Duality for Nonlinear Simply Laced Groups

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Abstract

We establish a character multiplicity duality for a certain natural class of nonlinear (nonalgebraic) groups arising as two-fold covers of simply laced real reductive algebraic groups. This allows us to extend part of the formalism of the Local Langlands Conjecture to such groups.

1. Introduction

One of the fundamental results in representation theory of real reductive algebraic groups is Vogan duality [20]. The purpose of this paper is to extend this duality to a certain natural class of nonalgebraic groups.

To be more precise, suppose G is the real points of a complex connected reductive algebraic group $G_{\mathbb{C}}$. In its original form, Vogan duality relates characters of representations of G with characters of representations of real forms of reductive subgroups of the complex Langlands dual group $G_{\mathbb{C}}^{\vee}$. In [4] this plays a key role is establishing a duality between the Grothendieck group of Harish-Chandra modules for G and the Grothendieck group of $G_{\mathbb{C}}^{\vee}$-equvariant perverse sheaves on a modification of the space of Langlands parameters for G. This form of the duality is especially enlightening, since it puts the local Langlands conjecture for the real group G on an equal footing with the local Langlands conjectures over other local fields, as discussed in [22].

The class of algebraic groups G described above is certainly very natural, but still rather restrictive. It doesn't include the universal cover of any split real group, for instance. On the other hand, one of the very few known constructions of automorphic forms of algebraic groups (via theta series) makes essential use of the metaplectic representation of the double cover of the symplectic group. So it is important to understand to what extent the local Langlands conjectures (which are formulated only for algebraic groups) can be extended to a broader class of nonalgebraic groups.

The purpose of the present paper is to give a unified duality theory for all nonlinear double covers of simply laced real reductive groups. Using the ideas of [4] it is then possible to extend some of the formalism of the local Langlands conjecture to such nonalgebraic groups. One key point is that the dual group $G_{\mathbb{C}}^{\vee}$ plays no role; this is consistent with [14], with $G = Sp(2n,\mathbb{R})$, $G_{\mathbb{C}}^{\vee} = Sp(2n,\mathbb{R})$.

To make these ideas more precise, we briefly describe the main result of [20]. Let $\mathcal{B}$ be a block of representations of G with regular infinitesimal character; this is a finite set of irreducible representations. (For terminology related to blocks and infinitesimal characters, see the discussions at the beginning of Section 4 and preceding Theorem 5.4 respectively.) Write $\mathcal{M}$ for the

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$\mathbb{Z}$-module spanned by the elements of $\mathcal{B}$ viewed as a submodule of the Grothendieck group. Then $\mathcal{M}$ has two distinguished bases: the irreducible representations in $\mathcal{B}$, and a corresponding basis of standard modules. We may thus consider the matrix relating these two bases. (The Kazhdan-Lusztig-Vogan algorithm of [19] provides a means to compute this matrix.) We say a block $\mathcal{B}'$ of representations is *dual* to $\mathcal{B}$ if there is a bijection between $\mathcal{B}$ and $\mathcal{B}'$ for which, roughly speaking, the two corresponding change of basis matrices for $\mathcal{B}$ and $\mathcal{B}'$ are inverse-transposes of each other. See Remark 1 for a stronger formulation in terms of certain Hecke modules.

Suppose the infinitesimal character λ of $\mathcal{B}$ is integral. According to [20, Theorem 1.15] there is a real form $G^\vee$ of $G^\vee_{\mathbb{C}}$ and a block $\mathcal{B}^\vee$ for $G^\vee$, so that $\mathcal{B}^\vee$ is dual to $\mathcal{B}$. If λ is not integral the same result holds with $G^\vee_{\mathbb{C}}$ replaced by a subgroup of $G^\vee_{\mathbb{C}}$. See Section 9 for more detail. (The case of singular infinitesimal character introduces some subtleties; see Remark 4.)

All of the ingredients entering the statement of [20, Theorem 1.15] still make sense for nonlinear groups. (The computability of the change basis matrix and Hecke module formalism in the nonlinear setting is established with Renard in [15].) Thus we seek a similar statement in the nonlinear case. Our main result is:

Theorem 1.1 cf. Theorem 9.8. *Suppose G is a real reductive linear group; see Section 2 for precise assumptions. We assume the root system of G is simply laced. Suppose $\widetilde{G}$ is an admissible two-fold cover of G (Definition 3.4). Let $\mathcal{B}$ be a block of genuine representations of $\widetilde{G}$, with regular infinitesimal character. Then there is a real reductive linear group G', an admissible two-fold cover $\widetilde{G'}$ of G', and a block of genuine representations of $\widetilde{G'}$ such that $\mathcal{B}'$ is dual to $\mathcal{B}$.*

The statement is not symmetric: when applied to a block of $\widetilde{G'}$, it may not produce a block of $\widetilde{G}$. This is similar to what happens in the linear case, at non-integral infinitesimal character.

The main simplification imposed by the simply laced hypothesis is explained in detail in Proposition 6.2. Roughly speaking, it amounts to the observation that in the double covers we consider every "root" subgroup with Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ is in fact isomorphic to the metaplectic double cover of $\mathrm{SL}(2,\mathbb{R})$. (This fails already in the metaplectic double cover of $\mathrm{Sp}(4,\mathbb{R})$.) Nonetheless, using less uniform case-by-case consideration, similar results have been obtained by other authors for groups which are not simply laced. The papers [13]–[14] give a duality theory for the metaplectic double cover of the symplectic group. In addition S. Crofts has made substantial progress on groups of type B and F_4 [7]. (The case of G_2 is partially covered by this paper, and is not difficult to work out by hand; see Example 10.8.) However, as is also true in [20], treatment of simple groups is inadequate to handle general reductive groups. In particular Theorem 9.8 cannot be reduced to the case of simple groups.

Often, but not always, the group $\widetilde{G'}$ in Theorem 9.8 is a nonlinear group. Let Δ be the root system of $G_{\mathbb{C}}$. The root system Δ' of $G'_{\mathbb{C}}$ is a subsystem of the dual root system $\Delta^\vee$, with equality if the infinitesimal character of $\mathcal{B}$ is half-integral (Definition 6.1). However this is misleading: one should think of Δ' as a root subsystem of Δ. This is possible since Δ is simply laced, so $\Delta \simeq \Delta^\vee$, and this isn't a meaningful distinction. This distinction does of course appear in the non simply-laced case. For example if $\widetilde{G}$ is the two-fold cover $\widetilde{Sp}(2n,\mathbb{R})$ of $Sp(2n,\mathbb{R})$, then (in the half-integral case) $\widetilde{G}'$ is also $\widetilde{Sp}(2n,\mathbb{R})$, rather than a cover of $SO(2n+1)$ [13]. In other words, the dual group $G^\vee_{\mathbb{C}}$ does not appear naturally in the setting nonalgebraic double covers.

It is worth noting that the genuine representation theory of a simply laced nonlinear group $\widetilde{G}$ is in many ways much simpler than that of a linear group. For example $\widetilde{G}$ has at most one genuine discrete series representation with given infinitesimal and central characters. The same result holds for principal series of a split group; this is the dual of the preceding assertion. See

Examples 2 and 3. Furthermore, while disconnectedness is a major complication in the proofs in [20], it plays essentially no role in this paper.

Finally, there is a close relationship between the duality of Theorem 1.1 and the lifting of characters developed in [6]. This is explained at the beginning of Section 9.

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2. Some notation and structure theory

A *real form* G of a complex algebraic group $G_{\mathbb{C}}$ is the fixed points of an antiholomorphic involution σ of $G_{\mathbb{C}}$.

By a *real reductive linear group* we mean a group in the category defined in [17, Section 0.1]. Thus $\mathfrak{g} = \mathrm{Lie}_{\mathbb{C}}(G)$ is a reductive Lie algebra, G is a real group in Harish-Chandra's class [9, Definition 4.29], has a faithful finite-dimensional representation, and has abelian Cartan subgroups. Examples include real forms of connected complex reductive groups, and any subgroup of finite index of such a group.

We will sometimes work in the greater generality of a *real reductive group in Harish-Chandra's class*: $\mathfrak{g}$ is reductive and G is a real group in Harish-Chandra's class. Examples include any two-fold cover of a real reductive linear group (see Section 3). A Cartan subgroup in G is defined to be the centralizer of a Cartan subalgebra, and may be nonabelian.

We will always assume we have chosen a Cartan involution θ of G and let $K = G^\theta$, a maximal compact subgroup of G. Fix a θ-stable Cartan subgroup H of G. Write $H = TA$ with $T = H \cap K$ and A connected and simply connected. We denote the Lie algebras of $G, H, K \dots$ by $\mathfrak{g}_0, \mathfrak{h}_0, \mathfrak{k}_0$, and their complexifications by $\mathfrak{g}, \mathfrak{h}, \mathfrak{k}$. The Cartan involution acts on the roots $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ of $\mathfrak{h}$ in $\mathfrak{g}$. We use r, i, cx, c, n to denote the real, imaginary, complex, compact, and noncompact roots, respectively (as in [17], for example). We say G is simply laced if all roots in each simple factor of Δ have the same length, and we declare all the roots to be long in this case. We only assume G is simply laced when necessary. We denote the center of G by $Z(G)$, and the identity component by G^0.

3. Nonlinear groups

We introduce some notation and basic results for two-fold covers. Throughout this section G is a real reductive linear group (Section 2).

Definition 3.1. We say a real Lie group $\widetilde{G}$ is a two-fold cover of G if $\widetilde{G}$ is a central extension of G by $\mathbb{Z}/2\mathbb{Z}$. Thus there is an exact sequence of Lie groups

$$1 \to A \to \widetilde{G} \to G \to 1$$

with $A \simeq \mathbb{Z}/2\mathbb{Z}$ central in $\widetilde{G}$. We say $\widetilde{G}$ is nonlinear if it is not a linear group, i.e. $\widetilde{G}$ does not admit a faithful finite-dimensional representation. A *genuine* representation of $\widetilde{G}$ is one which does not factor to G. If H is a subgroup of G, we let $\widetilde{H}$ denote the inverse image of H in G.

Suppose H is a θ-stable Cartan subgroup of G, and α is a real or imaginary root. Corresponding to α is the subalgebra $\mathfrak{m}_\alpha$ of $\mathfrak{g}$ generated by the root vectors $X_{\pm\alpha}$. The corresponding subalgebra of $\mathfrak{g}_0$ is isomorphic to $\mathfrak{su}(2)$ if α is imaginary and compact, or $\mathfrak{sl}(2,\mathbb{R})$ otherwise. Let M_α be the corresponding analytic subgroup of G. The following definition is fundamental for the study of nonlinear groups and appears in many places.

Definition 3.2. Suppose $\widetilde{G}$ is a two-fold cover of G. Fix a θ-stable Cartan subgroup H of G and a real or noncompact imaginary root $\alpha \in \Delta(\mathfrak{g},\mathfrak{h})$. We say that α is *metaplectic* if $\widetilde{M_\alpha}$ is a nonlinear group, i.e. is the nontrivial two-fold cover of $\mathrm{SL}(2,\mathbb{R})$.

See [3, Section 1] for the next result. Part (1) goes back to [16]; also see [11, Section 2.3].

Lemma 3.3. (i) *Fix a two-fold cover $\widetilde{G}$ of a real reductive linear group G and assume $\mathfrak{g}_0$ is simple. Suppose H is a Cartan subgroup of G and $\alpha \in \Delta(\mathfrak{g},\mathfrak{h})$ is a long real or long noncompact imaginary root. Then $\widetilde{G}$ is nonlinear if and only if α is metaplectic.*

(ii) *Suppose further that G is a real form of a connected, simply connected, simple complex group. Then G admits a nonlinear two-fold cover if and only if there is a θ-stable Cartan subgroup H with a long real root. The same conclusion holds with noncompact imaginary in place of real. This cover is unique up to isomorphism.*

(iii) *Suppose G is as in (2), and also assume it is simply laced. Let H be any θ-stable Cartan subgroup of G. Then G admits a nonlinear two-fold cover if and only if there is a real or noncompact imaginary root $\alpha \in \Delta(\mathfrak{g},\mathfrak{h})$.*

Definition 3.4. We say a two-fold cover $\widetilde{G}$ of a real reductive linear group G is *admissible* if for every θ-stable Cartan subgroup H every long real or long noncompact imaginary root is metaplectic.

Lemma 3.5. *Let G be a real reductive linear group, and retain the notation of Definition 3.1. Let $G_1,\dots,G_k$ denote the analytic subgroups of G corresponding to the simple factors of the Lie algebra of G. A two-fold cover $\widetilde{G}$ of G is admissible if and only if $\widetilde{G}_i$ is nonlinear for each i such that G_i admits a nonlinear cover.*

This is immediate from Lemma 3.3(1) and the definitions. Roughly speaking, the result says admissible covers are as nonlinear as possible.

Proposition 3.6. *Assume G is a real form of a connected, simply connected, semisimple complex algebraic group $G_{\mathbb{C}}$. Then G admits an admissible two-fold cover, which is unique up to isomorphism.*

Proof. It is well-known (for example see [12, Theorem 2.6(2)]) that $G_{\mathbb{C}} \simeq \prod_{i=1}^n G_{i,\mathbb{C}}$ (direct product) with each $G_{i,\mathbb{C}}$ simple, and $G \simeq \prod_{i=1}^n G_i$ accordingly. Let $\overline{G} = \prod \widetilde{G}_i$ where $\widetilde{G}_i$ is the unique nontrivial two-fold cover of G_i, if it exists, or the trivial cover otherwise (cf. Lemma 3.3(2).) Then $\overline{G}$ has a natural quotient which is an admissible cover of G.

Conversely if $\widetilde{G}$ is an admissible cover of G then $\widetilde{G} = \prod \widetilde{G_i}$ (the product taken in $\widetilde{G}$; the terms $\widetilde{G}_i$ necessarily commute). Then $\widetilde{G}$ is a quotient of the group $\overline{G}$ constructed above, and is isomorphic to the cover constructed in the previous paragraph. □

Example 1.

(i) Every two-fold cover of an abelian Lie group is admissible.

(ii) The group $GL(n,\mathbb{R})$ ($n \geqslant 2$) has two admissible covers, up to isomorphism, each with the same nontrivial restriction to $SL(n,\mathbb{R})$. These correspond to the two two-fold covers of $O(n)$, sometimes denoted $Pin^{\pm}$. The $\sqrt{\det}$ cover of $GL(n)$ is not admissible.

(iii) The group $U(p,q)$ $(pq>0)$ has three inequivalent admissible covers, corresponding to the three nontrivial two-fold covers of $K=U(p)\times U(q)$, all having the same nontrivial restriction to $SU(p,q)$.

(iv) Suppose each analytic subgroup of G corresponding to a simple factor of $\mathfrak{g}_0$ admits no nonlinear cover. (For example, suppose each such analytic subgroup is compact, complex, or $\mathrm{Spin}(n,1)$ for $n\geqslant 4$.) Then every two-fold cover of G is admissible, and nonlinear.

4. Definition of duality

We recall some definitions and notation from [17] and [20]. In this section we let G be any real reductive group in Harish-Chandra's class (cf. Section 2). We only consider representations with regular infinitesimal character.

We define block equivalence to be the equivalence relation on irreducible modules generated by $\eta\sim\eta'$ if η and η' are composition factors of a common standard module of the form $\pi(\gamma)$ (with notation as in Theorem 5.4 below). See [17, Proposition 9.2.10]. It is easy to see that all elements in a block have the same infinitesimal character, which we refer to as the infinitesimal character of the block, and therefore each block is a finite set. If η is an irreducible representation with regular infinitesimal character we write $\mathcal{B}(\eta)$ for the block containing η.

Fix a block $\mathcal{B}$, with regular infinitesimal character. Theorem 5.4 below provides a parameter set $\overline{\mathcal{B}}$ and for each $\gamma\in\overline{\mathcal{B}}$ a standard representation $\pi(\gamma)$, with a unique irreducible quotient $\overline{\pi}(\gamma)$. The map from $\overline{\mathcal{B}}$ to $\mathcal{B}$ given by $\gamma\to\overline{\pi}(\gamma)$ is a bijection.

Given a Harish-Chandra module X, we let $[X]$ denote its image in the Grothendieck group of all Harish-Chandra modules. Consider the $\mathbb{Z}$-module $\mathbb{Z}[\mathcal{B}]$ with basis $\{[\pi]\in\mathcal{B}\}=\{[\overline{\pi}(\gamma)]\,|\,\gamma\in\overline{\mathcal{B}}\}$, viewed as a subgroup of the Grothendieck group. Then $\mathbb{Z}[\mathcal{B}]$ is also spanned by the standard modules $\{[\pi(\gamma)]\,|\,\gamma\in\overline{\mathcal{B}}\}$. We may thus consider the change of basis matrices in the Grothendieck group:

$$(4.1)\qquad \begin{aligned}[\pi(\delta)]&=\sum_{\gamma\in\overline{\mathcal{B}}}m(\gamma,\delta)[\overline{\pi}(\gamma)];\\ [\overline{\pi}(\delta)]&=\sum_{\gamma\in\overline{\mathcal{B}}}M(\gamma,\delta)[\pi(\gamma)].\end{aligned}$$

For the linear groups considered in Section 2 the above matrices are computable by Vogan's algorithm [19]. For any two-fold cover of such a group they are computable by the main results of [15, Part I].

Definition 4.1. Suppose G and G' are real reductive groups in Harish-Chandra's class, with blocks $\mathcal{B}$ and $\mathcal{B}'$ having regular infinitesimal character. Write $\overline{\mathcal{B}}$ and $\overline{\mathcal{B}}'$ for the sets parametrizing $\mathcal{B}$ and $\mathcal{B}'$.

Suppose there is a bijection

$$\begin{aligned}\Phi\ :\ \overline{\mathcal{B}}&\longrightarrow\overline{\mathcal{B}}'\\ \delta&\longrightarrow\delta'\end{aligned}$$

and a parity function

$$\epsilon\ :\ \overline{\mathcal{B}}\times\overline{\mathcal{B}}'\longrightarrow\{\pm1\}$$

satisfying

$$\epsilon(\gamma,\delta')\epsilon(\delta,\eta')=\epsilon(\gamma,\eta'),\qquad \epsilon(\gamma,\gamma')=1.$$

Then we say that $\mathcal{B}$ is dual to $\mathcal{B}'$ (with respect to Φ) if for all $\delta,\gamma\in\overline{\mathcal{B}}$,

(4.2)(a) $$m(\delta,\gamma)=\epsilon(\delta,\gamma')M(\gamma',\delta');\ \text{or equivalently,}$$

(4.2)(b) $$M(\delta,\gamma)=\epsilon(\delta,\gamma')m(\gamma',\delta').$$

We say $\mathcal{B}'$ is *dual* to $\mathcal{B}$ if it is dual to $\mathcal{B}$ with respect to some bijection Φ.

