Decompositions of saturated models of stable theories

M. C. Laskowski*
Department of Mathematics
University of Maryland
College Park, MD 20742 USA
mcl@math.umd.edu

S. Shelah
Institute of Mathematics
Hebrew University
Jerusalem, Israel
Department of Mathematics
Rutgers University
New Brunswick, NJ 08903 USA †

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Abstract

We characterize the stable theories T for which the saturated models of T admit decompositions. In particular, we show that countable, shallow, stable theories with NDOP have this property.

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In [7], prior to his work classifying the uncountable models of certain theories, the second author proved a structure theorem for the class of a-models (i.e., $\mathbf{F}^a_{\kappa_r(T)}$-saturated models in the notation of [7]) of a superstable theory with NDOP. Specifically, in Chapter X of [7] he proved that an a-model of such a theory is a-prime and a-minimal over a normal tree of models, where each node is a-prime over its predecessor and the realization of a regular type. Thus, among superstable theories, the notion of NDOP provides a dichotomy: Either the number of nonisomorphic a-models in each cardinality $\geq 2^{|T|}$ is maximal, or every a-model is determined up to isomorphism by a tree of invariants. It is natural to ask whether a similar dichotomy can be found for the larger class of stable theories. The main obstruction is that an arbitrary stable theory need not have many regular types. Because of this we relax the regularity requirement in Definition 1.7. Our main result, Theorem 1.8, characterizes the stable theories for which large saturated models admit decompositions in this weaker sense.

The first section of the paper states our findings. Section 2 gives some preparatory lemmas that hold for arbitrary stable theories. In Section 3 we work over a single independent tree and characterize when the a-prime model is a-minimal. In Section 4 we prove Theorem 1.8. Finally, in Section 5 we investigate the effect of restricting to a countable language. By using methods of descriptive set theory we derive unexpected (to us) consequences of NDOP (Theorem 1.11 and Corollary 1.12).

We assume some familiarity with the notions and notational conventions of stability theory, specifically the forking calculus and orthogonality. Knowledge of the material in any of the basic articles or texts in stability (e.g., [1], [5], or [6]) should be sufficient. Also, since many of the arguments that appear here are variants of what occur in the superstable, NDOP situation, it might be helpful for the reader to skim Chapter X of [7]. We assume that we are working in a large, saturated structure $\mathfrak{C}$ and that our language admits elimination of quantifiers, so the notions of submodel and elementary submodel are interchangeable. To ease notation we do not distinguish between elements of $\mathfrak{C}$ and finite tuples. We write $S(A)$ to denote the union of the Stone spaces $S_n(A)$ of complete types over A in n free variables. For brevity we sometimes write AB in place of $A \cup B$.

1 Statement of results

Hyp: Throughout this paper all theories T are stable and κ always denotes the cardinal $\kappa_r(T)$.

We work in the category of a-models of T. That is, M is an *a-model* if and only if every type that is almost over a subset of M of size less than κ is realized in M. An a-model M is *a-prime* over a set X if M embeds over X into any a-model N that contains X. We rely heavily on Theorems IV 3.12 and 4.14 of [7], which assert that a-prime models exist over any set X, and are unique up to isomorphism over X. An a-model M is *a-minimal* over X if there is no proper a-submodel of M containing X.

We first describe two species of trees of a-models and characterize when the a-prime model over the union of such a tree is in fact a-minimal over the union.

Definition 1.1 A *tree* I is a nonempty, downward closed subset of ${}^{<\omega}\delta$ for some ordinal δ. As notation, for $\eta, \nu \in I$, we write $\eta \trianglelefteq \nu$ if η is an initial segment of ν. For $\eta \neq \langle\rangle$, η^- denotes the (unique) immediate predecessor of η.

Definition 1.2 An *independent tree of sets* is a set $\{X_\eta : \eta \in I\}$ indexed by a tree I such that $X_\eta \subseteq X_\nu$ whenever $\eta \trianglelefteq \nu$ and $X_\eta \underset{X_{\eta^-}}{\perp} \bigcup\{X_\nu : \eta \not\trianglelefteq \nu\}$ for all $\eta \neq \langle\rangle$. As notation, $X_J = \bigcup\{X_\eta : \eta \in J\}$ for any subtree $J \subseteq I$.

An independent tree is *normal* if, in addition, $\mathrm{tp}(X_\nu/X_\eta) \perp X_{\eta^-}$ for all $\eta, \nu \in I$ satisfying $\eta \neq \langle\rangle$ and $\eta = \nu^-$.

Theorem 1.3 *Let $\{M_\eta : \eta \in I\}$ be any independent tree of a-models and let M_I^* be a-prime over M_I. Then the following properties are equivalent:*

1. *M_I^* is a-minimal over M_I;*
2. *M_I^* does not contain any infinite indiscernible sequences over M_I;*
3. *For all nonalgebraic $p \in S(M_I^*)$, $p \not\perp M_\eta$ for some $\eta \in I$;*
4. *For all types p, if $p \not\perp M_I^*$ then $p \not\perp M_\eta$ for some $\eta \in I$.*

Two corollaries follow easily from this theorem.

Corollary 1.4 *Let $\{M_\eta : \eta \in I\}$ be any independent tree of a-models and let M_I^* be a-prime over M_I. If M_I^* is a-minimal over M_I then M_J^* is a-minimal over M_J for any subtree $J \subseteq I$.*

Corollary 1.5 *Fix a cardinal $\lambda \geq \kappa$. Let $\{M_\eta : \eta \in I\}$ be any independent tree of λ-saturated models and let M_I^* be a-prime over M_I. If M_I^* is a-minimal over M_I then M_I^* is λ-saturated.*

Next we describe classes of theories T for which a-prime models over certain species of trees are always a-minimal. The strongest such property is the *minimality property for independent trees*, which asserts that for a given theory T, the a-prime model over any independent tree of a-models of T is a-minimal. We say that T has the *minimality property for normal trees* if this holds for all normal trees. We will see below that these notions coincide.

The following definitions are weakenings of these global notions. They only require that a-prime models be a-minimal for independent trees indexed by some very simple index sets.

Definition 1.6 For α any ordinal, let I_α be the tree of height two with a unique root and whose successors are indexed by α. In particular, I_2 denotes the 3-element tree with two incomparable elements. Let J denote the linearly ordered tree of length ω.

A theory T has NDOP if a-prime models over any independent tree of a-models indexed by I_2 are necessarily a-minimal. For μ any infinite cardinal, T has μ-NDOP if for all $\alpha < \mu$, every a-prime model over every independent tree of a-models indexed by I_α is a-minimal. T has NDIDIP if a-prime models over independent trees of a-models indexed by J are a-minimal. T has *normal NDIDIP* if a-prime models over a normal tree of a-models indexed by J are a-minimal.

The reader who is disgusted with the phrase 'normal NDIDIP' can relax – For stable theories with κ-NDOP, it is equivalent to NDIDIP.

An easy inductive argument shows that if T has NDOP, then T has ω-NDOP. Additionally, since every type over an a-model is based and stationary over a set of size $< \kappa$, it follows from Theorem 1.3 that if T has κ-NDOP then T has μ-NDOP for all cardinals μ. In particular, when T is superstable the notions of NDOP and μ-NDOP coincide. However, when T

is strictly stable there may be a gap between NDOP and κ-NDOP. It was a surprise to us to discover (see Theorem 1.11) that in fact the gap does not exist when T is countable.

The following notions are central to our attempts at finding invariants for a-models of stable theories.

Definition 1.7 A *partial decomposition* of an a-model M is a normal tree of a-submodels $\{M_\eta : \eta \in J\}$ of M, where $M_{\langle\rangle}$ is a-prime over $\emptyset$ and for every $\eta \neq \langle\rangle$, M_η is a-prime over $M_{\eta^-} \cup \{a_\eta\}$ for some finite tuple a_η. A *decomposition of* M is a partial decomposition of M such that M is a-prime and a-minimal over M_J. A partial decomposition is *small* if $|M_J^*| < |M|$. We say that the partial decomposition $\{N_\eta : \eta \in I\}$ *extends* $\{M_\eta : \eta \in J\}$ simply if J is a subtree of I and $N_\eta = M_\eta$ for all $\eta \in J$.

Theorem 1.8 *The following are equivalent for a stable theory T:*

1. *T has the minimality property for independent trees;*

2. *Every small partial decomposition of every saturated N of size $> 2^{|T|}$ extends to a decomposition of N;*

3. *T has κ-NDOP and NDIDIP;*

4. *T has κ-NDOP and normal NDIDIP;*

5. *T has the minimality property for normal trees.*

Recall that a tree I is *well-founded* if it does not have an infinite branch.

Proposition 1.9 *Suppose that T has κ-NDOP and $\{M_\eta : \eta \in I\}$ is an independent tree of a-models where the index tree I is well-founded. Then every a-prime model over M_I is a-minimal over M_I.*

A (stable) theory T is *shallow* if there is no increasing sequence $\langle M_n : n \in \omega\rangle$ of a-models of T such that M_{n+1} is a-prime over $M_n \cup \{a_n\}$ for some tuple a_n for every n and $\mathrm{tp}(M_{n+1}/M_n) \perp M_{n-1}$ for all $n > 0$. Clearly, if T is shallow and $\{M_\eta : \eta \in I\}$ is a decomposition of an a-model M, then the indexing tree I is well-founded.

Corollary 1.10 *If T has κ-NDOP and is shallow, then T has the minimality property for independent trees. In particular, such a theory satisfies NDIDIP.*

Until this point, the cardinality of the language of T was not relevant. By contrast, the countability of T plays a crucial role in the following theorem, as it allows us to employ methods of descriptive set theory (specifically that every analytic subset of a Borel set has the property of Baire).

Theorem 1.11 *If T is countable, then NDOP implies ω_1-NDOP (hence μ-NDOP for all cardinals μ).*

Our final Corollary follows immediately from the two preceding results.

Corollary 1.12 *T countable, NDOP, shallow implies NDIDIP.*

2 Lemmas about saturation, nonforking and orthogonality

In this section we prove some assorted lemmas about stable theories that will be used in the following sections. The first is an easy characterization of λ-saturation of models when $\lambda \geq \kappa$.

Lemma 2.1 *Suppose that $\lambda \geq \kappa$ and M is an a-model such that for every subset $A \subseteq M$ with $|A| < \lambda$ and every nonalgebraic $p \in S(A)$, there is a forking extension $q \in S(M)$. Then M is λ-saturated.*

Proof. The definition of an a-model implies that M is κ-saturated, so assume that $\lambda > \kappa$. Choose any $A \subseteq M$ with $|A| < \lambda$ and choose any nonalgebraic $p \in S(A)$. Clearly, if there is any set B with $A \subseteq B \subseteq M$ and any type $p' \in S(B)$ extending p that is algebraic, then p is realized in M. But, if we assume by way of contradiction that this is not the case, there would be no difficulty in constructing (by induction on α) a continuous, increasing sequence $\langle A_\alpha : \alpha < \kappa \rangle$ of subsets of M, together with a sequence $\langle p_\alpha : \alpha < \kappa \rangle$ of types such that $A_0 = A$, $p_0 = p$, each $p_\alpha \in S(A_\alpha)$, $|A_\alpha| \leq |A| + \kappa$, and p_β is a forking extension of p_α for all $\alpha < \beta < \kappa$. As stability contradicts the existence of such a sequence, the lemma is proved. ∎

Definition 2.2 Let $\{X_\eta : \eta \in I\}$ be an independent tree of sets. A set B is *self-based on* $\{X_\eta : \eta \in I\}$ if $\mathrm{tp}(B/X_H)$ does not fork over $B \cap X_H$ for all subtrees $H \subseteq I$.

The following Lemma is straightforward.

Lemma 2.3 *If X and A are any sets and $|A| < \kappa$, then there is a set $B \supseteq A$ such that $|B| < \kappa$, $B \setminus A \subseteq X$, and* $\mathrm{tp}(B/X)$ *does not fork over $B \cap X$. Furthermore, if $B' \supseteq B$ and $B' \setminus B \subseteq X$, then* $\mathrm{tp}(B'/X)$ *does not fork over $B' \cap X$.*

Proof. Given A and X, let $C \subseteq X$ be such that $|C| < \kappa$ and $\mathrm{tp}(A/X)$ is based on C. Let $B = A \cup C$. ∎

The next Lemma is more substantial.

