{R}) = SU(2)$ (compact) of $SL(2,\mathbb{C})$, respectively.

Equivalently $SL(2,\mathbb{C})$ has two equivalence classes of holomorphic involutions. Let $\theta^s(g) = tgt^{-1}$ where $t = \mathrm{diag}(i,-i)$. Then $K(\mathbb{C}) = G^{\theta^s} = \mathbb{C}^\times$, so $K(\mathbb{R}) = S^1$, the maximal compact subgroup of the corresponding real form $SL(2,\mathbb{R})$. On the other hand let $\theta^c(g) = g$, so $K(\mathbb{C}) = SL(2,\mathbb{C})$, $K(\mathbb{R}) = SU(2)$, and the corresponding real form is $SU(2)$.

{r:oldrealforms}

Remark 3.5 By Theorem 3.2 there is a bijection between *equivalence classes of real forms* in the usual sense of $G(\mathbb{R})$ and antiholomorphic involutions, and *real forms* in the sense of Definition 3.3. We work almost entirely with the latter notion; in the few places we refer to the former the distinction will be clear since we will refer to a real form $G(\mathbb{R})$.

{d:inner}

Definition 3.6 *An involution $\theta \in Aut(G)$ (Definition 3.3) is in the inner class of $\gamma \in Out(G)$ if θ maps to γ in the exact sequence* (2.8)*. If θ, θ' are involutions of G we say θ is inner to θ' if θ and θ' have the same image in $Out(G)$.*

This corresponds to the usual notion of inner form [22, 12.3.7].

{r:realform}

Remark 3.7 The results of [3] are stated in terms of antiholomorphic, rather than holomorphic, involutions. See Remark 5.18.

Remark 3.8 As discussed at the beginning of Section 2, this is the one situation in which we need complex, as opposed to algebraic, groups. Once we are in the setting of Cartan involutions, we could replace G with the underlying algebraic group $\mathbb{G}$, and holomorphic involutions of G with *algebraic* involutions of $\mathbb{G}$. This is the setting of [20].

4 Basic Data

{s:basic}

The setting for our mathematical questions will be:

(1) A connected reductive complex group G,

(2) An involution $\gamma \in \mathrm{Out}(G)$.

We refer to (G,γ) as *basic data*. We say (G,γ) is equivalent to (G',γ') if there is an isomorphism $\phi : G \to G'$ such that $\gamma' \circ \phi = \phi \circ \gamma$.

On the other hand the software works entirely in the setting of based root data. Given (G,γ), let $(B, H, \{X_\alpha\})$ be a splitting datum (Section 2.1). From these we obtain:

(a) A based root datum $D_b = D_b(G, B, H)$,

(b) An involution γ of D_b (cf. (2.10)(b)).

The based root datum D_b and its involution are independent of the choice of splitting datum, up to canonical isomorphism. Conversely, given a based root datum D_b with an involution γ we can construct (G,γ), uniquely determined up to isomorphism.

Given basic data (G,γ), choose a splitting datum $(B, H, \{X_\alpha\})$. A key to our algorithm is that $(B, H, \{X_\alpha\})$ is fixed once and for all. This enables us to do all of our constructions on a fixed Cartan subgroup. Let $W = W(G, H)$ be the Weyl group.

The weight lattice for G is

(4.1) $$P = \{\lambda \in X^*(H) \otimes \mathbb{C} \,|\, \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Delta\}$$

and dually the co-weight lattice for G is

(4.2)(a) $$P^\vee = \{\lambda^\vee \in X_*(H) \otimes \mathbb{C} \,|\, \langle \alpha, \lambda^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Delta\}.$$

These are actually lattices only if G is semisimple. We may identify $2\pi i X_*(H)$ with the kernel of $\exp : \mathfrak{h} \to H(\mathbb{C})$. Under this identification

(4.2)(b) $$P^\vee = \{\lambda^\vee \in \mathfrak{h} \,|\, \exp(2\pi i \lambda^\vee) \in Z(G)\}.$$

We write $P(G, H)$ and $P^\vee(G, H)$ to indicate the dependence on G and H.

We also define

(4.2)(c) $$P_{\text{reg}} = \{\lambda \in P \,|\, \langle \lambda, \alpha^\vee \rangle \neq 0 \text{ for all } \alpha \in \Delta\}$$

and

(4.2)(d) $$P^\vee_{\text{reg}} = \{\lambda^\vee \in P^\vee \,|\, \langle \alpha, \lambda^\vee \rangle \neq 0 \text{ for all } \alpha \in \Delta\}.$$

By duality we obtain the dual based root datum $D_b^\vee$, an involution $\gamma^\vee$ of $D_b^\vee$ (cf. Definition 2.11), the dual group $G^\vee$, and thus *dual* basic data

$(G^\vee,\gamma^\vee)$. In particular $G^\vee$ comes equipped with fixed Cartan and Borel subgroups $H^\vee\subset B^\vee$. We have $X^*(H)=X=X_*(H^\vee)$ and $X_*(H)=X^\vee=X^*(H^\vee)$. These are canonical identifications. By (2.4) applied to H and $H^\vee$ these induce identifications $\mathfrak{h}=\mathfrak{h}^{\vee *}$ and $\mathfrak{h}^*=\mathfrak{h}^\vee$. Note that $P(G,H)=P^\vee(G^\vee,H^\vee)$, and by (4.2)(b) we have

$$P(G,H)=\{\lambda\in\mathfrak{h}^\vee \mid \exp(2\pi i\lambda)\in Z(G^\vee)\}. \tag{4.3}$$

We are also given the bijection $\Delta=\Delta(G,H)\ni\alpha\to\alpha^\vee\in\Delta^\vee=\Delta(G^\vee,H^\vee)$. We identify $W(G,H)$ with $W(G^\vee,H^\vee)$ by the map $W(G,H)\ni w\to w^t\in W(G^\vee,H^\vee)$ (cf. (2.3)); equivalently, $s_\alpha\to s_{\alpha^\vee}$.

5 Extended Groups and Strong Real Forms

{s:extended}

Fix basic data (G,γ) as in Section 4, and a splitting datum $(H,B,\{X_\alpha\})$ (cf. Section 2.1). This gives us a distinguished splitting of the exact sequence (2.8), taking γ to a distinguished involution of G (Definition 2.9) which we also denote γ. Let $\Gamma=\{1,\sigma\}$ be the Galois group of $\mathbb{C}/\mathbb{R}$.

{d:extended}

Definition 5.1 *The extended group for (G,γ) is the semi-direct product*

$$G^\Gamma=G\rtimes\Gamma \tag{5.2}$$

where $\sigma\in\Gamma$ acts on G by the distinguished involution γ.

We say the element $1\times\sigma$ is distinguished, and denote it δ.

Recall γ is a Cartan involution of a maximally compact form in this inner class.

{e:equalrank}

Example 5.3 Suppose $\gamma=1$, so $G^\Gamma=G\times\Gamma$. Note that γ is the Cartan involution of the compact real form of G, and this is the "equal rank" case. Every real form $G(\mathbb{R})$ in this inner class contains a compact Cartan subgroup, and every involution in this inner class is contained in $\mathrm{Int}(G)$.

In particular suppose G is semisimple and the Dynkin diagram of G has no non-trivial automorphisms. Then γ is necessarily trivial, and every involution of G is inner.

{e:slnc}

Example 5.4 Suppose $1\neq\gamma\in\mathrm{Out}(G)$. The most convenient way to compute the corresponding distinguished involution γ of G (cf. Definition 2.9) is to list the real forms in this inner class, and choose the most compact one.

For example let $G = SL(n,\mathbb{C})$ ($n \geq 3$). There is a unique outer automorphism of the Dynkin diagram, corresponding to the non-trivial element $\gamma \in \mathrm{Out}(G)$. If n is odd the only real form in this inner class is the split group $SL(n,\mathbb{R})$, and we take $\gamma(g) = {}^t g^{-1}$, the Cartan involution of $SL(n,\mathbb{R})$.

If n is even then G has two real forms, $SL(n,\mathbb{R})$ and $SL(n/2,\mathbb{H})$ where $\mathbb{H}$ is the quaternion algebra. We let γ act by a Cartan involution of $SL(n/2,\mathbb{H})$.

The extended group G^Γ encapsulates all of the real forms of G in the inner class defined by γ. That is if $\xi \in G^\Gamma\backslash G$ satisfies $\xi^2 \in Z(G)$ then $\mathrm{int}(\xi)$ is an involution in the inner class of γ. Conversely, if θ is in the inner class of γ, then there is an element $\xi \in G^\Gamma\backslash G$ with $\xi^2 \in Z(G)$ and $\theta = \mathrm{int}(\xi)$. For the algorithm it is important to keep track of ξ, and not just $\theta = \mathrm{int}(\xi)$.

{d:stronginvolutic

Definition 5.5 *A strong involution of G in the inner class of γ is an element $\xi \in G^\Gamma\backslash G$ such that $\xi^2 \in Z(G)$. The set of such strong involutions is denoted $\mathcal{I}(G,\gamma)$. We define a strong real form of G in the inner class of γ to be a G-conjugacy class of strong involutions.*

For $\xi \in \mathcal{I}(G,\gamma)$ let $\theta_\xi = int(\xi)$ and $K_\xi = Stab_G(\xi) = G^{\theta_\xi}$.

If γ is understood we refer to strong involutions and strong real forms of G.

{ex:sl2_1}

Example 5.6 Recall (Example 3.4) $SL(2,\mathbb{C})$ has two real forms, $\theta^c = 1$ (i.e. $SU(2)$) and $\theta^s = \mathrm{int}(\mathrm{diag}(i,-i))$ (i.e. $SL(2,\mathbb{R}) = SU(1,1)$). However $SL(2,\mathbb{C})$ has *three* strong real forms, i.e. conjugacy classes of *strong* involutions: $\xi = I, -I$ and $\mathrm{diag}(i,-i)$. (Here and elsewhere, when $\gamma = 1$, we write ξ for the element $(\xi,\sigma) \in G^\Gamma\backslash G$.) Then $\theta_\xi = \theta^c$ if $\xi = \pm I$, or $\theta_\xi = \theta^s$ if $\xi = \mathrm{diag}(i,-i)$. We can think of these strong real forms as $SU(2,0)$, $SU(0,2)$ and $SU(1,1) \simeq SL(2,\mathbb{R})$, respectively.

Now consider $PSL(2,\mathbb{C}) \simeq SO(3,\mathbb{C})$. This has two real forms: the compact group $SO(3,\mathbb{R})$ (with $K(\mathbb{R}) = SO(3,\mathbb{R})$, $K(\mathbb{C}) = SO(3,\mathbb{C})$) and the split one $SO(2,1)$ (with $K(\mathbb{R}) = O(2)$, $K(\mathbb{C}) = O(2,\mathbb{C})$). Up to conjugacy there are two strong involutions: I and $\mathrm{diag}(-1,-1,1)$.

The next Lemma is immediate from the definitions (cf. Definition 3.6).

Lemma 5.7 *We have*

$$\mathcal{I}(G,\gamma)/G = \{\textit{strong real forms in the inner class of } \gamma\}.$$

The map $\mathcal{I}(G,\gamma) \ni \xi \mapsto \theta_\xi$ is a surjection from $\mathcal{I}(G,\gamma)$ to the set of involutions in the inner class of γ. It factors to a surjection

$$\mathcal{I}(G,\gamma)/G \twoheadrightarrow \{\textit{real forms of } G \textit{ in the inner class of } \gamma\}.$$

If G is adjoint this is a bijection.

Example 5.8 In Example 5.6 the map from strong real forms to real forms is bijective for $PSL(2)$. If $G = SL(2)$ the fiber of the map is a single element for the split real form $SU(1,1)$, and two elements ($SU(2,0)$ and $SU(0,2)$, so to speak) for the compact real form $SU(2)$.

{ex:equalrank}

Example 5.9 This is generalization of Example 5.6. Suppose γ is the identity. The strong real forms of G are parametrized by

$$\{\xi \in G \,|\, \xi^2 \in Z(G)\}/G. \tag{5.10)(a}$$

(As in Example 5.6 since $\gamma = 1$ we write ξ instead of (ξ,σ).) Every strong involution ξ is conjugate to an element of H so this set is the same as

$$\{\xi \in H \,|\, \xi^2 \in Z(G)\}/W = (\frac{1}{2}P^\vee/X_*(H))/W \tag{5.10)(b}$$

where $P^\vee$ is the coweight lattice (cf. (4.2)(a)). In particular if G is adjoint the real forms of G are parametrized by

$$(\frac{1}{2}P^\vee/P^\vee)/W. \tag{5.10)(c}$$

Example 5.11 Continuing with the preceding example, let $G = PSp(2n)$. In the usual coordinates $P^\vee \simeq \mathbb{Z}^n \cup (\mathbb{Z}+\frac{1}{2})^n$, and for representatives of $(\frac{1}{2}P^\vee/P^\vee)/W$ we may take

$$\frac{1}{4}(1,\dots,1) \tag{5.12}$$

corresponding to $PSp(2n,\mathbb{R})$ (the split group) and

$$\frac{1}{2}(\overbrace{1,\dots,1}^{p},\overbrace{0,\dots,0}^{q}) \quad (p \le [n/2]) \tag{5.13}$$

corresponding to $PSp(p,q)$.

For $G=Sp(2n)$ we have $X_*(H)=R^\vee\simeq\mathbb{Z}^n$, and the strong real forms are parametrized by

$$(5.14)\qquad (\frac{1}{2}[\mathbb{Z}^n\cup(\mathbb{Z}+\frac{1}{2})^n]/\mathbb{Z}^n)/W.$$

For representatives we may take $\frac14(1,\dots,1)$ and $\frac12(\overbrace{1,\dots,1}^{p},\overbrace{0,\dots,0}^{q})$ as before, where now $0\le p\le n$. Thus there are two strong real forms mapping to each real form $Sp(p,q)$ for $p\ne q$.

See [1, Example 1-17].

