The Real Chevalley Involution

Jeffrey Adams

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1 Introduction

A Chevalley involution C of a connected reductive group G satisfies $C(h) = h^{-1}$ for all h in some Cartan subgroup of G. Furthermore, C takes any semisimple[1] element to a conjugate of its inverse. Consequently, for any algebraic representation π of G, π^C is isomorphic to the contragredient π^*.

We are interested in the existence, and properties, of rational Chevalley involutions.

Definition 1.1 *Suppose G is defined over a field F.*

(1) A rational Chevalley involution of G is one that is defined over F.

(2) We say an involution of $G(F)$ is dualizing if it takes every semisimple element to a $G(F)$-conjugate of its inverse.

Over an algebraically closed field every rational Chevalley involution is dualizing, and any two are conjugate by an inner automorphism. However, if F is not algebraically closed, since not all Cartan subgroups of $G(F)$ are conjugate, neither result is true in general. We are primarily interested in dualizing Chevalley involutions.

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[1] George Lusztig pointed out this is true for all elements [6].

For certain classical groups, over any local field, there is a dualizing Chevalley involution by [7, Chapitre 4].

Our main result is the existence of dualizing Chevalley involutions in general when $F = \mathbb{R}$. Not all of these are conjugate by an inner automorphism of $G(\mathbb{R})$ (see Example 2.3). In order to have a uniqueness result, we impose a further restriction. A Cartan subgroup of $G(\mathbb{R})$ is said to be fundamental if it is of minimal split rank. Such a Cartan subgroup is the "most compact" Cartan subgroup, and is unique up to conjugation by $G(\mathbb{R})$.

Theorem 1.2 *Suppose G is defined over $\mathbb{R}$. There is an involution C of $G(\mathbb{R})$ such that $C(h) = h^{-1}$ for all h in some fundamental Cartan subgroup of $G(\mathbb{R})$. Any such involution is the restriction of a rational Chevalley involution of G, and is dualizing (Definition 1.1). Any two such involutions are conjugate by an inner automorphism of $G(\mathbb{R})$.*

If G is semisimple and simply connected this is due to Vogan [3, Chapter I, Section 7]. The proof of the Theorem is similar, carried over to a more general setting.

Definition 1.3 *We refer to an involution of $G(\mathbb{R})$ satisfying the conditions of the Theorem as a fundamental Chevalley involution of G or of $G(\mathbb{R})$.*

Since all fundamental Chevalley involutions are conjugate by an inner automorphism of $G(\mathbb{R})$, we may safely refer to *the* fundamental Chevalley involution.

Corollary 1.4 *Suppose π is an irreducible representation of $G(\mathbb{R})$, and C is the fundamental Chevalley involution. Then $\pi^C \simeq \pi^*$.*

Over a p-adic field it is not always obvious, at least to this author, that there is a rational Chevalley involution, not to mention a dualizing one. In any event, the dualizing condition is quite restrictive. For example, if $G(F)$ is split, there are are many $G(F)$-conjugacy classes of involutions C such that $C(h) = h^{-1}$ for h in a split Cartan subgroup. Most of these are not dualizing. In fact, if G is a split exceptional group of type G_2, F_4 or E_8 over a p-adic field there is *no* dualizing involution. See Example 2.5.

The map $\pi \to \pi^*$ defines an involution on L-packets. The main result of [2] is that, on the dual side, this involution is given by the Chevalley involution of ${}^L G$. See Section 4. It follows that there is an elementary condition for every L-packet to be self-dual.

Proposition 1.5 *Every L-packet for $G(\mathbb{R})$ is self-dual if and only if $-1 \in W(G,H)$.*

Here H is any Cartan subgroup of G, and $W(G,H) = \mathrm{Norm}_G(H)/H$ is the (absolute) Weyl group.

Now consider the finer question, of whether every irreducible representation of $G(\mathbb{R})$ is self-dual. Let $H_f(\mathbb{R})$ be a fundamental Cartan subgroup of $G(\mathbb{R})$, and let $W(G(\mathbb{R}), H_f(\mathbb{R})) = \mathrm{Norm}_{G(\mathbb{R})}(H_f(\mathbb{R}))/H_f(\mathbb{R})$.

Theorem 1.6 *Every irreducible representation of $G(\mathbb{R})$ is self-dual if and only if $-1 \in W(G(\mathbb{R}), H_f(\mathbb{R}))$.*

The condition is equivalent to: every semisimple element of $G(\mathbb{R})$ is $G(\mathbb{R})$-conjugate to its inverse. We give some information about when this condition holds in Section 4. For example, suppose $G(\mathbb{R})$ is connected, $H(\mathbb{R})$ is compact, and let K be the complexification of a maximal compact subgroup of $G(\mathbb{R})$. Then $W(G(\mathbb{R}), H(\mathbb{R}))$ is the Weyl group of the root system of K. One can then look up this root system in a table, for example [8], and check if it contains -1.

Corollary 1.7 *Every irreducible representation of $G(\mathbb{R})$ is self-dual if and only if every irreducible representation of $K(\mathbb{R})$ is self dual, and, if $G(\mathbb{R})$ does not contain a compact Cartan subgroup, $-1 \in W(G,H)$.*

If $-1 \in W(G,H)$, and G is of adjoint type, then every irreducible representation of $G(\mathbb{R})$ is self dual.

For a more precise version, and some examples, see Section 4, especially Corollaries 4.7 and 4.9.

We next give an application to Frobenius Schur indicators. If π is an irreducible, self-dual representation of $G(\mathbb{R})$, set the Frobenius Schur indicator $\epsilon(\pi)$ of π, to be ± 1, depending on whether π admits an invariant symmetric, or skew-symmetric, bilinear form. Write χ_π for the central character of π. Let ${}^\vee\rho$ be one-half the sum of any set of positive co-roots. Then $z({}^\vee\rho) = \exp(2\pi i {}^\vee\rho)$ is in the center of $G(\mathbb{R})$.

Theorem 1.8 *Suppose every irreducible representation of $G(\mathbb{R})$ is self-dual. Then, for any irreducible representation π, $\epsilon(\pi) = \chi_\pi(z({}^\vee\rho))$.*

In particular, if $-1 \in W(G, H)$ and G is of adjoint type, then every irreducible representation of $G(\mathbb{R})$ is self-dual and orthogonal.

This paper is a complement to [2], which considers the action of the Chevalley involution on the dual group, and its relation to the contragredient. See [2, Remark 7.5].

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2 Split Groups

We start by defining Chevalley involutions. See [2, Section 2] for details. Throughout this paper G denotes a connected, reductive algebraic group. We may identify it with its complex points $G(\mathbb{C})$.

By a *Chevalley involution* of G we mean an involution C of G satisfying $C(h) = h^{-1}$ for all h in some Cartan subgroup H. Any two such involutions are conjugate by an inner automorphism.

Fix a pinning $\mathcal{P} = (H, B, \{X_\alpha\})$: $H \subset B$ are Cartan and Borel subgroups of G, respectively, and (for α a simple root) X_α is in the α-weight space of $\mathrm{Lie}(H)$ acting on $\mathrm{Lie}(G)$. For α a simple root let $X_{-\alpha}$ be the unique $-\alpha$-weight vector satisfying $[X_\alpha, X_{-\alpha}] = {}^\vee\alpha \in \mathrm{Lie}(H)$.