Lemma 2.4 *For every finite index tree I, for every independent tree $\{X_\eta : \eta \in I\}$ of models, and for every set A of size $< \kappa$, there is a set $B \supseteq A$ such that $|B| < \kappa$, $B \setminus A \subseteq X_I$, and B is self-based on $\{X_\eta : \eta \in I\}$.*

Proof. We argue by induction on $|I|$. If $|I| = 1$, this is immediate by Lemma 2.3. So assume that $|I| \geq 2$, $I = J \cup \{\eta^*\}$, where η^* is a leaf of I. Let $\{X_\eta : \eta \in I\}$ be any independent tree of sets. We assume that the conclusion of the Lemma holds for $\{X_\eta : \eta \in J\}$. Fix any set A with $|A| < \kappa$. By Lemma 2.3 choose $B_0 \supseteq A$ such that $|B_0| < \kappa$, $B_0 \setminus A \subseteq X_I$, and $\mathrm{tp}(B_0/X_I)$ does not fork over $B_0 \cap X_I$. Now apply the inductive hypothesis to B_0 to get $B \supseteq B_0$ such that $|B| < \kappa$, $B \setminus B_0 \subseteq X_J$, (hence $B \setminus A \subseteq X_I$) and B is self-based on $\{X_\eta : \eta \in J\}$. Finally, by employing Lemma 2.3 $lg(\eta^*)$ times, beginning at η^* and working downward to $\langle\rangle$, choose a set C such that $B \cap X_{\eta^*} \subseteq C \subseteq X_{\eta^*}$, $|C| < \kappa$, and $\mathrm{tp}(C/X_\nu)$ does not fork over $C \cap X_\nu$ for all $\nu \trianglelefteq \eta^*$.

We argue that the set BC is self-based on $\{X_\eta : \eta \in I\}$. To see this we set some notation. Let $\mu = (\eta^*)^-$. For $H \subseteq J$ a subtree, let H' be the smallest subtree of J containing H and μ, and let $H^* = H \cup \{\eta^*\}$. Note that for any subtree $H \subseteq J$, $X_{(H')^*} = X_{H^*}$ and $X_{H^*} = X_H X_{\eta^*} = X_{H'} X_{\eta^*}$. Furthermore, since $B \cap X_{\eta^*} \subseteq C$, $(B \cap X_H) \cup C = (B \cap X_{H'}) \cup C$. We begin with the following claim.

Claim. For all subtrees $H \subseteq J$, $B \underset{(B \cap X_H)C}{\perp} X_{H^*}$.

Proof. Fix a subtree $H \subseteq J$. From our observations above we can replace H by H' without changing X_{H^*} or $(B \cap X_H)C$. Thus, we may assume that $\mu \in H$. Since $X_I = X_J X_{\eta^*}$ and since $\mathrm{tp}(B/X_I)$ does not fork over $B \cap X_I$, we have $B \underset{(B\cap X_J)(B\cap X_{\eta^*})}{\perp} X_J X_{\eta^*}$. Let $D = (B \cap X_J) \setminus X_H$, so $B \cap X_J = D \cup (B \cap X_H)$. Thus

(1) $$X_{\eta^*} \underset{X_H D(B\cap X_{\eta^*})}{\perp} B$$

Since the tree $\{X_\eta : \eta \in I\}$ is independent, $\mathrm{tp}(X_J/X_{\eta^*})$ does not fork over X_μ. Since $\mu \in H$ $X_\mu \subseteq X_H$, so $X_H D \underset{X_\mu}{\perp} X_{\eta^*}$, so $\mathrm{tp}(D/X_H X_{\eta^*})$ does not fork over X_H. Combining this with (1) transitivity of nonforking yields

(2) $$X_{\eta^*} \underset{X_H(B\cap X_{\eta^*})}{\perp} B$$

Since B is self-based on $\{X_\eta : \eta \in J\}$, $\mathrm{tp}(B/X_H)$ does not fork over $B \cap X_H$, so $B \underset{(B\cap X_H)(B\cap X_{\eta^*})}{\perp} X_H$. Transitivity and (2) imply $B \underset{(B\cap X_H)(B\cap X_{\eta^*})}{\perp} X_H X_{\eta^*}$, so the Claim follows since $B \cap X_{\eta^*} \subseteq C \subseteq X_{\eta^*}$. ∎

Now fix an arbitrary subtree $H \subseteq J$. We will show that $\mathrm{tp}(BC/X_{H^*})$ does not fork over $(BC) \cap X_{H^*}$ and $\mathrm{tp}(BC/X_H)$ does not fork over $(BC) \cap X_H$. The former statement follows immediately from the Claim since $(BC) \cap X_{H^*} = (B \cap X_H)C$. For the latter statement, choose the shortest $\nu \trianglelefteq \eta^*$ such that $\mathrm{tp}(X_H/X_{\eta^*})$ does not fork over X_ν. Since $\mathrm{tp}(C/X_\nu)$ does not fork over $C \cap X_\nu$ and since $C \subseteq X_{\eta^*}$, $\mathrm{tp}(C/X_H)$ does not fork over $C \cap X_\nu$, hence

$$X_H \underset{(B\cap X_H)(C\cap X_\nu)}{\perp} C$$

So the Claim and the transitivity of nonforking gives $X_H \underset{(B\cap X_H)(C\cap X_\nu)}{\perp} BC$, which suffices since $(BC) \cap X_H = (B \cap X_H) \cup (C \cap X_\nu)$. ∎

Proposition 2.5 *Suppose that $\{X_\eta : \eta \in I\}$ is an independent tree of sets with $|I| < \kappa$ and suppose that $|A| < \kappa$. Then there is a set $B \supseteq A$ such that $|B| < \kappa$, $B \setminus A \subseteq X_I$, and B is self-based on $\{X_\eta : \eta \in I\}$.*

Proof. When $\kappa = \aleph_0$ this is precisely Lemma 2.4, so assume $\kappa > \aleph_0$. We begin by inductively constructing an increasing sequence $\langle B_n : n \in \omega \rangle$ of sets, each of size $< \kappa$ such that $B_0 = A$, $B_n \setminus A \subseteq X_I$, and $\mathrm{tp}(B_n/X_J)$ does not fork over $B_{n+1} \cap X_J$ for all finite subtrees $J \subseteq I$. This is possible by repeated use of Lemma 2.4, since there are fewer than κ *finite* subtrees of I.

Let $B^* = \bigcup\{B_n : n \in \omega\}$. Since κ is regular and uncountable $|B^*| < \kappa$. We argue that B^* is self-based on $\{X_\eta : \eta \in I\}$. Choose an arbitrary subtree $H \subseteq I$ and a finite tuple b from B^*. To show that $\mathrm{tp}(b/X_H)$ does not fork over $B^* \cap X_H$, choose a finite tuple c from X_H and a formula $\varphi(x, y)$ over $B^* \cap X_H$ such that $\varphi(b, c)$ holds. In order to show that $\varphi(x, c)$ does not fork over $B^* \cap X_H$ we show that $\varphi(x, c)$ does not k-divide over $B^* \cap X_H$ for any $k \in \omega$. If, by way of contradiction, $\varphi(x, c)$ did k-divide over $B^* \cap X_H$, then choose $n \in \omega$ and a finite subtree $J \subseteq H$ such that $b \in B_n$, $c \in X_J$, and $\varphi(x, y)$ is over $B_{n+1} \cap X_H$. If $\langle c_n : n \in \omega\rangle$ were a witness to $\varphi(x, c)$ k-dividing over $B^* \cap X_H$ (i.e., $\mathrm{tp}(c_n/B^* \cap X_H) = \mathrm{tp}(c/B^* \cap X_H)$ for all $n \in \omega$ and $\{\varphi(x, c_n) : n \in \omega\}$ is k-inconsistent) then the same sequence would witness $\varphi(x, c)$ k-dividing (hence forking) over $B_{n+1} \cap X_J$. But this would imply $\mathrm{tp}(B_n/X_J)$ forks over $B_{n+1} \cap X_J$, which is contrary to our construction of B_{n+1}. ∎

Our third group of results uses the ideas in [8] (which in turn were motivated by ideas in [2]) to prove a technical fact (Proposition 2.11) for arbitrary stable theories. Note that there is a much shorter proof of this when T is superstable, which is due to the ubiquity of regular types over a-models.

Definition 2.6 Let $\mathcal{P} \subseteq S(M)$ be a set of types over a model M. A set B is *weakly dominated by* $\mathcal{P}$ *over* M if there is an independent set I over M consisting of realizations of $\mathcal{P}$ such that B is dominated by I over M. (It is possible that I contains many realizations of the same type in $\mathcal{P}$.)

Definition 2.7 Let M be any a-model. A complete type p is an *a-type above* M if the domain of p is an a-model containing M. A class $\mathcal{P}$ of a-types above M *is* M*-determined* if for every $p \in \mathcal{P}$, either p does not fork over M or $p \perp M$. A class $\mathcal{P}$ of a-types above M is *dense above* M if, for all a-models $N \supseteq M$, every nonalgebraic type over N is nonorthogonal to some element of $\mathcal{P} \cap S(N)$.

Definition 2.8 Let $\mathcal{P}$ be a class of a-types above M. A $\mathcal{P}$*-sequence over* M is a sequence $\langle M_i, a_j : i \leq \alpha, j < \alpha\rangle$, where $\langle M_i : i \leq \alpha\rangle$ is an increasing sequence of a-models, $M_0 = M$, for all $i < \alpha$ $\mathrm{tp}(a_i/M_i) \in \mathcal{P}$ and M_{i+1} is a-prime over $M_i \cup \{a_i\}$, and M_i is a-prime over $\bigcup_{j<i} M_j$ for all limit ordinals $i \leq \alpha$.

Lemma 2.9 *If $\mathcal{P}$ is an M-determined class of a-types above M and $\langle M_i, a_j : i \leq \alpha, j < \alpha\rangle$ is a $\mathcal{P}$-sequence over M, then M_α is weakly dominated over M by $\{\mathrm{tp}(a_j/M_j)|M : j < \alpha, \mathrm{tp}(a_j/M_j)$ does not fork over $M\}$.*

Proof. Fix an M-determined class $\mathcal{P}$ of a-types above M. We will prove (by simultaneous induction on α) that if $\langle M_i, a_j : i \leq \alpha, j < \alpha\rangle$ is a $\mathcal{P}$-sequence over M, $I = \{a_j : \mathrm{tp}(a_j/M_j)$ does not fork over $M\}$ and $J = \{a_j : \mathrm{tp}(a_j/M_j) \perp M\}$, then

1. I is independent over M and
2. M_α is dominated by I over M.

The conclusions are vacuous when $\alpha = 0$ and are trivially verified when α is a limit ordinal. So assume that the two conditions hold for the $\mathcal{P}$-sequence $\langle M_i, a_j : i \leq \alpha, j < \alpha\rangle$. Choose any a^* such that $\mathrm{tp}(a^*/M_\alpha) \in \mathcal{P}$ and let M^* be a-prime over $M_\alpha a^*$. We argue that the two conditions also hold for the concatenation of the original $\mathcal{P}$-sequence with $\langle M^*, a^*\rangle$. Let $p = \mathrm{tp}(a^*/M_\alpha)$.

We first check that (1) continues to hold: If $p \perp M$, then there is nothing to check. On the other hand, if p does not fork over M, then $\mathrm{tp}(a^*/MI)$ does not fork over M, hence $I \cup \{a^*\}$ is independent over M.