{e:I}

We will make frequent use of the following construction. Choose a set of representatives $\{\xi_i\,|\,i\in I\}$ of the set of strong real forms. That is

$$(5.15)(a)\qquad \{\xi_i\,|\,i\in I\}\simeq\mathcal{I}(G,\gamma)/G.$$

If G is semisimple (in fact if $Z(G)^\Gamma$ is finite, cf. (11.2)(3)) this is a finite set. For $i\in I$ let

$$(5.15)(b)\qquad \theta_i=\mathrm{int}(\xi_i),\quad K_i=G^{\theta_i}$$

and

$$(5.15)(c)\qquad \mathcal{I}_i=\{\xi\in\mathcal{I}(G,\gamma)\,|\,\xi\text{ is }G\text{-conjugate to }\xi_i\}.$$

The stabilizer of ξ_i in G is K_i, so $\mathcal{I}_i\simeq G/K_i$, and we have

$$(5.15)(d)\qquad \mathcal{I}(G,\gamma)\simeq\coprod_{i\in I}G/K_i.$$

Recall the distinguished involution $\theta_\delta=\mathrm{int}(\delta)$ is the Cartan involution of a maximally compact real form in the inner class of γ. It is helpful to also invoke the most split form in this inner class. We say an involution of G is *quasisplit* if it is the Cartan involution of a quasisplit real form of G. For a characterization of quasisplit involutions see [4, Proposition 6.24] (where they are called *principal*). By [4, Theorem 6.14] there is a unique conjugacy class of quasisplit involutions in each inner class.

We emphasize the symmetry of the situation by summarizing this:

{l:dandq}

Lemma 5.16 *Let G^Γ be the extended group for (G, γ).*
(1) There exists a strong involution $\xi \in G^\Gamma$ so that θ_ξ is distinguished. The involution θ_ξ is unique up to conjugation by G.
(2) There exists a strong involution η so that θ_η is quasisplit. The involution θ_η is unique up to conjugation by G.

{r:lgroup}

Remark 5.17 The extended group G^Γ in [4, Definition 9.6] is defined in terms of a quasisplit involution, rather than a distinguished one. The equivalence of the two definitions is the content of [4, 9.7]. This discussion also shows that, applied to $(G^\vee, \gamma^\vee)$, the group $G^{\vee\Gamma}$ is isomorphic to the L-group of the real forms of G in the iner class of γ.

{r:stronginvolutio

Remark 5.18 Since in [3] we work with antiholomorphic involutions instead of holomorphic ones, (cf. Remark 3.7), the extended group G^Γ in [3, Chapter 3] is defined in terms of an antiholomorphic involution. The results are equivalent, but some translation is necessary between the two pictures.

6 Harish-Chandra modules for strong real forms

{s:representations

Fix basic data (G, γ) as in Section 2 and let $\mathcal{I} = \mathcal{I}(G, \gamma)$ be the corresponding set of strong involutions. For $\xi \in \mathcal{I}$ let $K_\xi = \mathrm{Stab}_G(\xi)$ as usual, and define $(\mathfrak{g}, K_\xi)$ modules as in [25].

{d:hc}

Definition 6.1 *A Harish-Chandra module for a strong involution is a pair (ξ, π) where $\xi \in \mathcal{I}$ and π is a $(\mathfrak{g}, K_\xi)$-module. A Harish-Chandra module for a strong real form of G is the G-orbit of a pair (ξ, π), where $g \cdot (\xi, \pi) = (g\xi g^{-1}, \pi^g)$.*

An infinitesimal character for $\mathfrak{g}$ may be identified, via the Harish-Chandra homomorphism with an orbit of W on $\mathfrak{h}^*$. For $\lambda \in \mathfrak{h}^*$ we write χ_λ for the corresponding infinitesimal character; so $\chi_\lambda = \chi_{w\lambda}$ for all $w \in W$. If π is a Harish-Chandra module with infinitesimal character χ_λ we may simply say π has infinitesimal character λ.

{d:reginfchar}

Definition 6.2 *We say λ and χ_λ are integral if $\lambda \in P$, and regular and integral if $\lambda \in P_{reg}$ (cf.* (4.2)(c)*).*

Definition 6.3 *Given $\xi \in \mathcal{I}$ let $\Pi(G,\xi)$ be the set of equivalence classes of irreducible $(\mathfrak{g}, K_\xi)$-modules with regular integral infinitesimal character. Let* {d:pig}

$$\Pi(G,\gamma) = \{(\xi,\pi) \,|\, \xi \in \mathcal{I}, \pi \in \Pi(G,\xi)\}/G. \tag{6.4}$$

If Λ is a subset of P_{reg} let $\Pi(G,\gamma,\Lambda) \subset \Pi(G,\gamma)$ be the set of (equivalence classes of) pairs (ξ,π) for which the infinitesimal character of π is an element of Λ.

If we fix a set of representatives I of $\mathcal{I}/G$ as in (5.15) we have

$$\Pi(G,\gamma) \simeq \coprod_{i\in I} \Pi(G,\xi_i). \tag{6.5}$$

Thus $\Pi(G,\gamma)$ parametrizes Harish-Chandra modules for strong real forms of (G,γ), with regular integral infinitesimal character. With the obvious notation we also have $\Pi(G,\gamma,\Lambda) = \coprod_{i\in I} \Pi(G,\xi_i,\Lambda)$.

Fix $\xi \in \mathcal{I}$. As a consequence of [15], associated to each $G^\vee$-conjugacy class of admissible homomorphisms $\phi : W_\mathbb{R} \to G^{\vee\Gamma}$ is an L-packet $\Pi_\phi(G,\xi)$ (see Section 7). This is a finite set of (equivalence classes of) $(\mathfrak{g}, K_\xi)$ modules, all having the same infinitesimal character. Each L-packet is finite, and non-empty if ξ is quasisplit. The non-empty L-packets partition the set of irreducible Harish-Chandra modules for the strong involution ξ.

We define the *large* L-packet of ϕ to be the union, over all strong involutions, of L-packets, modulo our notion of equivalence: {e:large}

$$\Pi_\phi(G,\gamma) = \{(\xi,\pi) \,|\, \xi \in \mathcal{I}, \pi \in \Pi_\phi(G,\xi)\}/G. \tag{6.6)(a}$$

With I as in (5.15) we have

$$\Pi_\phi(G,\gamma) \simeq \coprod_{i\in I} \Pi_\phi(G,\xi_i) \tag{6.6)(b}$$

and each set $\Pi_\phi(G,\xi_i)$ is finite.

7 L-data

{s:ldata}

Fix basic data (G,γ), with corresponding dual data $(G^\vee,\gamma^\vee)$ (cf. Section 4). Let $G^{\vee\Gamma}$ be the corresponding extended group (Definition 5.1). Recall

(Remark 5.17) that $G^{\vee\Gamma}$ is isomorphic to the L-group of G. We begin by parametrizing admissible maps of the Weil group into $G^{\vee\Gamma}$ [6].

Let $W_{\mathbb{R}}$ be the Weil group of $\mathbb{R}$. That is $W_{\mathbb{R}} = \langle \mathbb{C}^\times, j\rangle$ where $jzj^{-1} = \overline{z}$ and $j^2 = -1$. An admissible homomorphism $\phi : W_{\mathbb{R}} \to G^{\vee\Gamma}$ is a continuous homomorphism such that $\phi(\mathbb{C}^\times)$ consists of semisimple elements and $\phi(j) \in G^{\vee\Gamma}\backslash G^\vee$.

Suppose $\phi : W_{\mathbb{R}} \to G^{\vee\Gamma}$ is an admissible homomorphism. Then $\phi(\mathbb{C}^\times)$ is contained in a Cartan subgroup $H_1^\vee(\mathbb{C})$, and $\phi(e^z) = \exp(2\pi i(z\lambda + \overline{z}\nu))$ for some $\lambda, \nu \in \mathfrak{h}_1^\vee$. Choose an inner isomorphism $\mathfrak{h}_1^\vee \simeq \mathfrak{h}^\vee$. As explained in Section 4 we have $\mathfrak{h}^\vee = \mathfrak{h}^*$, so we may identify λ with an element (which we still call λ) of $\mathfrak{h}^*$, whose W orbit is well-defined. We define the *infinitesimal character* of ϕ to be the W-orbit of $\lambda \in \mathfrak{h}^*$.

Recall $P(G, H) = \{\lambda \in \mathfrak{h}^* \mid \exp(2\pi i\lambda) \in Z(G^\vee)\}$ (cf. (4.3)).

{d:onesidedcomplet

Definition 7.1 *A complete one-sided L-datum for* $(G^\vee, \gamma^\vee)$ *is a triple* $(\eta, B_1^\vee, \lambda)$ *where* η *is a strong involution of* $G^\vee$ *(Definition 5.5),* $B_1^\vee$ *is a Borel subgroup of* $G^\vee$ *and* $\lambda \in P(G, H)$ *satisfies* $\exp(2\pi i\lambda) = \eta^2$. *The group* $G^\vee$ *acts by conjugation on this data, and let*

$$\mathcal{P}_c(G^\vee, \gamma^\vee) = \{\textit{complete one-sided L-data}\}/G^\vee. \tag{7.2}$$

Fix a complete one-sided L-datum $S_c^\vee = (\eta, B_1^\vee, \lambda)$. By [16] (also see [3, Lemma 6.18]) there is a θ_η-stable Cartan subgroup $H_1^\vee$ of $B_1^\vee$, unique up to conjugacy by $K_\eta^\vee \cap B_1^\vee$. Choose $g \in G^\vee$ such that $gH^\vee g^{-1} = H_1^\vee$ and $\langle \mathrm{Ad}(g)\lambda, \alpha^\vee\rangle \geq 0$ for all $\alpha \in \Delta(B_1^\vee, H_1^\vee)$. Let $\lambda_1 = \mathrm{Ad}(g)\lambda \in \mathfrak{h}_1^\vee$. Define $\phi : W_{\mathbb{R}} \to G^{\vee\Gamma}$ by:

$$\begin{aligned} \phi(z) &= z^{\lambda_1}\overline{z}^{\eta\lambda_1} \quad (z \in \mathbb{C}^\times) \\ \phi(j) &= \exp(-\pi i\lambda_1)\eta. \end{aligned} \tag{7.3}$$

The first statement is shorthand for $\phi(e^z) = \exp(z\lambda_1 + \mathrm{Ad}(\eta)\overline{z}\lambda_1)$. It is easy to see ϕ is an admissible homomorphism, and the $G^\vee$ conjugacy class of ϕ is independent of the choices of H_1 and g. The next result follows readily.

{p:weil}

Proposition 7.4 *The map taking* $(\eta, B_1^\vee, \lambda)$ *to* ϕ *defined by* (7.3) *induces a bijection between* $\mathcal{P}_c(G^\vee, \gamma^\vee)$ *and the set of* $G^\vee$*-conjugacy classes of admissible homomorphisms* $W_{\mathbb{R}} \to G^{\vee\Gamma}$.

Recall (cf. 6.6) associated to an admissible homomorphism ϕ is a large L-packet $\Pi_\phi(G,\gamma)$. For $S_c^\vee \in \mathcal{P}_c(G^\vee,\gamma^\vee)$ define ϕ by (7.3) and let

$$\Pi_{S_c^\vee}(G,\gamma) = \Pi_\phi(G,\gamma). \tag{7.5}$$

Fix $(\eta, B_1^\vee)$. If $\lambda \in P_{\text{reg}}$, $S_c^\vee = (\eta, B_1^\vee, \lambda)$ is a complete one-sided L-datum if and only if $\exp(2\pi i\lambda) = \eta^2$. For λ satisfying this condition we may define the large L-packet $\Pi_{S_c^\vee}(G,\gamma)$. Any two such L-packets are related by a *translation functor* [27], and in this sense the choice of λ is not important.

We therefore drop the parameter λ from complete L-data:

{d:onesidedldatum}

Definition 7.6 *A one-sided L-datum for $(G^\vee,\gamma^\vee)$ is a pair $S^\vee = (\eta, B_1^\vee)$ where η is a strong involution of $G^\vee$ (Definition 5.5) and $B_1^\vee$ is a Borel subgroup of $G^\vee$. Let*

$$\mathcal{P}(G^\vee,\gamma^\vee) = \{\textit{one-sided L-data}\}/G^\vee. \tag{7.7}$$

There is a well defined map

$$\mathcal{P}(G^\vee,\gamma^\vee) \mapsto Z(G^\vee) \tag{7.8}$$

taking (the equivalence class of) $(\eta, B_1^\vee)$ to η^2.

For the purposes of Proposition 7.4 we have defined one-sided L-data $\mathcal{P}(G^\vee,\gamma^\vee)$ for $(G^\vee,\gamma^\vee)$. It is evident that the definition is symmetric, and applies equally to (G,γ). As discussed in Section 1.6 symmetrizing will give us the finer data which parametrizes individual representations, instead of L-packets. So let

$$\begin{aligned} \mathcal{P} &= \mathcal{P}(G,\gamma) \\ \mathcal{P}^\vee &= \mathcal{P}(G^\vee,\gamma^\vee). \end{aligned} \tag{7.9}$$

Define $\mathcal{P}_c$ and $\mathcal{P}_c^\vee$ similarly.

{d:L}

Definition 7.10 *An L-datum for (G,γ) is a quadruple*

$$\mathbf{S} = (\xi, B_1, \eta, B_1^\vee) \tag{7.11}$$

where $S = (\xi, B_1)$ is an L-datum for G, and $S^\vee = (\eta, B_1^\vee)$ is an L-datum for $G^\vee$. Let $\theta_{\xi,\mathfrak{h}}$ (respectively $\theta_{\eta,\mathfrak{h}^\vee}$) be θ_ξ restricted to $\mathfrak{h}$ (resp. θ_η restricted to $\mathfrak{h}^\vee$). Recalling (2.3), we require that these satisfy

$$(\theta_{\xi,\mathfrak{h}})^t = -\theta_{\eta,\mathfrak{h}^\vee}. \tag{7.12}$$

A complete L-datum is a set

$$\mathbf{S}_c = (\xi, B_1, \eta, B_1^\vee, \lambda) \tag{7.13}$$

where the same conditions hold, $\lambda \in P_{reg}$ *and* $\exp(2\pi i\lambda) = \eta^2$.

Let

$$\begin{aligned} \mathcal{L} &= \{\textit{L-data}\}/G \times G^\vee \subset \mathcal{P} \times \mathcal{P}^\vee \\ \mathcal{L}_c &= \{\textit{complete L-data}\}/G \times G^\vee \subset \mathcal{P} \times \mathcal{P}_c^\vee. \end{aligned} \tag{7.14}$$

Suppose $\mathbf{S}_c = (S, S_c^\vee) = (\xi, B_1, \eta, B_1^\vee, \lambda)$ is a complete L-datum for (G, γ). By [1, Theorem 2.12] associated to $\mathbf{S}_c$ is a $(\mathfrak{g}, K_\xi)$-module $I(\mathbf{S}_c)$. This is a standard module, with regular integral infinitesimal character λ, and has a unique irreducible quotient $J(\mathbf{S}_c)$. We obtain the following version of the Langlands classification. For more detail, in particular information on how to write $J(\mathbf{S}_c)$ in various classifications, see [4, Section 9].