The choice of $\mathcal{P}$ determines a unique Chevalley involution C, satisfying $C(h) = h^{-1}$ ($h \in H$) and $C(X_\alpha) = X_{-\alpha}$ (α simple).

Now suppose G is semisimple and simply connected, and $G(F)$ is split. Generators and relations for $G(F)$ are given by [12, Théorèm 3.2] (see [13]). The generators are $x_\alpha(u)$ for α a root, and $u \in F$, and these satisfy certain relations. It is easy to check that the map $C(x_\alpha(u)) = x_{-\alpha}(u)$ preserves the defining relations of $G(F)$, and the resulting automorphism satisfies $C(h) = h^{-1}$ for h in a split Cartan subgroup $H(F)$.

Lemma 2.1 *Suppose G is semisimple and simply connected, and $G(F)$ is split. Let $H(F)$ be a split Cartan subgroup. Then there is a rational Chevalley involution satisfying $C(h) = h^{-1}$ for all $h \in H(F)$.*

Remark 2.2 The same result holds *a fortiori* for the nonlinear covering group Δ of $G(F)$ of [12, Théorèm 3.1], which is obtained by dropping some relations from those for $G(F)$.

This is a somewhat weak result. Not every rational Chevalley involution is dualizing, and not all dualizing Chevalley involutions are conjugate by an inner automorphism of $G(F)$. Both these facts are illustrated by a simple example. For $g \in G$, let $int(g)$ be conjugation by g.

Example 2.3 Let $G(F) = SL(2,F)$. Let $H_s(F)$ be the diagonal (split) Cartan subgroup. Let $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and let $C = \mathrm{int}(\sigma)$. Then $C(g) = {}^t g^{-1}$ for all g, and in particular $C(g) = g^{-1}$ for all $g \in H_s(F)$.

Suppose $g \neq \pm I$ is contained in an anisotropic Cartan subgroup $H_a(F)$. Then if $-1 \notin F^{*2}$, not to g^{-1} (i.e. -1 is not in the Weyl group of $H_a(F)$). Therefore C is not dualizing.

On the other hand let $C' = \mathrm{int}(\mathrm{diag}(i,-i)\sigma)$. Then C' is rational and dualizing. Note that C' is an outer automorphism of $G(F)$ unless $-1 \in F^{*2}$.

Now replace $SL(2,F)$ with $G(F) = PGL(2,F)$. Both C, C' factor to inner automorphisms of $G(F)$. Since every semisimple element of $G(F)$ is $G(F)$-conjugate to its inverse, C, C' are both dualizing. However it is easy to see that C is not conjugate to C' by an inner automorphism of $G(F)$.

The fact that specifying an involution on the split Cartan subgroup is a weak condition is illustrated by the following result.

Lemma 2.4 *In the setting of Lemma 2.1, let $H_{ad}(F)$ be the split Cartan subgroup of the adjoint group $G_{ad}(F)$. The number of rational Chevalley involutions satisfying the conditions of Lemma 2.1, modulo conjugation by $G(F)$, is $|H_{ad}(F)/H_{ad}(F)^2|$.*

For example over $\mathbb{R}$ this number is $2^{rank(G)}$. Since we make no use of this we omit the proof.

Surprisingly, even for split groups, which have rational Chevalley involutions, there may be no dualizing involution. This is illustrated by the following example, which was pointed out by D. Prasad [9].

Example 2.5 Suppose F is p-adic and $G(F)$ is the split real form of G_2, F_4 or E_8. By Lemma 2.1 there is a rational Chevalley involution C of $G(F)$. However, $G(F)$ has no dualizing involution.

To see this, assume τ is a dualizing involution. Then $\pi^\tau \simeq \pi^*$ for all irreducible representations π. Every automorphism of $G(F)$ is inner (since $\mathrm{Out}(G) = 1$ and G is both simply connected and adjoint), so $\pi^\tau \simeq \pi$, and this would imply every irreducible representation is self-dual. However, there are irreducible representations of $G(F)$ which are not self-dual, coming from non-self dual cuspidal unipotent representations over the finite field.

Returning to our rational Chevalley involution C, if $g \in G(F)$ is semisimple, $C(g) \in G(F)$ is $G(\overline{F})$ conjugate to g^{-1}. So presumably every L-packet of $G(F)$ is self-dual. This is consistent with [2]. Also see [9].

3 Real Chevalley Involutions

Before discussing general real forms of G, we focus momentarily on the compact and split ones. In this section we identify G with its complex points.

Fix a pinning $\mathcal{P} = (H, B, \{X_\alpha\})$ and define $\{X_{-\alpha}\}$ as at the beginning of Section 2. Let σ_c be the unique antiholomorphic automorphism of G satisfying $\sigma(X_\alpha) = -X_{-\alpha}$. Then $G_c(\mathbb{R}) = G^{\sigma_c}$ is compact.

The Chevalley automorphism $C = C_{\mathcal{P}}$ commutes with σ_c. Therefore $\sigma_s = C\sigma_c$ is an antiholomorphic involution of G, and $G_s(\mathbb{R}) = G^{\sigma_s}$ is split. In other words, C is the Cartan involution of the split real form of G.

General real forms of G may be classified either by antiholomorphic or holomorphic involutions of G. The latter is provided by the theory of the Cartan involution. We make extensive use of this machinery. See [8, Section 5.1.4], [5], or [1, Section 3].

In particular, there is a bijection

(3.1)

$$\{\text{antiholomorphic involutions } \sigma\}/G \leftrightarrow \{\text{holomorphic involutions } \theta\}/G$$

(the quotients are by conjugation by G). If σ is an antiholomorphic involution, after conjugating by G we may assume it commutes with σ_c, and set $\theta = \sigma\sigma_c$. The other direction is similar.

Suppose $\sigma \leftrightarrow \theta$, and σ, θ commute. Let $G(\mathbb{R}) = G^\sigma$, $K = G^\theta$, and $K(\mathbb{R}) = K \cap G(\mathbb{R}) = G(\mathbb{R})^\theta = K^\sigma$. Then $K(\mathbb{R})$ is a maximal compact subgroup of $G(\mathbb{R})$, with complexification K.

$$\begin{tikzcd}
 & G \arrow[dl, "\theta"'] \arrow[dr, "\sigma"] & \\
G^\theta = K \arrow[dr, "\sigma"'] & & G(\mathbb{R}) = G^\sigma \arrow[dl, "\theta"] \\
 & K(\mathbb{R}) &
\end{tikzcd}$$

Write $\mathrm{Aut}(G), \mathrm{Out}(G)$ for the (holomorphic) automorphisms of G, and the outer automorphisms, respectively. The pinning $\mathcal{P}$ defines an injective map $\mathrm{Out}(G) \overset{s}{\hookrightarrow} \mathrm{Aut}(G)$. We say an automorphism of G is *distinguished* if it preserves $\mathcal{P}$, or equivalently, is in the image of s. If G is semisimple the distinguished automorphisms embed into the automorphisms of the Dynkin diagram, and this is a bijection if G is simply connected or adjoint.

Now fix a holomorphic involution θ of G. Let δ be the image of θ under the map $\mathrm{Aut}(G) \to \mathrm{Out}(G) \overset{s}{\hookrightarrow} \mathrm{Aut}(G)$. In particular δ is distinguished.