We now check that (2) continues to hold in both cases. First, assume that $p \perp M$. Then if any set X does not fork with I over M, then it follows from our inductive assumption that X does not fork with M_α over M. Since $p \perp M$, $\mathrm{tp}(a^*/M_\alpha X)$ does not fork over M_α. Since M^* is a-prime over $M_\alpha a^*$, this implies that X does not fork with M^* over M_α. Hence X does not fork with M^* over M by transitivity. On the other hand, suppose that p does not fork over M. In this case, assume that X does not fork with Ia^* over M. Then, since $I \cup \{a^*\}$ is independent over M, a^*X does not fork with I over M. By our inductive hypothesis this implies that a^*X does not fork with M_α over M. In particular, X does not fork with $M_\alpha a^*$ over M. So, X does not fork with M^* over M, since a^* dominates M^* over M_α. ∎

Lemma 2.10 *Suppose that a class $\mathcal{P}$ of a-types above M is dense above M. Then for every $b \in \mathfrak{C}$, there is a $\mathcal{P}$-sequence over M of length $\alpha < \kappa$ such that $b \in M_\alpha$.*

Proof. Construct a $\mathcal{P}$-sequence over M $\langle M_i, a_j : i \leq \alpha, j < \alpha\rangle$ of maximal length such that $\mathrm{tp}(a_j/M_j b)$ forks over M_j for every $j < \alpha$. For any such sequence $\mathrm{tp}(b/M_{j+1})$ forks over M_j for all $j < \alpha$, hence $\alpha < \kappa$. But, since $\mathcal{P}$ is dense above M, the only way the process can terminate is if $\mathrm{tp}(b/M_\alpha)$ is algebraic, so $b \in M_\alpha$. ∎

Proposition 2.11 *Suppose that $\{X_j : j \in \lambda\}$ is any collection of subsets of an a-model N. If a type p is not orthogonal to N but $p \perp X_j$ for all $j < \lambda$, then there is a type $q \in S(N)$ such that $q \not\perp p$, but $q \perp X_j$ for all j.*

Proof. Choose an a-model $N_0 \supseteq N$ with $\mathrm{dom}(p) \subseteq N_0$ and let p_0 be the nonforking extension of p to N_0. Choose $A_0 \subseteq N_0$ of size $< \kappa$ such that p_0 is definable over A_0. Choose $C \subseteq N$ of size $< \kappa$ such that $\mathrm{tp}(A_0/N)$ is definable over C. Choose a set $\{N_i : i < \kappa\}$ of a-models to be independent over N with $\mathrm{tp}(N_i/N) = \mathrm{tp}(N_0/N)$ for all $i < \kappa$. For each $0 < i < \kappa$ choose an automorphism σ_i of $\mathfrak{C}$ fixing N pointwise and sending N_0 onto N_i. As notation let $A_i = \sigma_i(A_0)$ and $p_i = \sigma_i(p_0)$. Since $p_0 \not\perp N$ it follows that $p_i \not\perp p_j$ for all $i < j < \kappa$ (see e.g., 1.4.3.3 of [6]). Let N^* be an a-model containing $\bigcup\{N_i : i < \kappa\}$ and let

$$\mathcal{P}_0 = \{r : r \text{ an a-type above } N \text{ and } \{i < \kappa : r \not\perp p_i\} \text{ has size } < \kappa\}.$$

Claim. Some nonalgebraic $q \in S(N)$ is orthogonal to every $r \in \mathcal{P}_0$.

Proof. We first argue that $\mathcal{P}_0$ is not dense above N^*. Suppose it were. Let p_0^+ denote the nonforking extension of p_0 to N^* and let b be any realization of p_0^+. By Lemma 2.10 there would be a $\mathcal{P}_0$-sequence $\langle M_i, a_j : i \le \alpha, j < \alpha\rangle$ over N^* of length $\alpha < \kappa$ such that $b \in M_\alpha$. For each $j < \alpha$ let $r_j = \mathrm{tp}(a_j/M_j)$. Since $\alpha < \kappa$ and each $r_j \in \mathcal{P}_0$ we could find $m < \kappa$ such that every $r_j \perp p_m$ for every $j < \alpha$. But now, if e is any realization of p_m^+ (the nonforking extension of p_m to N^*) then we argue by induction on $i \le \alpha$ that $\mathrm{tp}(e/M_i)$ does not fork over N^*. In particular, $\mathrm{tp}(e/M_\alpha)$ does not fork over N^*, hence p_m^+ and p_0^+ would be almost orthogonal over N^*. But this would contradict $p_0 \not\perp p_m$ since N^* is an a-model.

So $\mathcal{P}_0$ is not dense above N^*. Fix an a-model $N' \supseteq N^*$ and a nonalgebraic type $q' \in S(N')$ such that q' is orthogonal to every $r \in \mathcal{P}_0 \cap S(N')$. Choose D' of size $< \kappa$ satisfying $C \subseteq D' \subseteq N'$ over which q' is definable and choose $D \subseteq N$ such that there is an automorphism f of $\mathfrak{C}$ fixing C pointwise with $D = f(D')$. Let q be the nonforking extension of $f(q'|D')$ to $S(N)$.

To see that q satisfies the Claim, choose any $r \in \mathcal{P}_0$. Say $r \in S(N'')$. Choose any $E \subseteq N''$ of size $< \kappa$ on which r is defined, and choose an automorphism τ of $\mathfrak{C}$ such that $\tau|D = f^{-1}|D$ (so τ fixes C pointwise and $\tau(q)$ is parallel to q') and $\tau(E) \subseteq N'$. Let $r' \in S(N')$ be parallel to $\tau(r)$. Since $E \cup \tau(E)$ is independent from A_i over C for almost all $i < \kappa$ (i.e., fewer than κ exceptions) and since $r \in \mathcal{P}_0$, it follows that $\{i < \kappa : \tau(r) \not\perp p_i\}$ has size $< \kappa$, so $r' \in S(N') \cap \mathcal{P}_0$. If, by way of contradiction, $q \not\perp r$, then since nonorthogonality is parallelism invariant, it would follow that $q' \not\perp r'$, contradicting our choice of q'. Thus $q \perp r$ for all $r \in \mathcal{P}_0$. ∎

We argue that any such $q \in S(N)$ satisfies the conclusions of the Proposition. Fix such a q and choose any $j < \lambda$. Let $r \in S(N)$ be the nonforking extension of any strong type over X_j. Since $p \perp X_j$ and since $\{p_i : i \in \kappa\}$ are conjugate over N, $r \perp p_i$ for all i, hence $r \in \mathcal{P}_0$. Thus $q \perp r$. That is, $q \perp X_j$ for all $j < \lambda$.

It remains to show that $q \not\perp p$. Let q^+ and p_i^+ ($i < \kappa$) denote the nonforking extensions of q and p_i (respectively) to N^*. Let $\mathcal{P}^+ = \{p_i^+ : i < \kappa\}$, let

> $\mathcal{P}_0^{\perp\perp} = \{p \in S(N^*) : p$ is orthogonal to every type s that is orthogonal to every type in $\mathcal{P}_0\}$

and let

> $\mathcal{P}_1 = \{s : s$ is an a-type above N^* such that either $s \perp N^*$ or s is a nonforking extension of an element of $\mathcal{P}^+ \cup \mathcal{P}_0^{\perp\perp}\}$.

In a moment we will show that $\mathcal{P}_1$ is dense above N^*, but we first show that this suffices. Once it is, then since $\mathcal{P}_1$ is N^*-determined, it follows from Lemmas 2.9 and 2.10 that q^+ is weakly dominated over N^* by $\mathcal{P}^+ \cup \mathcal{P}_0^{\perp\perp}$. Since q^+ is nonalgebraic, q^+ (and hence q) is nonorthogonal to at least one element of $\mathcal{P}^+ \cup \mathcal{P}_0^{\perp\perp}$. Since q is orthogonal every element of $\mathcal{P}_0$, q is also orthogonal to every element of $\mathcal{P}_0^{\perp\perp}$, so $q \not\perp p_i$ for some $i < \kappa$. But, since the p_i's are all conjugate over N and since $q \in S(N)$, it follows that $q \not\perp p_0$, so $q \not\perp p$.

Thus, it suffices to show that $\mathcal{P}_1$ is dense above N^*. Choose any a-model $M' \supseteq N^*$ and any nonalgebraic $r \in S(M')$. We argue that r is nonorthogonal to some element of $\mathcal{P}_1 \cap S(M')$. We may assume that $r \not\perp N^*$ and $r \perp p_i$ for all $i < \kappa$, otherwise r itself would be a witness. We complete the proof by constructing a conjugate type $r^* \in \mathcal{P}_0^{\perp\perp}$ such that $r \not\perp r^*$. To accomplish this, first note that $r \in \mathcal{P}_0$, hence r is orthogonal to every type that is orthogonal to every type in $\mathcal{P}_0$. Since $r \not\perp N^*$ we can choose a type $t \in S(N^*)$ such that $r \not\perp t$. Next, choose sets $D \subseteq M'$ and $E \subseteq N^*$ such that $|D| < \kappa$, $E = D \cap N^*$, $C \subseteq E$, t is definable over E, and r is definable over D. Finally, choose $D' \subseteq N^*$ such that D and D' satisfy the same strong type over E and are independent over E and let $r^* \in S(N^*)$ be definable over D' in the same manner that r is over D. Since $r \not\perp E$, $r \not\perp r^*$. Also, since D and D' realize the same type over C, r^* is also orthogonal to every type that is orthogonal to every element of $\mathcal{P}_0$. Thus, $r^* \in \mathcal{P}_0^{\perp\perp}$, so s, the nonforking extension of r^* to $S(M')$ is nonorthogonal to r and is in $\mathcal{P}_1 \cap S(M')$. ∎

Our final group of results is aimed at proving Proposition 2.16, which is a variant on the more familiar fact that if $\{B_i : i \in \kappa\}$ are independent over a set A and a stationary type p is nonorthogonal to every B_i, then $p \not\perp A$. The buildup to the proof of this proposition develops the notion of nonforking in an ultrapower of the monster model. For the rest of this section

Fix a nonprincipal ultrafilter $\mathcal{D}$ on ω and let $\mathfrak{C}^* = \prod \mathfrak{C}/\mathcal{D}$.

We abuse notation slightly and consider $\mathfrak{C}^*$ to be an elementary extension of $\mathfrak{C}$. Specifically, we identify elements $a \in \mathfrak{C}$ with the diagonal element $\langle a : i \in \omega\rangle/\mathcal{D} \in \mathfrak{C}^*$. For a subset $X \subseteq \mathfrak{C}$ we let X^* denote $\prod X/\mathcal{D}$. By our notational convention $X \subseteq X^* \subseteq \mathfrak{C}^*$.

Lemma 2.12 *For any $a \in \mathfrak{C}$ and $B \subseteq \mathfrak{C}$, $\mathrm{tp}(a/B^*)$ does not fork over B.*

Proof. Choose any model M such that $B \subseteq M \subseteq \mathfrak{C}$ and $\mathrm{tp}(a/M)$ does not fork over B. It clearly suffices to show that $\mathrm{tp}(a/M^*)$ does not fork over M. So suppose that $\theta(a, b^*)$ holds (in $\mathfrak{C}^*$, where a is identified with its diagonal element) for some formula $\theta(x, y)$ with no hidden parameters. By finite satisfiability, it suffices to find some $b \in M$ such that $\theta(a, b)$ holds. Choose a representation $b^* = \langle b_i : i \in \omega\rangle/\mathcal{D}$ with each $b_i \in M$. Since $\theta(a, b^*)$ holds, $\{i \in \omega : \theta(a, b_i)\} \in \mathcal{D}$, so is nonempty. ∎

Lemma 2.13 *Suppose that $A \subseteq B_i \subseteq \mathfrak{C}$ for all $i \in \omega$, and that $\{B_i : i \in \omega\}$ is independent over A. Then $\mathfrak{C} \underset{A}{\perp} \overline{B}$, where $\overline{B} = \prod_{i\in\omega} B_i/\mathcal{D}$.*

Proof. Choose any $d \in \mathfrak{C}$ and model M satisfying $A \subseteq M \subseteq \mathfrak{C}$ and $M \underset{A}{\perp} \bigcup\{B_i : i \in \omega\}d$. Then $\{B_i : i \in \omega\}$ is independent over M and by transitivity it suffices to prove that $\mathrm{tp}(d/M\overline{B})$ does not fork over M. Let $\theta(x, y)$ be an $L(M)$-formula such that $\theta(d, b^*)$ holds for some $b^* \in \overline{B}$. By finite satisfiability it suffices to find some $m \in M$ such that $\theta(d, m)$ holds.