Remark 1. Typically the duality of two blocks $\mathcal{B}$ and $\mathcal{B}'$ is deduced from a stronger kind of duality. For instance, consider the setting of [20], and fix a block $\mathcal{B}$ at infinitesimal character λ. Then the $\mathbb{Z}[q,q^{-1}]$ span $\mathcal{M}$ of the elements of $\mathcal{B}$ is naturally a module for the Hecke algebra of the integral Weyl group of λ. The paper [20] constructs a block $\mathcal{B}'$ at infinitesimal character λ' such that the integral Weyl group of λ' is isomorphic as a Coxeter group to the integral Weyl group of λ. Denote the associated Hecke algebra $\mathcal{H}$, and write the $\mathcal{H}$ modules corresponding to $\mathcal{B}$ and $\mathcal{B}'$ as $\mathcal{M}$ and $\mathcal{M}'$. Each module has a natural basis corresponding to the standard modules (Section 5) attached to block elements. The $\mathbb{Z}[q,q^{-1}]$-linear dual $\mathcal{M}^*$ of $\mathcal{M}$ also has a natural structure of an $\mathcal{H}$ module, and it is equipped with the basis dual to the standard module basis for $\mathcal{M}$. Roughly speaking, Vogan defines a bijection from this basis of $\mathcal{M}^*$ to the basis of standard modules for $\mathcal{M}'$ (up to sign), and proves that this induces an isomorphism of $\mathcal{H}$ modules. His proof of the Kazhdan-Lusztig conjecture immediately implies that $\mathcal{B}$ and $\mathcal{B}'$ are dual in the sense of Definition 4.1. In this way the character multiplicity duality statement of [20, Theorem 1.15] is deduced from the duality of $\mathcal{H}$ modules in [20, Proposition 13.12].

For the simply laced groups we consider below, an (extended) Hecke algebra formalism is provided by [15]. We will ultimately prove Theorem 1.1 by proving a duality of certain modules for an extended Hecke algebra in the proof of Theorem 8.6 below.

Remark 2. In cases in which the matrices $M(\gamma,\delta)$ are computable, the computation depends in an essential way on a length function $l:\overline{\mathcal{B}}\to\mathbb{N}$. In the setting below, the correct notion in our setting is the extended integral length of [15]. (When G is linear, this reduces to the notion of integral length defined in [17].) Then the parity function appearing in Definition 4.1 may be taken to be

$$\epsilon(\gamma,\delta)=(-1)^{l(\gamma)+l(\delta)}.$$

5. Regular characters

In this section G is a real reductive linear group (Section 2) and $\widetilde{G}$ is a two-fold cover of G. Since $\widetilde{G}$ is a central extension of G, a Cartan subgroup $\widetilde{H}$ of $\widetilde{G}$ is the inverse image of a Cartan subgroup H of G. It is often the case that $\widetilde{H}$ is not abelian. If we write H^0 for the identity component of H, note that

(5.1) $$\widetilde{H^0}\subset Z(\widetilde{H})$$

since the map $(g,h)\to ghg^{-1}h^{-1}$ is a continuous map from $\widetilde{H}\times\widetilde{H}$ to ±1. In particular $|\widetilde{H}/Z(\widetilde{H})|$ is finite.

Equivalence classes of irreducible genuine representations of $\widetilde{H}$ are parametrized by genuine characters of $Z(\widetilde{H})$ according to the next lemma.

Lemma 5.1. *Write $\Pi(Z(\widetilde{H}))$ and $\Pi(\widetilde{H})$ for equivalence classes of irreducible genuine representations of $Z(\widetilde{H})$ and $\widetilde{H}$, respectively, and let $n=|\widetilde{H}/Z(\widetilde{H})|^{\frac12}$. For every $\chi\in\Pi(Z(\widetilde{H}))$ there is a* *unique representation* $\pi = \pi(\chi) \in \Pi(\widetilde{H})$ *for which* $\pi|_{Z(\widetilde{H})}$ *is a multiple of* χ. *The map* $\chi \to \pi(\chi)$ *is a bijection between* $\Pi(Z(\widetilde{H}))$ *and* $\Pi(\widetilde{H})$. *The dimension of* $\pi(\chi)$ *is* n, *and* $Ind^{\widetilde{H}}_{Z(\widetilde{H})}(\chi) = n\pi$.

The proof is elementary; for example see [5, Proposition 2.2]. The lemma shows that an irreducible representation of $\widetilde{H}$ is determined by a character of $Z(\widetilde{H})$. The fact that this is smaller than $\widetilde{H}$ makes the representation theory of $\widetilde{G}$ in many ways simpler than that of G. In fact for an admissible cover of a simply laced group we have the following important result.

Proposition 5.2 [6], Proposition 4.7. *Suppose* G *is simply laced,* $\widetilde{G}$ *is an admissible two-fold cover of* G, *and* H *is a Cartan subgroup of* G. *Then*

$$Z(\widetilde{H}) = Z(\widetilde{G})\widetilde{H^0}. \tag{5.2}$$

In particular a genuine character of $Z(\widetilde{H})$ *is determined by its restriction to* $Z(\widetilde{G})$ *and its differential.*

Definition 5.3 [21, Definition 2.2]. Suppose G is a real reductive group in Harish-Chandra's class. A regular character of G is a triple

$$\gamma = (H, \Gamma, \lambda) \tag{5.3}$$

consisting of a θ-stable Cartan subgroup H, an irreducible representation Γ of H, and $\lambda \in \mathfrak{h}^*$, satisfying the following conditions. Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$, let Δ^i be the imaginary roots, and $\Delta^{i,c}$ the imaginary compact roots. The first condition is

$$\langle \lambda, \alpha^\vee \rangle \in \mathbb{R}^\times \text{ for all } \alpha \in \Delta^i. \tag{5.4(a)}$$

Let

$$\rho_i(\lambda) = \frac{1}{2} \sum_{\substack{\alpha \in \Delta^i \\ \langle \lambda, \alpha^\vee \rangle > 0}} \alpha, \qquad \rho_{i,c}(\lambda) = \frac{1}{2} \sum_{\substack{\alpha \in \Delta^{i,c} \\ \langle \lambda, \alpha^\vee \rangle > 0}} \alpha. \tag{5.4(b)}$$

The second condition is

$$d\Gamma = \lambda + \rho_i(\lambda) - 2\rho_{i,c}(\lambda). \tag{5.4(c)}$$

Finally we assume

$$\langle \lambda, \alpha^\vee \rangle \neq 0 \text{ for all } \alpha \in \Delta. \tag{5.5}$$

We say γ is *genuine* if Γ is genuine.

Of course if G is linear then H is abelian, and Γ is a character of H. If G is nonlinear (and γ is genuine) by Lemma 5.1 we could replace the irreducible representation Γ of H with a character of $Z(H)$.

Write $H = TA$ as usual, and let M be the centralizer of A in G. The conditions on γ imply that there is a unique relative discrete series representation of M, denoted π_M, with Harish-Chandra parameter λ, whose lowest $M \cap K$-type has Γ as a highest weight. Equivalently π_M is determined by λ and its central character, which is Γ restricted to the center of M. Define a parabolic subgroup MN by requiring that the real part of λ restricted to $\mathfrak{a}_\circ$ be (strictly) positive on the roots of $\mathfrak{a}$ in $\mathfrak{n}$. Then set

$$\pi(\gamma) = \mathrm{Ind}_{MN}^G(\pi_M \otimes \mathbb{1}).$$

By the choice of N and (5.5), $\pi(\gamma)$ has a unique irreducible quotient which we denote $\overline{\pi}(\gamma)$. The central character of $\pi(\gamma)$ is Γ restricted to $Z(G)$, which we refer to as the central character of γ.

The maximal compact subgroup K acts in the obvious way on the set of regular characters, and the equivalence class of $\overline{\pi}(\gamma)$ only depends on the K orbit of γ. Write $cl(\gamma)$ for the K-orbit of γ, and $\gamma \sim \gamma'$ if $cl(\gamma) = cl(\gamma')$.

Fix a maximal ideal I in the center of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. We call I an infinitesimal character for $\mathfrak{g}$, and as usual say that a $U(\mathfrak{g})$ module has infinitesimal character I if it is annihilated by I. For $\mathfrak{h}$ a Cartan subalgebra and $\lambda \in \mathfrak{h}^*$ let I_λ be the infinitesimal character for $\mathfrak{g}$ determined by λ via the Harish-Chandra homomorphism. We say I_λ is regular if λ is regular, i.e. $\langle \lambda, \alpha^\vee \rangle \neq$ for all roots α. If $\gamma = (H, \Gamma, \lambda)$ is a regular character then $\pi(\gamma)$ has infinitesimal character I_λ.

Fix a regular infinitesimal character I. Let $\mathcal{S}_I$ be the set of K-orbits of regular characters $\gamma = (H, \Gamma, \lambda)$ with $I_\lambda = I$. In this setting the Langlands classification takes the following form. See [21, Section 2], for example.

Theorem 5.4. *Suppose G is a real reductive group in Harish-Chandra's class. Fix a regular infinitesimal character I for $\mathfrak{g}$. The map $\gamma \to \overline{\pi}(\gamma)$ is a bijection from $\mathcal{S}_I$ to the set of equivalence classes of irreducible admissible representations of G with infinitesimal character I.*

Proposition 5.2 implies the following important rigidity result for genuine regular characters of admissible two-fold covers.

Proposition 5.5. *Let $\widetilde{G}$ be an admissible two-fold cover of a simply laced real reductive linear group. Suppose $\gamma = (\widetilde{H}, \Gamma, \lambda)$ is a genuine regular character of $\widetilde{G}$. Then γ is determined by λ and the restriction of Γ to $Z(\widetilde{G})$.*

This follows immediately from Proposition 5.2 since the differential of Γ is determined by λ by (5.4)(c). Consequently $\widetilde{G}$ typically has few genuine irreducible representations.

Example 2. Suppose $\widetilde{G}$ is split and fix a genuine central character and an infinitesimal character for $\widetilde{G}$. Then there is precisely one minimal principal series representation with the given infinitesimal and central characters. This is very different from a linear group; for example if G is linear and semisimple of rank n it has $2^n/|Z(G)|$ minimal principal series representation with given infinitesimal and central character.

Example 3. This example is dual to the preceding one. Suppose $\widetilde{G}$ is an admissible cover of a real form G of a simply connected, semisimple complex group. Then $\widetilde{G}$ has at most one discrete series representation with given infinitesimal and central character. Again this is very different from the linear case.

Definition 5.6. We say two genuine regular characters γ and δ are block equivalent if $\overline{\pi}(\gamma)$ and $\overline{\pi}(\delta)$ are block equivalent as in Section 4. A block of regular characters is an equivalence class for this relation.

We write $\mathcal{B}$ for a block of regular characters. Clearly K acts on $\mathcal{B}$, and we set $\overline{\mathcal{B}} = \mathcal{B}/K$. If γ is a regular character write $\mathcal{B}(\gamma)$ for the block of regular characters containing γ, and $\overline{\mathcal{B}}(\gamma) = \mathcal{B}(\gamma)/K$. Fix γ_0, and let $\overline{\mathcal{B}} = \overline{\mathcal{B}}(\gamma_0)$ and $\mathcal{B}(\overline{\pi}(\gamma_0))$ be the corresponding blocks of regular characters and representations, respectively. Then the map $\gamma \to \overline{\pi}(\gamma)$ factors to $\overline{\mathcal{B}}$, and the map

$$\overline{\mathcal{B}} \ni \gamma \to \overline{\pi}(\gamma) \in \mathcal{B} \tag{5.6}$$

is a bijection.

To conclude, we note that Proposition 5.5 has a geometric interpretation. Revert for a moment to the general setting of G in Harish-Chandra's class. Let $\mathfrak{B}$ be the variety of Borel subalgebras of $\mathfrak{g}$. Given a regular character $\gamma = (H, \Gamma, \lambda)$ of G, let $\mathfrak{b}$ be the Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$ making λ dominant. This defines a map from K orbits of regular characters for G to $K_{\mathbb{C}}$ orbits on $\mathfrak{B}$ (see [19, Proposition 2.2(a)–(c)]). Using Theorem 5.4, we may interpret this as a map, which we denote *supp*, from irreducible representations of G (with regular infinitesimal character) to $K_{\mathbb{C}}$ orbits on $\mathfrak{B}$. Although we do not need it, we remark that *supp*(π) is indeed (dense in) the support of a suitable localization of π; see [19, Section 5].

Proposition 5.7. *Suppose $\widetilde{G}$ is an admissible cover of G, a simply laced linear real reductive group. Let $\mathcal{B}$ be a block of genuine representations of $\widetilde{G}$ with regular infinitesimal character. Then supp is injective, and hence $\mathcal{B} \hookrightarrow K_{\mathbb{C}} \backslash \mathfrak{B}$.*

Proof. Note that since $\widetilde{K}$ is a central extension of K, the orbits of $(\widetilde{K})_{\mathbb{C}}$ and $K_{\mathbb{C}}$ on $\mathfrak{B}$ coincide. Since any two representation in $\mathcal{B}$ have the same central and infinitesimal character, the proposition follows from Proposition 5.5. □

6. Bigradings

We state versions of some of the results of [20] in our setting. Throughout this section let G be a real reductive linear group and $\widetilde{G}$ an admissible two-fold cover of G (Section 2 and Definition 3.4).

Fix a θ-stable Cartan subgroup H of G. Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ and set

$$\Delta(\lambda) = \{\alpha \in \Delta \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\}, \tag{6.1}$$

the set of integral roots defined by λ. Define

$$m(\alpha) = \begin{cases} 2 & \alpha \text{ metaplectic} \\ 1 & \text{otherwise.} \end{cases} \tag{6.2}$$

See [15, Definition 6.5]. Define

$$\Delta_{\frac{1}{2}}(\lambda) = \{\alpha \in \Delta \mid \langle \lambda, m(\alpha)\alpha^\vee \rangle \in \mathbb{Z}\}. \tag{6.3}$$

A short argument shows that this is always a θ-stable root system. Note that if G is simply laced then $\Delta_{\frac{1}{2}}(\lambda) = \Delta(2\lambda)$. This also holds for G_2, since in this case all roots are metaplectic. Let $W_{\frac{1}{2}}(\lambda) = W(\Delta_{\frac{1}{2}}(\lambda))$.

Definition 6.1. We say λ is half-integral if $\Delta_{\frac{1}{2}}(\lambda) = \Delta$.

Let Δ^r be the set of real roots, i.e. those roots for which $\theta(\alpha) = -\alpha$. Let

$$\begin{aligned} \Delta^r(\lambda) &= \Delta^r \cap \Delta(\lambda) \\ \Delta^r_{\frac{1}{2}}(\lambda) &= \Delta^r \cap \Delta_{\frac{1}{2}}(\lambda). \end{aligned} \tag{6.4}$$

Let Δ^i be the imaginary roots, and define $\Delta^i(\lambda)$ and $\Delta^i_{\frac{1}{2}}(\lambda)$ similarly.

Now fix a genuine regular character $\gamma = (H, \Gamma, \lambda)$ (Definition 5.3). For $\alpha \in \Delta^r_{\frac{1}{2}}(\lambda)$ let $m_\alpha \in \widetilde{G}$ be defined as in [15, Section 5]. Thus m_α is an inverse image of the corresponding element of G of [17, 4.3.6]. Recalling the notion of metaplectic roots (Definition 3.2), it is well-known that

$$\alpha \text{ is metaplectic if and only if } m_\alpha \text{ has order 4.} \tag{6.5}$$

We say α satisfies the parity condition with respect to γ if

$$\text{the eigenvalues of } \Gamma(m_\alpha) \text{ are of the form } -\epsilon_\alpha e^{\pm\pi i\langle\lambda,\alpha^\vee\rangle} \tag{6.6}$$

where $\epsilon_\alpha = \pm 1$ as in [17, Definition 8.3.11]. Note that if $\alpha \in \Delta(\lambda)$ then $e^{\pm\pi i\langle\lambda,\alpha^\vee\rangle} = (-1)^{\langle\lambda,\alpha^\vee\rangle}$, and this definition agrees with [17, Definition 8.3.11].

Let

(6.7)(a) $$\Delta^{r,+}_{\frac{1}{2}}(\gamma) = \{\alpha \in \Delta^r_{\frac{1}{2}}(\lambda) \,|\, \alpha \text{ does not satisfy the parity condition}\}$$

(6.7)(b) $$\Delta^{r,-}_{\frac{1}{2}}(\gamma) = \{\alpha \in \Delta^r_{\frac{1}{2}}(\lambda) \,|\, \alpha \text{ satisfies the parity condition}\}.$$

Also define

(6.7)(c) $$\Delta^{i,+}_{\frac{1}{2}}(\gamma) = \{\alpha \in \Delta^i_{\frac{1}{2}}(\lambda) \,|\, \alpha \text{ is compact}\}$$

(6.7)(d) $$\Delta^{i,-}_{\frac{1}{2}}(\gamma) = \{\alpha \in \Delta^i_{\frac{1}{2}}(\lambda) \,|\, \alpha \text{ is noncompact}\}.$$

Finally let

(6.7)(e) $$\begin{aligned} \Delta^{i,\pm}(\gamma) &= \Delta^{i,\pm}_{\frac{1}{2}}(\gamma) \cap \Delta(\lambda) \\ \Delta^{r,\pm}(\gamma) &= \Delta^{r,\pm}_{\frac{1}{2}}(\gamma) \cap \Delta(\lambda). \end{aligned}$$

These are the usual integral imaginary compact and noncompact roots, and the integral real roots which satisfy (or do not satisfy) the parity condition.

It is a remarkable fact that $\Delta^{i,\pm}_{\frac{1}{2}}(\gamma)$ and $\Delta^{r,\pm}_{\frac{1}{2}}(\gamma)$ only depend on θ and λ, as the next proposition shows. This is very different from the linear case.