Lemma 3.2 *After conjugating by G we may assume*

$$\theta = int(h)\delta \quad \text{for some } h \in H^\delta \tag{3.3}$$

(where the superscript denotes the δ-fixed points).

For example if $\delta = 1$ this just amounts to the fact that every semisimple element is conjugate to an element of H.

Proof. By the definition of δ, $\theta = \mathrm{int}(g)\delta$ for some semisimple element $g \in G$. Then g is contained in some δ-stable Cartan subgroup H_1. Write $H_1 = T_1 A_1$ where T_1 (resp. A_1) is the identity component of H_1^θ (resp. $H_1^{-\theta}$). Since, for $h \in H_1$, $h(g\delta)h^{-1} = h\delta(h^{-1})g\delta$, we may assume the A_1 component of g is trivial, i.e. $h \in T_1$.

Let $K_\delta = G^\delta$. Use the subscript 0 to indicate the identity component. Then $H_0^\delta = (H^\delta)_0$ is Cartan subgroup of $K_{\delta,0}$. Now T_1 is a torus in $K_{\delta,0}$, and is therefore $K_{\delta,0}$-conjugate to a subgroup of H_0^δ. Therefore after conjugating by $K_{\delta,0}$ we may assume $\theta = \mathrm{int}(h)\delta$ for $h \in H_0^\delta$. □

With this choice of θ, H is defined over $\mathbb{R}$, and $H(\mathbb{R})$ is a fundamental Cartan subgroup of $G(\mathbb{R})$ (see the introduction). We say H is a fundamental Cartan subgroup of G with respect to θ.

For example $\delta = 1$ if and only if $H(\mathbb{R})$ is compact. For later use, we single out this class of groups. We say $G(\mathbb{R})$ is of equal rank if any of the following equivalent conditions hold: $G(\mathbb{R})$ contains a compact Cartan subgroup; $H(\mathbb{R})$ is compact; $\mathrm{rank}(K) = \mathrm{rank}(G)$; $\delta = 1$; or θ is an inner involution.

We now give the proof of Theorem 1.2, which we break up into steps. We first construct an involution of $G(\mathbb{R})$, restricting to -1 on a fundamental Cartan subgroup.

Lemma 3.4 *Let $H(\mathbb{R})$ be a fundamental Cartan subgroup. There is a rational Chevalley involution of G, satisfying $C(h) = h^{-1}$ for all $h \in H(\mathbb{R})$.*

Proof. Choose θ corresponding to σ by the bijection (3.1). By the Lemma, after conjugating σ and θ, we may assume $\theta = \mathrm{int}(h)\delta$, where δ is distinguished and $h \in H^\delta$.

Let $C = C_{\mathcal{P}}$, the Chevalley involution defined by the splitting $\mathcal{P}$. In particular $C(h) = h^{-1}$ for $h \in H$.

We claim:

(1) $\theta\sigma_c = \sigma_c\theta$;

(2) $\theta C = C\theta$.

Assume this for the moment. By (1) $\sigma = \theta\sigma_c$, so we compute:

$$C\sigma = C(\theta\sigma_c) = (\theta\sigma_c)C = \sigma C$$

since C commutes with θ (by (2)) and σ_c (see the beginning of the section). Therefore C is defined over $\mathbb{R}$.

We prove (1) and (2). For (1), compute:

$$\begin{aligned}(\theta\sigma_c)(X_\alpha) &= \mathrm{int}(h)\delta(-X_{-\alpha})\\ &= -\mathrm{int}(h)(X_{-\delta\alpha})\\ &= -(\delta\alpha)(h^{-1})X_{-\delta\alpha}\end{aligned} \tag{3.5)(a}$$

On the other hand

$$\begin{aligned}(\sigma_c\theta)(X_\alpha) &= \sigma_c(\mathrm{int}(h)X_{\delta\alpha})\\ &= \sigma_c((\delta\alpha)(h)X_{\delta\alpha})\\ &= -\overline{(\delta\alpha)(h)}X_{-\delta\alpha}\end{aligned} \tag{3.5)(b}$$

Now $\theta = \mathrm{int}(h)\delta$ is an involution, which implies $h\delta(h) \in Z(G)$ (here and elsewhere Z denotes the center). But $\delta(h) = h$, so $h^2 \in Z(G)$. This implies $\beta(h) = \pm 1$ for all roots. Therefore $(\delta\alpha)(h^{-1}) = \overline{(\delta\alpha)(h)}$, proving (1).

The proof of (2) is very similar:

$$(\theta C)(X_\alpha) = \mathrm{int}(h)\delta(X_{-\alpha}) = \mathrm{int}(h)X_{-\delta\alpha} = (\delta\alpha)(h^{-1})X_{-\delta\alpha} \tag{3.6)(a}$$

and

$$(C\theta)(X_\alpha) = C(\mathrm{int}(h)X_{\delta\alpha} = C(\delta(\alpha)(h)X_{\delta\alpha} = (\delta\alpha)(h)X_{-\delta\alpha} \tag{3.6)(b}$$

and these are equal since $h^2 \in Z(G)$. $\square$

Now we show the Chevalley involution just constructed is dualizing (Definition 1.1).

Lemma 3.7 *Suppose C satisfies the conditions of Lemma 3.4. Then C is dualizing, i.e. it takes every semisimple element to a $G(F)$-conjugate of its inverse.*

Proof. This is true for g in the fundamental Cartan subgroup $H(\mathbb{R})$. We obtain the result on other Cartan subgroups using Cayley transforms.

We proceed by induction, so change notation momentarily, and assume H is any θ and σ-stable Cartan subgroup, such that $C(h)$ is $G(\mathbb{R})$-conjugate to h^{-1} for all $h \in H(\mathbb{R})$. Taking h regular, we see there is $g \in \mathrm{Norm}_{G(\mathbb{R})}(H(\mathbb{R}))$ such that, if $\tau = \mathrm{int}(g) \circ C$, then $\tau|_{H(\mathbb{R})} = -1$.

Suppose α is a root of H. Let G_α be the derived group of the centralizer of the kernel of α, and set $H_\alpha = H \cap G_\alpha$. Thus G_α is locally isomorphic to $SL(2)$, and $H = \ker(\alpha) H_\alpha$.

Now assume α is a noncompact imaginary root, which amounts to saying that G_α is θ, σ stable, $G_\alpha(\mathbb{R})$ is split, and $H_\alpha(\mathbb{R})$ is a compact Cartan subgroup of $G_\alpha(\mathbb{R})$. Replace H_α with a θ, σ-stable split Cartan subgroup H'_α of G_α. Since τ normalizes G_α, and is defined over $\mathbb{R}$, $\tau(H'(\mathbb{R}))$ is another split Cartan subgroup of $G_\alpha(\mathbb{R})$. Therefore we can find $x \in G_\alpha(\mathbb{R})$ so that $x(\tau(h))x^{-1} = h^{-1}$ for all $h \in H'_\alpha(\mathbb{R})$.

Let $H' = \ker(\alpha) H'_\alpha$. Then $(\mathrm{int}(x) \circ \tau)(h) = h^{-1}$ for all $h \in H'(\mathbb{R})$.

Every Cartan subgroup of $H(\mathbb{R})$ is obtained, up to conjugacy by $G(\mathbb{R})$, by a series of Cayley transforms from the fundamental Cartan subgroup. The result follows. $\square$

Finally, the uniqueness statement of Theorem 1.2 comes down to the next Lemma.