Let $E = M \cup \{B_i : i \in \omega\}$. Since $\mathrm{tp}_\theta(d/E)$ is definable, there is an L-formula $\psi(y, z)$ and an $e \in E$ such that

$$\theta(d, c) \leftrightarrow \psi(c, e) \tag{3}$$

for all $c \in E$. Choose a representation $\langle b_i : i \in \omega\rangle/\mathcal{D}$ for b^* with $b_i \in B_i$ for all $i \in \omega$. Since $\theta(d, b^*)$ holds, $\{i \in \omega : \theta(d, b_i)\} \in \mathcal{D}$. Since $\mathcal{D}$ is nonprincipal, e is finite, and $\{B_i : i \in \omega\}$ is independent over M, there is an $i \in \omega$ such that both $\theta(d, b_i)$ holds and $e \underset{M}{\perp} b_i$. Since $b_i \in E$, (3) implies that $\psi(b_i, e)$ holds. Thus, by symmetry and finite satisfiability there is $m \in M$ such that $\psi(m, e)$ holds. By (3) again, $\theta(d, m)$ holds and we finish. ∎

Lemma 2.14 *Suppose that $\{a_i : i \in \omega\} \subseteq \mathfrak{C}$, $N \subseteq \mathfrak{C}$ is a model, and for each $i \in \omega$ $M_i \subseteq N$ is a model such that $\mathrm{tp}(a_i/N)$ does not fork over M_i. Then $\mathrm{tp}(a^*/N^*)$ does not fork over $\overline{M}$, where $a^* = \langle a_i : i \in \omega\rangle/\mathcal{D}$ and $\overline{M} = \prod_{i\in\omega} M_i/\mathcal{D}$.*

Proof. First, note that $\overline{M}$ is itself a submodel of $\mathfrak{C}^*$. Let $\theta(x, y)$ be any L-formula and let $c^* \in N^*$ be any element such that $\theta(a^*, c^*)$ holds. By finite satisfiability it suffices to find $b^* \in \overline{M}$ such that $\theta(a^*, b^*)$. Choose a representation $\langle c_i : i \in \omega\rangle/\mathcal{D}$ for c^* with each $c_i \in N$. Let $R = \{i \in \omega : \theta(a_i, c_i)\}$. Since $\theta(a^*, c^*)$ holds, $R \in \mathcal{D}$. We construct a sequence $\langle b_i : i \in \omega\rangle$ as follows: For each $i \in R$, choose $b_i \in M_i$ such that $\theta(a_i, b_i)$. (This is possible since $\mathrm{tp}(a_i/N)$ does not fork over M_i.) For any $i \notin R$, let b_i be an arbitrary element of M_i. Let $b^* = \langle b_i : i \in \omega\rangle/\mathcal{D}$. Then $b^* \in \overline{M}$ and $\theta(a^*, b^*)$ holds. ∎

We apply these three lemmas in the proof of Proposition 2.16.

Definition 2.15 Let Δ be a finite set of (partitioned) L-formulas and let B be any set. A stationary type p is Δ-*nonorthogonal to* B, written $p \not\perp^\Delta B$, if there is a set $D \supseteq \mathrm{dom}(p) \cup B$, $\varphi(x, yz) \in \Delta$, and elements a realizing $p|D$, $b \in D$, and $c \in \mathfrak{C}$ such that $\mathrm{tp}(c/D)$ does not fork over B, $\varphi(a, bc)$ holds, and $R_\Delta(p|D \cup \{\varphi(x, bc)\}) < R_\Delta(p)$.

Clearly, $p \not\perp B$ if and only if $p \not\perp^\Delta B$ for some finite Δ. Also, if $B \subseteq B'$ and $p \not\perp^\Delta B$ then $p \not\perp^\Delta B'$.

Proposition 2.16 *Let Δ be a finite set of formulas and let p be any stationary type. If $\{B_i : i \in \omega\}$ are independent over A and $p \not\perp^\Delta B_i$ for each $i \in \omega$, then $p \not\perp A$.*

Proof. To begin we inductively find submodels $\{M_i : i \in \omega\}$ of $\mathfrak{C}$ such that $B_i \subseteq M_i$ (hence $p \not\perp^\Delta M_i$) for each i, yet $\{M_i : i \in \omega\}$ are independent over A. For each i, choose D_i containing $\mathrm{dom}(p) \cup M_i$ as in the definition of Δ-nonorthogonality and let N be a substructure of $\mathfrak{C}$ containing $\bigcup\{D_i : i \in \omega\}$. By replacing p by its nonforking extension to N, we may assume that $p \in S(N)$. Let $\mathcal{D}$ be any nonprincipal ultrafilter on ω, let $\overline{M} = \prod_{i\in\omega} M_i/\mathcal{D}$ and let $N^* = \prod N/\mathcal{D}$. It follows immediately from Lemma 2.13 that $N \underset{A}{\downarrow} \overline{M}$. So, in light of X 1.1 of [7], in order to conclude that $p \not\perp A$ it suffices to show that $p \not\perp \overline{M}$. In fact we will show that p is Δ-nonorthogonal to $\overline{M}$ by demonstrating that N^* is a suitable choice of D in Definition 2.15.

Let a be any realization of p. It follows from Lemma 2.12 that a realizes the nonforking extension p^* of p to N^*. Let $k = R_\Delta(p) = R_\Delta(p^*)$. For each $i \in \omega$, since $D_i \subseteq N$ we can find $\varphi_i \in \Delta$, $b_i \in N$, and $c_i \in \mathfrak{C}$ such that $\varphi_i(a, b_ic_i)$ holds, $\mathrm{tp}(c_i/N)$ does not fork over M_i and $R_\Delta(p \cup \{\varphi_i(x, b_ic_i)\}) < k$. Since Δ is finite we may assume that φ_i is identically φ for all i. Let $b^* = \langle b_i : i \in \omega\rangle/\mathcal{D}$ and $c^* = \langle c_i : i \in \omega\rangle/\mathcal{D}$. Then $b^* \in N^*$ and $\varphi(a, b^*c^*)$ holds. Since p is stationary, its Δ-multiplicity is 1, hence $\{yz : R_\Delta(p \cup \{\varphi(x, yz)\}) < k\}$ is definable. So the Łoś theorem yields

$$R_\Delta(p^* \cup \{\varphi(x, b^*c^*)\}) < k = R_\Delta(p^*)$$

Finally, since $\mathrm{tp}(c_i/N)$ does not fork over M_i for each i, $\mathrm{tp}(c^*/N^*)$ does not fork over $\overline{M}$ by Lemma 2.14. So N^* witnesses $p \not\perp^\Delta \overline{M}$ and we finish. ∎

3 Local minimality: Proofs of 1.3–1.5

In this section we work over a specific independent tree and investigate the consequences of the a-prime model over it being a-minimal. In particular, we prove Theorem 1.3 and two corollaries that follow from it.

Lemma 3.1 *Let $\{M_\eta : \eta \in I\}$ be any independent tree of a-models, let $J \subseteq I$ be any subtree, and let $\bar{a} = \langle a_\alpha : \alpha < \beta\rangle$ be any a-construction sequence* *over M_J. Then $\bar{a}$ is an a-construction sequence over M_I and $\mathrm{tp}(\bar{a}/M_I)$ does not fork over M_J. In particular, if M_J^* is a-prime over M_J, then M_J^* is the universe of an a-construction sequence over M_I and $\mathrm{tp}(M_J^*/M_I)$ does not fork over M_J.*

Proof. Let K be a maximal subtree such that $J \subseteq K$ and $\mathrm{stp}(\bar{a}/M_J) \vdash \mathrm{stp}(\bar{a}/M_K)$. It follows that $\bar{a}$ is an a-construction sequence over M_K. By way of contradiction assume that $K \neq I$. Choose $\nu \in K$ and an immediate successor $\eta \in I \setminus K$. Now $M_K \underset{M_\nu}{\perp} M_\eta$ and M_ν is an a-model, so, using either V 3.2 of [7] or I 4.3.4 of [6], an easy induction on β shows that $K \cup \{\eta\}$ contradicts the maximality of K. The final sentence follows immediately. ∎

Proof of Theorem 1.3. The equivalences (1) $\Leftrightarrow$ (2) and (3) $\Leftrightarrow$ (4) have nothing to do with trees. (1) $\Leftrightarrow$ (2) is the content of IV 4.21 of [7], (4) $\Rightarrow$ (3) is trivial, and (3) $\Rightarrow$ (4) follows immediately from Proposition 2.11 (take the sets X_i to be the submodels M_η of M). The other two implications are generalizations of arguments that appear in the proof of X 2.2 of [7].

(**2**) $\Rightarrow$ (**3**) Let $r \in S(M_I^*)$ be nonalgebraic and assume that $r \perp M_\eta$ for all $\eta \in I$. Choose $A \subseteq M_I^*$ of size less than κ over which r is based and stationary. Fix a subtree $J \subseteq I$ of size $< \kappa$ and an a-prime submodel $M_J^* \subseteq M_I^*$ that contains A. Call a subset $B \subseteq M_J^*$ *suitable* if $A \subseteq B$, $|B| < \kappa$, and B is self-based on $\{M_\eta : \eta \in J\}$. It follows from Proposition 2.5 that for every set $C \subseteq M_J^*$ of size $< \kappa$, there is a suitable B containing C. Thus, by iterating the Claim below ω times we can construct an infinite Morley sequence $\mathbf{J}$ in r over A *inside* M_I^*, such that $\mathrm{tp}(\mathbf{J}/AM_I)$ does not fork over A. In particular, such a $\mathbf{J}$ is indiscernible over M_I. So, it suffices to prove the following:

Claim. If B is suitable and c realizes $r|B$, then $\mathrm{tp}(c/B) \vdash \mathrm{tp}(c/BM_I)$.

Proof. Fix a suitable B and let c denote any realization of $r|B$. As notation, we write B_η for $B \cap M_\eta$ and write $B_{J'} = B \cap M_{J'}$ for subtrees J' of J.

We first argue that $\mathrm{tp}(c/B) \vdash \mathrm{tp}(c/BM_{\langle\rangle})$. Choose any finite tuple a from $M_{\langle\rangle}$. Since B is suitable, $\mathrm{tp}(a/B)$ does not fork over $B_{\langle\rangle}$. But $\mathrm{tp}(a/B_{\langle\rangle})$ is parallel to a type over $M_{\langle\rangle}$, hence r is orthogonal to $\mathrm{tp}(a/B)$. This implies that $a \underset{B}{\perp} c$. Since c was an arbitrary realization of $r|B$, this implies $\mathrm{tp}(c/B) \vdash \mathrm{tp}(c/Ba)$, hence $\mathrm{tp}(c/B) \vdash \mathrm{tp}(c/BM_{\langle\rangle})$.

Now let J' be a maximal subtree of J such that $\mathrm{tp}(c/B) \vdash \mathrm{tp}(c/BM_{J'})$. We demonstrate that $J' = J$. From the previous paragraph J' is nonempty. If $J' \neq J$ then there is $\nu \in J \setminus J'$ such that its immediate predecessor, denoted by η is in J'. As above, choose $a \in M_\nu$. Since we know that $\mathrm{tp}(c/B) \vdash \mathrm{tp}(c/BM_{J'})$, it suffices to show that $\mathrm{tp}(c/BM_{J'}) \vdash \mathrm{tp}(c/BM_{J'}a)$.

Subclaim. $a \underset{M_\eta B_\nu}{\perp} M_{J'}B$

Proof. Since the original tree is independent $M_\nu \underset{M_\eta}{\perp} M_{J'}$. Since $aB_\nu \subseteq M_\nu$ this implies

(4)
$$a \underset{M_\eta B_\nu}{\perp} M_{J'}B_\nu$$

However, since B is suitable, $\mathrm{tp}(B/M_{J'}M_\nu)$ does not fork over $B_{J'}B_\nu$. Thus, $B \underset{M_{J'}B_\nu}{\perp} M_\nu$. Since $a \in M_\nu$, symmetry provides $a \underset{M_{J'}B_\nu}{\perp} B$, so the subclaim follows from (4) and transitivity. ∎

Now let $p = \mathrm{tp}(a/M_{J'}B)$. The type p does not fork over $M_\eta B_\nu \subseteq M_\nu$, so $p \perp r$. Thus, $\mathrm{tp}(c/BM_{J'}) \vdash \mathrm{tp}(c/BM_{J'}a)$. Hence $J' = J$.

We have now established that $\mathrm{tp}(c/B) \vdash \mathrm{tp}(c/BM_J)$. We argue that in fact $\mathrm{tp}(c/B) \vdash \mathrm{tp}(c/BM_I)$. To see this, let I' be a maximal subtree that contains M_J such that $\mathrm{tp}(c/B) \vdash \mathrm{tp}(c/BM_{I'})$. As above, if $I' \neq I$, then there would be $\nu \in I \setminus I'$ whose immediate predecessor $\eta \in I'$. Since the tree is independent, $M_\nu \underset{M_\eta}{\perp} M_{I'}$. Since $B \subseteq M_J^*$ and $J \subseteq I'$, Lemma 3.1 implies that B is a-constructible hence a-atomic over $M_{I'}$. Since M_η is a-saturated, $BM_{I'}$ is dominated by $M_{I'}$ over M_η. Thus, $M_\nu \underset{M_\eta}{\perp} M_{I'}B$. Also, for any finite tuple a from M_ν, $\mathrm{tp}(a/M_\eta) \perp r$. Thus, $a \underset{M_{I'}B}{\perp} c$ for any such a. It follows that $\mathrm{tp}(c/B) \vdash \mathrm{tp}(c/BM_{I'}M_\nu)$, contradicting the maximality of I'. Hence $I' = I$ and our proof is complete. ∎

$(\mathbf{4}) \Rightarrow (\mathbf{2})$ Let $\mathbf{J} \subseteq M_I^*$ be a countably infinite, indiscernible sequence over M_I. By stability, $\mathbf{J}$ is an indiscernible set over M_I. Partition $\mathbf{J}$ into two infinite sets $\mathbf{J_0}$ and $\mathbf{J_1}$. Then, by taking $B = \bigcup \mathbf{J_0}$ when $\kappa \geq \omega_1$ or to be a sufficiently large finite subset of $\mathbf{J_0}$ when $\kappa = \omega$, $|B| < \kappa$ and $\mathbf{J_1}$ is an infinite, independent sequence over B such that $\mathbf{J_1} \underset{B}{\perp} M_I$. Let $a \in \mathbf{J_1}$ and let $p = \mathrm{tp}(a/B)$. Without loss, we may assume that p is stationary.