{t:reps}

Theorem 7.15 ([1], Theorem 2-12) *The map*

$$\mathcal{L}_c \ni \mathbf{S}_c \mapsto J(\mathbf{S}_c) \in \Pi(G, \gamma) \tag{7.16}$$

is a bijection.

The large L-packet Π_{S_c} of (7.5) is the set of irreducible representations defined by L-data with second coordinate $S_c^\vee$:

$$\Pi_{S_c^\vee} = \{J(\mathbf{S}) \mid \mathbf{S} = (S, S_c^\vee)\}.$$

Recall the right hand side of (7.16) consists of the equivalence classes of pairs (ξ, π) where ξ is a strong involution and π is a $(\mathfrak{g}, K_\xi)$-module. This is a somewhat subtle space. Furthermore, as discussed after Proposition 7.4, we would like to replace $\mathcal{L}_c$ with $\mathcal{L}$ on the left side of 7.16. This amounts to considering λ only up to $X^*(H)$, and using the translation principle. With these considerations in mind we give several alternative formulations of Theorem 7.15.

Suppose $G(\mathbb{R})$ is a real form of G (cf. Remark 3.5) and $\Lambda \subset P_{\text{reg}}$. Recall (cf. 1.1) $\Pi(G(\mathbb{R}), \lambda)$ is the set of equivalence classes of irreducible admissible representations with infinitesimal character λ. By analogy with Definition 6.3 define $\Pi(G(\mathbb{R}), \Lambda) = \coprod_{\lambda\in\Lambda} \Pi(G(\mathbb{R}), \lambda)$.

{t:ldata}

Theorem 7.17

(1) Fix a set $\Lambda \subset P_{reg}$ *of representatives of* $P/X^*(H)$*. This is a finite set if* G *is semisimple. The map* (7.16) *induces a bijection*

$$\mathcal{L} \overset{1-1}{\longleftrightarrow} \Pi(G, \gamma, \Lambda) \tag{7.18}$$

(cf. Definition 6.3). If G *is semisimple and simply connected we may take* $\Lambda = \{\rho\}$*, the infinitesimal character of the trivial representation.*

(2) Let $I \simeq \mathcal{I}/G$ *be a set of representatives of the strong real forms as in* (5.15)*. For each* $i \in I$ *let* $G_i(\mathbb{R})$ *be the real form of* G *corresponding to the strong involution* ξ_i *(cf. Section 3). Choose* Λ *as in (1). The map* (7.16) *induces a bijection*

$$\mathcal{L} \overset{1-1}{\longleftrightarrow} \coprod_{i \in I} \Pi(G_i(\mathbb{R}), \Lambda). \tag{7.19}$$

(3) Suppose G *is adjoint. Write* $G_1(\mathbb{R}), \dots, G_m(\mathbb{R})$ *for the real forms of* G *in the given inner class, and choose representatives* $\lambda_1, \dots, \lambda_n \in P_{reg}$ *for* P/R*. The map* (7.16) *induces a bijection:*

$$\mathcal{L} \overset{1-1}{\longleftrightarrow} \coprod_{i,j} \Pi(G_i(\mathbb{R}), \lambda_j). \tag{7.20}$$

Sections 9 and 10 will be concerned with finding a combinatorial description of the set $\mathcal{L}$. Most of the work involves the one-sided parameter space $\mathcal{P}(G, \gamma)$ (7.9). Before turning to this we give a geometric interpretation of this space.

8 Relation with the flag variety

{s:kgb}

Recall (Definition 7.6) the one-sided parameter space $\mathcal{P} = \mathcal{P}(G, \gamma)$ is the set of conjugacy classes of one-sided L-data, i.e. pairs (ξ, B_1) where ξ is a strong involution (Definition 5.5) and B_1 is a Borel subgroup. The space $\mathcal{P}$ has a natural interpretation in terms of the flag variety. We see this by conjugating any pair (ξ, B_1) to one with ξ in a fixed set of representatives as in in (5.15).

In the next section we will instead conjugate B_1 to B, and thereby obtain a combinatorial model of $\mathcal{P}$.

Let $\mathcal{B}$ be the set of Borel subgroups of G. Then the set of one-sided L-data for (G,γ) is $\mathcal{I}\times\mathcal{B}$, and

$$\mathcal{P}=(\mathcal{I}\times\mathcal{B})/G. \tag{8.1)(a}$$

Every Borel subgroup is conjugate to B, and $\mathcal{B}\simeq G/B$. For $\xi\in\mathcal{I}$ let

$$\mathcal{P}[\xi]=\{(\xi',B_1)\,|\,\xi' \text{ is conjugate to } \xi\}/G. \tag{8.1)(b}$$

Then

$$\mathcal{P}[\xi]\simeq(G/K_\xi\times G/B)/G \tag{8.1)(c}$$

with G acting by left multiplication. It is an elementary exercise to see the map

$$(\xi',B_1)=(g\xi g^{-1},hBh^{-1})\mapsto K_\xi(g^{-1}h)B\quad(g,h\in G) \tag{8.1)(d}$$

gives a bijection

$$\mathcal{P}[\xi]\overset{1-1}{\longleftrightarrow}K_\xi\backslash G/B. \tag{8.1)(e}$$

As in (5.15) choose a set $\{\xi_i\,|\,i\in I\}$ of representatives of $\mathcal{I}/G$. Then

$$\mathcal{P}\simeq\coprod_{i\in I}(G/K_i\times G/B)/G \tag{8.1)(f}$$

and we see:

{p:kgb}

Proposition 8.2 *There is a natural bijection:*

$$\mathcal{P}(G,\gamma)\overset{1-1}{\longleftrightarrow}\coprod_{i\in I}K_i\backslash G/B. \tag{8.3}$$

9 The One Sided Parameter Space

{s:onesided}

We now turn to the question of formulating an effective algorithm for computing the space $\mathcal{L} \subset \mathcal{P} \times \mathcal{P}^\vee$ (7.14). This mainly comes down to computing the one-sided parameter space $\mathcal{P}$, which we do in this section. We put $\mathcal{P}$ and $\mathcal{P}^\vee$ together to define the parameter space $\mathcal{Z}$ of representations in Section 10.

We begin by looking for a normal form for one-sided L-data.

Fix basic data (G, γ) as usual and set $\mathcal{I} = \mathcal{I}(G, \gamma)$. Recall (cf. Section 8) $\mathcal{P} = (\mathcal{I} \times \mathcal{B})/G$. Since every Borel subgroup is conjugate to B, every element of $\mathcal{I} \times \mathcal{B}$ may be conjugated to one of the form (ξ, B). Therefore the map

(9.1)(a) $$\mathcal{I} \ni \xi \mapsto (\xi, B) \in (\mathcal{I} \times \mathcal{B})/G = \mathcal{P}$$

is surjective. Since B is its own normalizer, we see (ξ, B) is G-conjugate to (ξ', B) if and only if ξ is B-conjugate to ξ'. So we obtain a bijection

(9.1)(b) $$\mathcal{I}/B \xleftrightarrow{1-1} \mathcal{P}$$

from B-orbits on $\mathcal{I}$ to $\mathcal{P}$, sending the B-orbit of $\xi \in \mathcal{I}$ to the G-orbit of the pair (ξ, B).

Now suppose $\xi \in \mathcal{I}$. By ([3, Lemma 6.18], [16]), $\xi \in \mathrm{Norm}_{G^\Gamma}(H_1)$ for some Cartan subgroup $H_1 \subset B$. There exists $b \in B$ such that $bH_1b^{-1} = H$, so $b\xi b^{-1} \in N^\Gamma = \mathrm{Norm}_{G^\Gamma}(H)$. If b_1 is another such element then $b_1 = hb$ with $h \in H$, and $b_1\xi b_1^{-1} = h(b\xi b^{-1})h^{-1}$. Therefore

(9.1)(c) $$\mathcal{I}/B \simeq (\mathcal{I} \cap N^\Gamma)/H.$$

This gives our primary combinatorial construction:

{d:X}

Definition 9.2 *The one-sided parameter space for (G, γ) is the set*

(9.3) $$\begin{aligned}\mathcal{X}(G, \gamma) &= (\mathcal{I} \cap N^\Gamma)/H \\ &= \{\xi \in \mathit{Norm}_{G^\Gamma \backslash G}(H) \mid \xi^2 \in Z(G)\}/H.\end{aligned}$$

This is the set of strong involutions normalizing H, modulo conjugation by H. If (G, γ) is understood we write $\mathcal{X} = \mathcal{X}(G, \gamma)$.

From the preceding discussion we have

(9.4) $$\mathcal{X} = (\mathcal{I} \cap N^\Gamma)/H \simeq \mathcal{I}/B \simeq \mathcal{P}$$

and we conclude:

{p:X=S}

Proposition 9.5 *There is a canonical bijection*

$$\mathcal{X} \stackrel{1-1}{\longleftrightarrow} \mathcal{P} \tag{9.6}$$

taking the H*-orbit of an element* ξ *of* $\mathcal{I} \cap N^\Gamma$ *to the* G*-orbit of* (ξ, B) *in* $\mathcal{P}$.

Given $x \in \mathcal{X}$, let

$$\mathcal{X}[x] = \{x' \,|\, x' \text{ is } G\text{-conjugate to } x\}. \tag{9.7}$$

This is a slight abuse of notation: we say $x, x' \in \mathcal{X}$ are G-conjugate if ξ, ξ' are G-conjugate, where ξ, ξ' are pre-images of x, x' in $\mathcal{I} \cap N^\Gamma$, respectively.

By Proposition 9.5 and (8.1)(e) we see

$$\mathcal{X}[x] \simeq K_\xi \backslash G/B \tag{9.8}$$

where ξ is a preimage of x in $\mathcal{I} \cap N^\Gamma$.

Choose a set $\{\xi_i \,|\, i \in I\}$ of representatives of $\mathcal{I}/G$ as in (5.15). By Proposition 8.2 and (9.8) we obtain:

{c:I}

Corollary 9.9

$$\mathcal{X} \simeq \coprod_{i \in I} K_i \backslash G/B. \tag{9.10}$$

See Examples 12.20 and 12.25.

We need to understand the structure of $\mathcal{X}$ in some detail. We now give more information about it. At the same time we reiterate some earlier definitions and introduce the twisted involutions in the Weyl group.

We fix (G, γ) throughout and drop them from the notation.

Let

(9.11)(a) $$N = \mathrm{Norm}_G(H) \subset N^\Gamma = \mathrm{Norm}_{G^\Gamma}(H)$$

and

(9.11)(b) $$W = N/H \subset W^\Gamma = N^\Gamma/H.$$

Recall (Definition 5.5)

(9.11)(c) $$\mathcal{I} = \{\xi \in G^\Gamma \backslash G \,|\, \xi^2 \in Z(G)\},$$

and that G acts on $\mathcal{I}$ by conjugation. Let

$$\begin{aligned}\widetilde{\mathcal{X}} &= \mathcal{I} \cap N^\Gamma \\ &= \{\xi \in N^\Gamma\backslash N \mid \xi^2 \in Z(G)\}.\end{aligned} \tag{9.11(d)}$$

This is the set of strong involutions normalizing H.

Let

$$\mathcal{X} = \widetilde{\mathcal{X}}/H \tag{9.11(e)}$$

as in (9.3) (the quotient is by the conjugation action).

The group N acts naturally on $\widetilde{\mathcal{X}}$ and $\mathcal{X}$; the action of N on $\mathcal{X}$ factors to W. This action of W on $\mathcal{X}$ corresponds to the cross action of W on characters [25, Definition 8.3.1]. We therefore denote this action $\times$. That is for $w \in W$ and $x \in \mathcal{X}$, choose $n \in N$ mapping to w, $\xi \in \widetilde{\mathcal{X}}$ mapping to x and define

$$w \times x = \text{ image of } n\xi n^{-1} \text{ in } \mathcal{X}. \tag{9.11(f)}$$

Every strong involution is conjugate to one in $\widetilde{\mathcal{X}}$, and we see

$$\widetilde{\mathcal{X}}/N \simeq \mathcal{X}/W. \tag{9.11(g)}$$

See Proposition 12.9 for an interpretation of this space.

Let

$$\mathcal{I}_W = \{\tau \in W^\Gamma\backslash W \mid \tau^2 = 1\}. \tag{9.11(h)}$$

Write $\tilde{p} : \widetilde{\mathcal{X}} \mapsto W^\Gamma$ for the restriction of the map $N^\Gamma \mapsto W^\Gamma$ to $\widetilde{\mathcal{X}}$. It is immediate that $\mathrm{Im}(\tilde{p}) \subset \mathcal{I}_W$, and $\tilde{p} : \widetilde{\mathcal{X}} \mapsto \mathcal{I}_W$ factors to a map

$$p : \mathcal{X} \mapsto \mathcal{I}_W. \tag{9.11(i)}$$

Recall δ is the distinguished element of $\mathcal{X}$ (Definition 5.1); use the same notation for its image in $\mathcal{I}_W$.

{l:pX}

Lemma 9.12 ([20]) *The map $p : \mathcal{X} \mapsto \mathcal{I}_W$ is surjective.*

We prove this later; see Proposition 12.12 and the end of Section 14.

For $\xi \in \widetilde{\mathcal{X}}$ the restriction of θ_ξ to H only depends on the image x of ξ in $\mathcal{X}$. Therefore we may define

$$\theta_{x,H} = \theta_\xi \text{ restricted to } H. \tag{9.13(a)}$$

By Lemma 9.12 $\mathcal{I}_W$ may be thought of as the set of Cartan involutions of H for this inner class:

$$\mathcal{I}_W \xleftrightarrow{1-1} \{\theta_{x,H} \mid x \in \mathcal{X}\}. \tag{9.13(b)}$$

The map $\xi \mapsto \xi^2 \in Z(G)$ is constant on fibers of the map $\widetilde{\mathcal{X}} \mapsto \mathcal{X}$. For $x \in \mathcal{X}$ we define $x^2 \in Z(G)$ accordingly. For $z \in Z(G)$ let

$$\mathcal{X}(z) = \{x \in \mathcal{X} \mid x^2 = z\}. \tag{9.13(c)}$$

Note that $\mathcal{X}(z)$ is empty unless $z \in Z^\Gamma$.