Lemma 3.8 *Suppose τ is an automorphism of $G(\mathbb{R})$ such that the restriction of τ to a fundamental Cartan subgroup $H(\mathbb{R})$ is trivial. Then $\tau = int(h)$ for some $h \in H(\mathbb{R})$.*

Proof. Since both $\mathbb{R}$ and $\mathbb{C}$ play a role here we write $G(\mathbb{C})$ to emphasize the complex group. After complexifying, τ is an automorphism of $G(\mathbb{C})$ which is trivial on $H(\mathbb{C})$. It is well known that $\tau = \mathrm{int}(h)$ for some $h \in H(\mathbb{C})$ (for example see [2, Lemma 2.4]). It is enough to show that $h \in H(\mathbb{R})Z(G(\mathbb{C}))$.

Since τ normalizes $G(\mathbb{R})$, $\sigma(h) = hz$ for some $z \in Z(G(\mathbb{C}))$. Writing p for the map to the adjoint group, this says $p(h) \in H_{ad}(\mathbb{R})$. It is well known that $H_{ad}(\mathbb{R})$ is connected (this is where we use that $H(\mathbb{R})$ is fundamental), so the map $p : H(\mathbb{R}) \to H_{ad}(\mathbb{R})$ is surjective. Therefore we can find $h' \in H(\mathbb{R})$ with $p(h') = p(h)$, i.e. $h = h'z \in H(\mathbb{R})Z(G(\mathbb{C}))$. $\square$

Lemma 3.9 *Any two automorphisms of $G(\mathbb{R})$, restricting to -1 on a fundamental Cartan subgroup, are conjugate by an inner automorphism of $G(\mathbb{R})$.*

Proof. Suppose τ, τ' satisfying the conditions, with respect to a fundamental Cartan subgroup $H(\mathbb{R})$. By the previous Lemma $\tau' = \mathrm{int}(h) \circ \tau$ for some $h \in H(\mathbb{R})$. Since $H(\mathbb{R})$ is connected, choose $x \in H(\mathbb{R})$ with $x^2 = h$. Then $\tau' = \mathrm{int}(x) \circ \tau \circ \mathrm{int}(x^{-1})$. □

This completes the proof of Theorem 1.2.

4 Groups for which every representation is self-dual

We first consider the elementary question of when every L-packet is self-dual (Proposition 1.5).

Fix a real form $G(\mathbb{R})$ of G, and choose K as usual (see Section 2). Let $\mathfrak{g} = \mathrm{Lie}(G)$. By an irreducible representation of $G(\mathbb{R})$ we mean an irreducible $(\mathfrak{g}, K)$-module, or equivalently an irreducible admissible representation of $G(\mathbb{R})$ on a complex Hilbert space. See [14, Section 0.3].

Proof of Proposition 1.5. Suppose an L-packet Π is defined by an admissible homomorphism $\phi : W_{\mathbb{R}} \to {}^LG$. By [2, Theorem 1.3] the contragredient L-packet corresponds to $C \circ \phi$, where C is the Chevalley automorphism of ${}^LG$. Therefore every L-packet is self-dual if and only if this action is trivial, up to conjugation by ${}^\vee G$, i.e. the Chevalley automorphism is inner for ${}^\vee G$. This is the case if and only if $-1 \in W({}^\vee G, {}^\vee H) \simeq W(G, H)$. □

Remark 4.1 By the classification of root systems, $-1 \in W(R)$, the Weyl group of an irreducible root system, if and only if R is of type A_1, B_n, C_n, D_{2n}, F_4, G_2, E_7, or E_8. It is worth noting that if G is simple and simply connected, $-1 \in W(G, H)$ if and only if $Z(G)$ is a two-group (one direction is obvious, and the other is case-by-case).

We are interested in real groups $G(\mathbb{R})$ for which every irreducible representation is self-dual. By Proposition 1.5 an obvious necessary condition is $-1 \in W(G, H)$. We first give a necessary and sufficient condition, and then give more detail in some special cases.

Fix a Cartan involution θ for $G(\mathbb{R})$, and set $K = G^\theta$, as in Section 3. Let H_K be a Cartan subgroup K, and let $H_f = \mathrm{Cent}_G(H_K)$, a fundamental Cartan subgroup of G with respect to θ (see the discussion before Lemma 3.4). For example, choose θ as in Lemma 3.2. Then H_f is the fixed Cartan of the pinning $\mathcal{P}$, and $H_K = H_f \cap K = H_f^\delta$.

Consider the groups

$$W(K, H_f) = \mathrm{Norm}_K(H_f)/H_K \tag{4.2)(a}$$

and

$$W(G(\mathbb{R}), H_f(\mathbb{R})) = \mathrm{Norm}_{G(\mathbb{R})}(H_f(\mathbb{R}))/H_f(\mathbb{R}). \tag{4.2)(b}$$

These are isomorphic. Finally, consider

$$W(K, H_K) = \mathrm{Norm}_K(H_K)/H_K. \tag{4.2)(c}$$

This is defined solely in terms of K; the difference between (a) and (c) is whether we consider an element to be an automorphism of H_f or H_K (see the next Remark). This is also isomorphic to (a) and (b), and is useful in computing these groups.

Some care is required here due to the fact that K, equivalently $G(\mathbb{R})$, may be disconnected. If K is connected then $W(K, H_K)$ is the Weyl group of the root system of H_K in K, but otherwise it is not necessarily the Weyl group of a root system.

We are interested in the condition $-1 \in W(K, H_f)$. We need to keep in mind the following dangerous bend concerning the meaning of -1.

Remark 4.3 Suppose $-1 \in W(K, H_K)$. By definition this means there is an element $g \in \mathrm{Norm}_K(H_K)$ such that $ghg^{-1} = h^{-1}$ for all $h \in H_K$. However, although g normalizes H_f, it is not necessarily the case that $ghg^{-1} = h^{-1}$ for all $h \in H_f \supset H_K$.

In other words, if $\mathrm{rank}(K) \neq \mathrm{rank}(G)$, $-1 \in W(K, H_K)$ does not imply $-1 \in W(K, H_f)$, even though these two groups are isomorphic.

Example 4.4 Let $G = SL(3, \mathbb{C})$, $G(\mathbb{R}) = SL(3, \mathbb{R})$. Then $-1 \notin W(G, H)$, so *a fortiori* $-1 \notin W(K, H)$. On the other hand $K = SO(3, \mathbb{C})$, $W(K, H_K)$ is the Weyl group of type A_1, and $-1 \in W(K, H_K)$.

In natural coordinates $H = \{(z, w) \mid z, w \in \mathbb{C}^*\}$, the Weyl group action is $(z, w) \to (w, z)$, and this acts by -1 on $H_K = \{(z, z^{-1})\}$, but not on H.

If K is connected, it is an elementary root system check to determine if $-1 \in W(K, H_K)$. In the equal rank case this is all that is needed, although in the unequal rank case some care is required to determine if $-1 \in W(K, H)$.

Let K_0 be the identity component of K. It is often the case that $-1 \notin W(K_0, H)$, but $-1 \in W(K, H)$. See below for some examples.

We now state necessary and sufficient conditions for every representation of $G(\mathbb{R})$ to be self-dual.