Claim. $p \perp M_\eta$ for all $\eta \in I$.

Proof. By way of contradiction, choose η such that $p \not\perp r$ for some $r \in S(M_\eta)$. Since $p^{(n)}$ is not almost orthogonal to $r^{(n)}$ over BM_I, we can increase B by finitely many elements of $\mathbf{J_1}$ and replace r by $r^{(n)}$ and thereby assume that

$$p \not\perp^a_{BM_I} r$$

Choose $A \subseteq M_\eta$ of size less than κ such that r is based and stationary over A. Since M_I^* is a-prime over M_I, we can choose $C \subseteq M_I$, also of size less than κ, such that $A \subseteq C$ and $\mathrm{stp}(aB/C) \vdash \mathrm{stp}(aB/M_I)$. Note that this condition implies that

(5) $$a^*B \underset{C}{\perp} M_I$$

for any a^* such that $\mathrm{tp}(a^*/BC) = \mathrm{tp}(a/BC)$. Since forking is witnessed by a single formula, there is D, $C \subseteq D \subseteq M_I$ such that $D \setminus C$ is finite and $\mathrm{tp}(a/B) \not\perp^a_{BD} r$. Since M_η is a-saturated and r is based and stationary on A, there is $e \in M_\eta$ such that $\mathrm{tp}(e/A)$ is parallel to r and $\mathrm{tp}(e/BD)$ does not fork over A. So, by the non-almost orthogonality condition, there is a^* realizing $\mathrm{tp}(a/B)$ such that $a^* \underset{B}{\perp} D$ and $a^* \underset{BD}{\not\perp} e$.

But, since $\mathrm{tp}(a/M_I)$ does not fork and is stationary over B, this implies that a and a^* have the same type over BD, hence over BC. So (5) implies that $\mathrm{tp}(a^*B/M_I)$ does not fork over C. Since $De \subseteq M_I$ this would imply that $\mathrm{tp}(a^*/BDe)$ does not fork over BD, which is a contradiction. ∎

The proof of **Corollary 1.4** is straightforward. Fix an independent tree $\{M_\eta : \eta \in I\}$ of a-models such that the a-prime model M_I^* is a-minimal and fix a subtree $J \subseteq I$. To show that M_J^* is a-minimal over M_J it suffices to show that every nonalgebraic type $p \in S(M_J^*)$ is nonorthogonal to some M_η with $\eta \in J$. So fix such a type p. Since p has a nonforking extension to $S(M_I^*)$ and since M_I^* is a-minimal, $p \not\perp M_\eta$ for some $\eta \in I$. Choose such an η of least length and assume by way of contradiction that $\eta \notin J$. Then $lg(\eta) \neq 0$ and there is $\nu \trianglelefteq \eta$ of maximal length such that $\nu \in J$. Since the tree is independent, $\mathrm{tp}(M_\eta/M_J)$ does not fork over M_ν. Since M_ν is an a-model this implies that $\mathrm{tp}(M_\eta/M_J^*)$ does not fork over M_ν. But then, since $p \perp M_\nu$, forking symmetry and X 1.1 of [7] imply that $p \perp M_\eta$, which is a contradiction. ∎

Proof of Corollary 1.5. If $\lambda = \kappa$ there is nothing to prove since a-models are κ-saturated. So fix $\lambda > \kappa$ and an independent tree $\{M_\eta : \eta \in I\}$

of λ-saturated a-models. Suppose that the a-prime model M_I^* over M_I is a-minimal over M_I. Choose $A \subseteq M$ with $|A| < \lambda$ and choose a nonalgebraic $q \in S(A)$. Because of Lemma 2.1 it suffices to show that q has a forking extension in $S(M_I^*)$. Choose a subset $A_0 \subseteq A$ of size less than κ over which q is based and let q_0 denote the restriction of q to A_0. By appending a countable Morley sequence in q_0 to A_0, we may additionally assume that q_0 is stationary. Since M_I^* is a-minimal over M_I, $q_0 \not\perp M_\eta$ for some $\eta \in I$. Choose $p \in S(M_\eta)$ such that $p \not\perp q_0$ and choose $B \subseteq M_\eta$ of size less than κ over which p is based and stationary. Let p_0 denote the restriction of p to B. Since $p_0 \not\perp q_0$ there is an $n \in \omega$ such that $p_0^{(n+1)}$ is not almost orthogonal to $q_0^{(n+1)}$ over BA_0. Since M_I^* is an a-model there are finite sequences C and D in M_I^* realizing $p_0^{(n)}$ and $q_0^{(n)}$ respectively. Thus,

$$p_0 \underset{A_0BCD}{\not\perp^a} q_0$$

Since M_η is λ-saturated there is a Morley sequence $\langle e_i : i \in \lambda \rangle$ in M_I^* of (independent) realizations of p_0 over B of length λ. Since $|ABCD| < \lambda$ this implies that $\mathrm{tp}(e_i/ABCD)$ does not fork over B for some i. But then q has a forking extension to $S(ABCDe_i)$ and we finish. ∎

4 Global minimality: Proofs of 1.8—1.10

We begin with a definition and a series of lemmas.

Definition 4.1 A partial decomposition $\{M_\eta : \eta \in J\}$ is λ-*full* if for every $\eta \in J$ and every nonalgebraic $p \in S(M_\eta)$ satisfying $p \perp M_{\eta^-}$ (when $\eta \neq \langle\rangle$) there is a set $H_\eta \subseteq J$ of λ immediate successors of η such that M_ν realizes p for every $\nu \in H_\eta$.

The proof of the following lemma is a routine exercise in bookkeeping. (Note that if $\{M_\eta : \eta \in J\}$ is a partial decomposition of $\mathfrak{C}$, then for each η, $|M_\eta| \leq 2^{|T|}$, so $|S(M_\eta)| \leq 2^{|T|}$.)

Lemma 4.2 *If $\{M_\eta : \eta \in J\}$ is a partial decomposition of $\mathfrak{C}$ and $\lambda \geq 2^{|T|} + |J|$, then there is a tree I of size λ and a λ-full partial decomposition $\{M_\eta : \eta \in I\}$ of $\mathfrak{C}$ extending it.*

Lemma 4.3 *If $|I| = \lambda > 2^{|T|}$, $\{M_\eta : \eta \in I\}$ is a λ-full partial decomposition of $\mathfrak{C}$, and M_I^* is a-minimal over M_I, then M_I^* is λ-saturated. Moreover, if $\lambda^{<\kappa} = \lambda$, then M_I^* is saturated of power λ.*

Proof. Fix $A \subseteq M_I^*$ of size $< \lambda$ and a nonalgebraic, stationary type $p \in S(A)$. We argue that p has a forking extension in $S(M_I^*)$.

Let $\mu = |A| + 2^{|T|}$. Choose a subtree $J \subseteq I$, $|J| \leq \mu$, and an a-prime submodel $M_J^* \preceq M_I^*$ such that $A \subseteq M_J^*$. Since M_I^* is a-minimal, M_J^* is a-minimal by Corollary 1.4. Thus by Theorem 1.3(4) we can choose $\eta \in J$ of minimal length such that $p \not\perp M_\eta$. By Proposition 2.11, there is $q \in S(M_\eta)$ such that $p \not\perp q$ and $q \perp M_{\eta^-}$ when $\eta \neq \langle\rangle$.

Let p', q' denote the respective nonforking extensions of p, q to $S(M_I^*)$. Since M_I^* is an a-model $p' \underset{M_I^*}{\not\perp^a} q'$. Choose a subset D such that $AM_\eta \subseteq D \subseteq M_I^*$ such that $|D| \leq \mu$ and $p'' \underset{D}{\not\perp^a} q''$, where p'', q'' denote the respective restrictions of p, q to D. Since $\{M_\eta : \eta \in I\}$ is λ-full and $|D| < \lambda$, there is $b \in M_I^*$ realizing q''. Thus, p has a forking extension to M_I^*, which implies that M_I^* is λ-saturated by Lemma 2.1.

Finally, since $\{M_\eta : \eta \in I\}$ is λ-full, $|M_I| = \lambda$. Since $\lambda \geq 2^{|T|}$, the size of an a-prime model over a set of size λ has size at most $\lambda^{<\kappa}$. So, if $\lambda^{<\kappa} = \lambda$, then $|M_I^*| = \lambda$, hence is saturated. ∎

Lemma 4.4 *Fix an independent tree $\{M_\eta : \eta \in I\}$ of a-models. Suppose that $\langle J_\alpha : \alpha \leq \delta\rangle$ is a continuous, increasing sequence of subtrees of I and $\langle E_\alpha : \alpha < \delta\rangle$ is a sequence of sequences such that E_α is an a-construction sequence over M_{J_α} and E_α is an initial segment of E_β whenever $\alpha < \beta < \delta$. Then any a-prime model over $\bigcup E^*$ is a-prime over M_{J_δ}, where E^* is the shortest sequence such that each E_α is an initial segment.*

Proof. It follows from Lemma 3.1 that each E_α is a-constructible over M_{J_δ}, so E^* is a-constructible over M_{J_δ} as well. Thus, if N is a-prime (hence a-constructible) over $\bigcup E^*$, then N is a-constructible (hence a-prime) over M_{J_δ}. ∎

Proof of Theorem 1.8. The implications $(\mathbf{1}) \Rightarrow (\mathbf{3}) \Rightarrow (\mathbf{4})$ as well as $(\mathbf{5}) \Rightarrow (\mathbf{4})$ are trivial.

We begin by showing $(\mathbf{3}) \Rightarrow (\mathbf{1})$. Suppose (3) holds and fix an independent tree of a-models $\{M_\eta : \eta \in I\}$. Let M_I^* be any a-prime model over M_I. Form an increasing sequence $\langle N_n : n \in \omega\rangle$ of a-submodels of M_I^* as follows:

For each $n \in \omega$, let $I_n = \{\eta \in I : lg(\eta) \leq n\}$. Let $N_0 = M_{\langle\rangle}$. We inductively define N_{n+1} as any a-prime submodel of M_I^* over $N_n \cup M_{I_{n+1}}$. Let N^* be any a-prime submodel of M_I^* over $\bigcup\{N_n : n \in \omega\}$. By Lemma 3.1 N^* is also a-prime over M_I, hence N^* and M_I^* are isomorphic over M_I. So it suffices to show that N^* is a-minimal over M_I. By Theorem 1.3(3) it suffices to show that every nonalgebraic $p \in S(N^*)$ is nonorthogonal to some M_η. So fix such a nonalgebraic type p. By NDIDIP and Theorem 1.3(3) there is a smallest $n \in \omega$ such that $p \not\perp N_n$. If $n = 0$ then we finish since $N_0 = M_{\langle\rangle}$. So assume $n > 0$. Let $J_n = \{\eta \in I : lg(\eta) = n\}$. By Lemma 3.1 $\{M_\eta : \eta \in J_n\}$ are independent over N_{n-1}. Thus, we can find a set $\{M'_\eta : \eta \in J_n\}$ of submodels of N_n such that each M'_η is a-prime over $M_\eta \cup N_{n-1}$ and N_n is a-prime over $\bigcup\{M'_\eta : \eta \in J_n\}$. Since κ-NDOP implies μ-NDOP for any cardinal μ and since $p \not\perp N_n$, it follows from Theorem 1.3(4) that $p \not\perp M'_\eta$ for some $\eta \in J_n$. But now, since M_η and N_{n-1} are independent over M_{η^-}, it follows from another instance of NDOP that $p \not\perp M_\eta$.