Proposition 6.2 Corollaries 6.9 and 6.10 of [15]. *Let $\widetilde{G}$ be an admissible cover of a simply laced, real reductive linear group G. Let $\gamma = (H, \Gamma, \lambda)$ be a genuine regular character for $\widetilde{G}$. Then*

(6.8)(a) $$\Delta^{r,+}_{\frac{1}{2}}(\gamma) = \Delta^r(\lambda)$$

(6.8)(b) $$\Delta^{i,+}_{\frac{1}{2}}(\gamma) = \Delta^i(\lambda).$$

In other words if α is a real, half-integral root then it fails to satisfy the parity condition if and only if it is integral. Similarly an imaginary root (which is necessarily half-integral) is compact if and only if it is integral.

Proof. Suppose $\alpha \in \Delta_{\frac{1}{2}}(\lambda)$ is a real root. According to Definition 3.4, α is metaplectic, so by (6.5) m_α has order 4. Therefore, since Γ is genuine, $\Gamma(m_\alpha)$ has eigenvalues $\pm i$. On the other hand $-\epsilon_\alpha e^{\pi i\langle\lambda,\alpha^\vee\rangle} = \pm 1$ if $\alpha \in \Delta(\lambda)$, or $\pm i$ if $\alpha \in \Delta_{\frac{1}{2}}(\lambda) \setminus \Delta(\lambda)$. Therefore the parity condition (6.6) fails if $\alpha \in \Delta(\lambda)$, and holds otherwise. This proves (a). For (b) we may reduce to the case of a simply laced group, in which case it is an immediate consequence of the following lemma. □

Lemma 6.3. *Let $\widetilde{G}$ be an admissible cover of a simply laced, real reductive linear group G. Suppose H is a θ-stable Cartan subgroup of G, with inverse image $\widetilde{H}$, and let $\widetilde{\exp} : \mathfrak{h}_0 \to \widetilde{H}$ be the exponential map.*

(a) Suppose α is an imaginary root. Then

$$\widetilde{\exp}(2\pi i\alpha^\vee) = \begin{cases} 1 & \alpha \text{ compact} \\ -1 & \alpha \text{ noncompact.} \end{cases} \tag{6.9}$$

(b) Let α be a complex root. Then

$$\widetilde{\exp}(2\pi i(\alpha^\vee+\theta\alpha^\vee))=\begin{cases}1 & \langle\alpha,\theta\alpha^\vee\rangle=0\\ -1 & \langle\alpha,\theta\alpha^\vee\rangle\neq 0.\end{cases} \tag{6.10}$$

Proof. First assume α is imaginary. Recall the group M_α introduced just before Definition 3.2. The proposition reduces to the case of $G=M_\alpha$, which is locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SU}(2)$. If α is compact then M_α and the identity component of $\widetilde{M_\alpha}$ are either isomorphic to $\mathrm{SU}(2)$ or $\mathrm{SO}(3)$. The fact that $\widetilde{\exp}(2\pi i\alpha^\vee)=1$ reduces to the case of a linear group.

If α is noncompact then M_α is locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$ and by Lemma 3.3(1) and the admissibility hypothesis $\widetilde{M_\alpha}$ is the nonlinear cover $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. It is well-known that $\widetilde{\exp}(\pi i\alpha^\vee)$ has order 4 (cf. (6.5)), so $\widetilde{\exp}(2\pi i\alpha^\vee)=-1$.

Now suppose α is complex. If $\langle\alpha,\theta(\alpha^\vee)\rangle=-1$ then $\beta=\alpha+\theta(\alpha)$ is a noncompact imaginary root, $\beta^\vee=\alpha^\vee+\theta(\alpha^\vee)$, and we are reduced to the previous case.

If $\langle\alpha,\theta(\alpha^\vee)\rangle=1$, then $\beta=\alpha-\theta(\alpha)$ is a real root. By taking a Cayley transform by β we obtain a new Cartan subgroup with an imaginary noncompact root, and again reduce to the previous case (see Section 7.3).

Finally suppose $\langle\alpha,\theta(\alpha^\vee)\rangle=0$. Let $\mathfrak{l}_\alpha$ be the subalgebra of $\mathfrak{g}$ generated by root vectors $X_{\pm\alpha},X_{\pm\theta(\alpha)}$. Let L_α be the subgroup of G with Lie algebra $\mathfrak{l}_\alpha\cap\mathfrak{g}_0$. The assumption $\langle\alpha,\theta(\alpha^\vee)\rangle=0$ implies L_α is locally isomorphic to $\mathrm{SL}(2,\mathbb{C})$, which has no nontrivial cover. Then $\widetilde{\exp}(2\pi i(\alpha^\vee+\theta\alpha^\vee)=1$ by the corresponding fact for linear groups. □

We recall some definitions from [20, Section 3]. Suppose Δ is a root system. A *grading* of Δ is a map $\epsilon:\Delta\to\pm1$, such that $\epsilon(\alpha)=\epsilon(-\alpha)$ and $\epsilon(\alpha+\beta)=\epsilon(\alpha)\epsilon(\beta)$ whenever α,β and $\alpha+\beta\in\Delta$. A *cograding* of Δ is a map $\delta^\vee:\Delta\to\pm1$ such that the dual map $\delta:\Delta^\vee\to\pm1$ (defined by $\delta(\alpha^\vee)=\delta^\vee(\alpha))$ is a grading of the dual root system $\Delta^\vee$. We identify a grading ϵ with its kernel $\epsilon^{-1}(1)$, and similarly for a cograding.

A *bigrading* of Δ is a triple $g=(\theta,\epsilon,\delta^\vee)$ where θ is an involution of Δ, ϵ is a grading of $\Delta^i=\Delta^\theta$, and $\delta^\vee$ is a cograding of $\Delta^r=\Delta^{-\theta}$. (This is a *weak bigrading* of [20, Definition 3.22].) We write $g=(\Delta,\theta,\epsilon,\delta^\vee)$ to keep track of Δ. Alternatively we write $g=(\theta,\Delta^{i,+},\Delta^{r,+})$ where $\Delta^{i,+}\subset\Delta^i$ is the kernel of ϵ and $\Delta^{r,+}\subset\Delta^r$ is the kernel of $\delta^\vee$. The *dual bigrading* of g is the bigrading $g^\vee=(\Delta^\vee,-\theta^\vee,\delta,\epsilon^\vee)$.

It is easy to see that $\Delta^{i,+}_{\frac12}(\gamma)$ is a grading of $\Delta^i_{\frac12}(\lambda)$, and by (6.8)(a) $\Delta^{r,+}_{\frac12}(\gamma)$ is a grading of $\Delta^r_{\frac12}(\lambda)$. In the simply laced case gradings and cogradings coincide. We restrict to this case for the remainder of this section.

Definition 6.4. Assume G is simply laced. The bigrading of $\Delta_{\frac12}(\lambda)$ defined by a genuine regular character γ of $\widetilde{G}$ is

$$g_{\frac12}(\gamma)=(\Delta_{\frac12}(\gamma),\theta,\Delta^{i,+}_{\frac12}(\gamma),\Delta^{r,+}_{\frac12}(\gamma)).$$

With the obvious notation $g(\gamma)=g_{\frac12}(\gamma)\cap\Delta(\lambda)$ is a bigrading of $\Delta(\lambda)$.

By Proposition 6.2

$$\begin{aligned}g_{\frac12}(\gamma)&=(\Delta_{\frac12}(\lambda),\theta,\Delta^i(\lambda),\Delta^r(\lambda))\\ g(\gamma)&=(\Delta(\lambda),\theta,\Delta^i(\lambda),\Delta^r(\lambda))\end{aligned} \tag{6.11(a)}$$

Identifying Δ and $\Delta^\vee$, the dual bigrading to $g_{\frac{1}{2}}(\gamma)$ is

$$g^\vee_{\frac{1}{2}}(\gamma) = (\Delta_{\frac{1}{2}}(\lambda), -\theta, \Delta^r(\lambda), \Delta^i(\lambda)). \tag{6.11)(b}$$

The point of Definition 6.4 is that is contains the information necessary to prove a duality theorem for simply admissible covers of simply laced groups. To the reader familiar with [20] (where strong bigradings are needed), it is perhaps surprising that we can get away with so little. We offer a few comments explaining this.

Let G be a real reductive linear group. Suppose $\gamma = (H, \Gamma, \lambda)$ is a regular character of G (Definition 5.3). Write $H = TA$ as usual, and let M be the centralizer of A in G. Section 4 of [20] associates a strong bigrading of the integral root system $\Delta(\lambda)$ to γ. One component of the strong bigrading is the imaginary cross-stabilizer $W^i(\gamma)$ (see Section 7). This group (denoted $W^i_1(\gamma)$ in [20]) satisfies the following containment relations:

$$W(\Delta^{i,+}(\lambda)) \subset W^i(\gamma) \subset \mathrm{Norm}_{W(\Delta^i(\lambda))}(\Delta^{i,+}(\lambda)). \tag{6.12)(a}$$

The outer terms in (6.12)(a) depend only on Lie algebra data, while $W^i(\gamma)$ depends on the disconnectedness of G.

Dually, a strong bigrading also includes the real cross stabilizer $W^r(\gamma)$ satisfying

$$W(\Delta^{r,+}(\gamma)) \subset W^r(\gamma) \subset \mathrm{Norm}_{W(\Delta^r(\lambda))}(\Delta^{r,+}(\lambda)). \tag{6.12)(b}$$

Now suppose $\gamma = (H, \Gamma, \lambda)$ is a genuine regular character of an admissible cover $\widetilde{G}$ of G, with G simply laced. Then Proposition 6.2 implies that the outer two terms in (6.12)(a) and (b) are the same, so the middle terms are determined:

$$W^i(\gamma) = W(\Delta^i(\lambda)), \quad W^r(\gamma) = W(\Delta^r(\lambda)). \tag{6.13}$$

Thus the (weak) bigrading determines the strong bigrading in this setting. Furthermore, the only information contained in $g_{\frac{1}{2}}(\gamma)$ not already in $g(\gamma)$ (see (6.11)(a)) is the action of θ on the roots which are half-integral but not integral. See the proof of Proposition 7.14.

7. Cross action and Cayley transforms

We start by defining the cross action of the Weyl group on genuine regular characters. Throughout this section let G be a real reductive linear group and $\widetilde{G}$ an admissible two-fold cover of G (Section 2 and Definition 3.4). Starting in Section 7.2 we assume that G is simply laced.

7.1 The family $\mathcal{F}$

Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$, $W = W(\Delta)$ and let R be the root lattice. Fix a set Δ^+ of positive roots. As in Section 5 fix an infinitesimal character I, which determines a dominant element $\lambda_\circ \in \mathfrak{h}^*$. Recall (as in (6.1)) $\Delta(\lambda_0) = \{\alpha \,|\, \langle \lambda_0, \alpha^\vee\rangle \in \mathbb{Z}\}$, and let $W(\lambda_0) = W(\Delta(\lambda_0)) = \{w \,|\, w\lambda_0 - \lambda_0 \in R\}$.

The map $w \to w\lambda_0$ induces an isomorphism

$$W/W(\lambda_0) \simeq (W\lambda_0 + R)/R. \tag{7.1}$$

(The right-hand side is the set of orbits of R acting on $w\lambda_0 + R$ by translations, and the isomorphism respects the obvious W actions on both sides.) Let $\mathcal{F}_0$ be a set of representatives of

$(W\lambda_0 + R)/R$ consisting of dominant elements. For any $w \in W$ let λ_w be the element of $\mathcal{F}_0$ corresponding to w under the isomorphism (7.1). Thus

$$\lambda_w \text{ is } \Delta^+\text{-dominant,} \tag{7.2)(a}$$

$$\lambda_x = \lambda_y \text{ if and only if } y = xw \text{ with } w \in W(\lambda_0), \tag{7.2)(b}$$

$$\lambda_w - w\lambda_0 \in R. \tag{7.2)(c}$$

Let $\mathcal{F} = W\mathcal{F}_0$. Note that $|\mathcal{F}| = |W||W/W(\lambda_0)|$.

Definition 7.1. For $\lambda \in \mathcal{F}$ and $w \in W$ define $w \cdot \lambda$ to be the unique element of $\mathcal{F}$ satisfying: $w \cdot \lambda$ is in the same Weyl chamber as $w\lambda$, and $w \cdot \lambda - \lambda \in R$.

Lemma 7.2. *This is well-defined, and is an action of W on $\mathcal{F}$. Furthermore for any $\lambda \in \mathcal{F}$, $w \in W$, $w \cdot \lambda = w\lambda$ if and only if $w \in W(\lambda)$.*

Proof. This follows readily from an abstract argument, but perhaps it is useful to make it explicit.

Given λ choose $x, y \in W$ so that $\lambda = x\lambda_y$. We claim for any $w \in W$

$$w \cdot (x\lambda_y) = wx\lambda_{x^{-1}w^{-1}xy}. \tag{7.3}$$

Let $v = x^{-1}w^{-1}xy$. It is clear that $wx\lambda_v$ is in the same Weyl chamber as $w \cdot (x\lambda_y)$, i.e. the Weyl chamber of $wx\lambda_y$. We also have to show $wx\lambda_v - w \cdot (x\lambda_y) \in R$, which amounts to

$$wx\lambda_v - x\lambda_y \in R. \tag{7.4}$$

By (7.2)(c) write $\lambda_v = v\lambda_0 + \tau$ and $\lambda_y = y\lambda_0 + \mu$ with $\tau, \mu \in R$. We have to show

$$wx(v\lambda_0 + \tau) - x(y\lambda_0 + \mu) \in R. \tag{7.5}$$

This is clear since $wxv = xy$ and R is W-invariant.

Uniqueness is immediate, and the fact that this is an action is straightforward. The final assertion is also clear. □

7.2 Cross Action

For the remainder of section 7 we assume G is simply laced.

Fix a family $\mathcal{F}$ as in the preceding section. Now suppose H is a Cartan subgroup of $\widetilde{G}$ with complexified Lie algebra $\mathfrak{h}$, and suppose $\gamma = (H, \Gamma, \lambda)$ is a genuine regular character with $\lambda \in \mathcal{F}$. Recall (Section 5) the central character of γ is the restriction of Γ to $Z(\widetilde{G})$.

Definition 7.3. For $w \in W$ define

$$w \times \gamma = (H, \Gamma', w \cdot \lambda) \tag{7.6}$$

where $(H, \Gamma', w \cdot \lambda)$ is the unique genuine regular character of this form with the same central character as that of γ.

Lemma 7.4. *The regular character $(H, \Gamma', w\cdot\lambda)$ of Definition 7.3 exists and is unique. This defines an action of W on the set of genuine regular characters (H, Γ, λ) with $\lambda \in \mathcal{F}$. Finally γ and $w \times \gamma$ have the same infinitesimal character if and only if $w \in W(\lambda)$, in which case Definition 7.3 agrees with [17, Definition 8.3.1].*

Note that (for $w \notin W(\lambda)$) the definition of $w \times \gamma$ depends on the choice of $\mathcal{F}$.

Proof. Uniqueness is immediate from Proposition 5.5. For existence, let

$$\tau = (w \cdot \lambda - \lambda) + (\rho_i(w\lambda) - \rho_i(\lambda)) - (2\rho_{i,c}(w\lambda) - 2\rho_{i,c}(\lambda)) \tag{7.7}$$

(see Definition 5.3). By Definition 7.1, τ is in the root lattice, so we may view it as a character of H which is trivial on $Z(\widetilde{G})$. An easy calculation shows that $(H, \Gamma\otimes\tau, w\cdot\lambda)$ is a genuine regular character with the same central character as γ. This gives existence.

If $w \notin W(\lambda)$ then the infinitesimal characters of γ and $w\cdot\gamma$ are different by the final assertion of Lemma 7.2. If $w \in W(\lambda)$ then $w\times\gamma$ as defined in [17, Definition 8.3.1] is of the form $(H, *, w\lambda)$. By Lemma 7.2 $w\cdot\lambda = w\lambda$, so our definition of $w\times\gamma$ is also of this form. The final assertion of the lemma follows from Proposition 5.5 (or a direct comparison of the two definitions). □

7.3 Cayley Transforms

We continue to assume G is simply laced.

Suppose H is a θ-stable Cartan subgroup of G and α is a real root. Define the Cayley transform $H_\alpha = c_\alpha(H)$ as in [8, §11.15]. In particular $\ker(\alpha) = \mathfrak{h}_\alpha\cap\mathfrak{h}$ is of codimension one in $\mathfrak{h}$ and $\mathfrak{h}_\alpha$. We say a root β of H_α is a Cayley transform of α if $\ker(\beta) = \mathfrak{h}\cap\mathfrak{h}_\alpha$; there are two such roots $\pm\beta$, which are necessarily noncompact imaginary.

For $\lambda\in\mathfrak{h}^*$ and α,β as above define $c_{\alpha,\beta}(\lambda)\in\mathfrak{h}_\alpha^*$:

(7.8)(a) $$c_{\alpha,\beta}(\lambda)|_{\mathfrak{h}\cap\mathfrak{h}_\alpha} = \lambda|_{\mathfrak{h}\cap\mathfrak{h}_\alpha}$$

(7.8)(b) $$\langle c_{\alpha,\beta}(\lambda), \beta^\vee\rangle = \langle\lambda, \alpha^\vee\rangle.$$

If α is a noncompact imaginary root define H^α and $c^{\alpha,\beta}$ similarly. It is clear that $c^{\alpha,-\beta}(\lambda) = s_\beta c^{\alpha,\beta}(\lambda)$, $c_{\alpha,-\beta}(\lambda) = s_\beta c_{\alpha,\beta}(\lambda)$, and $c^{\beta,\alpha}c_{\alpha,\beta}(\lambda) = c_{\beta,\alpha}c^{\alpha,\beta}(\lambda) = \lambda$.

If $\widetilde{H}$ is the preimage of H in $\widetilde{G}$ then $c^\alpha(\widetilde{H})$ and $c_\alpha(\widetilde{H})$ are defined to be the inverse images of the corresponding Cartan subgroups of G.

Definition 7.5. Let $\gamma = (\widetilde{H}, \Gamma, \lambda)$ be a genuine regular character. Suppose α is a noncompact imaginary root. Let $\widetilde{H}^\alpha = c^\alpha(\widetilde{H})$ and suppose β is a Cayley transform of α. Then the Cayley transform $c^{\alpha,\beta}(\gamma)$ of γ with respect to α and β is any regular character of the form $(\widetilde{H}^\alpha, \Gamma', c^{\alpha,\beta}(\lambda))$ with the same central character as γ. (Existence and uniqueness are addressed in Proposition 7.6.) If α is a real root $c_{\alpha,\beta}(\gamma)$ is defined similarly.