Theorem 4.5 *Fix a real form $G(\mathbb{R})$ of G, and let H_f be a fundamental Cartan subgroup of G. Every irreducible representation of $G(\mathbb{R})$ is self-dual if and only if $-1 \in W(K, H_f)$.*

By the equality of (4.2)(a) and (b) this gives Theorem 1.6.

Proof. It is straightforward to see that every irreducible representation is self-dual if and only if

(4.6) every semisimple element is $G(\mathbb{R})$-conjugate to its inverse.

Assume $-1 \in W(K, H_K) \simeq W(G(\mathbb{R}), H_f(\mathbb{R}))$, so there is an inner automorphism τ of $G(\mathbb{R})$ acting by -1 on $H_f(\mathbb{R})$. By Theorem 1.2, if g is semisimple $\tau(g)$ is $G(\mathbb{R})$-conjugate to g^{-1}. Since τ is inner this gives (4.6).

Conversely suppose (4.6) holds. Let h be a regular element of $H_f(\mathbb{R})$. Then $h^{-1} = xhx^{-1}$ for some $x \in G(\mathbb{R})$, and by regularity x normalizes $H_f(\mathbb{R})$. Therefore $-1 \in W(G(\mathbb{R}), H_f(\mathbb{R})) \simeq W(K, H_K)$. $\square$

Here is a statement in terms of representations of K. This is the first part of Corollary 1.7.

Corollary 4.7 *Every irreducible representation of $G(\mathbb{R})$ is self-dual if and only if every irreducible representation of K is self-dual and, if $G(\mathbb{R})$ is not of equal rank, $-1 \in W(G, H)$.*

Remark 4.8 If every irreducible representation of K is self-dual then $-1 \in W(K, H)$. If $\mathrm{rank}(G) = \mathrm{rank}(K)$ this implies $-1 \in W(G, H)$, but not otherwise. This explains why the final condition is necessary.

For example, suppose $G(\mathbb{R}) = SL(2n+1, \mathbb{R})$, $K = SO(2n+1, \mathbb{C})$. Then $-1 \in W(K, H_K)$, and every irreducible representation of $SO(2n+1, \mathbb{C})$ is self-dual. However this is not the case (for example for minimal principal series) for $SL(2n+1, \mathbb{R})$, since $-1 \notin W(G, H)$.

Proof. Every irreducible representation μ of $K(\mathbb{R})$ is the unique lowest $K(\mathbb{R})$-type of an irreducible representation π of $G(\mathbb{R})$ [17]. Since the lowest $K(\mathbb{R})$-type of π^* is μ^*, $\pi \simeq \pi^*$ implies $\mu \simeq \mu^*$. This proves one direction.

Conversely, by Theorem 4.5 we need to show every irreducible representation of K is self-dual implies $-1 \in W(K, H)$.

We first show that $-1 \in W(K, H_K)$ and $-1 \in W(G, H)$ implies $-1 \in W(K, H)$. This is obvious if $H_K = H$ (the equal rank case). Otherwise (here we need the assumption that $-1 \in W(G, H)$) choose $g \in G$ such that $ghg^{-1} = h^{-1}$ for all $h \in H$. Also choose $k \in K$ satisfying $khk^{-1} = h^{-1}$ for all $h \in H_K$. Then $gk^{-1} \in \mathrm{Cent}_G(H_K) = H$. This implies $khk^{-1} = h^{-1}$ for all $h \in H$.

So it is enough to show that if every irreducible representation of K is self-dual then $-1 \in W(K, H_K)$. If K is connected this follows from Theorem 4.5 applied to K.

For $\lambda \in X^*(H_K)$ let π_λ be the irreducible representation of K_0 with extremal weight λ. Then $\pi^*_\lambda = \pi_{-\lambda}$.

Consider the induced representation $I = \mathrm{Ind}_{K_0}^{K}(\pi_\lambda)$. The restriction of I to K_0 contains π_λ. Since I is self-dual by hypothesis, this restriction also contains $\pi_{-\lambda}$.

It is easy to see that every extremal weight of the restriction of this representation to K_0 is $W(K, H_K)$-conjugate to λ (choose representatives of K/K_0 in $\mathrm{Norm}_K(H_K)$, and use the fact that K_0 is normal in K). Therefore $-\lambda$ is $W(K, H_K)$-conjugate to λ. Taking λ generic this implies $-1 \in W(K, H_K)$. $\square$

Here is a practical way to determine if every irreducible representation of $G(\mathbb{R})$ is self-dual.

First assume $G(\mathbb{R})$ is of equal rank (see the discussion after Lemma 3.2). Then θ is inner, so write $\theta = \mathrm{int}(x)$ for some $x \in G$, with $x^2 \in Z(G)$.

Assume for the moment that $-1 \in W(G, H)$; this implies $Z(G)$ is a two-group. We say the real form defined by θ is *pure* if $x^2 = 1$. Since $Z(G)$ is a two-group, this condition is independent of the choice of x such that $\theta = \mathrm{int}(x)$. (In other words, although purity is typically only well-defined as a property of *strong* real forms [1, Definition 5.5], it is a well-defined property of real forms provided $-1 \in W(G, H)$.) Every real form is pure if G is adjoint.

Corollary 4.9 *Assume $G(\mathbb{R})$ is simple. Every irreducible representation of*

$G(\mathbb{R})$ is self-dual if and only if $-1 \in W(G,H)$ and, if $G(\mathbb{R})$ is of equal rank, it is a pure real form.

The second statement of Corollary 1.7 is a special case.

Proof. First assume we are in the equal rank case. By Theorem 1.2 we have to show

$$-1 \in W(G,H), x^2 = 1 \Leftrightarrow -1 \in W(K,H). \tag{4.10}$$

After conjugating by G we may assume $x \in H$. Suppose $g \in G$ satisfies $ghg^{-1} = h^{-1}$ for all $h \in H$. Then $\theta_x(g) = xgx^{-1} = x(gx^{-1}g^{-1})g = x^2g$. Therefore $g \in K$ if and only if $x^2 = 1$.

Now suppose $G(\mathbb{R})$ is not of equal rank. We have to show

$$-1 \in W(G,H) \Leftrightarrow -1 \in W(K,H). \tag{4.11}$$

The implication $\Leftarrow$ is obvious.

First assume $G(\mathbb{R}) = G_1(\mathbb{C})$, i.e. a complex group, viewed as a real group by restriction of scalars. Then, if H_1 is a Cartan subgroup of G_1, $G = G_1 \times G_1, H = H_1 \times H_1, K = G_1^\Delta$ (embedded diagonally). It follows immediately that $-1 \in W(K,H)$ if and only if $-1 \in W(G,H)$.

Finally assume $G(\mathbb{R})$ is unequal rank, but not complex. Then G is of type A_n ($n \geq 2$), D_n or E_6. But then $-1 \in W(G,H)$ only in type D_{2n}. This leaves only the groups locally isomorphic to $SO(p,q)$ with p,q odd and $p+q = 0 \pmod 4$.

Let $G(\mathbb{R}) = Spin(p,q)$ with $p+q = 4n$. It is enough to show $-1 \in W(K,H)$, since $W(K,H)$ is, if anything, larger if G is not simply connected. Note that K is connected, of type $B_r \times B_s$, and $-1 \in W(K,H_K)$. The only remaining issue is to check that $-1 \in W(K,H)$; here $\text{rank}(H) = \text{rank}(H_K)+1$. This is a straightforward check. It essentially comes down to the case of $Spin(3,1)$, for which it is easy to see, since $Spin(3,1) \simeq SL(2,\mathbb{C})$. $\square$

With a little effort we can deduce the following list from Corollary 4.9.