We can make these constructions more concrete using the distinguished element δ of Definition 5.1. Let $\theta = \mathrm{int}(\delta)$. Then

$$\begin{aligned}
\widetilde{\mathcal{X}} &= \{x \in N\delta \mid x^2 \in Z(G)\}\\
&= \{g\delta \mid g \in N,\ g\theta(g) \in Z(G)\}\\
&\xleftrightarrow{1-1} \{g \in N \mid g\theta(g) \in Z(G)\}\\
\mathcal{X} &= \widetilde{\mathcal{X}}/\{g\delta \to hg\theta(h^{-1})\delta \mid h \in H\}\\
&\xleftrightarrow{1-1} \{g \in N \mid g\theta(g) \in Z(G)\}/\{g \to hg\theta(h^{-1}) \mid h \in H\}\\
\mathcal{I}_W &= \{\tau \in W\delta \mid \tau^2 = 1\}\\
&= \{w\delta \mid w \in W, w\theta(w) = 1\}\\
&\xleftrightarrow{1-1} \{w \in W \mid w\theta(w) = 1\}
\end{aligned} \tag{9.14}$$

The last equality identifies $\mathcal{I}_W$ with the *twisted involutions* in the Weyl group. Also note that conjugation in W^Γ becomes *twisted conjugation*:

$$y \cdot w = yw\theta(y^{-1}) \quad (w, y \in W). \tag{9.15}$$

10 The Space $\mathcal{Z}$ and the Main Theorem

{s:Z}

We now describe the parameter space for Harish-Chandra modules of strong real forms of G. By Theorem 7.17 we need to describe the set $\mathcal{L}$ (Definition

7.10). We have done all of the work describing the one-sided parameter space $\mathcal{X}(G,\gamma)$ (Definition 9.2), and now we merely need to put the two sides together.

Fix basic data (G,γ), and let $(G^\vee,\gamma^\vee)$ be the dual data (cf. Section 4). Let $\mathcal{X}=\mathcal{X}(G,\gamma)$ and $\mathcal{X}^\vee=\mathcal{X}(G^\vee,\gamma^\vee)$.

{d:Z}

Definition 10.1 *The two-sided parameter space is the set*

$$\mathcal{Z}(G,\gamma)=\{(x,y)\in\mathcal{X}\times\mathcal{X}^\vee \mid (\theta_{x,H})^t=-\theta_{y,H^\vee}\}. \tag{10.2}$$

We may now state the main result on the parametrization of admissible representations of real forms of G. This is an immediate consequence of Theorem 7.17 and the Definitions of $\mathcal{L}$ and $\mathcal{Z}$.

Fix a set $\Lambda\subset P_{\mathrm{reg}}$ of representatives of $P/X^*(H)$.

{t:main}

Theorem 10.3
(1) There is a natural bijection

$$\mathcal{Z}(G,\gamma)\overset{1-1}{\longleftrightarrow}\Pi(G,\gamma,\Lambda) \tag{10.4}$$

(cf. Definition 6.3).

(2) Let $I\simeq\mathcal{I}/G$ be a set of representatives of the strong real forms as in (5.15)*. For each $i\in I$ let $G_i(\mathbb{R})$ be the real form of G corresponding to the strong involution ξ_i (cf. Section 3). There is a natural bijection*

$$\mathcal{Z}(G,\gamma)\overset{1-1}{\longleftrightarrow}\coprod_{i\in I}\Pi(G_i(\mathbb{R}),\Lambda). \tag{10.5}$$

(3) Suppose G is adjoint. Write $G_1(\mathbb{R}),\dots,G_m(\mathbb{R})$ for the (equivalence classes of) real forms of G in the given inner class, and choose representatives $\lambda_1,\dots,\lambda_n\in P_{reg}$ for P/R. There is a natural bijection:

$$\mathcal{Z}(G,\gamma)\overset{1-1}{\longleftrightarrow}\coprod_{i,j}\Pi(G_i(\mathbb{R}),\lambda_j). \tag{10.6}$$

For some examples see the end of Section 12. Many more examples are worked out in detail in [2].

We note that $\mathcal{Z}(G,\gamma)$ can be viewed as a space of orbits as follows. Let I (resp. $I^\vee$) be a set of representatives of $\mathcal{I}/G$ (resp. $\mathcal{I}^\vee/G^\vee$), as in (5.15)(a). For $i\in I$ (resp. $j\in I^\vee$) let K_i (resp. $K_j^\vee$) be as in (5.15)(b). Then

(10.7) $$\mathcal{Z} \subset \coprod_{i\in I} K_i\backslash G/B \times \coprod_{j\in I^\vee} K_j^\vee\backslash G^\vee/B^\vee.$$

We now give the statement of Vogan Duality [26] in this setting. It is evident that the definition of $\mathcal{Z}(G,\gamma)$ is entirely symmetric in G and $G^\vee$: the map $(x,y)\mapsto(y,x)$ is a bijection between $\mathcal{Z}(G,\gamma)$ and $\mathcal{Z}(G^\vee,\gamma^\vee)$.

{c:duality}

Corollary 10.8 (Vogan Duality) *Fix a set of representatives $\Lambda\subset P_{reg}$ of $P(G,H)/X^*(H)$, and a set $\Lambda^\vee\subset P^\vee_{reg}$ of representatives of $P(G^\vee,H^\vee)/X^*(H^\vee)$. There is a natural bijection*

(10.9) $$\Pi(G,\gamma,\Lambda)\leftrightarrow\Pi(G^\vee,\gamma^\vee,\Lambda^\vee).$$

This bijection is compatible, in a precise sense, with the duality of [26, Theorem 13.13]. See [1] and [3] for details. See the Table in example 12.20 for the case of $SL(2)/PSL(2)$, and [2] for some more elaborate examples.

11 Fibers of the map $p:\mathcal{X}\to\mathcal{I}$

{s:fibers}

In Sections 11 through 14 we study the space $\mathcal{X}$ in more detail, and relate it to structure and representation theory of real groups. We begin with a study of the fibers of $\tilde{p}$ and p. We work in the setting of Section 9.

Fix $\tau\in\mathcal{I}_W$. Let $\widetilde{\mathcal{X}}_\tau=\widetilde{p}^{-1}(\tau)$ and $\mathcal{X}_\tau=p^{-1}(\tau)$. For $z\in Z(G)$ let $\mathcal{X}_\tau(z)=\mathcal{X}_\tau\cap\mathcal{X}(z)=\{x\in\mathcal{X}\mid p(x)=\tau,x^2=z\}$. Let

(11.1) $$\begin{aligned}H'_{-\tau}=&\{h\in H\mid h\tau(h)\in Z(G)\}\\ H_{-\tau}=&\{h\in H\mid h\tau(h)=1\}\\ A_\tau=&\{h\tau(h^{-1})\mid h\in H\}.\end{aligned}$$

Note that A_τ is a connected torus, and is the identity component of $H_{-\tau}$.

{p:fibers}

Proposition 11.2

(1) $H'_{-\tau}$ acts simply transitively on $\widetilde{\mathcal{X}}_\tau$,

(2) $H'_{-\tau}/A_\tau$ acts simply transitively on $\mathcal{X}_\tau$,

(3) Fix $z \in Z(G)$. If $\mathcal{X}_\tau(z)$ is non-empty then $H_{-\tau}/A_\tau$ acts simply transitively on $\mathcal{X}_\tau(z)$. If $z \notin Z(G)^\Gamma$ (the Γ-invariants of $Z(G)$) then $\mathcal{X}_\tau(z)$ is empty.

In particular $|\mathcal{X}_\tau(z)|$ is a power of 2 (or 0). If $Z(G)^\Gamma$ is finite then $\mathcal{X}$ is a finite set.

Proof. Choose $\xi \in \widetilde{\mathcal{X}}_\tau$. Then $\widetilde{\mathcal{X}}_\tau = \{h\xi \,|\, h \in H, (h\xi)^2 \in Z(G)\} = \{h\xi \,|\, h\tau(h)\xi^2 \in Z(G)\}$. The first claim follows.

For $h \in H$ we have $h\xi h^{-1} = h\tau(h^{-1})\xi$. This shows that the stabilizer in $H'_{-\tau}$, for the left multiplication action of H, of the image of ξ in $\mathcal{X}$ is $\{h\tau(h^{-1}) \,|\, h \in H\} = A_\tau$. This proves (2), and (3) follows immediately from the fact that $(h\xi)^2 = h\tau(h)\xi^2$, and the fact that $\xi^2 \in Z(G)^\Gamma$. The assertion about $|\mathcal{X}_\tau(z)|$ is clear, since $H_{-\tau}/A_\tau$ is an elementary abelian two-group.

By (3) $\mathcal{X}$ is the union of the finite sets $\mathcal{X}_\tau(z)$ for $\tau \in \mathcal{I}_W$ and $z \in Z(G)^\Gamma$, and the final conclusion follows. □

{r:H2}

Remark 11.3 Let T_τ be the identity component of the fixed points of τ acting on H. Then T_τ and A_τ are connected tori, $H = T_\tau A_\tau$ and $A_\tau \cap T_\tau$ is an elementary abelian two group. The group in Proposition 11.2 (3) is

$$H_{-\tau}/A_\tau \simeq T_\tau(2)/A_\tau \cap T_\tau. \tag{11.4}$$

If we write the real torus corresponding to τ as $(\mathbb{R}^\times)^a \times (S^1)^b \times (\mathbb{C}^\times)^c$ then $T_\tau(2) \simeq (\mathbb{Z}/2\mathbb{Z})^{b+c}$, $A_\tau \cap T_\tau \simeq (\mathbb{Z}/2\mathbb{Z})^c$ and $T_\tau(2)/A_\tau \cap T_\tau \simeq (\mathbb{Z}/2\mathbb{Z})^b$.

{r:dualH}

Remark 11.5 It is helpful to note that

$$H_{-\tau}/A_\tau \simeq [H^\vee(\mathbb{R})/H^\vee(\mathbb{R})^0]^\wedge \tag{11.6}$$

where $H^\vee$ is the dual torus to H, with real form $H^\vee(\mathbb{R})$ defined by the Cartan involution $-\tau^\vee$.

12 Action of W on $\mathcal{X}$

{s:waction}

We now study the action of W on $\mathcal{X}$, which plays an important role. We begin with some definitions and terminology.

Fix $\tau \in \mathcal{I}_W$. Let

$$(12.1)\qquad \begin{aligned} \Delta_i &= \{\alpha \in \Delta \,|\, \tau(\alpha) = \alpha\} \text{ (the imaginary roots)} \\ \Delta_r &= \{\alpha \in \Delta \,|\, \tau(\alpha) = -\alpha\} \text{ (the real roots)} \\ \Delta_{cx} &= \{\alpha \in \Delta \,|\, \tau(\alpha) \neq \pm\alpha\} \text{ (the complex roots)} \\ \Delta_i^+ &= \Delta_i \cap \Delta^+,\ \Delta_r^+ = \Delta_r \cap \Delta^+ \\ W_i &= W(\Delta_i) \\ W_r &= W(\Delta_r). \end{aligned}$$

We will write $\Delta_{i,\tau}, W_{i,\tau}$ etc. to indicate the dependence on τ. We also refer to the τ-imaginary, τ-real roots, etc.

Let $\rho_i = \frac{1}{2}\sum_{\alpha\in\Delta_i^+}\alpha$, and $\rho_r^\vee = \frac{1}{2}\sum_{\alpha\in\Delta_r^+}\alpha^\vee$. As in [26, Proposition 3.12] let

$$(12.2)\qquad \begin{aligned} \Delta_C &= \{\alpha \in \Delta \,|\, \langle \rho_i, \alpha^\vee\rangle = \langle \alpha, \rho_r^\vee\rangle = 0\} \subset \Delta_{cx} \\ W_C &= W(\Delta_C) \end{aligned}$$

This is a complex root system.

Now τ acts on W, and we let W^τ be the fixed points. By [26, Proposition 3.12]

$$(12.3)\qquad W^\tau = (W_C)^\tau \ltimes (W_i \times W_r).$$

Note that W_i and W_r are Weyl groups of the root systems Δ_i and Δ_r respectively; also $(W_C)^\tau$ is isomorphic to the Weyl group of the root system $(\Delta_C)^\tau$ [26].

Fix $\xi \in \widetilde{\mathcal{X}}_\tau$ and let $\theta = \theta_\xi$, $K = K_\xi$. For $\alpha \in \Delta_i$ let X_α be an α-root vector. Then $\theta_\xi(X_\alpha)$ only depends on the image x of ξ in $\mathcal{X}$. We say $\mathrm{gr}_x(\alpha) = 0$ if $\theta_\xi(X_\alpha) = X_\alpha$ and 1 if $\theta_x(X_\alpha) = -X_\alpha$. This is a $\mathbb{Z}/2\mathbb{Z}$-grading of Δ_i in the sense that if $\alpha, \beta, \alpha+\beta \in \Delta_i$ then $\mathrm{gr}_x(\alpha+\beta) = \mathrm{gr}_x(\alpha) + \mathrm{gr}_x(\beta)$ (mod 2). These are the compact and noncompact imaginary roots.

Let $W(K,H) = \mathrm{Norm}_K(H)/H\cap K$. This is isomorphic to $W(G(\mathbb{R}), H(\mathbb{R}))$ where $G(\mathbb{R})$ is the real form of G corresponding to θ, and we call it the real Weyl group. (This contains the Weyl group of the root system of real roots.) Clearly $W(K,H) \subset W^\tau$. Let $M = \mathrm{Cent}_G(A_\tau)$ (cf. 11.1). By [26, Proposition 4.16],

$$(12.4)\qquad W(K,H) = (W_C)^\tau \ltimes (W(M\cap K, H) \times W_r).$$

We have

(12.5) $$W_{i,c} \subset W(M\cap K,H) \simeq W_{i,c} \ltimes \mathcal{A}(H) \subset W_i$$

where $W_{i,c}$ is the Weyl group of the compact imaginary roots (i.e. $\mathrm{gr}_\xi(\alpha)=0$) and $\mathcal{A}(H)$ is a certain two-group [26]. This describes $W(K,H)$ in terms of the Weyl groups $(W_C)^\tau$, W_r and $W_{i,c}$, which are straightforward to compute, and the two-group $\mathcal{A}(H)$. For more information on $W(K,H)$ see Proposition 12.14.