Define real roots of type I and II as in [20, Definition 8.3.8].

Proposition 7.6. *In the setting of Definition 7.5, suppose α is a noncompact imaginary root and β is a Cayley transform of α. Then $c^{\alpha,\beta}(\gamma)$ exists and is unique. In this case $s_\beta\in W(G,H^\alpha)$ and $c^{\alpha,-\beta}(\gamma) = s_\beta c^{\alpha,\beta}(\gamma)$.*

Suppose α is real and β is a Cayley transform of α. If $\langle\lambda,\alpha^\vee\rangle\notin\mathbb{Z}+\frac{1}{2}$ then $c_{\alpha,\beta}(\gamma)$ does not exist.

Assume $\langle\lambda,\alpha^\vee\rangle\in\mathbb{Z}+\frac{1}{2}$. If α is type II then $c_{\alpha,\beta}(\gamma)$ exists and is unique, $s_\beta\in W(G,H_\alpha)$, and $c_{\alpha,-\beta}(\gamma) = s_\beta c_{\alpha,\beta}(\gamma)$. If α is type I, then precisely one of $c_{\alpha,\pm\beta}(\gamma)$ exists, and is unique.

Proof. If the indicated Cayley transforms exist, they are unique by Proposition 5.5. Existence is more subtle, however. As in Section 5 let Δ^i be the set of all imaginary (not necessarily integral) roots. We say an imaginary root α is *imaginary-simple* (with respect to λ) if it is simple for $\{\alpha\in\Delta^i \mid \langle\lambda,\alpha^\vee\rangle > 0\}$.

First assume α is noncompact imaginary and imaginary-simple. Then the existence of $c^{\alpha,\beta}(\gamma)$ is given by [18, Section 4]. We can be more explicit in our setting as follows. Write $c^{\alpha,\beta}(\gamma) =$

$(\widetilde{H}^\alpha, \Gamma', c^{\alpha,\beta}(\lambda))$. By Lemma 5.1, it is enough to describe the restriction of Γ' to $Z(\widetilde{H})$. Set

(7.9)(a) $$\Gamma'(g) = \Gamma(g) \quad (g \in Z(\widetilde{H}) \cap Z(\widetilde{H}^\alpha))$$

(7.9)(b) $$\langle d\Gamma', \beta^\vee\rangle = \langle \lambda, \alpha^\vee\rangle.$$

The first condition amounts to specifying Γ' on $Z(\widetilde{G})$ and $d\Gamma'|_{\mathfrak{h}\cap\mathfrak{h}^\alpha}$. It is straightforward to see that such Γ' exists; see [6, Lemma 4.32]. To show that $(\widetilde{H}^\alpha, \Gamma', c^{\alpha,\beta}(\lambda))$ is a regular character we invoke that α is imaginary-simple and apply [17, Definition 8.3.6] and [17, Lemma 8.3.7].

Now suppose α is not necessarily imaginary-simple. Choose $w \in W(\mathfrak{g}, \mathfrak{h})$ making α imaginary-simple. Then $w^{-1}\times\gamma$ is defined (Section 7.3) and is of the form $(\widetilde{H}, *, w^{-1}\cdot\lambda)$, where $*$ represents some representation we need not specify. By definition $w^{-1}\cdot\lambda$ is in the same Weyl chamber as $w^{-1}\lambda$ so α is imaginary-simple with respect to $w\lambda$. Then, letting $c = c^{\alpha,\beta}$, $c(w^{-1}\times\gamma)$ is defined, and is of the form $(\widetilde{H}^\alpha, *, c(w^{-1}\cdot\lambda))$. It is clear that $\widetilde{w}c(w^{-1}\cdot\lambda) = c(\lambda)$ for some $\widetilde{w} \in W(\mathfrak{g}, \mathfrak{h}^\alpha)$. Let $\mathcal{F}^\alpha = c\mathcal{F}$ use it to define the cross action for H^α. It is clear that $c(w^{-1}\cdot\lambda)$ and $c(\lambda)$ differ by a sum of roots, and $\widetilde{w}\cdot c(w^{-1}\cdot\lambda) = c(\lambda)$. Consider

(7.10) $$\widetilde{w} \times c(w^{-1}\times\gamma).$$

This is a regular character of the form $(\widetilde{H}, *, c(\lambda))$, and has the same central character as γ. This establishes existence in this case.

Now suppose α is real. As in the preceding case, using the cross action we may assume β is imaginary-simple. in which case Γ' is constructed in [18, Section 4], although less explicitly. In this case Γ' must satisfy

(7.11)(a) $$\Gamma'(g) = \Gamma(g) \quad (g \in Z(\widetilde{H}) \cap Z(\widetilde{H}_\alpha))$$

(7.11)(b) $$\langle d\Gamma', \beta^\vee\rangle = \langle \lambda, \alpha^\vee\rangle + \langle \rho_i(c(\lambda)) - 2\rho_{i,c}(c(\lambda)), \beta^\vee\rangle$$

with notation as in (5.4)(b), and with $c(\lambda) = c_{\alpha,\beta}(\lambda)$. See [17, Definition 8.3.14]. Let $\eta = \langle\lambda, \alpha^\vee\rangle$ and $d = \langle \rho_i(c(\lambda)) - 2\rho_{i,c}(c(\lambda), \beta^\vee\rangle \in \mathbb{Z}$.

Let $m = \widetilde{\exp}(\pi i \beta^\vee)$ where $\widetilde{\exp} : \mathfrak{h}_0 \to \widetilde{H}$ is the exponential map. (This is the element m_α of (6.5)). By Lemma 6.3 $m^2 = -1$, so $\Gamma(m) = \pm i$ since Γ is genuine. On the other hand (7.11)(b) gives

(7.11)(c) $$\Gamma'(m) = \exp(\pi i(\eta + d)).$$

Since $d \in \mathbb{Z}$ this forces $\eta \in \mathbb{Z} + \frac{1}{2}$, so assume this holds.

In our setting there is no further condition if $\widetilde{m}$ is not contained in $Z(\widetilde{H})$, i.e. if α is type II. If α is of type I then $\Gamma'(m)$ is given by both (a) and (b). Changing β to $-\beta$ does not change the right hand side of (c), but replaces m with m^{-1} and therefore $\Gamma'(m)$ with $-\Gamma'(m)$. Therefore (c) holds for precisely one choice of β or $-\beta$. □

The proposition shows that, unlike the linear case, $c^{\alpha,\beta}(\gamma)$ is never multivalued. Like the linear case, the choice of β can only affect the Cayley transform up to K-conjugacy.

Definition 7.7. Fix a genuine regular character $\gamma = (\widetilde{H}, \Gamma, \lambda)$, and suppose $\alpha \in \Delta^{i,-}_{\frac{1}{2}}(\gamma)$. Choose a Cayley transform β of α and let $c^\alpha(\gamma)$ denote $c^{\alpha,\beta}(\gamma)$. Define $c^\alpha(cl(\gamma))$ to be $cl(c^{\alpha,\beta}(\gamma))$; here, as in discussion after Definition 5.3, $cl(\gamma)$ denotes the K orbit of γ. Although there is a choice of β in the definition of $c^\alpha(\gamma)$, Proposition 7.6 shows that $c^\alpha(cl(\gamma))$ independent of this choice.

Dually, suppose $\alpha \in \Delta^{r,-}_{\frac{1}{2}}(\gamma)$. Choose a Cayley transform β of α, let $c_\alpha(\gamma)$ denote $c_{\alpha,\beta}(\gamma)$, and define $c_\alpha(cl(\gamma))$ to be $cl(c_{\alpha,\beta}(\gamma))$.

We now introduce the *abstract* Cartan subalgebra and Weyl group as in [20, 2.6]. Thus we fix once and for all a Cartan subalgebra $\mathfrak{h}_a$, a set of positive roots Δ_a^+ of $\Delta_a = \Delta(\mathfrak{g}, \mathfrak{h}_a)$, and let $W_a = W(\mathfrak{g}, \mathfrak{h}_a)$. Suppose $\mathfrak{h}$ is any Cartan subalgebra of $\mathfrak{g}$ and λ is a regular element of $\mathfrak{h}^*$. Let $\phi_\lambda : \mathfrak{h}_a^* \to \mathfrak{h}^*$ be the unique isomorphism which is inner for the complex group, such that $\phi_\lambda^{-1}(\lambda)$ is Δ_a^+-dominant. This induces isomorphisms $\Delta_a \simeq \Delta$ and $W_a \simeq W$, written $\alpha \to \alpha_\lambda$ and $w \to w_\lambda$ respectively.

Fix a regular infinitesimal character and let λ_a be the corresponding Δ_a^+-dominant element of $\mathfrak{h}_a^*$ via the Harish-Chandra homomorphism. Choose a set $\mathcal{F}_a \subset \mathfrak{h}_a^*$ as in Section 7.1.

The cross action of W_a is defined using $\mathcal{F}_a$ as follows. Suppose $\gamma = (\widetilde{H}, \Gamma, \lambda)$ is a genuine regular character and $\phi_\lambda^{-1}(\lambda) \in \mathcal{F}_a$. Then for $w \in W_a$ define

$$w \times \gamma = w_\lambda^{-1} \times \gamma. \tag{7.12}$$

(The cross action on the right hand side is defined using $\phi_\lambda(\mathcal{F}_a)$). Compare [20, Definition 4.2]. It is easy to see this is an action on regular characters. If $\gamma \sim \gamma'$ then it is easy to see that $w \times \gamma \sim w \times \gamma'$, so this induces an action on K-orbits of regular characters.

Now fix $\gamma = (\widetilde{H}, \Gamma, \lambda)$ and suppose $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}_a)$. We say α is real, imaginary, etc. for γ if this holds for α_λ. Write $\Delta^{i,\pm}(\gamma)_a$ and $\Delta^{i,\pm}(\gamma)_a$ for the subsets of Δ_a, pulled back from the corresponding subsets of Δ. Suppose $\alpha \in \Delta^{i,-}_{\frac{1}{2}}(\gamma)_a$. Define $c^\alpha(\gamma) = c^{\alpha_\lambda}(\gamma)$; recall there are two choices $c^{\alpha,\pm\beta}$ of $c^\alpha(\gamma)$, and $c^\alpha(cl(\gamma))$ is well defined. For $\alpha \in \Delta^{r,-}_{\frac{1}{2}}(\gamma)_a$ define c_α similarly. Then

$$c^\alpha(c_\alpha(\gamma)) \sim \gamma \quad (\alpha \in \Delta^{i,-}_{\frac{1}{2}}(\gamma)_a) \tag{7.13a}$$

$$c_\alpha(c^\alpha(\gamma)) \sim \gamma \quad (\alpha \in \Delta^{r,-}_{\frac{1}{2}}(\gamma)_a). \tag{7.13b}$$

To be precise these equalities hold for both choices of $c^{\alpha,\pm\beta}$ of c^α, and $c_{\alpha,\pm\beta}$ of c_α. Similar equations will arise below (for example, (7.14)) and they are to be understood in the same way.

Definition 7.8. A real admissible subspace for γ is a subspace $\mathfrak{s} \subset \mathfrak{h}_a^*$ satisfying the following conditions.

(1) $\mathfrak{s}$ is spanned by a set of real roots $\alpha_1, \dots, \alpha_n$,

(2) the iterated Cayley transform $c_{\alpha_1} \circ \cdots \circ c_{\alpha_n}(\gamma)$ is defined for some (equivalently, any) choices.

Lemma 7.9. *A subspace $\mathfrak{s}$ of $\mathfrak{h}_a^*$ is a real-admissible subspace for γ if and only if it has a basis consisting of orthogonal roots of $\Delta^{r,-}_{\frac{1}{2}}(\gamma)_a$. This basis is unique (up to ordering and signs).*

Proof. The discussion after [20, Definition 5.3] shows that any real admissible sequence must be orthogonal. It is clear that the span of a set of strongly orthogonal elements of $\Delta^{r,-}_{\frac{1}{2}}$ is admissible. In the simply laced case orthogonal is equivalent to strongly orthogonal, and the first part follows easily. It is easy to see the only roots in the span of a set of strongly orthogonal roots $\{\alpha_1, \dots, \alpha_n\}$ are $\pm\alpha_i$, and the second assertion follows from this. □

Definition 7.10. Suppose $\mathfrak{s} \subset \mathfrak{h}_a^*$ is a real-admissible subspace for γ. Let $c_{\mathfrak{s}}(\gamma)$ denote $c_{\alpha_1} \circ \cdots \circ c_{\alpha_n}(\gamma)$ where $\{\alpha_1, \dots, \alpha_n\} \subset \Delta^{r,-}_{\frac{1}{2}}(\gamma)_a$ is a strongly orthogonal basis of $\mathfrak{s}$. Define $c_{\mathfrak{s}}(cl(\gamma)) = cl(c_{\mathfrak{s}}(\gamma))$. By Proposition 7.6 and Lemma 7.9, this is well-defined independent of all choices.

Imaginary admissible subspaces and iterated imaginary Cayley transforms are defined similarly:

Definition 7.11. An imaginary admissible subspace of $\mathfrak{h}_a^*$ is one which has a basis of strongly orthogonal roots of $\Delta^{i,-}_{\frac{1}{2}}(\gamma)_a$. If $\mathfrak{s}$ is such a subspace let $c^{\mathfrak{s}}(\gamma)$ denote $c^{\alpha_1}\circ\cdots\circ c^{\alpha_n}(\gamma)$ where $\{\alpha_1,\dots,\alpha_n\}\subset\Delta^{i,-}_{\frac{1}{2}}(\gamma)_a$ is an orthogonal basis of $\mathfrak{s}$. Define $c^{\mathfrak{s}}(cl(\gamma))=cl(c^{\mathfrak{s}}(\gamma))$.

The analogue of [20, Lemma 7.11] is straightforward.

Lemma 7.12. *Let $\gamma=(\widetilde{H},\Gamma,\lambda)$ be a genuine regular character for $\widetilde{G}$, with $\phi_\lambda^{-1}(\lambda)\in\mathcal{F}_a$. Fix $w\in W_a$ and $\mathfrak{s}\subset\mathfrak{h}_a^*$. Then $\mathfrak{s}$ is an imaginary admissible subspace for γ if and only if $w\mathfrak{s}$ is an imaginary admissible subspace for $w\times\gamma$. If this holds then*

(7.14)(a) $$c^{w\mathfrak{s}}(w\times\gamma)\sim w\times c^{\mathfrak{s}}(\gamma).$$

Similarly $\mathfrak{s}$ is a real admissible subspace for γ if and only if $w\mathfrak{s}$ is a real admissible subspace for $w\times\gamma$, in which case

(7.14)(b) $$c_{w\mathfrak{s}}(w\times\gamma)\sim w\times c_{\mathfrak{s}}(\gamma).$$

Proof. The facts about admissible subspaces are elementary, and similar to the linear case. Thus the left hand side of (7.14)(a) is defined, and by Proposition 5.5 the two sides of (7.14)(a) are actually equal after making the obvious choices. Equivalently the two sides are K-conjugate for any choices. The same holds for (7.14)(b). ☐

Define the cross stabilizer of γ to be

(7.15) $$\begin{aligned}W(\gamma)&=\{w\in W(\lambda)\,|\,w\times\gamma\sim\gamma\}\\&=\{w\in W(\lambda)\cap W(G,H)\,|\,w\times\gamma=w\gamma\}.\end{aligned}$$

See [20, Definition 4.13]. The next result is quite different from what one encounters in the linear case [20, Proposition 4.14].

Lemma 7.13.

(7.16) $$W(\gamma)=W(\lambda)\cap W(G,H).$$

Proof. If $w\in W(\lambda)$ then $w\times\gamma$ is of the form $(H,*,w\lambda)$ by Lemma 7.2. If $w\in W(G,H)$ the same holds for $w\gamma$. Since $w\times\gamma$ and $w\gamma$ have the same central character Proposition 5.5 implies $w\times\gamma=w\gamma$. ☐

We need a more explicit version of the lemma, for which we use some notation from [20, Section 3]. Recall (Section 6) $\Delta(\lambda)$ is the set of integral roots. Let $\Delta^+=\{\alpha\in\Delta(\lambda)\,|\,\langle\lambda,\alpha^\vee\rangle>0\}$. Let Δ^r,Δ^i be the real and imaginary roots as usual (Section 5), and let $\Delta^i(\lambda)=\Delta^i\cap\Delta(\lambda)$, let $\Delta^r(\lambda)=\Delta^r\cap\Delta(\lambda)$ as in Section 6. Let

(7.17)(a) $$\rho_i=\frac{1}{2}\sum_{\alpha\in\Delta^+\cap\Delta^i(\lambda)}\alpha,\quad \rho_r^\vee=\frac{1}{2}\sum_{\alpha\in\Delta^+\cap\Delta^r(\lambda)}\alpha^\vee$$

and

(7.17)(b) $$\Delta^C(\lambda)=\{\alpha\in\Delta(\lambda)\,|\,\langle\alpha,\rho_r^\vee\rangle=\langle\rho_i,\alpha^\vee\rangle=0\}.$$

Let

(7.17)(c) $$W^i(\lambda)=W(\Delta^i(\lambda)),W^r(\lambda)=W(\Delta^r(\lambda)),W^C(\lambda)=W(\Delta^C(\lambda)).$$

Finally let $W^C(\lambda)^\theta$ be the fixed points of θ acting on $W^C(\lambda)$.