First assume G is simple, and $G(\mathbb{R})$ is equal rank. If G is adjoint it is only a question of whether $-1 \in W(G,H)$. If G is simply connected we need to check if -1 is in the Weyl group of the root system of K, which is easy, for example by the tables [8, pp. 312-317]. This leaves only the intermediate groups of type D_n, which require some case-by-case checking.

In the unequal rank case, we only need to consider complex groups, and (up to isogeny) $SO(p,q)$ with p,q odd.

Suppose $G(\mathbb{R})$ is simple. Then every irreducible representation of $G(\mathbb{R})$ is self-dual if and only if it is on the following list.

(1) A_n: $SO(2,1)$, $SU(2)$ and $SO(3)$.

(2) B_n: Every real form of the adjoint group, $Spin(2p, 2q+1)$ (p even).

(3) C_n: Every real form of the adjoint group, all $Sp(p, q)$.

(4) D_{2n+1}: none.

(5) D_{2n}, equal rank: $Spin(2p, 2q)$ (p, q even); all $SO(2p, 2q)$ ($p + q = 2n$) $\overline{SO}(2p, 2q)$ (p, q even); $\overline{SO}^*(4n)$; all adjoint groups: $PSO(2p, 2q)$ ($p+q = 2n$) and $PSO^*(4n)$.

(6) D_{2n}, unequal rank: all real forms, i.e. all groups locally isomorphic to $SO(2p + 1, 2q + 1)$ ($p + q$ odd).

(7) E_6: none.

(8) E_7: Every real form of the adjoint group, the simply connected compact group.

(9) G_2, F_4, E_8: every real form.

(10) complex groups of type $A_1, B_n, C_n, D_{2n}, G_2, F_4, E_7, E_8$ (see Remark 4.1).

Here $\overline{SO}(2n, \mathbb{C})$ denotes the group $Spin(2n, \mathbb{C})/A$ where $A \simeq \mathbb{Z}/2\mathbb{Z}$ is the "non-diagonal" subgroup of $Z(Spin(2n, \mathbb{C})) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and $\overline{SO}(p, q)$ and $\overline{SO}^*$ are its real forms.

5 Frobenius Schur Indicators

Suppose π is an irreducible self-dual representation of G. Choosing an isomorphism $T : \pi \to \pi^*$, $\langle v, w\rangle = T(v)(w)$ a non-degenerate, invariant, bilinear form, unique up to scalar. It is either symmetric or skew-symmetric. The Frobenius Schur indicator $\epsilon(\pi)$ of π is defined to be 1 or -1, accordingly. It is of some interest to compute this invariant. For example see [10].

It is well known that if π is finite dimensional $\epsilon(\pi)$ is given by a particular value of its central character [4, Ch. IX, §7.2, Proposition 1]. Here is an elementary proof. This is a refinement of one of the proofs of [11, Section 1, Lemma 2]; we use the Tits group to identify the central element in question.

Let ${}^\vee\rho$ be one-half the sum of any set of positive co-roots, and set

$$z({}^\vee\rho) = \exp(2\pi i {}^\vee\rho). \tag{5.1}$$

Not only is $z({}^\vee\rho)$ central in G, it is fixed by every automorphism of G. In particular $z \in Z(G(\mathbb{R}))$ for any real form of G. If it is necessary to specify the group in question we will write $z({}^\vee\rho_G)$.

Lemma 5.2 *Let w_0 be the long element of $W(G,H)$. There is a representative $g \in Norm_G(H)$ of w_0 satisfying $g^2 = z({}^\vee\rho)$. Furthermore, if $w_0 = -1$, this holds for any representative of w_0.*

Proof. We use the Tits group. Fix a pinning $\mathcal{P} = (H, B, \{X_\alpha\})$ for G (see Section 2). This defines the Tits group $\mathcal{T}$, a subgroup of $\mathrm{Norm}_G(H)$ mapping surjectively to $W(G,H)$. Every element w of the Weyl group has a canonical inverse image $\sigma(w) \in \mathcal{T}$. See [2, Section 5].

Let $g = \sigma(w_0)$. By [2, Lemma 5.4], $g^2 = z({}^\vee\rho)$. Any other representative is of the form hg for some $h \in H$. If $w_0 = -1$ then $(hg)^2 = h(ghg^{-1})g^2 = (hh^{-1})g^2 = g^2$. $\square$

Lemma 5.3 *Assume G is a connected, reductive complex group. Suppose π is an irreducible, finite dimensional, self-dual representation of G. Then*

$$\epsilon(\pi) = \chi_\pi(z({}^\vee\rho)). \tag{5.4}$$

Proof. For any vectors u, w in the space V of π we have

$$\langle u, w\rangle = \epsilon(\pi)\langle w, u\rangle. \tag{5.5(a)}$$

Suppose $g \in G, g^2 \in Z(G)$, and $v \in V$. Set $u = \pi(g^2)v, w = \pi(g)v$:

$$\begin{aligned} \chi_\pi(g^2)\langle v, \pi(g)v\rangle &= \langle \pi(g^2)v, \pi(g)v\rangle \quad \text{(since } g^2 \text{ is central} \\ &= \langle \pi(g)v, v\rangle \quad \text{(by invariance)} \\ &= \epsilon(\pi)\langle v, \pi(g)v\rangle \quad \text{(by (a))}. \end{aligned} \tag{5.5(b)}$$

We conclude

$$g^2 \in Z(G),\ \langle v, \pi(g)v\rangle \neq 0 \Rightarrow \epsilon(\pi) = \chi_\pi(g^2). \tag{5.5(c)}$$

Fix a Cartan subgroup H, and for $\lambda \in X^*(H)$ write V_λ for the corresponding weight space. It is easy to see $\langle V_\lambda, V_{-\lambda}\rangle \neq 0$.

Let λ be the highest weight, so V_λ is one-dimensional. Let w_0 be the long element of the Weyl group. Then π^* has highest weight $-w_0\lambda$; since π is self-dual this implies $-\lambda = w_0\lambda$.

Choose $g \in \mathrm{Norm}_G(H)$ as in Lemma 5.2, so $g^2 = z({}^\vee\rho)$, and $0 \neq v \in V_\lambda$. Then $\pi(g)(v) \in V_{-\lambda}$. Since $V_{\pm\lambda}$ are one-dimensional $\langle v, \pi(g)v\rangle \neq 0$, so apply (5.5)(c). $\square$

We now consider the Frobenius Schur indicator for infinite dimensional representations. The basic technique is the following elementary observation, which appears in [10].

Suppose $H \subset G$ are groups, π is a self-dual representation of G, π_H is a self-dual representation of H, and π_H occurs with multiplicity one in $\pi|_H$. Then $\epsilon(\pi) = \epsilon(\pi_H)$. We first apply this to G and K, and later to K and its identity component.

The next Lemma is a special case of the main result of this section (Theorem 5.16), but it is worth stating separately since it clearly illustrates the main idea.

As usual given a real form $G(\mathbb{R})$ fix a corresponding Cartan involution θ and let $K = G^\theta$.