Let

(12.6) $$\mathcal{H} = \{(\xi, H_1)\,|\,\xi\in\mathcal{I}, H_1 \text{ a } \theta_\xi\text{-stable Cartan subgroup}\}/G.$$

With I, θ_i and K_i as in (5.15) we may conjugate ξ to some ξ_i, and this shows

(12.7) $$\mathcal{H} \simeq \coprod_{i\in I}\{\theta_i\text{-stable Cartan subgroups of } G\}/K_i.$$

On the other hand every Cartan subgroup is conjugate to H, and the normalizer of H is N, so

(12.8) $$\mathcal{H} \simeq \mathcal{I}\cap N^\Gamma/N = \widetilde{\mathcal{X}}/N \simeq \mathcal{X}/W$$

(cf. (9.11)(g)).

With notation as in Corollary 9.9 for $i\in I$ we may assume $\xi_i\in\widetilde{\mathcal{X}}$, and define $x_i\in\mathcal{X}$ and $\mathcal{X}_i = \mathcal{X}[x_i]$ accordingly. We conclude

{p:X/W}

Proposition 12.9 *For each $i\in I$ we have*

(12.10) $$\mathcal{X}_i/W \leftrightarrow \{\theta_i\textit{-stable Cartan subgroups of } G\}/K_i.$$

Taking the union over $i\in I$ gives

(12.11) $$\mathcal{X}/W \leftrightarrow \coprod_i\{\theta_i\textit{-stable Cartan subgroups of } G\}/K_i.$$

Recall that by (9.8) and Proposition 8.2 $\mathcal{X}_i \simeq K_i\backslash G/B$.

{p:X_i/W}

Proposition 12.12 ([20]) *The map $p:\mathcal{X}_i/W \mapsto \mathcal{I}_W/W$ is injective. If θ_i is quasisplit it is a bijection.*

Remark 12.13 This says that the conjugacy classes of Cartan subgroups of any real form of G embed in those of the quasisplit form. See [19, page 340].

Proof. For injectivity we have to show that $\xi, \xi' \in \widetilde{\mathcal{X}}$, $\widetilde{p}(\xi) = \widetilde{p}(\xi')$ and $\xi' = g\xi g^{-1}$ ($g \in G$) implies $\xi' = n\xi n^{-1}$ for some $n \in N$. The condition $\widetilde{p}(\xi) = \widetilde{p}(\xi')$ implies $\xi' = h\xi$ for some $h \in H$, so $g\xi g^{-1} = h\xi$, i.e. $g\theta_\xi(g^{-1}) = h$. By [20, Proposition 2.3] there exists $n \in N$ satisfying $h = n\theta_\xi(n^{-1})$, and then $\xi' = n\xi n^{-1}$.

We defer the proof of surjectivity in the quasisplit case to the end of Section 14. $\square$

The real Weyl group (see the discussion preceding (12.4)) appears naturally in our setting. Fix $\xi \in \widetilde{\mathcal{X}}$, let $K = K_\xi$, and let x be the image of ξ in $\mathcal{X}$.

{p:realweyl}

Proposition 12.14 *$W(K,H) \simeq Stab_W(x)$.*

Proof. We have

$$
\begin{aligned}
W(K,H) &= \mathrm{Norm}_K(H)/H \cap K \\
&= \mathrm{Stab}_N(\xi)/\mathrm{Stab}_H(\xi) \\
&= \mathrm{Stab}_N(\xi)H/H.
\end{aligned} \tag{12.15}
$$

It is easy to see that $\mathrm{Stab}_N(\xi)H = \mathrm{Stab}_N(x)$, so this equals

$$\mathrm{Stab}_N(x)/H \simeq \mathrm{Stab}_{N/H}(x) = \mathrm{Stab}_W(x).$$

$\square$

Now fix $\tau \in \mathcal{I}_W$. By Proposition 12.14 and (12.4) we see $(W_C)^\tau$ and W_r act trivially on $\mathcal{X}_\tau$. It is worth noting that we can see this directly.

{p:trivially}

Proposition 12.16 *Both $(W_C)^\tau$ and W_r act trivially on $\mathcal{X}_\tau$.*

This proof was communicated to us by David Vogan.

Proof. Fix $\xi \in \widetilde{\mathcal{X}}_\tau$. The group $(W_C)^\tau$ is generated by elements $s_\alpha s_{\tau\alpha}$ where $\alpha \in \Phi_C$. So suppose $\alpha \in \Phi_C$ and let $\sigma_\alpha \in N$ be a preimage of $s_\alpha \in W$. Let $\sigma_{\tau(\alpha)} = \xi\sigma_\alpha\xi^{-1}$. Note that $\alpha + \tau(\alpha)$ is not a root, since it would have to be imaginary, and (by (12.2)) orthogonal to ρ_i. Therefore the root subgroups G_α and $G_{\tau(\alpha)}$ commute. Then $\xi\sigma_\alpha\sigma_{\tau(\alpha)}\xi^{-1} = \sigma_{\tau(\alpha)}\sigma_\alpha = \sigma_\alpha\sigma_{\tau(\alpha)}$.

If α is a τ-real root this reduces easily to a computation in $SL(2)$. We omit the details. $\square$

Fix $\tau \in \mathcal{I}_W$ and suppose $x, x' \in \mathcal{X}_\tau$. As a consequence of Propositions 12.12 and 12.14 we have

(12.17) $\quad x'$ is G-conjugate to $x \Leftrightarrow x' = w \times x$ for some $w \in W_{i,\tau}$.

Another useful result obtained from the action of W^τ is the computation of strong real forms.

{p:Xstrongrealform

Proposition 12.18 *Every element $x \in \mathcal{X}$ is G-conjugate to an element of $\mathcal{X}_\delta$, and there is a canonical bijection between $\mathcal{X}_\delta/W_{i,\delta}$ and the set of strong real forms of* (G, γ).

(See the remark after (9.7) for the notion of G-conjugacy.) This corresponds to the fact that every real form in the given inner class contains a fundamental (i.e. most compact) Cartan subgroup. Note that if G is adjoint this gives a bijection between $\mathcal{X}_\delta/W_{i,\delta}$ and real forms in the given inner class.

We defer the proof until the end of Section 14. See Examples 12.20 and 12.25.

12.1 Recapitulation

{s:recap}

We summarize the main results on the translation between the structure of $\mathcal{X}$ and some standard objects for G.

Fix basic data (G, γ) as in Section 4 and define G^Γ as in Definition 5.1. Let $\mathcal{X} = \mathcal{X}(G, \gamma)$ (cf. (9.11)(e)) and define $\mathcal{I}_W$ as in (9.11)(h). Fix a set $\{\xi_i \,|\, i \in I\}$ of representatives of the strong real forms, i.e. $\mathcal{I}/G$, as in (5.15), and for $i \in I$ let $\theta_i = \theta_{\xi_i}$ and $K_i = K_{\xi_i}$. Recall δ is the distinguished element of G^Γ, or its image in $\mathcal{I}_W$. Also $W_{i,\delta}$ is the Weyl group of the δ-imaginary roots. Let θ_{qs} be a quasisplit involution in this inner class (cf. Lemma 5.16) and let $K_{qs} = G^{\theta_{qs}}$.

{p:recapitulation}

Proposition 12.19 *We have bijections:*

(1) $\mathcal{X} \overset{1-1}{\longleftrightarrow} \coprod_{i\in I} K_i \backslash G/B$ *(Corollary 9.9),*

(2) $\mathcal{X}_\delta/W_{i,\delta} \overset{1-1}{\longleftrightarrow} \{$*strong involutions*$\}/G = \{$*strong real forms*$\}$ *(Prop. 12.18),*

(3) $\mathcal{I}_W/W \overset{1-1}{\longleftrightarrow} \{\theta_{qs}$*-stable Cartan subgroups in* $G\}/K_{qs}$ *(Prop. 12.12),*

(4) $\mathcal{X}/W \overset{1-1}{\longleftrightarrow} \coprod_i \{\theta_i$*-stable Cartan subgroups in* $G\}/K_i$ *(Prop. 12.9),*

(5) $\mathcal{X}_\tau(z) \simeq [H^\vee(\mathbb{R})/H^\vee(\mathbb{R})^0]^\wedge$ *or is empty* $(\tau \in \mathcal{I}_W, z \in Z, H^\vee(\mathbb{R})$ *as in Remark 11.5),*

(6) $Stab_W(x) \simeq W(K_\xi, H)$ $(\xi \in \widetilde{\mathcal{X}},$ *with image* $x \in \mathcal{X})$ *(Prop. 12.14).*

{ex:sl2_2}

Example 12.20 We illustrate each part of the Proposition in the case of $SL(2,\mathbb{C})$. In this case $\mathrm{Out}(G) = 1$ so there is only one inner class of involutions, and we drop δ from the notation. See examples 3.4 and 5.6.

Write $H = \{\mathrm{diag}(z, \frac{1}{z}) \mid z \in \mathbb{C}^\times\}$, $W = \{1, s\}$, and let $t = \mathrm{diag}(i, -i)$. Let n be any element of $\mathrm{Norm}_G(H)$ mapping to $s \in W$. Then $\mathrm{Norm}_G(H) = H \cup Hn$, and

$$\begin{aligned} \widetilde{\mathcal{X}} &= \{\xi \in H \cup Hn \mid \xi^2 = \pm\mathrm{Id}\} \\ &= \{\pm\mathrm{Id}, \pm t\} \cup Hn \\ \mathcal{X} &= \{\pm\mathrm{Id}, \pm t\}/H \cup Hn/H \end{aligned} \tag{12.21}$$

Note that H acts trivially on $\{\pm\mathrm{Id}, \pm t\}$, and $hnh^{-1} = h^2 n$ for all $h \in H$, so Hn/H is a singleton. Therefore

$$\mathcal{X} = \{\pm\mathrm{Id}, \pm t, n\}. \tag{12.22}$$

Strictly speaking these are elements of $\widetilde{\mathcal{X}}$ representing $\mathcal{X}$.

Since $\delta = 1$ we have $\mathcal{X}_\delta = \{\pm\mathrm{Id}, \pm t\}$, and $W_{i,\delta} = W$. Part (2) of the Proposition says we can take I to be a set of representatives of $\mathcal{X}_\delta/W = \{\pm\mathrm{Id}, t\}$. Recall (5.6) we think of these as "strong real forms" $SU(2,0), SU(0,2)$ and $SU(1,1) \simeq SL(2,\mathbb{R})$, respectively. Then

$$\mathcal{X}[\mathrm{Id}] = \{\mathrm{Id}\}, \quad \mathcal{X}[-\mathrm{Id}] = \{-\mathrm{Id}\}, \quad \mathcal{X}[t] = \{t, -t, n\}. \tag{12.23}$$

Now G/B is isomorphic to the complex projective plane $\mathbb{C} \cup \{\infty\}$. We have $K_{\pm\mathrm{Id}} = G$ and $K_{\pm\mathrm{Id}}\backslash G/B$ is a point. We label these orbits $\mathcal{O}_{2,0}$ and $\mathcal{O}_{0,2}$, respectively. On the other hand $K_t \simeq \mathbb{C}^\times$, which acts on G/B by $z : u \mapsto z^2 u$. Therefore there are three orbits of this action: $\mathcal{O}_0 = \{0\}, \mathcal{O}_\infty = \{\infty\}$ and $\mathcal{O}_* = \mathbb{C}^\times$.

So the bijection of (1) is

(12.24)

$x \in \mathcal{X}$	Id	-Id	t	-t	n
K_x	G	G	$\mathbb{C}^\times$	$\mathbb{C}^\times$	$\mathbb{C}^\times$
Orbit	$\mathcal{O}_{2,0}$	$\mathcal{O}_{0,2}$	$\mathcal{O}_0$	$\mathcal{O}_\infty$	$\mathcal{O}_*$

In this case $\mathcal{I}_W = W = \{1, s\}$. The quasisplit group is $SL(2, \mathbb{R})$, which has two conjugacy classes of Cartan subgroups. The compact Cartan subgroup $T \simeq S^1$ corresponds to $1 \in W$, and the split Cartan subgroup $A \simeq \mathbb{R}^\times$ corresponds to $s \in W$. This is (3) in this case.

Now $\mathcal{X}/W$ has four elements $I, -I, t$ and n, corresponding to the compact Cartan subgroups of $SU(2,0), SU(0,2), SU(1,1)$, and the split Cartan subgroup of $SU(1,1)$, respectively. This is the content of (4).

For (5) we have $\mathcal{X}_1(\mathrm{Id}) = \{\pm\mathrm{Id}\}$ and $\mathcal{X}_1(-\mathrm{Id}) = \{\pm t\}$. In this case $H(\mathbb{R}) \simeq S^1$, and $H^\vee(\mathbb{R}) \simeq \mathbb{R}^\times$, so $H^\vee(\mathbb{R})/H^\vee(\mathbb{R})^0 \simeq \mathbb{Z}/2\mathbb{Z}$. On the other hand $\mathcal{X}_s(\mathrm{Id}) = \emptyset$ and $\mathcal{X}_s(-\mathrm{Id}) = \{n\}$. In this case $H(\mathbb{R}) = \mathbb{R}^\times$ and $H^\vee(\mathbb{R}) = S^1$ is connected.

Finally consider (6). We have $\mathrm{Stab}_W(\pm\mathrm{Id}) = W$, and $\mathrm{Stab}_W(\pm t) = 1$. This corresponds to the fact that $W(SU(2), S^1)) = W$, and $W(SL(2,\mathbb{R}), S^1) = 1$. On the other hand $\mathrm{Stab}_W(n) = W$, i.e. $W(SL(2,\mathbb{R}), \mathbb{R}^\times) = W$.

Most of the conclusions of the Proposition seen in Figure 1. Projection $p : \mathcal{X} \mapsto \mathcal{I}_W$ is written vertically.