Proposition 7.14.

$$W(\gamma) = W^C(\lambda)^\theta \ltimes (W^i(\lambda) \times W^r(\lambda)). \tag{7.18}$$

Proof. By Lemma 7.13 $W(\gamma) = W(\lambda) \cap W(G,H)$, and as in [20, Proposition 4.14] this equals $W(\lambda)^\theta \cap W(G,H)$. Propositions 3.12 and 4.16 of [20] compute the two terms on the right hand side; the result is

$$W(\gamma) = W^C(\lambda)^\theta \ltimes ([W^i(\lambda) \cap W(M,H)] \times W^r(\lambda)), \tag{7.19}$$

where M is the centralizer in G of the split part A of $H = TA$. As in [20, Proposition 4.16(d)]

$$W(\Delta^{i,+}(\gamma)) \subset W(M,H) \subset \mathrm{Norm}_{W^i}(\Delta^{i,+}(\gamma)). \tag{7.20}$$

Intersecting with $W^i(\lambda)$ gives

$$W(\Delta^{i,+}(\gamma)) \subset W(M,H) \cap W^i(\lambda) \subset \mathrm{Norm}_{W^i(\lambda)}(\Delta^{i,+}(\gamma)). \tag{7.21}$$

By (6.8)(b), $\Delta^{i,+}(\gamma) = \Delta^i(\lambda)$, so this gives

$$W^i(\lambda) \subset W(M,H) \cap W^i(\lambda) \subset \mathrm{Norm}_{W^i(\lambda)}(\Delta^i(\lambda)). \tag{7.22}$$

The outer terms are the same, so $W(M,H) \cap W^i(\lambda) = W^i(\lambda)$. This completes the proof. □

In the setting of the abstract Weyl group the abstract cross stabilizer of γ is

$$W(\gamma)_a = \{w \in W(\lambda_a) \cap W(G,H)_a \,|\, w \times \gamma = w\gamma\}. \tag{7.23}$$

where $W(G,H)_a = \{w \in W_a \,|\, w_\lambda \in W(G,H)\}$. The obvious analogues of Lemma 7.13 and Proposition 7.14 hold.

7.4 Blocks

We now turn to blocks of regular characters (Definition 5.6) still assuming G is simply laced.

Lemma 7.15. *Blocks of genuine regular characters are closed under K-conjugacy, the cross action of $W(\lambda_a)$, and the Cayley transforms of Definitions 7.10 and 7.11.*

Proof. Closure under K-conjugacy follows from the definition (and Theorem 5.4), and the case of the cross action of $W(\lambda_a)$ and real Cayley transforms are covered by [17]. The case of noncompact imaginary Cayley transforms then follows from (7.13). □

Proposition 7.16. *Blocks of genuine regular characters with infinitesimal character λ_a are the smallest sets closed under K-conjugacy, the cross action of $W(\lambda_a)$, and the Cayley transforms of Definitions 7.10 and 7.11.*

Proof. We argue as in [17, 9.2.11]. More precisely, using the definition of blocks given in the second paragraph of Section 4 above, we are reduced to showing that if $\overline{\pi}(\gamma')$ is a composition factor of $\pi(\gamma)$, then γ and γ' may be related by the relations listed in the proposition. We proceed by induction on a modification of the usual integral length of $\gamma = (\widetilde{H}, \Gamma, \lambda)$:

$$l(\gamma) = \frac{1}{2}\left|\{\alpha \in \Delta^+_{\frac{1}{2}}(\lambda) \mid \theta(\alpha) \notin \Delta^+_{\frac{1}{2}}(\lambda)\}\right| + \dim(\mathfrak{a}_{\mathbb{R}}).$$

(Recall we are in the simply laced case so $\Delta^+_{\frac{1}{2}}(\lambda) = \Delta^+(2\lambda)$.)

Suppose there is a root α which is simple in $\Delta^+_{\frac{1}{2}}(\lambda)$, and either $\alpha \in \Delta^{r,-}_{\frac{1}{2}}(\lambda)$ or α is complex, integral, and $\theta(\alpha) < 0$. In the first case the argument of [17, 9.2.11] holds. In the second, since α is also simple for $\Delta^+(\lambda)$, the argument of [17, 9.2.11] applies directly.

So suppose $\pi(\gamma)$ is reducible, but no roots of the kind just described exist. Then (see [15, Remark 7.7]) there exists a (nonintegral) complex root α which is simple in $\Delta^+_{\frac{1}{2}}(\lambda)$ such that $\theta(\alpha) \notin \Delta^+_{\frac{1}{2}}(\lambda)$. Choose a family $\mathcal{F}$ and define the abstract cross action in W_a as in (7.12). Consider $\eta = s_{\alpha_\lambda} \times \gamma$ (where now s_{α_λ} is the reflection in the abstract Weyl group through the abstract root α_λ corresponding to α; see the discussion about Definition 7.5). Then $l(\eta) < l(\gamma)$, but η has infinitesimal character $s_{\alpha_\lambda} \cdot \lambda_a \neq \lambda_a$. The proof of [18, Corollary 4.8] shows that $\pi(\eta) = \psi(\pi(\gamma))$ for a nonintegral wall crossing functor ψ which is an equivalence of categories. Since ψ is an equivalence $\psi(\overline{\pi}(\gamma'))$ is irreducible; using Theorem 5.4, $\psi(\overline{\pi}(\gamma')) = \overline{\pi}(\eta')$ for some η'. Using the equivalence again, $\overline{\pi}(\gamma')$ is a composition factor of $\pi(\gamma)$ if and only if $\overline{\pi}(\eta')$ is a composition factor of $\pi(\eta)$. By induction, η and η' are related by a sequence of the kind described in the statement of the proposition with $W(s_{\alpha_\lambda} \cdot \lambda_a)$ in place of $W(\lambda_a)$. From Lemma 7.12, we conclude that γ and γ' are related by conjugation, Cayley transforms, and the cross action in W_a. But the last sentence of Lemma 7.4 implies that the cross action must in fact be in $W(\lambda_a)$, as claimed. □

We now introduce unions of blocks as in [15]. Fix $\lambda_a \in \mathfrak{h}_a$ and a block $\mathcal{B}$ of genuine regular characters of infinitesimal character λ_a. Choose a family $\mathcal{F} \subset \mathfrak{h}_a^*$ as above containing λ_a and use it to define the cross action as in (7.12). Recall the definition (Section 6) of $W_{\frac{1}{2}}(\lambda_a)$; since G is simply laced this equals $W(2\lambda_a)$. Set

$$\mathbf{B} = \{w \times \gamma \mid w \in W_{\frac{1}{2}}(\lambda_a),\ \gamma \in \mathcal{B}\}. \tag{7.24}$$

Then $\mathbf{B}$ is a union of the blocks discussed above, each of whose infinitesimal character is in $\mathcal{F}$.

Corollary 7.17. *The set* $\mathbf{B}$ *of* (7.24) *is the smallest set containing* $\mathcal{B}$ *which is closed under* K*-conjugacy, the cross action of* $W_{\frac{1}{2}}(\lambda_a)$*, and the Cayley transforms of Definitions 7.10 and 7.11.*

8. Duality

Throughout this section let $\widetilde{G}$ be an admissible double cover of a simply laced real reductive linear group G (Section 2 and Definition 3.4). The representation theory of $\widetilde{G}$ is sufficiently rigid to reduce the existence of a duality theory to the duality of bigradings (Definition 6.4). Theorem 8.6 makes this precise.

Definition 8.1. A genuine regular character γ (Definition 5.3) is called *weakly minimal* if $\Delta^{r,-}_{\frac{1}{2}}(\gamma)$ (cf. (6.7)(a)) is empty, i.e. there are no real, half-integral roots satisfying the parity condition. Similarly we say γ is weakly maximal if $\Delta^{i,-}_{\frac{1}{2}}(\gamma)$ is empty.

It is easy to see that every block $\mathcal{B}$ of genuine regular characters contains a weakly minimal element: if $\gamma \in \mathcal{B}$ and $\mathfrak{s}$ is a maximal real-admissible subspace for γ then $c_{\mathfrak{s}}(\gamma)$ is weakly minimal. In the same way weakly maximal elements exist. The following uniqueness statement for these elements will be important for the proof of Proposition 8.4. The analogous result in the linear case is [20, Lemma 8.10].

Fix an abstract Cartan subalgebra $\mathfrak{h}_a$ and a regular element $\lambda_a \in \mathfrak{h}_a^*$.

Proposition 8.2. *Let* $\mathcal{B}$ *be a block of genuine regular characters for* $\widetilde{G}$ *with infinitesimal character* λ_a*. Suppose* γ *and* γ' *are weakly minimal elements in* $\mathcal{B}$*. Then there is an element* $w \in W(\lambda_a)$ *such that* $\gamma' \sim w \times \gamma$*. An analogous statement holds for weakly maximal elements.*

Lemma 8.3. *Suppose γ is weakly minimal, $\mathfrak{s}\subset\mathfrak{h}_a^*$ is an imaginary admissible subspace for γ, and $\mathfrak{u}\subset\mathfrak{h}_a^*$ is a real admissible subspace for $c^{\mathfrak{s}}(\gamma)$. Then there is an imaginary admissible subspace $\mathfrak{w}$ for γ such that*

$$c_{\mathfrak{u}}(c^{\mathfrak{s}}(\gamma))\sim c^{\mathfrak{w}}(\gamma). \tag{8.1}$$

Proof. Suppose $\mathfrak{s}$ is the span of a set S of orthogonal roots of $\Delta^{i,-}_{\frac12}(\gamma)_a$. Since $\Delta^{r,-}_{\frac12}(\gamma)_a$ is empty, it follows easily that $\Delta^{r,-}_{\frac12}(c^{\mathfrak{s}}(\gamma)_a)=S\cup(-S)$. Therefore $\mathfrak{u}$ is spanned by a subset T of S; take $\mathfrak{w}$ to be the span of the complement of T in S. □

Proof of Proposition 8.2. By Proposition 7.16 we can move from γ to γ' using a sequence of cross actions and Cayley transforms. Using Lemma 7.12 repeatedly, we can thus write $w\times\gamma\sim c\gamma'$ where $w\in W(\lambda_a)$ and c is a sequence of Cayley transforms. Writing $c=c^{\mathfrak{s}_1}\circ c_{\mathfrak{s}_2}\circ\cdots\circ c^{\mathfrak{s}_n}$, and using Lemma 8.3 repeatedly, we may assume $n=1$ and $c=c^{\mathfrak{s}}$ for some imaginary admissible subspace $\mathfrak{s}$ for γ. Thus $w\times\gamma\sim c^{\mathfrak{s}}(\gamma')$, or by (7.13), $c_{\mathfrak{s}}(w\times\gamma)\sim\gamma'$. By Lemma 7.12 $w^{-1}\mathfrak{s}$ is a real-admissible subspace for γ, contradicting the assumption that γ is weakly minimal unless $\mathfrak{s}=0$ and $w\times\gamma\sim\gamma'$, as claimed. □

Recall $\overline{\mathcal{B}}$ denotes the set of K-orbits in $\mathcal{B}$, and fix a weakly minimal element γ of $\mathcal{B}$. Let $S_i(\gamma)_a$ be the set of subspaces of $\mathfrak{h}_a^*$ which are imaginary admissible subspaces for γ. If $u\in W(\gamma)_a$ (cf. 7.23) then $u\times\gamma\sim\gamma$, so by Lemma 7.12 if $\mathfrak{s}$ is contained in $S_i(\gamma)_a$ then so is $u\mathfrak{s}$. Therefore $W(\gamma)_a$ acts on on $S_i(\gamma)_a\times W(\lambda_a)$:

$$u\cdot(\mathfrak{s},w)=(u\mathfrak{s},wu^{-1})\quad(u\in W(\gamma)_a). \tag{8.2}$$

The map

$$\psi_i(\mathfrak{s},w)=cl(c^{w\mathfrak{s}}(w\times\gamma))\in\overline{\mathcal{B}} \tag{8.3}$$

is easily seen to be well-defined, and factors to $(S_i(\gamma)_a\times W(\lambda_a))/W(\gamma)_a$. This gives a well-defined map

$$\psi_i:(S_i(\gamma)_a\times W(\lambda_a))/W(\gamma)_a\to\overline{\mathcal{B}}. \tag{8.4)(a}$$

If γ is a weakly maximal element of $\mathcal{B}$, define $S_r(\gamma)_a$ similarly (as real admissible subspaces), and the analogous map

$$\psi_r:(S_r(\gamma)_a\times W(\lambda_a))/W(\gamma)_a\to\overline{\mathcal{B}}. \tag{8.4)(b}$$

Proposition 8.4. *If γ is a weakly minimal (respectively weakly maximal) element of $\mathcal{B}$, the map of* (8.4)(a) *(respectively (b)) is a bijection.*

Proof. We only consider the first case, the second is similar. Fix a weakly minimal element $\gamma\in\mathcal{B}$.

We first prove injectivity. It is clear that the cross action of W_a preserves the properties of being weakly minimal or weakly maximal (for example by Lemma 7.12). Suppose $\psi_i(\mathfrak{s},w)=\psi_i(\mathfrak{s}',w')$, so $c^{w\mathfrak{s}}(w\times\gamma)\sim c^{w'\mathfrak{s}'}(w'\times\gamma)$. By repeated applications of (7.13) this is equivalent to

$$c_{w'\mathfrak{s}'}c^{w\mathfrak{s}}(w\times\gamma)\sim w'\times\gamma. \tag{8.5}$$

By Lemma 8.3 this gives

$$c^{\mathfrak{w}}(w\times\gamma)\sim w'\times\gamma \tag{8.6}$$

for some imaginary admissible subspace $\mathfrak{w}$ for γ. Then $\mathfrak{w}$ is a real admissible subspace for $w'\times\gamma$. This contradicts the assumption that γ (and therefore $w'\times\gamma$) is weakly minimal unless $\mathfrak{w}=0$, in which case $w\mathfrak{s} = w'\mathfrak{s}'$ and $w \times \gamma \sim w' \times \gamma$. Let $u = (w')^{-1}w \in W(\lambda_a)$. Then $u \times \gamma \sim \gamma$, i.e. $u \in W(\gamma)_a$ (cf. 7.23). Then $(\mathfrak{s}', w') = (u\mathfrak{s}, wu^{-1})$. This proves injectivity.

For surjectivity, fix $\delta \in \mathcal{B}$. Let $\mathfrak{s}$ be a maximal real admissible subspace for δ, so $c_{\mathfrak{s}}(\delta)$ is weakly minimal. By Proposition 8.2, $c_{\mathfrak{s}}(\delta) \sim w \times \gamma$ for some $w \in W(\lambda_a)$. Then $\delta \sim \psi_i(w^{-1}\mathfrak{s}, w)$. □

We next obtain an analogous result for the set $\mathbf{B}$ of (7.24) by replacing $W(\lambda_a)$ with $W_{\frac{1}{2}}(\lambda_a)$ in (8.4)(a) and (b). Let $\overline{\mathbf{B}}$ denote the set of K-conjugacy classes in $\mathbf{B}$. For γ a weakly minimal element of $\mathbf{B}$ define a map

(8.7)(a) $$\psi_i : (S_i(\gamma)_a \times W_{\frac{1}{2}}(\lambda_a))/W(\gamma)_a \to \overline{\mathbf{B}}$$

taking a representative $(\mathfrak{s}, w)$ on the left-hand side to $c^{w\mathfrak{s}}(w \times \gamma)$. If γ is weakly maximal define

(8.7)(b) $$\psi_r : (S_r(\gamma)_a \times W_{\frac{1}{2}}(\lambda_a))/W(\gamma)_a \to \overline{\mathbf{B}}$$

similarly.

Proposition 8.5. *If γ is a weakly minimal (respectively weakly maximal) element of $\mathbf{B}$, the map of* (8.7)(a) *(respectively (b)) is a bijection.*

Proof. The proof is essentially the same as that of Proposition 8.4. For injectivity (see the proof of Proposition 8.4) we need that if $u \in W_{\frac{1}{2}}(\lambda_a) \cap W(G,H)_a$ satisfies $u \times \gamma = u\gamma$ then $u \in W(\gamma) = \{w \in W(\lambda_a) \cap W(G,H)_a \,|\, w \times \gamma = \gamma\}$ (cf. 7.15). This is almost obvious, except that $u \in W_{\frac{1}{2}}(\lambda_a)$, and not necessarily $W(\lambda_a)$. It suffices to show that if $u \in W_{\frac{1}{2}}(\lambda_a)\backslash W(\lambda_a)$ then $u \times \gamma \neq u\gamma$. Write $\gamma = (\widetilde{H}, \Gamma, \lambda)$. The left hand side is of the form $(\widetilde{H}, *, u \cdot \lambda)$, the right hand side is of the form $(\widetilde{H}, *, u\lambda)$. The result is now immediate from Lemma 7.2, which says that $u \cdot \lambda \neq u\lambda$.

Surjectivity also follows as in Proposition 8.4, using a version of Proposition 8.2 for $\mathbf{B}$. □

Let $\mathfrak{g}_d$ be the derived algebra of $\mathfrak{g}$, and for $\mathfrak{h}$ a Cartan subalgebra let $\mathfrak{h}_d = \mathfrak{h} \cap \mathfrak{g}_d$.

Theorem 8.6. *Recall $\widetilde{G}$ is an admissible two-fold cover of a simply laced, real reductive linear group. Let $\mathcal{B}$ be a block of genuine representations of $\widetilde{G}$ with regular infinitesimal character. Fix a weakly minimal element $\gamma = (H, \Gamma, \lambda)$ such that $\overline{\pi}(\gamma) \in \mathcal{B}$. Suppose we are given:*

(a) an admissible cover $\widetilde{G}'$ of a simply laced, real reductive linear group G';

(b) a genuine regular character $\gamma' = (H', \Gamma', \lambda')$ of $\widetilde{G}'$.

Let

$$\Delta(\lambda') = \{\alpha \in \Delta(\mathfrak{g}', \mathfrak{h}') \,|\, \langle \lambda', \alpha^\vee \rangle \in \mathbb{Z}\}$$

and

$$\Delta_{\frac{1}{2}}(\lambda') = \{\alpha \in \Delta(\mathfrak{g}', \mathfrak{h}') \,|\, \langle 2\lambda', \alpha^\vee \rangle \in \mathbb{Z}\}.$$

Let θ and θ' denote the Cartan involutions of $\mathfrak{g}$ and $\mathfrak{g}'$ respectively. We also assume we are given

(c) an isomorphism ϕ from $\mathfrak{h}'_d$ to $\mathbb{C}\langle \Delta_{\frac{1}{2}}(\lambda)^\vee \rangle \subset \mathfrak{h}$ satisfying

(8.8)(a) $$\phi(\theta'(X')) = -\theta(\phi(X')) \quad \textit{for all } X' \in \mathfrak{h}'_d,$$

(8.8)(b) $$\phi^*(\Delta(\lambda)) = \Delta(\lambda'),$$

(8.8)(c) $$\phi^*(\Delta_{\frac{1}{2}}(\lambda)) = \Delta_{\frac{1}{2}}(\lambda').$$

Let $\mathcal{B}' = \mathcal{B}(\overline{\pi}(\gamma'))$. Then $\mathcal{B}'$ is dual to $\mathcal{B}$ (Definition 4.1).

Note that the construction is not symmetric: when applied to $\mathcal{B}'$ it may not give $\mathcal{B}$, even in case of half-integral infinitesimal character.