Lemma 5.6 *Suppose every irreducible representation of $G(\mathbb{R})$ is self-dual (see Theorem 4.5). Also assume $G(\mathbb{R})$ is connected. If π is an irreducible representation then $\epsilon(\pi) = \chi_\pi(z({}^\vee\rho))$.*

Proof. The connected assumption is equivalent to K being connected. By Theorem 1.2 and Corollary 4.7 the self-duality assumption implies $-1 \in W(K, H)$, and every K-type is self-dual.

Let μ be a lowest K-type of π. Then μ has multiplicity one, and is self-dual, so by the comment above $\epsilon(\pi) = \epsilon(\mu)$. By Lemma 5.3, $\epsilon(\mu) = \chi_\mu(z({}^\vee\rho_K))$, where $z({}^\vee\rho_K)$ is defined by (5.1) applied to K. Let $g \in \mathrm{Norm}_G(H)$ be a representative of $-1 \in W(G, H)$, so by Lemma 5.2 $g^2 = z({}^\vee\rho)$. Now view g as a representative of $-1 \in W(K, H)$, in which case (by Lemma 5.2 applied to to K) we see $g^2 = z({}^\vee\rho_K)$.

Therefore $z({}^\vee\rho) = z({}^\vee\rho_K)$, and since $z({}^\vee\rho) \in Z(G)$, $\chi_\mu(z({}^\vee\rho)) = \chi_\pi(z({}^\vee\rho))$, independent of μ. Thus

$$\epsilon(\pi) = \epsilon(\mu) = \chi_\mu(z({}^\vee\rho_K)) = \chi_\mu(z({}^\vee\rho)) = \chi_\pi(z({}^\vee\rho)).$$

$\square$

A crucial aspect of the proof is that, for K connected, $-1 \in W(K,H)$ implies $z({}^\vee\rho) = z({}^\vee\rho_K)$. We need the surprising fact that, this is true without the first assumption.

Lemma 5.7 *Assume* $-1 \in W(K,H)$. *Then* $z({}^\vee\rho) = z({}^\vee\rho_K)$.

This is a bit subtle, as a a simple example shows.

Example 5.8 Let $G(\mathbb{R}) = SL(2,\mathbb{R})$, so $K = SO(2)$. Then $-1 \notin W(K,H)$, and $-I = z({}^\vee\rho) \neq z({}^\vee\rho_K) = I$.

On the other hand suppose $G(\mathbb{R}) = PSL(2,\mathbb{R}) = SO(2,1)$. Then $K = O(2)$, so $-1 \in W(K,H)$, and now $I = z({}^\vee\rho) = z({}^\vee\rho_K)$.

Proof. We may assume $G(\mathbb{R})$ is simple.

First assume $G(\mathbb{R})$ is equal rank. Recall (see the discussion in Section 2) $K = \mathrm{Cent}_G(x)$ for some $x \in H$. We will show x is of a particular form. We need a short digression on the Kac classification of real forms [8],[5].

Let $\widetilde{D}$ be the extended Dynkin diagram for G, with nodes $0,\dots,m$; roots $\alpha_0,\dots,\alpha_m$ ($-\alpha_0$ is the highest root); and labels $n_0 = 1, n_1, \dots, n_m$. The Dynkin diagram of K is obtained from $\widetilde{D}$ by deleting node j with label 2, or nodes j,k with label 1. In the second case, without loss of generality, we may assume $k = 0$, so both cases may be combined, as specifying a single node j with label $n_j = 1$ or 2.

Let ${}^\vee\lambda_j$ be the j^{th} fundamental weight for G. Then we can take $x = \exp(\pi i {}^\vee\lambda_j)$, i.e. $\mathrm{Cent}_G(x) = K$. Now let

(5.9)(a)
$$c = \begin{cases} \frac{N}{2} & n_j = 2 \\ N-1 & n_j = 1. \end{cases}$$

Except in type A_{2n}, which is ruled out since $-1 \in W(G,H)$, N is even, so $c \in \mathbb{Z}$.

It is an exercise in root systems to see that

(5.9)(b)
$${}^\vee\rho - {}^\vee\rho_K = c\,{}^\vee\lambda_j.$$

(For $i \neq 0, j$, both sides are 0 when paired with α_i, so this amounts to computing the pairing with α_0 and α_j.) Therefore

(5.9)(c)
$$x = \exp(\frac{\pi i}{c}({}^\vee\rho - {}^\vee\rho_K)).$$

Then $x^{2c} = z({}^\vee\rho)/z({}^\vee\rho_K)$.

By (4.10) we have:

$$-1 \in W(K,H) \Leftrightarrow x^2 = 1 \Rightarrow x^{2c} = 1 \Rightarrow z({}^\vee\rho) = z({}^\vee\rho_K). \tag{5.10}$$

A similar, but more elaborate, argument holds in the unequal rank case. Alternatively, we proceed in a more case-by-case fashion. If $G(\mathbb{R})$ is complex, then K is connected, and we have already treated this case, so assume G is of type D_{2n} (see the previous section). If G is simply connected then (assuming unequal rank) $-1 \in W(K,H)$, and K is connected, so again we have $z({}^\vee\rho) = z({}^\vee\rho_K)$. The result is then true *a fortiori* if G is not simply connected. This completes the proof. $\square$

We also need a generalization of Lemma 5.3.

Lemma 5.11 *Assume G is a connected, reductive complex group. Let $G^\dagger = G \rtimes \langle\delta\rangle$ where $\delta^2 \in Z(G)$ and δ acts on G by a Chevalley involution.*

Every irreducible finite dimensional representation $\pi^\dagger$ of $G^\dagger$ is self-dual, and if π is an irreducible constituent of $\pi^\dagger|_G$, then

$$\epsilon(\pi^\dagger) = \begin{cases} \epsilon(\pi) & \pi \simeq \pi^* \\ \chi_\pi(\delta^2) & \pi \not\simeq \pi^*. \end{cases} \tag{5.12}$$

Proof. The restriction of $\pi^\dagger$ is irreducible if and only if $\pi \simeq \pi^\delta$. Since δ acts by the Chevalley involution, this is equivalent to $\pi \simeq \pi^*$.

If $\pi \simeq \pi^*$ the result is clear. Otherwise, let λ be the highest weight of π. Then π^δ has extremal weight $-\lambda$, i.e. highest weight $-w_0\lambda$ where w_0 is the long element of the Weyl group. Since $\pi \not\simeq \pi^*$, $-w_0\lambda \neq \lambda$, so the λ-weight space of $\pi^\dagger$ is one-dimensional. The proof of Lemma 5.3 now carries through using δ, which interchanges the λ and $-\lambda$ weight spaces of $\pi^\dagger$. $\square$

We need to consider finite dimensional representations of the possibly disconnected group $K = G^\theta$. These groups are not badly disconnected, for example the component group is a two-group, and we need the following property of their representations.

Lemma 5.13 *Let μ be an irreducible, finite-dimensional, representation of K. Then $\mu|_{K_0}$ is multiplicity free.*

Proof. Suppose μ_0 is any irreducible summand of $\mu|_{K_0}$, and let $K_1 = \mathrm{Stab}_K(\mu_0)$. It is enough to show that μ_0 extends to an irreducible representation μ_1 of K_1. For then, by Mackey theory, $\mathrm{Ind}_{K_1}^K(\mu_1)$ is irreducible, so isomorphic to μ, and restricts to the sum of the distinct irreducible representations $\{\pi_0^x \mid x \in S\}$, where S is a set of representatives of K/K_1.