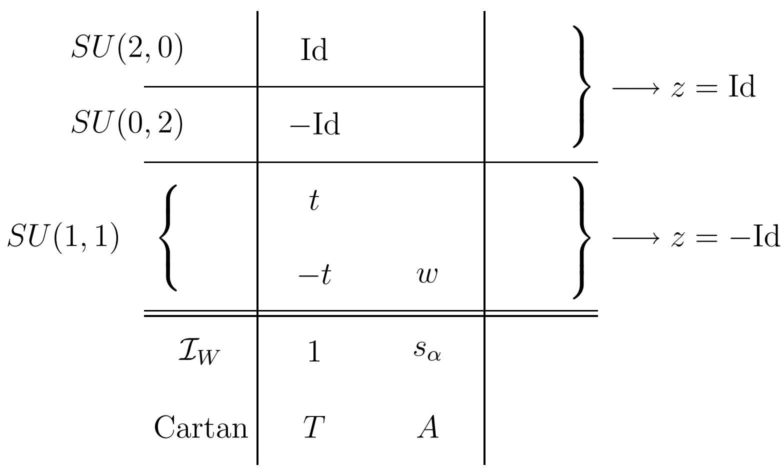

Figure 1: $\mathcal{X}$ for $SL(2)$

{ex:psl2_1}

Example 12.25 We reconsider the previous example with $PSL(2,\mathbb{C})$ in place of $SL(2,\mathbb{C})$. Again $\gamma = 1$ and we drop it from the notation.

Recall $PSL(2,\mathbb{C}) \simeq SO(3,\mathbb{C})$, and it is easier to work with the latter realization, with respect to the form $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. We take $H = \{\mathrm{diag}(z, \frac{1}{z}, 1) \mid z \in \mathbb{C}^\times\}$, write $W = \{1, s\}$ and let n be a representative in $\mathrm{Norm}_G(H)$ of s. Let $t = \mathrm{diag}(-1,-1,1)$.

As in the previous example we have:

$$\begin{aligned} \mathcal{X} &= \{\xi \in H \cup Hn \mid \xi^2 = \mathrm{Id}\}/H \\ &= \{\mathrm{Id}, t\}/H \cup (Hn)/H \\ &= \{\mathrm{Id}, t, n\}. \end{aligned} \tag{12.26}$$

In this case we can take $I = \{\mathrm{Id}, t\}$, and

$$\mathcal{X}[\mathrm{Id}] = \{\mathrm{Id}\}, \mathcal{X}[t] = \{t, n\}. \tag{12.27}$$

Then $K_{\mathrm{Id}} = G$ and $K_{\mathrm{Id}} \backslash G/B$ is a point. On the other hand $K_t \simeq O(2,\mathbb{C})$, which has two orbits on the projective plane: $\{0, \infty\}$ and $\mathbb{C}^\times$. This is (1) of the Proposition in this case.

In this case (2) says that $\mathcal{X}_1/W = \{\mathrm{Id}, t\}$, corresponding to the two real forms of G.

The analogue of (12.24) is

$x \in \mathcal{X}$	Id	t	n
K_x	G	$\mathbb{C}^\times$	$\mathbb{C}^\times$
Orbit	$\mathcal{O}_{2,0}$	$\mathcal{O}_0$	$\mathcal{O}_*$

Statement (3) is the same as for $SL(2,\mathbb{C})$: $\mathcal{I}_W/W$ has two elements, corresponding to the two conjugacy classes of Cartan subgroups of $SO(2,1)$.

For (4), $\mathcal{X}/W = \mathcal{X} = \{\mathrm{Id}, t, n\}$; corresponding to the compact Cartan subgroups of $SO(3), SO(2,1)$, and the split Cartan subgroup of $SO(2,1)$, respectively.

Next, $\mathcal{X}_1(\mathrm{Id}) = \{\mathrm{Id}, t\}$ and $\mathcal{X}_s(I) = \{n\}$, corresponding to $H^\vee(\mathbb{R}) = \mathbb{R}^\times$ and S^1 as in the previous example. This gives (5).

Finally note that $\mathrm{Stab}_W(1) = \mathrm{Stab}(n) = W$ as in the previous example. However $\mathrm{Stab}_W(t) = W$; i.e. $W(SO(2,1), S^1) = W$. Comparing this case with the fact that $\mathrm{Stab}_W(t) = 1$ in the case of $SL(2,\mathbb{C})$ illustrates how $\mathrm{Stab}_W(\xi)$ depends in a subtle way on isogenies. See (12.5).

{ex:sl2psl2reps}

Example 12.28 Having described $\mathcal{X}$ for $SL(2,\mathbb{C})$ and $PSL(2,\mathbb{C})$ we can now describe the representation theory of real forms of these groups in terms of the space $\mathcal{Z}$. See Theorem 10.3, and Examples 12.20 and 12.25. Note that the representations are parametrized by pairs of orbits as in (10.7).

Write $\mathbb{C}$ for the trivial representation.

For $SU(1,1) \simeq SL(2,\mathbb{R})$, at infinitesimal character ρ, write $DS_\pm$ for the discrete series representations and PS_{odd} for the irreducible (non-spherical) principal series representation.

Consider $SO(2,1)$. Let sgn be the sign representation of $SO(2,1)$, and DS be the unique discrete series representation with infinitesimal character ρ. At infinitesimal character 2ρ $SO(2,1)$ has two irreducible principal series representations denoted $PS_\pm$.

Table of representations of $SL(2)$ and $PGL(2)$

Orbit	x	x^2	θ_x	$G_x(\mathbb{R})$	λ	rep	Orbit	y	y^2	θ_y	$G_y^\vee(\mathbb{R})$	λ	rep
$\mathcal{O}_{2,0}$	Id	Id	1	$SU(2,0)$	ρ	$\mathbb{C}$	$\mathcal{O}'_*$	n	Id	-1	$SO(2,1)$	2ρ	PS_+
$\mathcal{O}_{0,2}$	-Id	Id	1	$SU(0,2)$	ρ	$\mathbb{C}$	$\mathcal{O}'_*$	n	Id	-1	$SO(2,1)$	2ρ	PS_-
$\mathcal{O}_0$	t	-Id	1	$SU(1,1)$	ρ	DS_+	$\mathcal{O}'_*$	n	Id	-1	$SO(2,1)$	ρ	$\mathbb{C}$
$\mathcal{O}_\infty$	-t	-Id	1	$SU(1,1)$	ρ	DS_-	$\mathcal{O}'_*$	n	Id	-1	$SO(2,1)$	ρ	sgn
$\mathcal{O}_*$	n	-Id	1	$SU(1,1)$	ρ	$\mathbb{C}$	$\mathcal{O}'_+$	t	Id	-1	$SO(2,1)$	ρ	DS
$\mathcal{O}_*$	n	Id	1	$SU(1,1)$	ρ	PS_{odd}	$\mathcal{O}'_{3,0}$	Id	Id	1	$SO(3)$	ρ	$\mathbb{C}$

See [2] for more detail on this example.

13 The reduced parameter space

{s:reduced}

If G is adjoint we saw in Section 10 that the parameter space $\mathcal{X} = \mathcal{X}(G,\gamma)$ is perfectly suited to parametrizing representations of real forms of G. If G is not adjoint then strong involutions play an essential role, and the difference between involutions and strong involutions is unavoidable. Nevertheless in some respects the space $\mathcal{X}$ is larger than necessary, and a satisfactory theory is obtained with a smaller set, the *reduced one-sided parameter space.* While $\mathcal{X}$ may be infinite, this is always a finite set.

The most economical possibility would be to keep a single orbit of strong involutions over each orbit of real forms. However there is no canonical way to make this choice. The reduced parameter space provides a more canonical way to reduce to a small finite number of choices related to the center.

Let $Z = Z(G)$, and recall (cf. 11.2(3)) Z^Γ is the Γ-invariants in Z. There is a natural action of Z on $\mathcal{X}$ by left multiplication. This preserves the fibers $\mathcal{X}_\tau$, and commutes with the conjugation action of G. For $x \in \mathcal{X}$ recall (9.7) $\mathcal{X}[x]$ is the set of elements of $\mathcal{X}$ conjugate to x. If $z \in Z$ multiplication by z

is a bijection:

$$\mathcal{X}[x] \overset{1-1}{\longleftrightarrow} \mathcal{X}[zx] \tag{13.1}$$

(cf. (9.7)). In other words the orbit pictures for x and zx are identical. Suppose $\xi \in \widetilde{\mathcal{X}}$ lies over x. Harish-Chandra modules for ξ are $(\mathfrak{g}, K_\xi)$ modules; since $K_\xi = K_{z\xi}$, Harish-Chandra modules for ξ are exactly the same as Harish-Chandra modules for $z\xi$, with the same notion of equivalence.

For example suppose $G = SL(2)$ and take $\xi = I$ and $z = -I$. Then $\xi = I$ and $z\xi = -I$ both correspond to the compact group $SU(2)$. See Example 5.6.

Write θ for the action of the non-trivial element of Γ on Z. For $z \in Z$ recall $\mathcal{X}(z)$ is the set of elements of $\mathcal{X}$ satisfying $x^2 = z$ (9.13)(c), and $\mathcal{X}(z)$ is empty unless $z \in Z^\Gamma$. If $x \in \mathcal{X}(z')$ and $z \in Z$ then

$$zx \in \mathcal{X}(z'z\theta(z)). \tag{13.2}$$

It is easy to see that

$$Z^\Gamma/\{z\theta(z) \mid z \in Z\} \simeq H^2(\Gamma, Z) \tag{13.3}$$

is a finite set. This comes down to the fact that if Z is a torus then $Z^\Gamma/\{z\theta(z)\} \simeq (\mathbb{Z}/2\mathbb{Z})^n$ where n is the number of $\mathbb{R}^\times$ factors in the corresponding real torus (cf. Remark 11.3).

{d:reduced}

Definition 13.4 *Choose a set of representatives $Z_0 \subset Z^\Gamma$ for $Z^\Gamma/\{z\theta(z)\} \simeq H^2(\Gamma, Z)$. The reduced parameter space is*

$$\mathcal{X}^r(G, \gamma) = \coprod_{z \in Z_0} \mathcal{X}(z). \tag{13.5}$$

Example 13.6 Let $G = SL(n, \mathbb{C})$, and let $\gamma = 1$. Suppose $p + q = n$ and $\alpha^n = (-1)^q$. Let

$$\xi_\alpha = \mathrm{diag}(\overbrace{\alpha, \dots, \alpha}^{p}, \overbrace{-\alpha, \dots, -\alpha}^{q})$$

These are representatives of the equivalence classes of strong involutions in this inner class. For fixed $p \neq q$ there are n strong real forms, all mapping to the real form $SU(p, q)$. In other words we obtain n identical orbit pictures. If $p = q$ a similar statement holds, except that ξ_α is conjugate to $\xi_{-\alpha}$.

If n is odd then $Z^\Gamma/\{z\theta(z)\} = Z/Z^2$ is trivial, so we take $Z_0 = \{I\}$, and the equivalence classes of strong involutions in $\mathcal{X}_0(G,\gamma)$ are represented by

$$(\overbrace{1,\dots,1}^{p},\overbrace{-1,\dots,-1}^{q})\quad (q \text{ even}). \tag{13.7}$$

For each p,q there is a unique strong involution mapping to the real form $SU(p,q)$, instead of n as we had earlier.

If n is even then $Z^\Gamma/\{z\theta(z)\}$ has order 2. We can take $Z_0 = \{I,\zeta I\}$ where ζ is a primitive n^{th} root of 1. Let τ be a primitive $2n^{th}$ root of 1. Then the strong real forms in $\mathcal{X}_0(G,\gamma)$ are

$$\begin{aligned} &\pm\mathrm{diag}(\overbrace{1,\dots,1}^{p},\overbrace{-1,\dots,-1}^{q})\quad (q \text{ even}, p\ne q)\\ &\mathrm{diag}(\overbrace{1,\dots,1}^{p},\overbrace{-1,\dots,-1}^{p})\\ &\pm\mathrm{diag}(\overbrace{\tau,\dots,\tau}^{p},\overbrace{-\tau,\dots,-\tau}^{q})\quad (q \text{ odd}). \end{aligned} \tag{13.8}$$

In this case there are two strong real forms mapping to the real form $SU(p,q)$ if $p\ne q$, and 1 if $p=q$.

The calculations needed to understand representation theory (see Section 10) take place entirely in a fixed set $\mathcal{X}(z)$. The sets $\mathcal{X}(z')$ and $\mathcal{X}(z'z\theta(z))$ are canonically identified, so it is safe to think of $\mathcal{X}(z)$ as being defined for $z\in Z_0$. The `atlas` software takes this approach.

14 Cayley Transforms and the Cross Action

{s:cayley}

We continue to work with the one-sided parameter space $\mathcal{X} = \mathcal{X}(G,\gamma)$. We begin with some formal constructions.

Fix $x\in\mathcal{X}$ and let $\tau = p(x)\in\mathcal{I}_W$. Recall (Section 12) τ defines the real, imaginary and complex roots, and x defines a grading gr_x of the imaginary roots. Suppose α is an imaginary noncompact root, i.e. $\tau(\alpha)=\alpha$ and $\mathrm{gr}_x(\alpha)=1$.

Let G_α be the derived group of $\mathrm{Cent}_G(\ker(\alpha))$, and $H_\alpha\subset G_\alpha$ the one-parameter subgroup corresponding to α. Then G_α is isomorphic to $SL(2)$ or $PSL(2)$ and H_α is a Cartan subgroup of G_α. Choose $\sigma_\alpha\in\mathrm{Norm}_{G_\alpha}(H_\alpha)\backslash H_\alpha$; then $\sigma_\alpha(\alpha)=-\alpha$.

{d:cayley}

Definition 14.1 *Suppose $x \in \mathcal{X}$ and α is a noncompact imaginary root. Choose a pre-image ξ of x in $\widetilde{\mathcal{X}}$, and define $c^\alpha(x)$ to be the image of $\sigma_\alpha\xi$ in $\mathcal{X}$.*

{l:cayley1}

Lemma 14.2

(1) $c^\alpha(x)$ is well defined, independent of the choices of σ_α and ξ.

(2) $c^\alpha(x)$ is G-conjugate to x, and $c^\alpha(x)^2 = x^2$.

(3) $p(c^\alpha(x)) = s_\alpha p(x) \in \mathcal{I}_W$.

(Here and in Lemma 14.9 keep in mind the remark after (9.7) regarding the notion of G-conjugacy.)