Note that (8.8) implies that the the bigradings $g_{\frac{1}{2}}(\gamma)$ and $g_{\frac{1}{2}}(\gamma')$ of Definition 6.4 are dual. The key point is that this is enough to imply $\mathcal{B}'$ is dual to $\mathcal{B}$.

Before turning to the proof of the theorem, we define the bijection $\Phi : \mathcal{B} \to \mathcal{B}'$. By (8.8)(a) and (b), we obtain a bijection

$$\Delta^i(\lambda) \leftrightarrow \Delta^r(\lambda'). \tag{8.9}$$

By Proposition 6.2, $\Delta^{i,+}_{\frac{1}{2}}(\gamma) = \Delta^i(\lambda)$ and $\Delta^{r,+}_{\frac{1}{2}}(\gamma') = \Delta^r(\lambda')$, so there is a bijection

$$\Delta^{i,-}_{\frac{1}{2}}(\gamma) \leftrightarrow \Delta^{r,-}_{\frac{1}{2}}(\gamma'). \tag{8.10}$$

Fix an abstract Cartan subalgebra $\mathfrak{h}_a$ of $\mathfrak{g}$, and $\lambda_a \in \mathfrak{h}_a^*$ giving the infinitesimal character of $\mathcal{B}$. Choose $\mathfrak{h}'_a$ and λ'_a for $\mathfrak{g}'$ similarly. Pulling (8.10) back to $\mathfrak{h}_a$ and $\mathfrak{h}'_a$ as usual we obtain a bijection

$$\Delta^{i,-}_{\frac{1}{2}}(\gamma)_a \leftrightarrow \Delta^{r,-}_{\frac{1}{2}}(\gamma')_a. \tag{8.11}$$

By Lemma 7.9 we obtain a bijection

$$S_i(\gamma)_a \leftrightarrow S_r(\gamma')_a \tag{8.12}$$

which we will denote by $\mathfrak{s} \mapsto \mathfrak{s}'$.

Pulling back to $\mathfrak{h}_a$ and $\mathfrak{h}'_a$, (8.8)(b) gives bijections

$$\Delta(\lambda_a) \simeq \Delta(\lambda'_a), \quad W(\lambda_a) \simeq W_a(\lambda'_a), \tag{8.13}$$

which we denote

$$\alpha \mapsto \alpha', \quad w \mapsto w'. \tag{8.14}$$

It is clear from (8.8)(a) that the isomorphism $W(\lambda_a) \simeq W(\lambda'_a)$ interchanges $W^i(\lambda_a)$ and $W^r(\lambda'_a)$, $W^r(\lambda_a)$ and $W^i(\lambda'_a)$, and takes $W^C(\lambda_a)^\theta$ to $W^C(\lambda'_a)^{\theta'}$. Thus Proposition 7.14 implies $W(\gamma)_a \simeq W(\gamma')_a$. Therefore there is a natural isomorphism

$$(S_i(\gamma)_a \times W(\lambda_a))/W(\gamma)_a \leftrightarrow (S_r(\gamma)_a \times W(\lambda_a))/W_a(\gamma'). \tag{8.15}$$

The following result is an immediate consequence of this and Proposition 8.4. Write ψ_i, ψ'_r for the maps of (8.4) applied to $\overline{\mathcal{B}}$ and $\overline{\mathcal{B}}'$, respectively.

Proposition 8.7. *In the setting of Theorem 8.6, let $\mathcal{B} = \mathcal{B}(\gamma), \mathcal{B}' = \mathcal{B}(\gamma')$. Recall (Section 5) $\overline{\mathcal{B}} = \mathcal{B}/K$, and $\overline{\mathcal{B}}'$ similarly. Using bijections* (8.12) *and* (8.14), *the map*

$$\Psi \;:\; \overline{\mathcal{B}} \longrightarrow \overline{\mathcal{B}}'$$
$$\psi_i(\mathfrak{s}, w) \mapsto \psi'_r(\mathfrak{s}', w')$$

is a bijection. By Theorem 5.4 this gives a bijection $\Phi : \mathcal{B} \to \mathcal{B}'$. Furthermore Ψ commutes with the cross action and Cayley transforms in the following sense. Fix $\delta \in \mathcal{B}$.

(a) $\Psi(c^\alpha(cl(\delta))) = c_{\alpha'}(\Psi(cl(\delta)))$ *for all* $\alpha \in \Delta^{i,-}_{\frac{1}{2}}(\delta)$,

(b) $\Psi(c_\alpha(cl(\delta))) = c^{\alpha'}(\Psi(cl(\delta)))$ *for all* $\alpha \in \Delta^{r,-}_{\frac{1}{2}}(\delta)$,

(c) $\Psi(cl(w \times \delta)) = cl(w' \times \Psi(cl(\delta)))$ *for all* $w \in W_a(\lambda)$.

The proposition establishes most of the key properties needed to prove that Ψ satisfies the conditions of Definition 4.1. The Kazhdan-Lusztig algorithm of [15] does not work within a fixed block of representations, but instead with the larger set $\mathbf{B}$ of (7.24). So we need to extend the bijection of Proposition 8.7.

By (8.8)(c) $W_{\frac{1}{2}}(\lambda) \simeq W_{\frac{1}{2}}(\lambda')$. With λ_a and λ'_a as in the discussion following the theorem we also have $W_{\frac{1}{2}}(\lambda_a) \simeq W_{\frac{1}{2}}(\lambda'_a)$. Choose $\mathcal{F}$ containing λ_a as in Section 7. Use this to define the cross action of W_a, and to define $\mathbf{B}$ as in (7.24). Choose $\mathcal{F}'$ for $\mathfrak{g}'$, and define $\mathbf{B}'$, similarly. The analogue of Proposition 8.7 for $\mathbf{B}$ follows from Proposition 8.5.

Proposition 8.8. *Retain the hypotheses and notation of Theorem 8.6. With* $\overline{\mathbf{B}}$ *and* $\overline{\mathbf{B}}'$ *denoting* K*-conjugacy classes in* $\mathbf{B}$ *and* $\mathbf{B}'$ *as above, there is a bijection*

$$\Psi\ :\ \overline{\mathbf{B}} \longrightarrow \overline{\mathbf{B}}'$$

extending the bijection of Proposition 8.7. Furthermore properties (a-c) of Proposition 8.7 hold, with $W_{\frac{1}{2}}(\lambda_a)$ *in place of* $W(\lambda_a)$ *in (c).*

Proof of Theorem 8.6. We are now in a position to work within the extended Hecke algebra formalism of [15, Section 9]. Proposition 9.5 of [15] defines an algebra $\mathcal{H}$ over the ring of formal Laurent polynomials $\mathbb{Z}[q, q^{-1}]$ whose structure depends only on $\Delta(\lambda)$ and $\Delta_{\frac{1}{2}}(\lambda)$. (By definition, $\mathcal{H}$ contains the Hecke algebra of the integral Weyl group $W(\lambda)$.) The operators defined in [15, Definition 9.4], generalizing those of [20, Definition 12.4], give rise to an $\mathcal{H}$ module $\mathcal{M}$ with a $\mathbb{Z}[q, q^{-1}]$ basis $\{m_\gamma \mid \gamma \in \overline{\mathbf{B}}\}$. Since $\Delta(\lambda)$ and $\Delta_{\frac{1}{2}}(\lambda)$ identify with $\Delta(\lambda')$ and $\Delta_{\frac{1}{2}}(\lambda')$ by the hypotheses of (8.8)(b) and (8.8)(c), we also obtain an $\mathcal{H}$ module $\mathcal{M}'$ with $\mathbb{Z}[q, q^{-1}]$ basis $\{m'_{\gamma'} \mid \gamma' \in \overline{\mathbf{B}}'\}$.

We define an $\mathcal{H}$ module structure on $\mathcal{M}^*$, the $\mathbb{Z}[q, q^{-1}]$ linear dual of $\mathcal{M}$, as in [20, Definition 13.3] or [15, Equation 11.3]. (Since $\mathcal{H}$ is nonabelian, a little care is required.) The dual module $\mathcal{M}^*$ is equipped with the basis $\{\mu_\gamma\}$ dual to $\{m_\gamma\}$. Write the bijection $\Psi : \overline{\mathbf{B}} \to \overline{\mathbf{B}}'$ of Proposition 8.8 as $\gamma \mapsto \gamma'$. It thus gives a $\mathbb{Z}[q, q^{-1}]$-linear isomorphism

$$\mathcal{M}^* \longrightarrow \mathcal{M}' \tag{8.16}$$

$$\mu_\gamma \longrightarrow (-1)^{l(\gamma)} m'_{\gamma'}, \tag{8.17}$$

where $l(\gamma)$ is described in Remark 2. As in the proofs of [20, Theorem 13.13] and [15, Theorem 11.1], the existence of the duality between elements of $\mathcal{B}$ and $\mathcal{B}'$ follows from the assertion that the map of (8.16) is in fact an $\mathcal{H}$ module isomorphism. Again like the proofs of [20, Theorem 13.13] and [15, Theorem 11.1], this follows formally from the definition of the $\mathcal{H}$-module structure given in [15, Definition 9.4] and the key symmetry properties summarized in parts (a)–(c) of Proposition 8.8. (As an example of the formal calculations involved one may consult the proof of [15, Theorem 11.1].) This completes the proof. □

For use in the next section we need the concept of isomorphism of blocks.

Suppose $\widetilde{G}$ is a two-fold cover of a real reductive linear group, and $\mathcal{B}$ is a block of genuine representations of $\widetilde{G}$ with regular infinitesimal character. Let $\mathscr{B}$ be the corresponding block of genuine regular characters. Fix a family $\mathcal{F}$ as in Section 7.1, and use it to define $\mathbf{B}$ as in (7.24). As in the proof of Theorem 8.6, the set $\overline{\mathbf{B}}$ of K-orbits in $\mathbf{B}$ index a basis of a module $\mathcal{M}$ for the extended Hecke algebra $\mathcal{H}$. It is easy to see that that the structure of $\mathcal{M}$ as a based module for $\mathcal{H}$ does not depend on the choice of $\mathcal{F}$.

Now suppose $\mathcal{B}_1, \mathcal{B}_2$ are blocks of genuine representations of groups $\widetilde{G}_1, \widetilde{G}_2$ as in the preceding paragraph. Write λ_i for the infinitesimal character of $\mathcal{B}_i$, and assume we are given an isomorphism $\Delta_{\frac{1}{2}}(\lambda_1) \simeq \Delta_{\frac{1}{2}}(\lambda_2)$ taking $\Delta(\lambda_1)$ to $\Delta(\lambda_2)$, and $\mathcal{H}_1 \simeq \mathcal{H}_2$. For $i = 1, 2$ let $\mathbf{B}_i$, $\mathcal{M}_i$ and $\mathcal{H}_i$ be as above.

Definition 8.9. We say $\mathcal{B}_1$ is *isomorphic* to $\mathcal{B}_2$ if there is a bijection $\overline{\mathbf{B}}_1 \to \overline{\mathbf{B}}_2$ which induces an isomorphism $\mathcal{M}_1 \simeq \mathcal{M}_2$ as $\mathcal{H}_1 \simeq \mathcal{H}_2$ modules.

As in the discussion after (8.16) an isomorphism of blocks preserves multiplicity matrices:

Lemma 8.10. *Suppose $\mathcal{B}_1 \simeq \mathcal{B}_2$. Writing the isomorphism $\gamma_1 \to \gamma_2$, for all $\delta_1, \gamma_1 \in \mathcal{B}_1$ we have:*

$$\begin{aligned} m(\delta_1, \gamma_1) &= m(\delta_2, \gamma_2), \\ M(\delta_1, \gamma_1) &= M(\delta_2, \gamma_2). \end{aligned} \tag{8.18}$$

As in the proof of Theorem 8.6, rigidity of genuine representations gives the following result.

Proposition 8.11. *In the setting of Definition 8.9, suppose $\gamma_i \in \mathcal{B}_i$ are weakly minimal, and $g_{\frac{1}{2}}(\gamma_1) \simeq g_{\frac{1}{2}}(\gamma_2)$ (Definition 6.4). Then $\mathcal{B}_1$ is isomorphic to $\mathcal{B}_2$ in the sense of Definition 8.9.*

Sketch. Since $g_{\frac{1}{2}}(\gamma_1) \simeq g_{\frac{1}{2}}(\gamma_2)$, Proposition 8.5 parametrizes $\overline{\mathbf{B}}_1$ and $\overline{\mathbf{B}}_2$ in terms of the same set, and gives a bijection between them. To conclude $\mathcal{B}_1$ and $\mathcal{B}_2$ are isomorphic, we need to verify that the induced map $\mathcal{M}_1 \to \mathcal{M}_2$ is an isomorphism of $\mathcal{H}_1 \simeq \mathcal{H}_2$ modules. This follows from the formulas of [15, Proposition 9.4] defining the $\mathcal{H}_i$ module structure. □

9. Definition of the dual regular character

Theorem 8.6 reduces the duality for a block $\mathcal{B}$ to the existence of a single regular character with prescribed properties. The point of this section is to construct that character. This may be done directly, but instead we choose to use results from [6]. There is a good reason for doing so: as discussed in the introduction there is a close connection between the results of the current paper and those of [6]. We begin by describing this relationship.

Suppose for the moment that G is the real points of a connected complex group $G_{\mathbb{C}}$, and $\mathcal{B}$ is a block of irreducible representations of G with regular infinitesimal character. As in [20] let $\mathcal{B}^\vee$ be a dual block for $G^\vee$, a real form of $G^\vee_{\mathbb{C}}$. Write $\pi \to \pi^\vee$ for the duality map on the level of irreducible representations. Define an equivalence relation on elements of $\mathcal{B}$ by $\pi \sim \eta$ if $supp(\pi^\vee) = supp(\eta^\vee)$, where $supp$ is discussed before Proposition 5.7. Equivalence classes for this relation are classical L-packets for G [20, Section 15]. From this point of view, they are parametrized by a subset of the orbits of the complexification, say $U_{\mathbb{C}}$, of the maximal compact subgroup of $G^\vee$ on the flag variety $X^\vee$ for $\mathfrak{g}^\vee$.

Now fix a block of genuine representations $\widetilde{\mathcal{B}}$ for a nonlinear cover $\widetilde{H}$ of a linear group H. Suppose, for example in the setting of Theorem 1.1, there is group H', with (typically nonlinear) cover $\widetilde{H}'$, and a block of genuine representations of $\widetilde{H}'$ which is dual to $\widetilde{\mathcal{B}}$. Let $\widetilde{K}'$ be a maximal compact subgroup of $\widetilde{H}'$, with complexification $\widetilde{K}'_{\mathbb{C}}$. Then we may define an equivalence relation on $\widetilde{\mathcal{B}}$ in the same way as in the previous paragraph, to partition $\widetilde{\mathcal{B}}$ into subsets, analogous to L-packets for a linear group. As before, these are parametrized by the orbits of $\widetilde{K}'_{\mathbb{C}}$ on the flag variety X' for $\mathfrak{h}'$. (In spite of this analogy we do not refer to these subset as L-packets for $\widetilde{H}'$. See [6, Section 19].) Note that Proposition 5.7 implies that these subsets are singletons if $\widetilde{H}$ is simply laced.

Let $K'_{\mathbb{C}}$ be the complexification of a maximal compact subgroup of H'. There is a surjection $\widetilde{K}'_{\mathbb{C}} \to K'_{\mathbb{C}}$ with central kernel; it follows that the orbits of $\widetilde{K}'_{\mathbb{C}}$ and $K'_{\mathbb{C}}$ on X' are the same. Now suppose $H' = G^\vee$, the group appearing in Vogan duality for the linear group G. We conclude that the L-packets in $\mathcal{B}$ and the subsets of $\mathcal{B}'$ defined above are in natural bijection, parametrized by $\widetilde{K}'_{\mathbb{C}}$ or $U_{\mathbb{C}}$ orbits on the flag variety for $\mathfrak{g}^\vee = \mathfrak{h}'$. Examples of this phenomenon include [15]:

$$G = GL(n,\mathbb{R}), G^\vee = U(p,q), \widetilde{H} = \widetilde{GL}(n,\mathbb{R}), \widetilde{H}' = \widetilde{U}(p,q)$$

and [14]: and

$$G = SO(2p, 2q+1), G^\vee = Sp(2n,\mathbb{R}), \widetilde{H} = \widetilde{Sp}(2n,\mathbb{R}), \widetilde{H}' = \widetilde{Sp}(2n,\mathbb{R}).$$

According to Langlands and Shelstad, associated to each packet Π in $\mathcal{B}$ is a certain interesting *stable* virtual character Θ_Π. (For tempered L-packets this is the sum of the representations in the packet.) This sum can be defined as in [4, Definition 1.28] using Vogan duality $\mathcal{B} \leftrightarrow \mathcal{B}^\vee$. Suppose $\widetilde{\Pi}$ is the corresponding subset of $\widetilde{\mathcal{B}}'$. Definition 1.28 of [4] applies in this situation to give a distinguished genuine virtual character $\Theta_{\widetilde{\Pi}}$ of $\widetilde{H}'$, although (since the notion of stability is not defined for $\widetilde{H}'$) it is not clear what properties it should have. In any event, since duality is closely related to character theory, it is reasonable to expect that the map $\Theta_\Pi \to \Theta_{\widetilde{\Pi}}$ has nice properties. In fact in the simply laced case [6], and for $\widetilde{H}' = \widetilde{Sp}$ [2], there is a theory of *lifting of characters* which takes Θ_Π to $\Theta_{\widetilde{\Pi}'}$.

This reasoning may be turned around: a theory of lifting of characters can be used to give information about Vogan duality. This is what happens here: we use some parts of the theory of [6] to solve some technical issues arising here in defining duality of characters. We turn to this now.

Fix an admissible two-fold cover $\widetilde{G}$ of a simply laced, real reductive linear group G (cf. Sections 2 and 3), and a block $\mathcal{B}$ of genuine regular characters of $\widetilde{G}$. Fix a weakly minimal element γ of $\mathcal{B}$, and let $\mathcal{B} = \mathcal{B}(\overline{\pi}(\gamma))$ be the corresponding block of representations. We will construct a group G', with admissible cover $\widetilde{G}'$, and a genuine regular character γ' of $\widetilde{G}'$ such that Theorem 8.6 applies to prove that $\mathcal{B}' = \mathcal{B}(\overline{\pi}(\gamma'))$ is dual to $\mathcal{B}$.