Choose Cartan and Borel subgroups $T \subset B_{K_0}$ of K_0. (We can arrange that $B_{K_0} = B \cap K_0$ and $T = H \cap K_0$).

Lemma 5.14 *We can choose elements $x_, \dots, x_n \in K$ such that:*

(1) $K = \langle K_0, x_1, \dots, x_n\rangle$;

(2) x_i normalizes B_{K_0} and T;

(3) The x_i commute with each other.

Remark 5.15 By a standard argument it is easy to arrange (1) and (2), the main point is (3). Alternatively, it is well known that we could instead choose the x_i to satisfy (1), (3) and that each x_i has order 2. It would be interesting to prove that one can satisfy all four conditions simultaneously, and perhaps even that conjugation by x_i is a distinguished involution of K_0.

Proof. Choose $x \in K\backslash K_0$. Then conjugation by x takes B_{K_0} to another Borel subgroup, which we may conjugate back to B_{K_0}. So after replacing x with another element in the same coset of K_0 we may assume x normalizes B_{K_0}. Conjugating again by an element of B_{K_0} we may assume x_i normalizes T. By induction this gives (1) and (2).

For (3), it is straightforward to reduce to the case when $G(\mathbb{R})$ is simple. Then a case-by-case check shows that $|K/K_0| \le 2$ except in type D_n. Furthermore the only exception is the adjoint group $PSO(2n, 2n)$, in which case the result can be easily checked. This is essentially [15, Proposition 9.7]. $\square$

Let $\lambda \in X^*(T)$ be the highest weight of μ_0 with respect to B_{K_0}. Then $\mu_0^{x_i}$ has highest weight $x_i\lambda$. So, after renumbering, we may write $K_1 = \langle K_0, x_1, \dots, x_r\rangle$ where $x_i\lambda = \lambda$ for $1 \le i \le r$.

Let V_λ be the (one-dimensional) highest weight space of μ_0. The group $T_1 = \langle T, x_1, \dots, x_r\rangle$ acts on V_λ. In the terminology of [16, Definition 1.14(e)], T_1 is a large Cartan subgroup of K_1, and [16, Theorem 1.17] implies that there is an irreducible representation μ_1 of K_1, containing the one-dimensional representation V_λ of T_1. Then $\mu_1|_{K_0} = \mu_0$. $\square$

Theorem 5.16 *Suppose every irreducible representation of $G(\mathbb{R})$ is self-dual (see Theorem 4.5). If π is an irreducible representation then*

$$\epsilon(\pi) = \chi_\pi(z({}^\vee\rho)). \tag{5.17}$$

Every irreducible representation is orthogonal if and only if $z({}^\vee\rho) = 1$. This holds if G is adjoint.

Proof. By Corollary 4.7 every K-type is self-dual, and $-1 \in W(K, H)$. Choose a minimal K-type μ. Since μ is self-dual and has multiplicity one, $\epsilon(\pi) = \epsilon(\mu)$.

Let μ_0 be an irreducible summand of $\mu|_{K_0}$. By Lemma 5.13 μ_0 has multiplicity one. If μ_0 is self-dual then $\epsilon(\mu) = \epsilon(\mu_0)$, and by Lemma 5.3 $\epsilon(\mu_0) = \chi_{\mu_0}(z({}^\vee\rho_K))$. By Lemma 5.7 this equals $\chi_{\mu_0}(z({}^\vee\rho))$.

Suppose μ_0 is not self-dual. Since $-1 \in W(K, H)$, choose a representative $g \in \mathrm{Norm}_K(H)$ of $-1 \in W(K, H)$, and let $K^\dagger = \langle K, g\rangle$. By Lemma 5.11 $\mu^\dagger = \mathrm{Ind}_{K_0}^{K^\dagger}(\mu_0)$ is irreducible, self-dual, and of multiplicity one in μ, so $\epsilon(\mu) = \epsilon(\mu^\dagger)$. Since $\mu_0 \not\simeq \mu_0^*$, by Lemma 5.3, $\epsilon(\mu^\dagger) = \chi_{\mu_0}(g^2)$. We can also think of g as a representative of $-1 \in W(G, H)$. Since G (unlike K) is (necessarily) connected, by Lemma 5.2, $g^2 = z({}^\vee\rho)$, so again $\epsilon(\mu) = \chi_{\mu_0}(z({}^\vee\rho))$.

As in the proof of Lemma 5.6, since $z({}^\vee\rho) \in Z(G(\mathbb{R}))$, $\chi_{\mu_0}(z({}^\vee\rho)) = \chi_\pi(z({}^\vee\rho))$. This completes the proof. □

References

[1] J. Adams and Fokko du Cloux. Algorithms for representation theory of real reductive groups. *J. Inst. Math. Jussieu*, 8(2):209–259, 2009.

[2] J. Adams and D. A. Vogan, Jr. The Contragredient. 2012. `arXiv:1201.0496`.

[3] A. Borel and N. Wallach. *Continuous cohomology, discrete subgroups, and representations of reductive groups*, volume 67 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2000.

[4] Nicolas Bourbaki. *Lie groups and Lie algebras. Chapters 7–9*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2005. Translated from the 1975 and 1982 French originals by Andrew Pressley.

[5] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 34 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. Corrected reprint of the 1978 original.

[6] G. Lusztig. Remarks on Springer's representations. *Represent. Theory*, 13:391–400, 2009.

[7] Colette Mœglin, Marie-France Vignéras, and Jean-Loup Waldspurger. *Correspondances de Howe sur un corps p-adique*, volume 1291 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1987.

[8] A. L. Onishchik and È. B. Vinberg. *Lie groups and algebraic groups*. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1990. Translated from the Russian and with a preface by D. A. Leites.

[9] D. Prasad. A 'relative' local langlands correspondence. preprint.

[10] D. Prasad and D. Ramakrishnan. Self-dual representations of division algebras and weil groups: A contrast. preprint.

[11] Dipendra Prasad. On the self-dual representations of a p-adic group. *Internat. Math. Res. Notices*, (8):443–452, 1999.

[12] Robert Steinberg. Générateurs, relations et revêtements de groupes algébriques. In *Colloq. Théorie des Groupes Algébriques (Bruxelles, 1962)*, pages 113–127. Librairie Universitaire, Louvain, 1962.

[13] Robert Steinberg. *Robert Steinberg*, volume 7 of *Collected Works*. American Mathematical Society, Providence, RI, 1997. Edited and with a foreword by J.-P. Serre.

[14] David A. Vogan, Jr. *Representations of real reductive Lie groups*, volume 15 of *Progress in Mathematics*. Birkhäuser Boston, Mass., 1981.

[15] David A. Vogan, Jr. Irreducible characters of semisimple Lie groups. IV. Character-multiplicity duality. *Duke Math. J.*, 49(4):943–1073, 1982.

[16] David A. Vogan, Jr. *Unitary representations of reductive Lie groups*, volume 118 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1987.

[17] David A. Vogan, Jr. Branching to a maximal compact subgroup. In *Harmonic analysis, group representations, automorphic forms and invariant theory*, volume 12 of *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.*, pages 321–401. World Sci. Publ., Hackensack, NJ, 2007.