Proof. Choose ξ, and let $t = \alpha^\vee(i) \in H_\alpha$. Suppose $h \in H_\alpha$. We have a few elementary identities, essentially in $SL(2)$:

$$\begin{aligned} \sigma_\alpha h \sigma_\alpha^{-1} &= h^{-1}, \quad h\sigma_\alpha h^{-1} = h^2\sigma_\alpha \\ \xi h \xi^{-1} &= h \\ tgt^{-1} &= \xi g \xi^{-1} \quad (g \in G_\alpha) \\ \xi\sigma_\alpha\xi^{-1} &= \sigma_\alpha^{-1}. \end{aligned} \tag{14.3}$$

The first two lines follow from $\sigma_\alpha(\alpha^\vee) = -\alpha^\vee$ and $\theta_\xi(\alpha^\vee) = \alpha^\vee$. For the third, $\mathrm{int}(t)$ and θ_ξ agree on G_α, since they agree on H_α and the $\pm\alpha$ root spaces. The final assertion follows from the third and a calculation in $SL(2)$.

Now $\sigma_\alpha\xi$ clearly normalizes H, and

$$(\sigma_\alpha\xi)^2 = \sigma_\alpha(\xi\sigma_\alpha\xi^{-1})\xi^2 = \xi^2 \in Z(G),$$

so $\sigma_\alpha\xi \in \widetilde{\mathcal{X}}$.

Given a choice of σ_α any other choice is of the form $h^2\sigma_\alpha$ for some $h \in H_\alpha$, and

$$(h^2\sigma_\alpha)\xi = (h\sigma_\alpha h^{-1})\xi = h\sigma_\alpha(h^{-1}\xi h)h^{-1} = h(\sigma_\alpha\xi)h^{-1}. \tag{14.4}$$

Therefore the image of $\sigma_\alpha\xi$ in $\mathcal{X}$ is independent of the choice of σ_α.

We need to show that $\sigma_\alpha\xi$ and $\sigma_\alpha h\xi h^{-1}$ have the same image in $\mathcal{X}$ for all $h \in H$. Write $H = H_\alpha(\ker(\alpha))$. If $h \in H_\alpha$ then $h\xi h^{-1} = \xi$ so the assertion is obvious. If $h \in \ker(\alpha)$ then $\sigma_\alpha h = h\sigma_\alpha$, and $\sigma_\alpha(h\xi h^{-1}) = h(\sigma_\alpha\xi)h^{-1}$.

For the second assertion, we actually show $c^\alpha(x)$ is conjugate to x by an element of G_α. By a calculation in $SL(2)$ it is easy to see $g(\sigma_\alpha t)g^{-1} = t$ for some $g \in G_\alpha$. Therefore

$$g(\sigma_\alpha\xi)g^{-1} = g(\sigma_\alpha tt^{-1}\xi)g^{-1} = g(\sigma_\alpha t)g^{-1}g(t^{-1}\xi)g^{-1} = tt^{-1}\xi = \xi. \tag{14.5}$$

The fact that $c^\alpha(x)^2 = x^2$ follows immediately, and the final assertion is obvious. □

We now define inverse Cayley transforms. Suppose $\xi \in \widetilde{\mathcal{X}}$, and let $\tau = \tilde{p}(\xi)$. Suppose α is a τ-real root. Define G_α and H_α as before. Let $m_\alpha = \alpha^\vee(-1)$.

{l:sigmaalpha}

Lemma 14.6 *There exists $\sigma_\alpha \in Norm_{G_\alpha}(H_\alpha)\backslash H_\alpha$ so that $\sigma_\alpha\xi = g\xi g^{-1}$ for some $g \in G_\alpha$. The only other element satisfying these conditions is $m_\alpha\sigma_\alpha$.*

Proof. This is similar to the previous case. The involution θ_ξ restricted to G_α is inner for G_α, and acts by $h \mapsto h^{-1}$ for $h \in H_\alpha$. Therefore we may choose $y \in \mathrm{Norm}_{G_\alpha}(H_\alpha)\backslash H_\alpha$ so that $ygy^{-1} = \xi g\xi^{-1}$ for all $g \in G_\alpha$. By a calculation in $SL(2)$ we may choose σ_α so that $g(\sigma_\alpha y)g^{-1} = y$ for some $g \in G_\alpha$. Then

$$g(\sigma_\alpha\xi)g^{-1} = g(\sigma_\alpha yy^{-1}\xi)g^{-1} = g(\sigma_\alpha y)g^{-1}g(y^{-1}\xi)g^{-1} = yy^{-1}\xi = \xi. \tag{14.7}$$

We have $\sigma_\alpha y \in H_\alpha$, and $\alpha(\sigma_\alpha y) = -1$. Therefore any two such choices differ by m_α. □

{d:cayley2}

Definition 14.8 *Suppose $\xi \in \widetilde{\mathcal{X}}$ and α is a real root with respect to θ_ξ. Let $c_\alpha(\xi) = \{\sigma_\alpha\xi, m_\alpha\sigma_\alpha\xi\}$.*

If $x \in \mathcal{X}$ choose $\xi \in \widetilde{\mathcal{X}}$ mapping to x, and define $c_\alpha(x)$ to be the image of $c_\alpha(\xi)$ in $\mathcal{X}$. This is a set with one or two elements.

The analogue of Lemma 14.2 is immediate:

{l:cayley2}

Lemma 14.9 *Suppose $x \in \mathcal{X}$ and α is a real root with respect to θ_x.*

(1) $c_\alpha(x)$ is well defined, independent of the choice of ξ.

(2) If $y \in c_\alpha(x)$ then y is G-conjugate to x, and $y^2 = x^2$.

(3) $p(c_\alpha(x)) = s_\alpha p(x) \in \mathcal{I}_W$.

We deduce some simple formal properties of Cayley transforms. Fix $\tau \in \mathcal{I}_W$. If α is τ-imaginary let

$$\begin{aligned}\mathcal{X}_\tau(\alpha) &= \{x \in \mathcal{X}_\tau \mid \alpha \text{ is noncompact with respect to } \theta_x\} \\ &= \{x \in \mathcal{X}_\tau \mid \mathrm{gr}_x(\alpha) = 1\} \\ &= \{x \in \mathcal{X}_\tau \mid c^\alpha(x) \text{ is defined}\}.\end{aligned} \tag{14.10}$$

{l:cayley3}

Lemma 14.11

(1) If $\tau(\alpha) = -\alpha$ then for all $x \in \mathcal{X}_\tau$, $c^\alpha(c_\alpha(x)) = x$,

(2) If $\tau(\alpha) = \alpha$ and $x \in \mathcal{X}_\tau(\alpha)$ then $c_\alpha(c^\alpha(x)) = \{x, m_\alpha x\}$.

(3) The map $c^\alpha : \mathcal{X}_\tau(\alpha) \mapsto \mathcal{X}_{s_\alpha \tau}$ is surjective, and at most two-to-one

(4) Suppose α is imaginary. The following conditions are equivalent:

(a) $c^\alpha : \mathcal{X}_\tau(\alpha) \to \mathcal{X}_{s_\alpha \tau}$ is a bijection;
(b) $c_\alpha : \mathcal{X}_{s_\alpha \tau} \to \mathcal{X}_\tau(\alpha)$ is a bijection;
(c) $c_\alpha(x)$ is single valued for all $x \in \mathcal{X}_{s_\alpha \tau}$;
(d) $m_\alpha \in A_\tau$ (cf. (11.1)*);*
(e) $s_\alpha \in W(K_\xi, H)$ for all $\xi \in \widetilde{\mathcal{X}}$ with image in $\mathcal{X}_\tau(\alpha)$,
(f) $x = m_\alpha x$ for all $x \in \mathcal{X}_\tau(\alpha)$.

If these conditions fail then $c^\alpha : \mathcal{X}_\tau(\alpha) \to \mathcal{X}_{s_\alpha \tau}$ is two to one, and $c_\alpha(x)$ is double valued for all $x \in \mathcal{X}_{s_\alpha \tau}$.

(5) Suppose α is imaginary with respect to τ. If there exists $h \in H'_{-\tau}$ (cf. (11.1)*) such that $\alpha(h) = -1$ then $\mathcal{X}_\tau$ is the disjoint union of $\mathcal{X}_\tau(\alpha)$ and $h\mathcal{X}_\tau(\alpha)$. Otherwise $\mathcal{X}_\tau(\alpha) = \mathcal{X}_\tau$.*

We leave the straightforward proof to the reader.

{r:computeX}

Remark 14.12 Using this Lemma it is straightforward to compute the space $\mathcal{X}$, starting with $\mathcal{X}_\delta$, and computing the fibers $\mathcal{X}_\tau$ inductively. This shows that it is in fact easier to understand the entire space $\mathcal{X}$ rather than the individual pieces $K_i\backslash G/B \subset \mathcal{X}$ (cf. Corollary 9.9). We describe the latter in more detail in the next section.

It is important to understand the effect of Cayley transforms on the grading of the imaginary roots. This is due to Schmid [21]; also see [26, Definition 5.2 and Lemma 10.9].

{l:grading}

Lemma 14.13 *Suppose $\tau \in \mathcal{I}_W$ and $x \in \mathcal{X}_\tau(\alpha)$. Then the $s_\alpha\tau$-imaginary roots are the τ-imaginary roots orthogonal to α, and for such a root β,*

$$gr_{c^\alpha x}(\beta) = \begin{cases} gr_x(\beta) & \text{if } \alpha+\beta \text{ is not a root} \\ gr_x(\beta)+1 & \text{if } \alpha+\beta \text{ is a root.} \end{cases} \tag{14.14}$$

{r:grading}

Remark 14.15 Choose a pre-image ξ of x in $\widetilde{\mathcal{X}}$. By (14.3) we have

$$\xi\sigma_\alpha\xi^{-1} = \begin{cases} \sigma_\alpha & \alpha \text{ compact} \\ m_\alpha\sigma_\alpha & \alpha \text{ noncompact.} \end{cases} \tag{14.16}$$

If (the derived group of) G is simply connected then $m_\alpha \neq 1$, so $\mathrm{gr}_x(\alpha) = 0$ if $\xi\sigma_\alpha\xi^{-1} = \sigma_\alpha$, and 1 otherwise. A calculation in rank 2 shows that if α and β are orthogonal then $\sigma_\alpha\sigma_\beta\sigma_\alpha^{-1} = \sigma_\beta^{\pm 1}$ depending on whether $\alpha+\beta$ is a root or not (see the next section). The Lemma follows readily from this.

Recall (9.11)(f) W acts on $\mathcal{X}$, and we refer to this as the cross action.

{l:cross}

Lemma 14.17 *Suppose α is imaginary with respect to x. Then $w\alpha$ is imaginary with respect to $w \times x$ and*

$$gr_{w\times x}(w\alpha) = gr_x(\alpha). \tag{14.18)(a}$$

Suppose $gr_x(\alpha) = 1$. Then

$$c^{w\alpha}(w \times x) = w \times c^\alpha(x). \tag{14.18)(b}$$

We leave the elementary proof, and the statement of the corresponding facts for real roots, to the reader. See [26], Lemmas 4.15 and 7.11.

With Cayley transforms in hand we can complete the proof of Proposition 12.12.

Proof of Propositions 12.12 and 12.18. Fix $\tau \in \mathcal{I}_W$. Assume there is a τ-imaginary root α. By Lemma 14.11(5) there exists $x \in \mathcal{X}_\tau(\alpha)$, so $c^\alpha(x) \in \mathcal{X}_{s_\alpha\tau}$ is defined. Now suppose β is an imaginary root with respect to $s_\alpha\tau$. By the same argument we may choose $x' \in \mathcal{X}_{s_\alpha\tau}$ so that $x'' = c_\beta(x')$ is defined.

Replacing $x \in \mathcal{X}_\tau$ with $c_\alpha(x') \in \mathcal{X}_\tau$ we now have $\mathcal{X}_\tau \ni x \to c^\beta c^\alpha x \in \mathcal{X}_{s_\beta s_\alpha \tau}$. By Lemma 14.2(2) $c^\beta c^\alpha(x)$ is G-conjugate to x.

Continue in this way until we obtain $x \in \mathcal{X}_\tau, x' \in \mathcal{X}_{\tau'}$, where x' is G-conjugate to x, and there are no imaginary roots with respect to τ'. (This corresponds to the most split Cartan subgroup of the quasisplit form of G.) By [4, Proposition 6.24] $\theta_{\xi'}$ is quasisplit, for any $\xi' \in \widetilde{\mathcal{X}}$ lying over x'.

This completes the proof of Proposition 12.12. The proof of Proposition 12.18 is similar, using inverse Cayley transforms, Proposition 12.16 and (12.17). We omit the details. □

{ex:sp4}

Example 14.19 We conclude this section with some details in the case of $G = Sp(4)$ (of type C_2). We give a picture of the space $\mathcal{X}$ and describe the action of Cayley transforms on $\mathcal{X}$.

There are four G-conjugacy classes of strong involutions, which we think of as corresponding to $Sp(2,0), Sp(0,2), Sp(1,1)$ and the split group $Sp(4,\mathbb{R})$. See Example 5.9. Write (9.10) as

$$\mathcal{X} = \mathcal{X}_{2,0} \cup \mathcal{X}_{0,2} \cup \mathcal{X}_{1,1} \cup \mathcal{X}_s. \tag{14.20}$$

There are 4 conjugacy classes in $\mathcal{I}_W$, corresponding to the 4 Cartan subgroups of $Sp(4,\mathbb{R})$, isomorphic to $S^1 \times S^1, S^1 \times \mathbb{R}^\times, \mathbb{C}^\times$ and $\mathbb{R}^\times \times \mathbb{R}^\times$.

Let α_1, α_2 be the long positive roots, and β_1, β_2 the short ones. Then $\mathcal{I}_W = \{1, s_{\alpha_1}, s_{\alpha_2}, s_{\beta_1}, s_{\beta_2}, w_0\}$ where $w_0 = -I$ is the long element.

Here is the output of the `kgb` command of the `atlas` software for the real form $Sp(4,\mathbb{R})$:

```
 0:  0  0  [n,n]    1   2     6   4
 1:  0  0  [n,n]    0   3     6   5
 2:  0  0  [c,n]    2   0     *   4
 3:  0  0  [c,n]    3   1     *   5
 4:  1  2  [C,r]    8   4     *   *  2
 5:  1  2  [C,r]    9   5     *   *  2
 6:  1  1  [r,C]    6   7     *   *  1
 7:  2  1  [n,C]    7   6    10   *  2,1,2
 8:  2  2  [C,n]    4   9     *  10  1,2,1
 9:  2  2  [C,n]    5   8     *  10  1,2,1
10:  3  3  [r,r]   10  10     *   *  1,2,1,2
```

Thus $\mathcal{X}_s$ has 11 elements, labelled by the first column. Each row corresponds to an orbit $\mathcal{O}$, which maps to a twisted involution τ. In this case (since $\gamma = 1$) we may view τ as an involution in W; the last column gives τ as a product of simple reflections. The conjugacy class of τ corresponds to a Cartan subgroup; the number of this Cartan subgroup, given by the output of the `cartan` command, is given in column 3. The length of $\mathcal{O}$, i.e. $\dim(\mathcal{O}) - \dim(\mathcal{O}_{\min})$ where $\mathcal{O}_{\min}$ is a minimal orbit, is given in column 2.