Since we will be passing back and forth between linear and nonlinear groups we change notation and write $\widetilde{\gamma} = (\widetilde{H}, \widetilde{\Gamma}, \widetilde{\lambda})$ for a genuine regular character of $\widetilde{G}$. Let $\mathcal{B} = \mathcal{B}(\overline{\pi}(\widetilde{\gamma}))$.

We first make an elementary reduction. Recall a connected complex group $G_{\mathbb{C}}$ is said to be *acceptable* if (for any Cartan subgroup $H_{\mathbb{C}}$ and set of positive roots) one-half the sum of the positive roots exponentiates to $H_{\mathbb{C}}$.

Lemma 9.1. *There is a connected, reductive complex group $G^\dagger_{\mathbb{C}}$, with acceptable derived group, real form $G^\dagger$, admissible cover $\widetilde{G^\dagger}$, and block $\mathcal{B}^\dagger$ of genuine representations of $\widetilde{G^\dagger}$ such that $\mathcal{B}^\dagger$ is isomorphic to $\mathcal{B}$ (Definition 8.9).*

Proof. Thanks to Proposition 8.11 this isn't hard, and there are various ways to construct $G^\dagger$. Here is one.

Let $G^{sc}_{\mathbb{C}}$ be the simply connected cover of the derived group of $G_{\mathbb{C}}$, with real points G^{sc}. Then G^{sc} maps onto the identity component G^0_d of the derived group G; let $\widetilde{G}^{sc}$ be the pullback of the restriction of the cover $\widetilde{G} \to G$ to G_d^0. This is the admissible cover of G^{sc}. Let $\widetilde{H}_d = \widetilde{H} \cap \widetilde{G}_d^0$, and let $\widetilde{H}^{sc}$ be the inverse image of this in $\widetilde{G}^{sc}$.

Let $\widetilde{\lambda}^{sc}$ be the restriction of $\widetilde{\lambda}$ to the Lie algebra of $\widetilde{H}^{sc}$. Let $\widetilde{\Gamma}_d$ be an irreducible representation of $\widetilde{H}_d$ contained in the restriction of $\widetilde{\Gamma}$ to $\widetilde{H}_d$, and let $\widetilde{\Gamma}^{sc}$ be the pullback of this to $\widetilde{H}^{sc}$.

It is easy to check that $\widetilde{\gamma}^{sc} = (\widetilde{H}^{sc}, \widetilde{\Gamma}^{sc}, \widetilde{\lambda}^{sc})$ is a genuine regular character of $\widetilde{G}^{sc}$. It is also clear that it has the same bigrading as $\widetilde{\gamma}$ (cf. Proposition 6.2). It follows from Proposition 8.11 that $\mathcal{B}(\overline{\pi}(\widetilde{\gamma}^{sc}))$ is isomorphic to $\mathcal{B}(\overline{\pi}(\widetilde{\gamma}))$. □

Therefore, after replacing G by $G^\dagger$ if necessary, we assume G is the real points of $G_{\mathbb{C}}$, where $G_{\mathbb{C}}$ is connected reductive and the derived group of $G_{\mathbb{C}}$ is acceptable.

We recall some notation from [6]. We consider the *admissible triple* $(\widetilde{G}, G, G)$ [6, Definition 3.14]. This amounts to saying that $G_{\mathbb{C}}$ is simply laced, with acceptable derived group, and $\widetilde{G}$ is an admissible cover of G. We fix a set of lifting data for this triple [6, Definition 7.1]. We don't need to spell out this construction, including the choices involved, but merely summarize the properties that we need.

Associated to this data is a character μ of the center $\mathfrak{z}$ of $\mathfrak{g}$ [6, Definition 6.35].

Let H be the Cartan subgroup of G corresponding to $\widetilde{H}$. Lifting data for $(\widetilde{H}, H, H)$ is chosen as in [6, Section 17]. If Γ is a character of H then $\mathrm{Lift}_H^{\widetilde{H}}(\Gamma)$ is 0 or a sum of irreducible representations of $\widetilde{H}$. See [6, Section 10].

Lifting of character data is defined as follows. Suppose $\gamma = (H, \Gamma, \lambda)$ is a regular character of G. Write

$$\mathrm{Lift}_H^{\widetilde{H}}(\Gamma e^{-2\rho_i + 2\rho_{i,c}}) = \sum_i \widetilde{\Gamma}_i e^{-2\rho_i + 2\rho_{i,c}} \tag{9.1}$$

Here $\rho_i = \rho_i(\lambda), \rho_{i,c}(\lambda)$ are as in Definition 5.3, and occur here due to the difference between character data (cf. Section 5) and *modified* character data of [6, Section 16]. By [6, Section 17] and (9.1) we have

$$2d\widetilde{\Gamma}_i = d\Gamma + \rho_i - 2\rho_{i,c} - \mu. \tag{9.2}$$

Then $\mathrm{Lift}_G^{\widetilde{G}}(\gamma) = \{(\widetilde{H}, \widetilde{\Gamma}_i, \widetilde{\lambda}) \mid 1 \leqslant i \leqslant n\}$ where $\widetilde{\lambda} = \frac{1}{2}(\lambda - \mu)$, and $\widetilde{\Gamma}_i$ are given by (9.1).

An important role is played by the character ζ_{cx} of [6, Section 2]. Let $\Gamma_r(H)$ be the subgroup of H generated by the m_α for all real roots α. Let G_d be the derived group of G, and let H_d^0 be the identity component of $H_d = H \cap G_d$. It is well-known that $H_d = \Gamma_r(H) H_d^0$. Let S be a set of complex roots of H such that the set of all complex roots is $\{\pm\alpha, \pm\theta\alpha \mid \alpha \in S\}$. Let $\zeta_{cx}(h) = \prod_{\alpha \in S} \alpha(h)$ $(h \in \Gamma_r(H))$. Then ζ_{cx} is independent of the choice of S.

Fix a genuine regular character $\widetilde{\gamma}$ of $\widetilde{G}$. We do not yet need to assume it is weakly minimal.

Lemma 9.2. *There is a character Γ of H such that $\gamma = (H, \Gamma, 2\widetilde{\lambda} + \mu)$ is a regular character, and $\widetilde{\gamma} \in \mathrm{Lift}_G^{\widetilde{G}}(\gamma)$.*

This is an immediate consequence of [6, Lemma 17.13]. In fact we only need the following weaker statement, whose proof is fairly self-contained.

Choose a set of positive roots satisfying:

$$\alpha \text{ complex}, \alpha > 0 \Rightarrow \theta(\alpha) < 0, \tag{9.3(a)}$$

$$\alpha \text{ imaginary}, \alpha > 0 \Rightarrow \langle \lambda, \alpha^\vee \rangle > 0. \tag{9.3(b)}$$

Define ρ, ρ_i and $\rho_{i,c}$ accordingly.

Define:

$$\Gamma(h) = (\widetilde{\Gamma}^2 e^{\rho})(h) e^{-2\rho_i+2\rho_{i,c}}(h)|e^{\rho_i-\rho}(h)| \quad (h \in H_d^0), \tag{9.4)(a}$$

$$\Gamma(h) = \zeta_{cx}(h) \quad (h \in \Gamma_r(H)). \tag{9.4)(b}$$

To be precise in (a), since the derived group is acceptable $e^{\rho}(h)$ is well defined, and clearly $e^{-2\rho_i+2\rho_{i,c}}(h)$ and $|e^{\rho_i-\rho}(h)| = |e^{2\rho_i-2\rho}(h)|^{\frac{1}{2}}$ are well defined since the exponents are sums of roots.

Lemma 9.3. *There is a regular character $\gamma = (H, \Gamma, \lambda)$ where Γ restricted to H_d is given by (9.4), and λ satisfies $\langle \lambda, \alpha^\vee \rangle = \langle 2\widetilde{\lambda}, \alpha^\vee \rangle$ for all roots α.*

Proof. By [6, (6.21)] (9.4)(a) and (b) agree on $\Gamma_r(H) \cap H_d^0$, and so define $\Gamma|_{H_d}$. Choose any extension of Γ to H and set $\lambda = d\Gamma - \rho_i(\widetilde{\lambda}) + 2\rho_{i,c}(\widetilde{\lambda})$ (cf. (5.4)(b)). It is easy to check that (H, Γ, λ) is a regular character, and that λ and $2\widetilde{\lambda}$ have the same restriction to $\mathfrak{h}_d$. $\square$

The proof of Lemma 9.2 in [6] is the same, except that we are more careful in choosing the extension of Γ to $Z(G)^0 H_d$, so that $\lambda = 2\widetilde{\lambda} + \mu$ where $\mu \in \mathfrak{z}^*$ is given. This isn't necessary for our purposes.

A crucial property of γ is the following.

Proposition 9.4. *Fix $\alpha \in \Delta(2\widetilde{\lambda})$.*

(a) Suppose α is imaginary. Then α is compact if and only if $\langle \widetilde{\lambda}, \alpha^\vee \rangle \in \mathbb{Z}$.

(b) Suppose α is real. Then α does not satisfy the parity condition with respect to γ if and only if $\langle \widetilde{\lambda}, \alpha^\vee \rangle \in \mathbb{Z}$.

In other words

$$\Delta^{r,+}(\gamma) = \Delta^r(\widetilde{\lambda}) \tag{9.5)(a}$$

$$\Delta^{i,+}(\gamma) = \Delta^i(\widetilde{\lambda}). \tag{9.5)(b}$$

Proof. If α is imaginary the assertion follows immediately from (6.8)(b), so assume α is real. We need to show (see Section 6):

$$\Gamma(m_\alpha) = \begin{cases} \epsilon_\alpha (-1)^{\langle 2\widetilde{\lambda}, \alpha^\vee \rangle} & \langle \widetilde{\lambda}, \alpha^\vee \rangle \in \mathbb{Z} \\ -\epsilon_\alpha (-1)^{\langle 2\widetilde{\lambda}, \alpha^\vee \rangle} & \langle \widetilde{\lambda}, \alpha^\vee \rangle \in \mathbb{Z} + \frac{1}{2}, \end{cases} \tag{9.6)(a}$$

i.e.

$$\Gamma(m_\alpha) = \epsilon_\alpha. \tag{9.6)(b}$$

Lemma 9.5. *We have $\zeta_{cx}(m_\alpha) = \epsilon_\alpha$.*

Proof. Let $d = \sum_X \langle \beta, \alpha^\vee \rangle$ where X is the set of positive roots which become non-compact imaginary roots on the Cayley transform H_α (cf. Section 7). According to ([17], Lemma 8.3.9) $\epsilon_\alpha = (-1)^d$. A root β of H is imaginary for H_α if $s_\alpha \theta \beta = \beta$, in which case $s_\alpha \beta$ is also imaginary for H_α. If $\langle \beta, \alpha^\vee \rangle \neq 0$ is even β and $s_\alpha \beta$ are both compact or both non-compact for H_α, otherwise one is compact and one is non-compact. Therefore the contribution of $\{\pm\beta, \pm\theta\beta\}$ to d is $\langle \beta, \alpha^\vee \rangle$ (mod 2), and we can replace X with

$$\{\beta \in S \mid s_\alpha \beta = \theta\beta, \langle \beta, \alpha^\vee \rangle \text{ odd}\}. \tag{9.7}$$

On the other hand $\zeta_{cx}(m_\alpha) = (-1)^e$ where $e = \sum_S \langle\beta, \alpha^\vee\rangle$. If $\langle\beta, \alpha^\vee\rangle = 0$ then β does not contribute to the sum; if $s_\alpha\beta \neq \pm\theta\beta$ the total contribution of $\beta, s_\alpha\beta$ is even. It is easy to see the case $s_\alpha\beta = -\theta\beta$ does not arise. Therefore we can replace S with the set (9.7). □

The proposition now follows from the Lemma, (9.4)(b) and (9.6)(b). □

We now want to construct a group G' with the following properties:

(1) G' is the real points of a connected, complex reductive group $G'_{\mathbb{C}}$,
(2) There is a regular character γ' for G' such that the bigrading of γ' is dual to that of γ,
(3) the derived group of $G'_{\mathbb{C}}$ is simply connected,
(4) G' admits an acceptable cover.

The main point is (2), and is provided by duality for G [20]. A little tinkering may be required to satisfy conditions (3) and (4).

We first construct a connected, reductive complex group $G'_{\mathbb{C}}$, with real points G', and a regular character γ' of G', such that $\mathcal{B}(\overline{\pi}(\gamma'))$ is dual to $\mathcal{B}(\overline{\pi}(\gamma))$, with $\overline{\pi}(\gamma)$ going to $\overline{\pi}(\gamma')$. This essentially follows from [20, Theorem 13.13], although the group G' constructed there is not necessarily the real points of a connected complex group. That we can construct such a group G' follows from [4, Theorem 1.24 and Chapter 21]. For the case of integral infinitesimal character see [1].

Thus condition (1) holds, and (2) holds as in [20, Theorem 11.1].

It is well-known that we can choose a connected, reductive complex group $G''_{\mathbb{C}}$, with real points G'', with simply connected derived group, such that there is a surjection $G''_{\mathbb{C}} \to G'_{\mathbb{C}}$ taking G'' onto G'. Pulling back $\overline{\pi}(\gamma')$ to G'' we obtain an isomorphic block for G''. After making this change we may assume conditions (1-3) hold.

If G' is semisimple then condition (4) is automatic. For G' reductive this condition may fail (although this is unusual). If this is the case, replace $G'_{\mathbb{C}}$ with its derived group, with real points G'_d, and γ' with its restriction to G'_d. This regular character has the same bigrading, and now conditions (1-4) hold.

So we now assume we are given G' and γ' satisfying (1-4). Write $\gamma' = (H', \Gamma', \lambda')$. We spell out the essential condition (2) (cf. [20, Theorem 11.1]).

There is an isomorphism $\phi : \mathfrak{h}' \cap \mathfrak{g}'_d \to \mathbb{C}\langle\Delta(2\lambda)^\vee\rangle \subset \mathfrak{h}$, satisfying

$$\phi(\Delta^\vee(\mathfrak{g}', \mathfrak{h}')) = \Delta^\vee(2\widetilde{\lambda}). \tag{9.8)(a}$$

This satisfies

$$\phi^*(\theta(\alpha)) = -\theta'(\phi^*(\alpha)) \tag{9.8)(b}$$

where θ' is the Cartan involution for G'. Furthermore λ' is integral for $\Delta(\mathfrak{g}', \mathfrak{h}')$. Since the derived group of $G'_{\mathbb{C}}$ is simply connected we may assume

$$\langle 2\widetilde{\lambda}, \alpha^\vee\rangle = \langle\lambda', \phi^*(\alpha)^\vee\rangle \tag{9.8)(c}$$

for all $\alpha \in \Delta(2\widetilde{\lambda})$. Finally we have

$$\phi^*(\Delta^{r,+}(\gamma)) = \Delta^{i,+}(\gamma'), \tag{9.8)(d}$$

$$\phi^*(\Delta^{i,+}(\gamma)) = \Delta^{r,+}(\gamma'). \tag{9.8)(e}$$

Now let $\widetilde{G}'$ be an admissible cover of G'. We want to lift γ' to a genuine regular character of $\widetilde{G}'$.

Proposition 9.6. *There is a choice of transfer data for $\widetilde{G}'$ (as in [6]) so that $Lift_{G'}^{\widetilde{G}'}(\gamma')$ is nonzero.*

Proof. This follows immediately from [6, Lemma 17.14], the hypothesis of which holds by the following key result. □

Lemma 9.7. *Let $\widetilde{\chi}$ be any genuine character of $\widetilde{H}'$. Choose positive roots satisfying* (9.3) *with respect to λ', and define ρ', ρ'_i and $\rho'_{i,c}$ accordingly. Suppose $h \in H_d$, $h^2 = 1$. Then*

$$\Gamma'(h)e^{-2\rho'_i+2\rho'_{i,c}}(h) = \begin{cases} (\widetilde{\chi}^2 e^{\rho'})(h) & (h \in H_d'^0), \\ \zeta_{cx}(h) & (h \in \Gamma_r(H')). \end{cases} \tag{9.9}$$

Proof. First suppose $\beta \in \Delta(\mathfrak{g}', \mathfrak{h}')$ is a real root, so $m_\beta \in \Gamma_r(H')$. We need to show $\Gamma'(m_\beta) = \zeta_{cx}(m_\beta)$. By Lemma 9.5 we can write this as

$$\Gamma'(m_\beta) = \begin{cases} \epsilon_\beta(-1)^{\langle\lambda',\beta^\vee\rangle} & \langle\lambda', \beta^\vee\rangle \in 2\mathbb{Z} \\ -\epsilon_\beta(-1)^{\langle\lambda',\beta^\vee\rangle} & \langle\lambda', \beta^\vee\rangle \in 2\mathbb{Z}+1. \end{cases} \tag{9.10}$$

This says

$$\beta \in \Delta^{r,+}(\gamma') \Leftrightarrow \langle\lambda', \beta^\vee\rangle \in 2\mathbb{Z}. \tag{9.11}$$

By (9.8)(c) if we let $\alpha = \phi^{*-1}(\beta) \in \Delta(2\widetilde{\lambda})$ we can write this as

$$\beta \in \Delta^{r,+}(\gamma') \Leftrightarrow \langle\widetilde{\lambda}, \alpha^\vee\rangle \in \mathbb{Z}. \tag{9.12}$$

By (9.8)(b) α is imaginary, and by by (9.5)(b) equality holds on the right hand side if and only if $\alpha \in \Delta^{i,+}(\gamma)$, so we need to show

$$\phi^*(\Delta^{i,+}(\gamma)) = \Delta^{r,+}(\gamma') \tag{9.13}$$

which is precisely (9.8)(e).