The verification of (**4**) $\Rightarrow$ (**5**) is identical once one checks that if the original tree M_I was normal, then the sequence $\langle N_n : n \in \omega\rangle$ defined above is normal as well.

(**5**) $\Rightarrow$ (**2**) Fix a cardinal $\lambda > 2^{|T|}$, a saturated model N of size λ, and a small partial decomposition $\{M_\eta : \eta \in J\}$ of N. The existence of a saturated model of size $\lambda \geq 2^{|T|}$ implies that $\lambda^{<\kappa} = \lambda$ (see VIII 4.7 of [7]). Now $\{M_\eta : \eta \in J\}$ is also a partial decomposition of $\mathfrak{C}$, so by Lemma 4.2 there is a tree I of size λ and a λ-full partial decomposition $\{M_\eta : \eta \in I\}$ of $\mathfrak{C}$ extending it. By (5) M_I^* is a-minimal over M_I, so Lemma 4.3 asserts that M_I^* is saturated of power λ. Thus, there is an isomorphism $h : M_I^* \to N$ over M_J. Then $\{h(M_\eta) : \eta \in I\}$ is a decomposition of N extending $\{M_\eta : \eta \in J\}$.

(**2**) $\Rightarrow$ (**3**) Assume that (2) holds. The heart of the argument is contained in the proof of the following claim.

Claim. If $\{M_\eta : \eta \in H\}$ is any partial decomposition of $\mathfrak{C}$, then any a-prime model M_H^* over M_H is a-minimal over M_H.

Proof. Fix $\mu > |M_H^*| + 2^{|T|}$ such that $\mu^{<\kappa} = \mu$ and choose a saturated model N of size μ containing M_H^*. By (2) there is a decomposition $\{M_\eta : \eta \in H'\}$ of N extending $\{M_\eta : \eta \in H\}$. Since N is a-minimal over $M_{H'}$, M_H^* is a-minimal over M_H by Corollary 1.4. ∎

We first verify that NDIDIP holds. Choose an increasing sequence $\langle M_n : n \in \omega\rangle$ of a-models. Let $M_\omega = \bigcup\{M_n : n \in \omega\}$ and let M_ω^* be a-prime over $M_\omega = \bigcup\{M_n : n \in \omega\}$. We will show that every nonalgebraic type over

M_ω^* is nonorthogonal to some M_n. Fix a regular cardinal $\lambda > |M_\omega^*| + 2^{|T|}$ satisfying $\lambda^{<\kappa} = \lambda$. Note that $(\lambda^{+n})^{<\kappa} = \lambda^{+n}$ for each $n \in \omega$. Inductively construct an increasing sequence $\langle N_n : n \in \omega \rangle$ of models such that each N_n is saturated of size λ^{+n}, contains M_n, $\mathrm{tp}(N_0/M_\omega^*)$ does not fork over M_0, and $\mathrm{tp}(N_{n+1}/M_\omega^* N_n)$ does not fork over $M_{n+1}N_n$ for each $n \in \omega$. It is an easy exercise in nonforking (using X 1.1 of [7]) to see that if a nonalgebraic type in $S(M_\omega^*)$ were nonorthogonal to some N_n, then it would be nonorthogonal to M_n. So let $N_\omega = \bigcup\{N_n : n \in \omega\}$, let N_ω^* be a-prime over N_ω and let $p \in S(N_\omega^*)$ be nonalgebraic. It suffices to show that $p \not\perp N_n$ for some $n \in \omega$.

Let $M_{\langle\rangle} \subseteq N_0$ be any a-prime submodel over $\emptyset$. Since $\{M_{\langle\rangle}\}$ is a small, partial decomposition of N_0, (2) implies that there is an extension $\{M_\eta : \eta \in J_0\}$ that is a decomposition of N_0. Continuing inductively, since a decomposition $\{M_\eta : \eta \in J_n\}$ of N_n is a small, partial decomposition of the saturated model N_{n+1}, (2) implies that there is an extension $\{M_\eta : \eta \in J_{n+1}\}$ that is a decomposition of N_{n+1}.

Let $J_\omega = \bigcup\{J_n : n \in \omega\}$. Let E_0 be an a-construction sequence for N_0 over M_{J_0}. By Lemma 3.1, E_0 is an a-construction sequence over M_{J_1}, so as N_1 is both a-prime and a-minimal over M_{J_1}, there is an a-construction sequence E_1 end extending E_0 for N_1 over M_{J_1}. Continuing inductively, we construct a sequence $\langle E_n : n \in \omega \rangle$ of sequences such that E_n is an a-construction sequence over M_{J_n} and E_n is an initial segment of E_{n+1} for all $n \in \omega$. By Lemma 4.4 N_ω^*, which was chosen to be a-prime over $N_\omega = \bigcup E^*$, is also a-prime over M_{J_ω}. The Claim above implies that N_ω^* is a-minimal over M_{J_ω}, so $p \not\perp M_\eta$ for some $\eta \in J_\omega$. Thus $p \not\perp N_n$ for some $n \in \omega$.

Next we argue that T has κ-NDOP. Fix any a-model M and any set $\{M_i : i < \alpha < \kappa\}$ of a-models that each contain and collectively are independent over M. Let M^* be a-prime over $\bigcup\{M_i : i < \alpha\}$ and choose $\lambda > |M^*| + 2^{|T|}$ such that $\lambda^{<\kappa} = \lambda$. Arguing as above, first choose a saturated model N containing M of size λ such that $\mathrm{tp}(N/M^*)$ does not fork over M (so $\{M_i : i < \alpha\}$ are independent over N) and then inductively choose a set $\{N_i : i < \alpha\}$ of saturated models, each of size λ^+ such that each N_i contains $M_i \cup N$ and $\mathrm{tp}(N_i/M^* \cup N \cup \{N_j : j < i\})$ does not fork over $M_i \cup N$. Thus $\{N_i : i < \alpha\}$ are independent over N. As in the case above, if a type in $S(M^*)$ is nonorthogonal to some N_i, then it is nonorthogonal to M_i. So let N^* be a-prime over $\bigcup\{N_i : i < \alpha\}$ and fix a nonalgebraic type $p \in S(N^*)$. It is certainly sufficient to show that $p \not\perp N_i$ for some $i < \alpha$.

As before, use (2) to choose a decomposition $\{M_\eta : \eta \in H\}$ of N. Then for each $i < \alpha$ use (2) to get an extension $\{M_\eta : \eta \in J_i\}$ that is a decomposition of N_i. Without loss assume that $J_i \cap J_j = H$ for all $i \neq j$. As notation, let $I_i = H \cup \bigcup\{J_j : j < i\}$ for each $i < \alpha$ and let $I = \bigcup\{I_i : i < \alpha\}$. Since $\{N_i : i < \alpha\}$ are independent over N, $\{M_\eta : \eta \in I\}$ is a partial decomposition of $\mathfrak{C}$. As in the NDIDIP case above, Lemmas 3.1 and 4.4 imply that N^* is a-prime over $M_I = \bigcup\{M_\eta : \eta \in I\}$. By the Claim, N^* is a-minimal over M_I. Thus $p \not\perp M_\eta$ for some $\eta \in I$ by Theorem 1.3(3), which implies that $p \not\perp N_i$ for some $i < \alpha$. ∎

Proof of Proposition 1.9. Fix a theory T with κ-NDOP. We recall the usual definition of the depth dp_I of a node η of a well-founded tree I, namely

$$dp_I(\eta) = \sup\{dp_I(\nu) + 1 : \ \nu \text{ an immediate successor of } \eta\}$$

and we define the depth of I to be $dp_I(\langle\rangle)$. We prove Proposition 1.9 by induction on the depth of I. Fix an ordinal α and assume that every a-prime model over a well-founded, independent tree of a-models of depth less than α is a-minimal over the tree of a-models.

Suppose that I is well-founded of depth α and that $\{M_\eta : \eta \in I\}$ is an independent tree of a-models indexed by I. Let M_I^* be any a-prime model over M_I and choose any type $p \not\perp M_I^*$. We will show that $p \not\perp M_\eta$ for some $\eta \in I$, whence M_I^* is a-minimal over M_I by Theorem 1.3. If $I = \{\langle\rangle\}$ then there is nothing to prove. Otherwise, let $A = \{\beta : \langle\beta\rangle \in I\}$. For each $\beta \in A$, let $I(\beta) = \{\nu : \langle\beta\rangle\hat{\ }\nu \in I\}$ and let $M_\nu^\beta = M_{\langle\beta\rangle\hat{\ }\nu}$ for each $\nu \in I(\beta)$. Choose $\{N_\beta : \beta \in A\}$ such that each N_β is an a-prime submodel of M_I^* over $\bigcup\{M_\nu^\beta : \nu \in I(\beta)\}$ and M_I^* is a-prime over $\bigcup\{N_\beta : \beta \in A\}$. Since the original tree of a-models was independent, $\{N_\beta : \beta \in A\}$ is independent over $M_{\langle\rangle}$. So, since κ-NDOP implies μ-NDOP for any cardinal μ, we can choose $\beta^* \in A$ so that $p \not\perp N_{\beta^*}$. By our definition of depth, $dp(I(\beta^*)) < dp(I) = \alpha$, so N_β is a-minimal over $\bigcup\{M_\nu^{\beta^*} : \nu \in I(\beta^*)\}$. So, by Theorem 1.3 $p \not\perp M_\eta$ for some $\eta \in I$. ∎

Proof of Corollary 1.10. Suppose that T has κ-NDOP and is shallow. Fix any saturated model N with $|N| > 2^{|T|}$ and any small partial decomposition $\{M_\eta : \eta \in J\}$ of N. We will show that this partial decomposition can be extended to a decomposition of N, which suffices by Theorem 1.8. Let $\lambda = |N|$. By VIII 4.7 of [7] the existence of a saturated model of size $\lambda > 2^{|T|}$ implies that $\lambda^{<\kappa} = \lambda$. Let $\{M_\eta : \eta \in I\}$ be a λ-full partial decomposition of $\mathfrak{C}$ extending $\{M_\eta : \eta \in J\}$, which exists by Lemma 4.2. Since T is shallow, the index tree I is well-founded. Since T has κ-NDOP as well, Proposition 1.9 implies that M_I^* is a-minimal over M_I, hence M_I^* is saturated of power λ by Lemma 4.3. So M_I^* and N are both saturated of size λ and contain M_J. Choose an isomorphism $h : M_I^* \to N$ over M_J. Then $\{h(M_\eta) : \eta \in I\}$ is our desired decomposition of N. ∎

5 Countable theories and the proof of Theorem 1.11

Until now, the cardinality of the language was irrelevant. In this section we restrict ourselves to countable languages and prove Theorem 1.11. The assumption of countability allows us to bring in some results from classical descriptive set theory. In particular, the proof given here relies on the fact that analytic subsets of Polish spaces have the property of Baire, i.e., for every analytic A there is an open U such that $A \triangle U$ is meagre (see e.g., [3]). At its heart, the proof presented here is similar to the argument that every $\mathbf{\Sigma_1^1}$-definable ultrafilter on ω is principal. The similarity between these two arguments is expounded upon in [4].

Theorem 5.1 *If T is countable and has NDOP, then T has μ-NDOP for all infinite cardinals μ.*

Proof. As noted in the remarks following Definition 1.6, the theorem follows immediately if T is superstable. Consequently, we assume for the whole of this section that

T **is countable, stable, but not superstable, with NDOP.**

In particular, $\kappa(T) = \aleph_1$ and the class of a-models of T is precisely the class of $\aleph_1$-saturated models of T. The first three subsections provide the requisite background and Theorem 5.1 is proved in Subsection 5.4.

5.1 On stable systems

In this subsection we set notation and prove an extension theorem for stable systems and an embedding theorem for pairs of stable systems.

Definition 5.2 A *good index set* I is a nonempty, countable set of finite sets that is closed under subsets, i.e., $u \in I$ and $v \subseteq u$ implies $v \in I$. An I*-system* $\overline{X} = \{X_u : u \in I\}$ is a family of sets indexed by I such that $X_u \subseteq X_v$ whenever $u \subseteq v$. As notation, for any I-system $\overline{X}$ and any $u \in I$

$$X_{\subsetneq u} = \bigcup\{X_v : v \subsetneq u\} \quad \text{and} \quad X_{\not\supseteq u} = \bigcup\{X_v : v \not\supseteq u\}$$

Throughout this section J denotes the set of finite subsets of ω and $\mathbf{K} = \{\mathbf{u} \in \mathbf{J} : |\mathbf{u}| \leq \mathbf{1}\}$.