The type of each simple root (r=real, c=compact imaginary, n=noncompact imaginary, C=complex) is given in brackets. Following this the next two columns give the cross actions of the simple roots, followed by give Cayley transforms by the simple noncompact imaginary roots.

Of course $\mathcal{X}_{2,0}$ and $\mathcal{X}_{0,2}$ are singletons. Finally here is the output of `kgb` for $Sp(1,1)$.

```
0:  0  0  [n,c]   1  0    2  *
1:  0  0  [n,c]   0  1    2  *
2:  1  1  [r,C]   2  3    *  *  1
3:  2  1  [c,C]   3  2    *  *  2,1,2
```

Figure 2 gives a picture of $\mathcal{X}$ [3]. As before the vertical columns are the fibers $\mathcal{X}_\tau$. The numbering of the points of $\mathcal{X}$ is from the output of the `kgb` commands above. The arrows $\to$ indicate Cayley transforms.

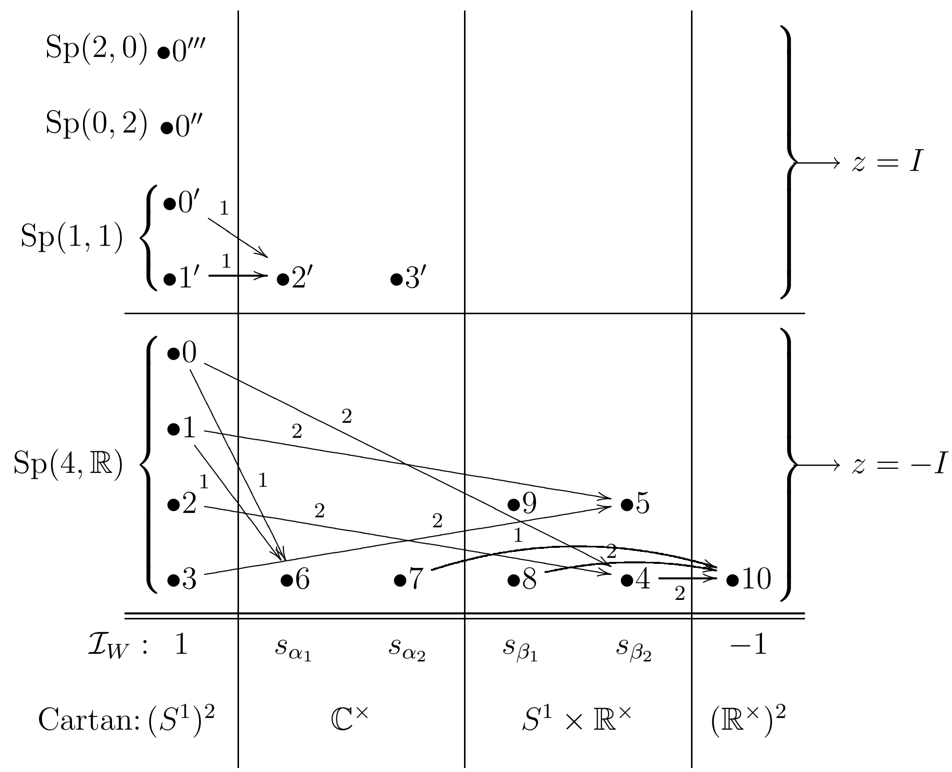

Figure 2: $\mathcal{X}$ and Cayley transforms for Sp(4)

See [2] for more detail about the representation theory of real forms of $Sp(4, \mathbb{C})$.

[3] Thanks to Leslie Saper for producing this diagram.

15 The Tits group and the algorithmic enumeration of parameters

{s:tits}

The combinatorial enumeration of $K\backslash G/B$ is in terms of the *Tits group* $\widetilde{W}$, introduced by Jacques Tits in [24] with the name *extended Coxeter group.*

We begin by fixing (G,γ) and a choice of of splitting datum $S=(H,B,\{X_\alpha\})$ (cf. Section 2). For each simple root α let $\phi_\alpha : SL(2)\rightarrow G$ be defined by $\phi_\alpha(\mathrm{diag}(z,\frac{1}{z}))=\alpha^\vee(z)$ and $d\phi_\alpha\begin{pmatrix}0&1\\0&0\end{pmatrix}=X_\alpha$. Let $\sigma_\alpha=\phi_\alpha\begin{pmatrix}0&1\\-1&0\end{pmatrix}$. This is consistent with the definition of σ_α in Section 14.

{d:tits}

Definition 15.1 *The Tits group $\widetilde{W}$ is the subgroup of N generated by $\{\sigma_\alpha\}$ for α simple.*

For each simple root α let $m_\alpha=\sigma_\alpha^2=\alpha^\vee(-1)$. Let H_0 be the subgroup of H generated by the elements m_α.

{t:tits}

Theorem 15.2 (Tits [24])
(1) The kernel of the natural map $\widetilde{W}\rightarrow W$ is H_0,
(2) The elements σ_α satisfy the braid relations,
(3) There is a canonical lifting of W to a subset of $\widetilde{W}$: take a reduced expression $w=s_{\alpha_1}\dots s_{\alpha_n}$, and let $\tilde{w}=\sigma_{\alpha_1}\dots\sigma_{\alpha_n}$.

It is a remarkable fact that the computations needed for the enumeration of the K-orbits on G/B can be carried out in the Tits group. This is due to the fact that Cayley transforms and cross actions can be described entirely in terms of the σ_α.

{l:normalizes}

Lemma 15.3 *Given $x\in\mathcal{X}$, there exists a pre-image ξ of x in $\widetilde{\mathcal{X}}$ such that ξ normalizes $\widetilde{W}$.*

Proof. Recall (following (12.17)) $\delta\in\mathcal{I}_W$ is the image of the distinguished element δ of $\mathcal{X}$. Using the fact that $p(x)=w\delta$ for some $w\in W$, it is easy to reduce to the case $x\in\mathcal{X}_\delta$. Let $\theta=\delta\in\mathrm{Aut}(H)$.

Since δ is an automorphism of the based root datum used to define $\widetilde{W}$, it follows easily that $\delta\sigma_\alpha\delta^{-1}=\sigma_{\theta(\alpha)}$ for all roots α, and therefore δ normalizes $\widetilde{W}$. Choose $\xi\in\widetilde{\mathcal{X}}$ mapping to x, so $\xi=h\delta$ for some $h\in H$; we have $h\theta(h)=\xi^2\in Z(G)$.

Recall (Remark 11.3) $H = T_\tau A_\tau$. We may replace ξ with $g\xi g^{-1} = g\theta(g^{-1})h\delta$ for any $g \in H$. The map $g \mapsto g\theta(g^{-1})$ has image A_τ, so we may assume $h \in T_\tau$.

We have $\xi\sigma_\alpha\xi^{-1} = (h\delta)\sigma_\alpha(h\delta)^{-1} = h\sigma_\beta(h^{-1})\sigma_\beta$ where $\beta = \theta(\alpha)$. Now $h\theta(h) = h^2 \in Z(G)$, so $\beta(h) = \pm 1$. It follows easily that $h\sigma_\beta(h^{-1})$ is equal to 1 or m_β, and is therefore contained in $\widetilde{W}$. □

Fix $x \in \mathcal{X}$, and choose a pre-image $\xi \in \widetilde{\mathcal{X}}$. We now describe an algorithm for computing $\mathcal{X}[x] \simeq K_\xi \backslash G/B$ (cf. (9.8)). By Proposition 12.19(2) we may assume $x \in \mathcal{X}_\delta$, and by Lemma 15.3 that ξ normalizes $\widetilde{W}$. For $\tilde{w} \in \widetilde{W}$ let $\theta(\tilde{w}) = \xi\tilde{w}\xi^{-1}$.

We will maintain an intermediate first-in-first-out list of pairs $(\tau, \tilde{w})$ where $\tau \in \mathcal{I}_W$, $\tilde{w} \in \widetilde{W}$, and $\tilde{w}\xi \in \widetilde{\mathcal{X}}_\tau$. We also maintain a store of elements $\widetilde{W}$. If $\tilde{w}$ is in the store then the image of $\tilde{w}\xi$ in $\mathcal{X}$ is contained in $\mathcal{X}[x]$; denote this element $\tilde{w}x$. Initialize the list with $(\delta, 1)$.

For each $(\tau, \tilde{w})$ occuring we will keep a record of of $\mathrm{gr}_{\tilde{w}x}$; we assume we are given gr_x. Let M_τ be the subgroup of H_0 generated by $\{m_\alpha \,|\, \tau(\alpha) = \alpha\}$. For each τ occuring we will compute $M_\tau \cap A_\tau$ (cf. (11.4)). We assume we are are given $M_\delta \cap A_\delta$.

If the list is non-empty, remove from it the first element $(\tau, \tilde{w})$.

First add $\tilde{w}$ to the store. At the first step ($\tilde{w} = 1$), record gr_x. Otherwise $\tilde{w}$ is either of the form $\sigma_\alpha\tilde{u}\theta(\sigma_\alpha)^{-1}$ or $\sigma_\alpha\tilde{u}$ for some $\tilde{u}$ already in the store (see below). Compute $\mathrm{gr}_{\tilde{w}x}$ by Lemmas 14.13 and 14.17, and record this information.

Next, compute the orbit of $\tilde{w}x$ under the cross action of $W_{i,\tau}$ as follows. Suppose $\tau(\alpha) = \alpha$ and $\mathrm{gr}_{\tilde{w}x}(\alpha) = 1$. Choose a representative $\sigma_\alpha \in \widetilde{W}$ of s_α (α is not necessarily simple). By (14.16) $\sigma_\alpha\tilde{w}\xi\sigma_\alpha^{-1} = \sigma_\alpha^2\tilde{w}\xi$. Repeating this we obtain a collection of elements of the form $\{t\tilde{w}\xi \,|\, t \in S\}$ where S is a subset of M_τ. The $W_{i,\tau}$-orbit of x is in bijection with the image of S in the group $M_\tau/M_\tau \cap A_\tau$. Choose representatives of this set.

For each such representative t let $\widetilde{u} = t\tilde{w}$ and add $\widetilde{u}$ to the store. Record $\mathrm{gr}_{\widetilde{u}x}$ by writing $\widetilde{u}x = w \times x$ for $w \in W_{i,\tau}$ and using Lemma 14.17. For each simple root α satisfying $\tau(\alpha) = \alpha$ and $\mathrm{gr}_{\tilde{u}x}(\alpha) = 1$, see if any element of the form $(s_\alpha\tau, *)$ is on the list. If not, add $(s_\alpha\tau, \sigma_\alpha\tilde{u})$ to the list. Compute the set $M_{s_\alpha\tau} \cap A_{s_\alpha\tau} = \langle M_\tau \cap A_\tau, m_\alpha\rangle$ (cf. Lemma 14.11).

Next, for each simple root α, check if $(s_\alpha\tau s_\alpha, *)$ is on the list. If not add $(s_\alpha\tau s_\alpha, \sigma_\alpha\tilde{w}\theta(\sigma_\alpha)^{-1})$ to the list and compute $M_{s_\alpha\tau s_\alpha} \cap A_{s_\alpha\tau s_\alpha} = s_\alpha \cdot M_\tau \cap A_\tau$.

Continue until the list is empty, at which point $\mathcal{X}[x]$ is the set of elements $\tilde{w}x$ for $\tilde{w}$ in the store.

Remark 15.4 Fix $x \in \mathcal{X}_\delta$ and choose ξ as in the Lemma. The argument shows that the following information suffices to compute $\mathcal{X}[x]$: $\widetilde{W}$, the involution $\theta = \mathrm{int}(\xi)$ of $\widetilde{W}$, the grading gr_x of the δ-imaginary roots, and the two-group $M_\delta \cap A_\delta \subset \widetilde{W}$.

Note that if (the derived group of) G is simply connected then the grading gr_x is determined by $\widetilde{W}$ and θ (cf. Remark 14.15).

Remark 15.5 We may also compute the entire space $\mathcal{X}$ by similar (in fact somewhat easier) methods, using Proposition 11.2 to describe the fiber $\mathcal{X}_\delta$, and Lemma 14.11 to compute Cayley transforms. See Remark 14.12. We omit the details.

Carrying out the algorithm uses computations in the Tits group, and we briefly sketch how to carry these out. According to Theorem 15.2, each element of the Tits group can be written uniquely as $\widetilde{w}h$, with $\widetilde{w}$ the canonical representative of $w \in W$ and $h \in H_0$.

Fix $w \in W$ and a simple root α, with corresponding reflection $s = s_\alpha$. We first compute $\widetilde{w}\widetilde{s}$. If $l(ws) > l(w)$, then $\widetilde{w}\widetilde{s}$ is the canonical representative of ws, and we have $\widetilde{w}\widetilde{s} = \widetilde{ws}$. Otherwise $w = vs$ for $v \in W$ with $l(v) = l(w) - 1$. In this case $\widetilde{w}\widetilde{s} = \widetilde{v}\widetilde{s}^2 = \widetilde{v}m_\alpha$. There is a similar formula for $\widetilde{s}\widetilde{w}$.

Now for $h \in H_0$ we have

$$(\widetilde{w}h)\widetilde{s} = (\widetilde{w}\widetilde{s})h^s, \quad \widetilde{s}(\widetilde{w}h) = (\widetilde{s}\widetilde{w})h \tag{15.6}$$

with $\widetilde{w}\widetilde{s}$ or $\widetilde{s}\widetilde{w}$ computed as above.

In addition, for $h_i \in H_0$ we have

$$h_1(\widetilde{w}h_2)h_3 = \widetilde{w}(h_1^w h_2 h_3). \tag{15.7}$$

Therefore multiplication in $\widetilde{W}$ can be computed from multiplication in W, multiplication in H_0, and the action of W on H_0.

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