Now suppose $Z' \in i\mathfrak{h}'_0$, $\exp(2\pi i Z') = 1$, and $h' = \exp(\pi i Z') \neq 1$. We have to show

$$\Gamma'(h')e^{-2\rho'_i+2\rho'_{i,c}}(h') = (\widetilde{\chi}^2 e^{\rho'})(h'). \tag{9.14}$$

The left hand side is $e^{\pi i\langle d\Gamma'-2\rho'_i+2\rho'_{i,c}, Z'\rangle}$. Using (5.4)(c) the exponent is $\pi i\langle\lambda' - \rho'_i, Z'\rangle$, so we have to show

$$e^{\pi i\langle\lambda'-\rho'_i,Z'\rangle} = (\widetilde{\chi}^2 e^{\rho'})(h'). \tag{9.15}$$

Write $\rho' = \rho'_r + \rho'_i + \rho'_{cx}$. Then $\langle\rho'_r, Z'\rangle = 0$, and $\langle\rho'_{cx}, Z'\rangle = 0$ by (9.3). Therefore we can replace ρ'_i with ρ' on the left hand side, and we are reduced to showing

$$e^{\pi i\langle\lambda',Z'}\rangle = \widetilde{\chi}^2(h'). \tag{9.16}$$

Write $\widetilde{\exp}$ for the exponential map $\mathfrak{h}'_0 \to \widetilde{H}'$. Then the right hand side is $\widetilde{\chi}^2(\exp(\pi i Z')) = \widetilde{\chi}(\widetilde{\exp}(2\pi i Z'))$, so we have to show

$$e^{\pi i\langle\lambda',Z'\rangle} = \widetilde{\chi}(\widetilde{\exp}(2\pi i Z')). \tag{9.17}$$

Suppose β is an imaginary root. Since the derived group of $G'_{\mathbb{C}}$ is simply connected we can take $Z' = \beta^\vee$. By Lemma 6.3(a) the right hand side is 1 if β is compact, and -1 otherwise. In other words the right hand side is 1 if and only if $\beta \in \Delta^{i,+}(\gamma')$.

By (9.8)(a) $\alpha = \phi^{*-1}(\beta)$ is real, and by (9.8)(c) the left hand side is equal to $e^{2\pi i\langle\widetilde{\lambda},\alpha^\vee\rangle}$. Therefore by (9.5)(a) the left hand side is 1 if and only if $\alpha \in \Delta^{r,+}(\gamma)$. So we have to show $\phi^*(\Delta^{r,+}(\gamma)) = \Delta^{i,+}(\gamma')$, which is (9.8)(d). This proves (9.17) in this case.

Now suppose β is a complex root, in which case we can take $Z' = \beta^\vee + \theta'\beta^\vee$. We have to show

$$e^{\pi i\langle\lambda',\beta^\vee+\theta'\beta^\vee\rangle} = \widetilde{\chi}(\widetilde{\exp}(2\pi i(\beta^\vee + \theta'\beta^\vee))) \tag{9.18}$$

By (9.8)(b) and (9.8)(c) we can write the exponent on the left hand side as

$$\pi i\langle 2\widetilde{\lambda}, \alpha^\vee - \theta\alpha^\vee\rangle = \pi i\langle 2\widetilde{\lambda}, \alpha^\vee + \theta\alpha^\vee\rangle - 2\pi i\langle 2\widetilde{\lambda}, \theta\alpha^\vee\rangle. \tag{9.19}$$

The final term is an integral multiple of $2\pi i$, so the left hand side is $e^{2\pi i\langle\widetilde{\lambda},\alpha^\vee+\theta\alpha^\vee\rangle}$. Now $\widetilde{\lambda}$ is the differential of a genuine character $\widetilde{\tau}$ of $\widetilde{H}^0$ so we can write this as

$$\widetilde{\tau}(\widetilde{\exp}(2\pi i(\alpha^\vee + \theta\alpha^\vee))). \tag{9.20}$$

Therefore we need to show

$$\widetilde{\tau}(\widetilde{\exp}(2\pi i(\alpha^\vee + \theta\alpha^\vee))) = \widetilde{\chi}(\widetilde{\exp}(2\pi i(\beta^\vee + \theta'\beta^\vee))). \tag{9.21}$$

This follows from Lemma 6.3(b): both sides are 1 if and only if $\langle\alpha, \theta\alpha^\vee\rangle = 0$.

Every element of H^0_d of order 2 is of the form $\exp(\pi i Z')$ for Z' as above, so this completes the proof. □

Let $\widetilde{\gamma}'$ be any constituent of $\mathrm{Lift}_{G'}^{\widetilde{G}'}(\gamma')$. We can write $\widetilde{\gamma}' = (\widetilde{H}', \widetilde{\Gamma}', \widetilde{\lambda}')$ for some $\widetilde{\Gamma}'$, and $\widetilde{\lambda}' = \frac{1}{2}(\lambda' - \mu')$. Here μ' is an element of the dual of the center of $\mathfrak{g}'$, depending on the choice of lifting data. Recall by (9.8)(c) for any root α we have $\langle 2\widetilde{\lambda}, \alpha^\vee\rangle = \langle\lambda', \phi^*(\alpha)^\vee\rangle$. Therefore

$$\langle\widetilde{\lambda}, \alpha^\vee\rangle = \langle\widetilde{\lambda}', \phi^*(\alpha)^\vee\rangle. \tag{9.22}$$

Therefore conditions 8.8(b) and (c) of Theorem 8.6 hold.

We now assume $\widetilde{\gamma}$ is weakly minimal. Then the conditions of Theorem 8.6 hold, and we conclude $\mathcal{B}(\overline{\pi}(\widetilde{\gamma}'))$ is dual to $\mathcal{B}(\overline{\pi}(\widetilde{\gamma}))$.

We summarize the preceding discussion.

Theorem 9.8. *Assume:*

(i) *G is the real points of a connected, reductive, simply laced complex group with acceptable derived group, and $\widetilde{G}$ is an admissible cover of G;*

(ii) *$\mathcal{B}$ is a block of genuine regular characters of $\widetilde{G}$, and $\widetilde{\gamma}$ is a weakly minimal element of $\mathcal{B}$;*

(iii) *γ is a regular character of G so that $\widetilde{\gamma}$ is a constituent of $Lift_G^{\widetilde{G}}(\gamma)$ for some choice of lifting data;*

(iv) *G' is the real points of a connected, reductive complex group, with simply connected derived group, and $\widetilde{G}'$ is an admissible cover of G';*

(v) *γ' is a regular character of G' such that γ and γ' have dual (weak) bigradings;*

Choose lifting data so $Lift_{G'}^{\widetilde{G}'}(\gamma')$ is nonzero, and let $\widetilde{\gamma}'$ be any constituent of this lift. Then $\mathcal{B}(\overline{\pi}(\widetilde{\gamma}'))$ is dual to $\mathcal{B}(\overline{\pi}(\widetilde{\gamma}))$.

Remark 3. It is clear from the proof that the assumptions are stronger than necessary, and various weaker versions of the theorem are possible.

Remark 4. We have assumed that the infinitesimal character is regular. There are several possibilities for formulating a version of Definition 4.1 in the case of singular infinitesimal character. This is explained in [4], and we omit the details here.

10. Examples

10.1 Example: $SL(2,\mathbb{R})$

The unique nontrivial two-fold cover of G is an admissible cover. At infinitesimal character $\rho/2$ (a typical half-integral infinitesimal character) there are four irreducible genuine representations: the two oscillator representations, each of which has two irreducible summands. It is easy to see that at this infinitesimal character $\widetilde{G}$ has two genuine blocks with distinct central characters, each containing two irreducible representations. Each block is self-dual. See [13, Section 4].

10.2 Example: $GL(n,\mathbb{R})$

A two-fold cover $\widetilde{G}$ of $G = \mathrm{GL}(n,\mathbb{R})$ is admissible if it restricts to the (unique) nontrivial cover of $\mathrm{SL}(n,\mathbb{R})$; assume this is the case. Suppose λ is a half-integral infinitesimal character. Then $\Delta(\lambda)$ is of type $A_{p-1} \times A_{q-1}$. If n is even $\widetilde{G}$ has a unique block at infinitesimal character λ; if n is odd, there are two isomorphic blocks. Fix such a block $\mathcal{B}$. Consider an admissible cover $\widetilde{G}'$ of $G' = \mathrm{U}(p,q)$; there are three, corresponding to the three double covers of $\mathrm{U}(p) \times \mathrm{U}(q)$, and the exact one we choose is not important. Fix an infinitesimal character λ' for which $\widetilde{G}'$ has genuine discrete series. This implies that $\Delta_{\frac{1}{2}}(\lambda)$ is of type $A_{p-1} \times A_{q-1}$. Then $\widetilde{G}'$ has a unique block $\mathcal{B}'$ at infinitesimal character λ', and $\mathcal{B}$ is dual to $\mathcal{B}'$.

Two cases of the previous paragraph are worth keeping in mind. If λ is integral $\mathcal{B}$ consists of a single irreducible principal series, G' is compact, the cover $\widetilde{G}'$ splits, and $\mathcal{B}'$ consists of a single finite-dimensional representation of the compact linear group $\widetilde{G}'$. On the other hand, consider $\lambda = \rho/2$. Then each block $\mathcal{B}$ for $\widetilde{G}$ has an interesting representation, say π, that is as small as the infinitesimal character permits. (The representation π is the unique Langlands quotient of a principal series in the block.) The group G' is either isomorphic to $\mathrm{U}(n/2,n/2)$ (if n is even) or $\mathrm{U}\left((n+1)/2,(n-1)/2\right)$ if n is odd. In either case $\widetilde{G}'$ is quasisplit and hence has genuine large discrete series. Each such is characterized by its infinitesimal character. Fix λ' for which such a large discrete series π' exists. Let $\mathcal{B}'$ denote the block containing π'. Then $\mathcal{B}$ is dual to $\mathcal{B}'$ and the duality maps π to π'. For further details see [15, Part II].

10.3 Example: Minimal Principal Series of Split Groups

Suppose G is the split real form of a connected, simply connected reductive complex group $G_{\mathbb{C}}$. Suppose $\widetilde{G}$ be an admissible cover of G and H is a split Cartan subgroup. Fix $\lambda \in \mathfrak{h}^*$. Suppose π is a genuine principal series representation of $\widetilde{G}$ with infinitesimal character λ. By Example 2 there is such a representation, determined by its central character. Choosing a Weyl group invariant bilinear form on $\mathfrak{h}$ we may identify λ with an element $\lambda^\vee \in \mathfrak{h}$. Let $h = \exp(\pi i \lambda^\vee)$ and

$$G'_{\mathbb{C}} = Cent_{G_{\mathbb{C}}}(h^2).$$

Let $\theta' = \mathrm{int}(h)$ considered as an involution of $G'_{\mathbb{C}}$. Note that $G'_{\mathbb{C}} = G_{\mathbb{C}}$ if λ is half-integral. Let G' be the corresponding real form, and let $\widetilde{G}'$ be an admissible cover. Let $K'_{\mathbb{C}}$ be the fixed points of θ' acting on $G'_{\mathbb{C}}$. Thus

$$\Delta(\mathfrak{g}',\mathfrak{h}) = \Delta(2\lambda), \quad \Delta(\mathfrak{k}',\mathfrak{h}) = \Delta(\lambda).$$

The Cartan subgroup of G' corresponding to $\mathfrak{h}$ is compact, and G' has a discrete series representation with infinitesimal character λ, determined by its central character. Then π and π' have dual bigradings, and the map $\pi \to \pi'$ extends to a duality of blocks.

For example if λ is integral the principal series representation is irreducible. Dually G' is compact, the trivial cover of G' is admissible, and π' is a finite dimensional representation of $\widetilde{G}'$.

If π is any genuine irreducible representation of $\widetilde{G}$ then we may apply a sequence of Cayley transforms to obtain a minimal principal series representation. This reduces the computation of the dual of the block containing π to the previous case; in particular $G'_{\mathbb{C}}$ and G' are computed from the infinitesimal character as above.

10.4 Example: Minimal Principal Series of Split Groups (continued)

The duality of Theorem 9.8 is for a single block, as in [20]. If G is a real form of a connected reductive algebraic group this duality can be promoted, roughly speaking, to a duality on all blocks simultaneously. See [4] for details. It would be very interesting to do this also in the nonlinear case. We limit our discussion here to minimal principal series of (simple) split groups.

For simplicity we assume G is the split real form of a connected, semisimple complex group, and let $\widetilde{G}$ be its admissible cover. Fix an infinitesimal character for G. Then the genuine minimal principal series representations of G with this infinitesimal character are parametrized by the genuine characters of $Z(\widetilde{G})$.

There are a finite number $\pi_1, \dots, \pi_n$ of such representations, generating distinct blocks $\mathcal{B}_1, \dots, \mathcal{B}_n$. It follows from Section 8 that there are natural bijections between these blocks, and the corresponding representations of the extended Hecke algebra are isomorphic. Here are the number of such blocks for the simple groups [5, Section 6, Table 1]:

$$\begin{aligned} A_{2n}, E_6, E_8 &: 1 \\ A_{2n+1}, D_{2n+1}, E_7 &: 2 \\ D_{2n} &: 4 \end{aligned} \tag{10.1}$$

In fact it can be shown that the group of *outer automorphisms* of $\widetilde{G}$ acts transitively on $\{\pi_1, \dots, \pi_n\}$.

10.5 Example: Discrete Series

This is dual to the previous example. Suppose G is a real form of a connected, semisimple group, which contains a compact Cartan subgroup. Let $\widetilde{G}$ be the admissible cover of G, and fix an infinitesimal character λ for which $\widetilde{G}$ has a genuine discrete series representation.

The number of genuine discrete series representations of $\widetilde{G}$ with infinitesimal character λ depends on the real form. If G is quasisplit it can be shown, for example by a case-by-case analysis, that the genuine discrete series representations of $\widetilde{G}$ with infinitesimal character λ are in bijection with the genuine principal series representation of the split real form of G discussed in the previous section. However if G is not quasisplit there may be fewer genuine discrete series of $\widetilde{G}$.

For example suppose $G_{\mathbb{C}} = SL(2n, \mathbb{C})$. As in Example 10.2 if $G = SU(p,q)$ then $\widetilde{G}$ has 1 genuine discrete series representation with infinitesimal character λ, if $p \neq q$, and 2 if $p = q$.

10.6 Example: Genuine Discrete Series for E_7

There are three noncompact real forms of E_7: split, Hermitian ($\mathfrak{k} = \mathbb{R} \times D_5$) and *quaternionic* ($\mathfrak{k} = A_1 \times A_5$). All have a compact Cartan subgroup. We label these $E_7(s), E_7(h)$ and $E_7(q)$, respectively. Then at the appropriate infinitesimal character (for example $\rho - \frac{1}{2}\lambda_i$ where λ_i is an appropriate fundamental weight), $E_7(s)$ and $E_7(h)$ have two genuine discrete series representations. On the other hand $E_7(q)$ has only 1 genuine discrete series representation.

It is worth noting that $Z(\widetilde{G}) = \mathbb{Z}/4\mathbb{Z}$ in the split and Hermitian cases, and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in the quaternionic real form. Let π be a genuine discrete series representation of $\widetilde{G}$. If G is split or quaternionic π and π^* have distinct central characters, and are therefore the two genuine discrete series representations. If G is quaternionic then $\pi \simeq \pi^*$.

10.7 Example: Genuine discrete series for D_n

Finally we specialize the discussion in Example 10.5 to type D_n.

Write $\lambda = (\lambda_1, \dots, \lambda_n)$ in the usual coordinates. Then λ is half-integral if

$$\lambda_i \in \frac{1}{2}\mathbb{Z} \quad \text{for all i}$$

or

$$\lambda_i \in \pm\frac{1}{4} + \mathbb{Z} \quad \text{for all i.}$$

In the first case $\Delta(\lambda)$ is of type $D_p \times D_q$, $G = Spin(2p, 2q)$, and $\widetilde{K} \simeq Spin(2p) \times Spin(2q)$. Write $\lambda = (a_1, \dots, a_p, b_1 \dots, b_q)$ with $a_i \in \mathbb{Z}$ and $b_j \in \mathbb{Z} + \frac{1}{2}$. Then $\widetilde{G}$ has discrete series representations with Harish-Chandra parameter (with the obvious notation) $(a_1, \dots, a_{p-1}, \epsilon a_p; b_1, \dots, b_{q-1}, \epsilon b_q)$ with $\epsilon = \pm 1$. If $p = q$ it also has two more, with Harish-Chandra parameter $(b_1, \dots, b_{p-1}, \epsilon b_p; a_1, \dots, a_{p-1}, \epsilon a_p)$. These two or four genuine discrete series representations have distinct central characters.

In the second case $\Delta(\lambda)$ is of type A_{n-1} and $\widetilde{K}$ is a two-fold cover of $U(n)$. In this case G the real form of $Spin(2n, \mathbb{C})$ corresponding to the real form $\mathfrak{so}^*(2n)$ of $\mathfrak{so}(2n, \mathbb{C})$, and a two-fold cover of $SO^*(2n)$.

10.8 Example: G_2

Let G be the connected split real group of type G_2, and let $\widetilde{G}$ an admissible two-fold cover of G (which is unique up to isomorphism). Since all real roots for $\widetilde{G}$ are metaplectic in the sense of Definition 3.2, most of the results of this paper apply to this case. We will use Proposition 5.7 to visualize the duality of Theorem 9.8. The closure order for the orbits of $K_{\mathbb{C}}$ (or $(\widetilde{K})_{\mathbb{C}}$) on the flag variety $\mathfrak{B}$ for $\mathfrak{g}$ are depicted in Figure 10.8. The darkened nodes in the figure are irrelevant at this point, and will be explained in a moment.

If we fix trivial infinitesimal character ρ and the block for G containing the trivial representation, the fibers of the map *supp* (described before Proposition 5.7) for G are all singletons, with the exception of the open obit which supports three representations. The duality of [20] interchanges these three with the three discrete series (supported on the three closed orbits) and interchanges the unique representations supported on each of the intermediate orbits in a way consistent with the obvious symmetry of those orbits in the closure order of Figure 10.8.

Now consider the unique block $\mathcal{B}$ of genuine representations of $\widetilde{G}$ with infinitesimal character $\rho/2$. The image of the injective map *supp* in this case is represented by the darkened nodes in Figure 10.8. The duality of Theorem 9.8 takes $\mathcal{B}$ to itself and corresponds to an order-reversing involution of the darkened nodes.

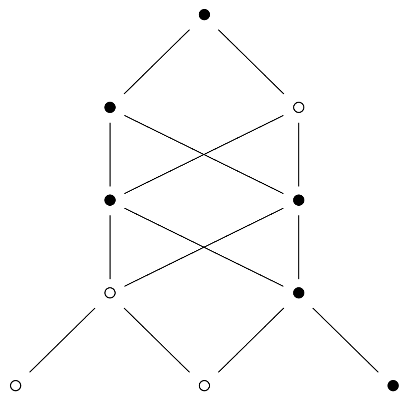

Figure 1. The closure order for $K_{\mathbb{C}}$ orbits on the flag variety of type G_2.

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