The following notion is the major theme of Section XII.2 of [7].

Definition 5.3 A *stable system* $\overline{M}$ *of models indexed by* I is an I-system $\overline{M} = \{M_u : u \in I\}$ of models such that $M_u \underset{M_{\subsetneq u}}{\perp} M_{\not\supseteq u}$ for all $u \in I$.

As a simple special case, note that $\{M_u : u \in K\}$ is a stable system of models if and only if $M_\emptyset \subseteq M_{\{i\}}$ for each $i \in \omega$ and $\{M_{\{i\}} : i \in \omega\}$ are independent over $M_\emptyset$.

The following Lemma is our primary tool for constructing stable systems.

Lemma 5.4 *Suppose that I is a good index set, u finite, $u \notin I$, but every proper subset of u is an element of I. If $\{M_v : v \in I\}$ is a stable system of a-saturated models and M_u is a-prime over $M_{\subsetneq u}$, then $\{M_v : v \in I \cup \{u\}\}$ is a stable system of a-saturated models. Moreover, if M_u is the union of an a-construction sequence $\bar{a} = \langle a_\alpha : \alpha < \beta\rangle$ over $M_{\subsetneq u}$, then $\bar{a}$ is also an a-construction sequence over $\bigcup\{M_v : v \in I\}$.*

Proof. Let $I_u = \{v \in I : v \subseteq u\}$. Then I_u is a finite, good index set, so by XII, Conclusion 2.11 of [7], M_u is ℓ-isolated over $M_{\subsetneq u}$. Also, by XII, Lemma 2.3(2) of [7], the pair $(M_{\subsetneq u}, \bigcup\{M_v : v \in I\})$ satisfies the Tarski-Vaught property, hence $\mathrm{tp}(M_u/M_{\subsetneq u})$ has a unique (nonforking) extension to a type in $S(\bigcup\{M_v : v \in I\})$ (see e.g., XII, Lemma 1.12(2) of [7]). In particular, $M_u \underset{M_{\subsetneq u}}{\perp} M_{\not\supseteq u}$ and the 'Moreover' clause follows immediately. In order to complete the proof that $\{M_v : v \in I \cup \{u\}\}$ is a stable system it suffices to show that that

$$M_v \underset{M_{\subsetneq v}}{\perp} M_{\not\supseteq v} M_u$$

where $M_{\not\supseteq v} = \bigcup\{M_r : r \in I, v \not\subseteq r\}$. for every $v \in I$ satisfying $v \not\subseteq u$ (for $v \subseteq u$ the appropriate requirement is satisfied since $\{M_v : v \in I\}$ is a stable system). So fix $v \not\subseteq u$, hence $M_{\subsetneq u} \subseteq M_{\not\supseteq v}$. From above,

$$M_u \underset{M_{\subsetneq u}}{\perp} M_v M_{\subsetneq v} M_{\not\supseteq v}$$

so $M_v \underset{M_{\subsetneq u} M_{\subsetneq v} M_{\not\supseteq v}}{\perp} M_u$. Thus $M_v \underset{M_{\subsetneq v} M_{\not\supseteq v}}{\perp} M_u$ and the result follows by the transitivity of nonforking. ∎

Proposition 5.5 *Suppose $\overline{M}_K = \{M_u : u \in K\}$ is a stable system of a-saturated models indexed by K and M is a-prime over $\bigcup \overline{M}_K$. Then there is a stable system $\overline{M}_J = \{M_u : u \in J\}$ indexed by J such that*

1. *Each $M_u \subseteq M$ and the u'th entry of $\overline{M}_J =$ the u'th entry of $\overline{M}_K$ for each $u \in K$;*
2. *M is a-prime over $\bigcup \overline{M}_J$;*
3. *For each $u \in J \setminus K$, M_u is a-prime over $\bigcup\{M_v : v \subsetneq u\}$; and*
4. *For all pairs of good index sets $I \subseteq I^* \subseteq J$, $\bigcup\{M_u : u \in I^*\}$ is the union of an a-construction sequence over $\bigcup\{M_u : u \in I\}$.*

Proof. Let $\langle u_j : j \in \omega\rangle$ be an enumeration of $J \setminus K$ such that for every $j \in \omega$, if $v \subseteq u_j$, then $v \in K \cup \{u_\ell : \ell < j\}$. As notation, let $J_j = K \cup \{u_\ell : \ell < j\}$ for each $j \in \omega$. Note that each J_j is a good index set. We construct $\mathcal{N}_J = \{N_u : u \in J\}$ as follows. First, let $N_u = M_u$ for each $u \in K$. Then for each $j \in \omega$ inductively choose N_{u_j} to be any a-prime model over $\bigcup\{N_v : v \subsetneq u\}$. Let N^* be any a-prime model over $\bigcup \mathcal{N}_J$. By successively applying Lemma 5.4 to each of the good index sets J_j we obtain that $\{N_u : u \in J_j\}$ is a stable system indexed by J_j such that N_{u_j} is a-constructible over $\bigcup\{N_v : v \in J_j\}$ for every $j \in \omega$. It follows that $\bigcup\{N_u : u \in J\}$ is a-constructible over $\bigcup\{N_u : u \in K\}$. Since $N_u = M_u$ for all $u \in K$, this implies that N^* is a-constructible (hence a-prime) over $\bigcup\{M_u : u \in K\}$. By the uniqueness of a-prime models there is an isomorphism $h : N^* \to M$ fixing $\bigcup\{M_u : u \in K\}$ pointwise. Define $M_u = h(N_u)$ for each $u \in J$. It is easy to see that $\overline{M}_J = \{M_u : u \in J\}$ satisfies Clauses (1)–(3).

As for (4), fix good index sets $I \subseteq I^* \subseteq J$. Let $\langle u_j : j < \alpha \leq \omega\rangle$ be an enumeration of $I^* \setminus I$ such that for every $j < \alpha$, if $v \subseteq u_j$, then

$v \in I \cup \{u_\ell : \ell < j\}$. As notation, let $I^*_j = I \cup \{u_\ell : \ell < j\}$ for each $j < \alpha$. Each I^*_j is a good index set, so it follows from Lemma 5.4 and induction on $j < \alpha$ that N_{u_j} is a-constructible over $\bigcup\{N_v : v \in I^*_j\}$ for each j. Clause (4) follows from this by the transitivity of a-constructibility. ∎

The next Definition is not given explicitly in [7], but the notion is inherent in the proof of Lemma XII 2.3 there.

Definition 5.6 Given a good index set I and $* \notin \bigcup I$, let $I^* = I \cup \{u \cup \{*\} : u \in I\}$. A *linked pair of stable systems* $(\overline{A}, \overline{B})$ is a stable system $\overline{C}$ indexed by I^* where for each $v \in I^*$, $C_v = A_v$ when $* \notin v$ and $C_v = B_u$ when $v = u \cup \{*\}$.

By unraveling the definitions, if $(\overline{A}, \overline{B})$ is a linked pair of stable systems then both $\overline{A}$ and $\overline{B}$ are stable systems indexed by I, $A_u \preceq B_u$ and $A_u \underset{A_{\subsetneq u}}{\perp} B_{\not\supseteq u}$ for all $u \in I$. Moreover, within the proof of Lemma 2.3 of Chapter XII of [7], the second author shows that these consequences characterize this notion. More precisely, if $\overline{A}, \overline{B}$ are stable systems indexed by I and for each $u \in I$, $A_u \preceq B_u$ and $A_u \underset{A_{\subsetneq u}}{\perp} B_{\not\supseteq u}$, then $(\overline{A}, \overline{B})$ are a linked pair of stable systems.

By using this characterization, the proof of the following Lemma is just like the proof of the Downward Löwenheim-Skolem theorem and is left to the reader.

Lemma 5.7 *Let $\overline{M}$ be any stable system of models indexed by I and let $\overline{X}$ be any I-system of sets in which each X_u is a countable subset of M_u. Then there is a stable system $\overline{A}$ such that for each $u \in I$, A_u is countable, $X_u \subseteq A_u \preceq M_u$, and $(\overline{A}, \overline{M})$ is a linked pair of stable systems.*

More interesting is Proposition 5.9. Its proof uses the following very general lemma, which is also left to the reader.

Lemma 5.8 *Suppose M is a-saturated, A, C are countable, $A \subseteq M$, and q is a type in countably many variables over AC that does not fork over A. Then q is realized in M.*

Proposition 5.9 *Suppose that $(\overline{A}, \overline{B})$ and $(\overline{A}, \overline{M})$ are both linked pairs of stable systems such that each B_u is countable and each M_u is a-saturated. Then there is an elementary map $f : \bigcup \overline{B} \to \bigcup \overline{M}$ such that $f|\bigcup \overline{A} = id$ and $f(B_u) \preceq M_u$ for each $u \in I$.*

Proof. Fix an enumeration $\{u_j : j < j^* \leq \omega\}$ of I such that $u_i \subseteq u_j$ implies $i \leq j$. To ease notation, write B_j in place of B_{u_j}, $B_{\subsetneq j}$ in place of $B_{\subsetneq u_j}$ and $B_{\not\supseteq j}$ in place of $B_{\not\supseteq u_j}$. Note that the condition on our enumeration ensures that $\bigcup\{B_k : k < j\} \subseteq B_{\not\supseteq j}$. We construct f as the union of a chain of increasing elementary maps

$$f_j : \bigcup \overline{A} \cup \bigcup\{B_k : k < j\} \to \bigcup \overline{A} \cup \bigcup\{M_k : k < j\}$$

that satisfy $f_j | \bigcup \overline{A} = id$ and $f_j(B_k) \preceq M_k$ for all $k < j$.

To begin, let f_0 be the identity map on $\bigcup \overline{A}$. Now assume that $0 < j < j^*$ and that f_{j-1} has been defined. Since $(\overline{A}, \overline{B})$ is a linked pair of stable systems

$$B_j \underset{A_j B_{\subsetneq j}}{\perp} \bigcup \overline{A} B_{\not\supseteq j}$$

Also, $A_j B_{\subsetneq j} \subseteq \mathrm{dom}(f_{j-1})$, so $f_{j-1}(A_j B_{\subsetneq j}) \subseteq M_j$. Since M_j is a-saturated, Lemma 5.8 ensures the existence of an elementary map $f_j \supseteq f_{j-1}$ with $f_j(B_j) \preceq M_j$, and our proof is complete. ∎

5.2 Pseudo ℓ-isolation

If the index set I is finite and $\overline{M} = \{M_u : u \in I\}$ is a stable system of a-saturated models, then a type $p \in S(\bigcup\{M_u : u \in I\})$ is a-isolated if and only if it is ℓ-isolated (see XII 2.11 of [7]). When one is analyzing a type over the union of a stable system of models of a superstable theory, the restriction that I be finite is inconsequential since the type is based on the union of a finite subsystem. However, here our theory is strictly stable, so we need an analogue of this result that holds for stable systems over infinite index sets as well. The notion of pseudo ℓ-isolation satisfies our needs.

Definition 5.10 A formula $\psi(x)$ (possibly with hidden parameters) *decides* the formula $\varphi(x, e)$ if either $\psi(x) \vdash \varphi(x, e)$ or $\psi(x) \vdash \neg\varphi(x, e)$. For M any model, $\psi(x)$ *decides* $\varphi(x, M)$ if $\psi(x)$ decides $\varphi(x, e)$ for all $e \in M$.

Lemma 5.11 *Suppose $M \subseteq A$, M is an a-saturated model, and $p \in S(A)$ is an a-isolated type. Then for any L-formula $\varphi(x, y)$ there is $\psi(x) \in p$ that decides $\varphi(x, M)$.*

Proof. Fix an L-formula $\varphi(x, y)$. Since p is a-isolated we can choose $q = \{\psi_n(x) : n < n^* \leq \omega\} \subseteq p$ such that $q \vdash p$ and $\psi_n \vdash \psi_{n-1}$ for all $0 < n < n^*$. For each $n < n^*$ let

$$Z_n = \{e \in M : \psi_n \text{ decides } \varphi(x, e)\}$$

Since T is stable each Z_n is M-definable. Furthermore, since $q \vdash p$ and $p \in S(A)$ is a complete type, $\bigcup_{n\in\omega} Z_n = M$. Since M is a-saturated this implies $M = Z_m$ for some $m < n^*$. That is, ψ_m decides $\varphi(x, M)$. ∎

Definition 5.12 Suppose that $\overline{M}$ is an I-system of models. A type $p \in S(\bigcup \overline{M})$ is *pseudo ℓ-isolated over $\overline{M}$* (not over $\bigcup \overline{M}$!) if for every $u \in I$ and every L-formula $\varphi(x, y)$, there is $\psi(x) \in p$ deciding $\varphi(x, M_u)$.

A set D is *pseudo ℓ-atomic over $\overline{M}$* if $\mathrm{tp}(d/\bigcup \overline{M})$ is pseudo ℓ-isolated over $\overline{M}$ for all finite tuples d from D.

The following Lemma connects these notions with a-atomicity.

Lemma 5.13 *Let $\overline{M}$ be an I-system of a-saturated models. For any set D, D is a-atomic over $\bigcup \overline{M}$ if and only if D is pseudo ℓ-atomic over $\overline{M}$.*

Proof. Left to right is immediate by Lemma 5.11. For the converse let $p \in S(\bigcup \overline{M})$ be pseudo ℓ-isolated over $\overline{M}$. For each L-formula $\varphi(x, y)$ and each $u \in I$, choose $\psi_{\varphi,u}(x) \in p$ that decides $\varphi(x, M_u)$. Then $q = \{\psi_{\varphi,u}(x) : \varphi, u\}$ witnesses that p is a-atomic over $\bigcup \overline{M}$. ∎

Lemma 5.14 *Suppose that I is a good index set that is closed under unions, i.e., $u, v \in I$ implies $u \cup v \in I$. Let $\overline{M}$ and $\overline{M}'$ be I-systems such that $M_u \subseteq M'_u$ and $\mathrm{tp}(M'_u/\bigcup \overline{M})$ is finitely satisfiable in M_u for all $u \in I$. If $p \in S(\bigcup \overline{M})$ is pseudo ℓ-isolated over $\overline{M}$ then p has a unique extension to $p' \in S(\bigcup \overline{M}')$ (which is pseudo ℓ-isolated over $\overline{M}'$).*

Proof. For each $\varphi(x, y)$ and $u \in I$ choose $\psi(x) \in p$ that decides $\varphi(x, M_u)$. We argue that $\psi(x)$ decides $\varphi(x, M'_u)$ as well. To see this, choose $v \in I$ such that $\psi(x)$ is over M_v. By our constraint on I we may assume that $u \subseteq v$. If $\psi(x, a_v)$ did not decide $\varphi(x, M'_u)$ then for some $b \in M'_u$ $\theta(a_v, b)$ would hold, where $\theta(y, z)$ is $\exists x_1 \exists x_2[\psi(x_1, y) \wedge \psi(x_2, y) \wedge (\varphi(x_1, z) \not\leftrightarrow \varphi(x_2, z))]$. But then finite satisfiability would imply that $\theta(a_v, a_u)$ would hold for some $a_u \in M_u$, which would contradict the fact that $\psi(x)$ decides $\varphi(x, M_u)$. Thus $\{\psi_{\varphi,u}(x) : \varphi, u\}$ has a unique extension to $p' \in S(\bigcup \overline{M}')$ and the same formulas witness the pseudo ℓ-isolation of p'. ∎

As notation, if $\overline{M} = \{M_u : u \in J\}$ is a stable system of models indexed by J and $X \subseteq \omega$, $\overline{M}_X$ denotes the stable system (also indexed by J) $\{M_{u\cap X} : u \in J\}$, while M_X denotes the model with universe $\bigcup\{M_u : u \in J \cap \mathcal{P}(X)\}$. The fact that M_X is a model follows from the fact that the index set J is closed under finite unions. It is readily checked that $M_X = \bigcup \overline{M}_X$. In particular, $M_\omega = \bigcup \overline{M}$. The following Lemma is a stable system analogue of Lemma 3.1.

Lemma 5.15 *Let $\overline{M}$ be any stable system of a-saturated models indexed by J, let $X \subseteq \omega$ and let $Y = \omega \setminus X$.*

1. *If D is pseudo ℓ-atomic over $\overline{M}_X$ then $\mathrm{tp}(D/M_X)$ has a unique extension to a type over M_ω, and D is also pseudo ℓ-atomic over $\overline{M}$.*
2. *Every a-construction sequence $\bar{a} = \langle a_\alpha : \alpha < \beta \rangle$ over M_X is an a-construction sequence over M_ω and $\mathrm{tp}(\bar{a}/M_\omega)$ does not fork over M_X.*
3. *If N_X, N_Y are a-prime over M_X, M_Y respectively, then*
   - *(a)* $\mathrm{tp}(N_X/M_X) \vdash \mathrm{tp}(N_X/M_\omega N_Y)$ *and*
   - *(b)* $\mathrm{tp}(M_\omega/M_X M_Y) \vdash \mathrm{tp}(M_\omega/N_X N_Y)$.

Proof. (1) Since $\overline{M}$ is a stable system $\mathrm{tp}(M_u/M_X)$ is finitely satisfiable over $M_{u\cap X}$, so we can apply Lemma 5.14 to the stable systems $\overline{M}_X$ and $\overline{M}$.

Using (1) and Lemma 5.13, (2) follows by induction on β.

For both parts of (3) choose $\varphi(x, y, z)$ and tuples c_1 from N_X, c_2 from N_Y, and d from M_ω such that $\varphi(c_1, c_2, d)$ holds.

We first show that there is $\psi(x, e) \in \mathrm{tp}(c_1/M_X)$ such that $\psi(x, e) \vdash \varphi(x, c_2, d)$. Choose a finite u such that d is from M_u. Since $\overline{M}_X$ is a stable system and N_X/M_X is a-atomic, $\mathrm{tp}(c_1/M_X)$ is pseudo ℓ-isolated over $\overline{M}_X$. So there is $\psi(x, e) \in \mathrm{tp}(c_1/M_X)$ such that $\psi(x, e) \vdash \mathrm{tp}_\varphi(c_1/M_{u\cap X})$. We argue that this $\psi(x, e) \vdash \varphi(x, c_2, d)$.

Let $Z = Y \cup u$. Since $X \cap Z = u \cap X$ and since $\overline{M}$ is a stable system, $\mathrm{tp}(M_Z/M_X)$ does not fork over $M_{u\cap X}$. As well, $\mathrm{tp}(c_2/M_Y)$ is a-isolated, hence $\mathrm{tp}(c_2/M_Z)$ is a-isolated as in (2). Since $M_{u\cap X}$ is an a-model, $c_2 M_Z$

is dominated by M_Z over $M_{u\cap X}$, hence $M_X \underset{M_{u\cap X}}{\perp} M_Z c_2$ follows by symmetry. Since $M_{u\cap X}$ is a model, the pair $(M_{u\cap X}, M_Z c_2)$ has the Tarski-Vaught property, hence $\mathrm{tp}_\varphi(c_1/M_{u\cap X})$ has a unique extension $q_\varphi \in S_\varphi(M_Z c_2)$ and $\psi(x,e) \vdash q_\varphi$. In particular, $\psi(x,e)$ decides $\varphi(x,c_2,d)$. But since $\varphi(c_1,c_2,d)$ holds, it decides it positively, i.e., $\psi(x,e) \vdash \varphi(x,c_2,d)$. Thus, (3a) holds.

To establish (3b) choose d' such that $\mathrm{tp}(d'/M_X M_Y) = \mathrm{tp}(d/M_X M_Y)$. It suffices to show that $\varphi(c_1,c_2,d)$ holds. So choose $\psi(x,e)$ as above and let

$$\theta(y,d,e) := \forall x[\psi(x,e) \to \varphi(x,y,d)]$$

By our choice of $\psi(x,e)$, $\theta(y,\bar{d},e) \in \mathrm{tp}(c_2/M_\omega)$. By (1) $\mathrm{tp}(c_2/M_Y) \vdash \mathrm{tp}(c_2/M_\omega)$, so there is $\delta(y,e') \in \mathrm{tp}(c_2/M_Y)$ such that $\delta(y,e') \vdash \theta(y,d,e)$. Since $e,e' \in M_X \cup M_Y$, it follows that $\delta(y,e') \vdash \theta(y,d',e)$, hence $\theta(c_2,d',e)$. Thus, $\varphi(c_1,c_2,d')$ holds as required. ∎

5.3 The standard topology on $\mathcal{P}(\omega)$

The standard topology on $\mathcal{P}(\omega)$ is obtained by positing that the sets

$$U_{F,G} = \{X \in \mathcal{P}(\omega) : F, G \text{ are finite subsets of } \omega,\ F \subseteq X,\ X \cap G = \emptyset\}$$

form a basis of open sets. Topologized in this way, the natural mapping between subsets of ω and characteristic functions is a homeomorphism between $\mathcal{P}(\omega)$ and the Cantor set ${}^\omega 2$.

Note that $U_{F,G} = \emptyset$ if and only if $F \cap G \neq \emptyset$. As notation, let $\mathcal{D} = \{(F,G) : F, G \text{ are finite subsets of } \omega \text{ and } F \cap G = \emptyset\}$. For $(F,G), (F',G') \in \mathcal{D}$ we write $(F,G) \leq (F',G')$ if and only if $F \subseteq F'$ and $G \subseteq G'$.

It is easily checked that a set $R \subseteq \mathcal{P}(\omega)$ is nowhere dense if and only if for every $(F,G) \in \mathcal{D}$ there is $(F',G') \in \mathcal{D}$ such that $(F',G') \geq (F,G)$ and $U_{F',G'} \cap R = \emptyset$. Recall that a set $Z \subseteq \mathcal{P}(\omega)$ is *meagre* if it is a countable union of nowhere dense subsets.

The following Lemma is routine, but is included for completeness.

Lemma 5.16 *Let Z be any meagre subset of $\mathcal{P}(\omega)$. Then:*

1. *There is $X \in \mathcal{P}(\omega)$ such that $X, \omega \setminus X \notin Z$.*
2. *There are $\{X_i : i \in \omega\} \subseteq \mathcal{P}(\omega) \setminus Z$ with $X_i \cap X_j = \emptyset$ when $i < j < \omega$.*

Proof. Suppose that $Z = \bigcup_{n\in\omega} R_n$, where each R_n is nowhere dense.

(1) Using the characterization of nowhere denseness given above, inductively construct a sequence $\langle (F_n, G_n) : n \in \omega\rangle$ from $\mathcal{D}$ that satisfies $(F_n, G_n) \leq (F_{n+1}, G_{n+1})$, $U_{F_{2n},G_{2n}} \cap R_n = \emptyset$, and $U_{G_{2n+1},F_{2n+1}} \cap R_n = \emptyset$. Take $X = \bigcup_{n\in\omega} F_n$. Then $X \in U_{F_n,G_n}$ for all n, so $X \notin Z$. Furthermore, $\omega \setminus X \in U_{G_n,F_n}$ for all n, so $\omega \setminus X \notin Z$ as well.

(2) Fix a bijection $\Phi : \omega \to \omega\times\omega$. Call an ω-sequence $\mathcal{F} = \langle F_i : i \in \omega\rangle$ of (finite) subsets of ω an *approximating sequence* if $\{F_i : i \in \omega\}$ are pairwise disjoint and $\bigcup\{F_i : i \in \omega\}$ is finite. We say that an approximating sequence $\mathcal{F} = \langle F_i : i \in \omega\rangle$ *satisfies Condition* k if, writing $\Phi(k) = (i, j)$,

$$U_{F_i,G_i} \cap R_j = \emptyset$$

where $G_i = \bigcup\{F_l : l \neq i\}$.

We inductively construct approximating sequences $\mathcal{F}^n = \langle F^n_i : i \in \omega\rangle$ for each $n \in \omega$ such that $F^n_i \subseteq F^m_i$ for all i and all $n < m < \omega$ and $\mathcal{F}^n$ satisfies Condition k for all $k < n$.

To start, define $\mathcal{F}^0 = \langle F^0_i : i \in \omega\rangle$, where each $F^0_i = \emptyset$. Now assume that $\mathcal{F}^n$ has been defined and let $\Phi(n) = (i^*, j^*)$. Let $G = \bigcup\{F^n_l : l \neq i^*\}$. Since R_{j^*} is nowhere dense there is $(F', G') \geq (F^n_{i^*}, G)$ such that $U_{F',G'} \cap R_{j^*} = \emptyset$. Let m be any integer $\neq i^*$ such that $F^n_m = \emptyset$. Let $F^{n+1}_{i^*} = F'$, $F^{n+1}_m = G' \setminus G$, and $F^{n+1}_l = F^n_l$ for all $l \neq i^*, m$. Then $\mathcal{F}^{n+1} = \langle F^{n+1}_i : i \in \omega\rangle$ satisfies Condition k for all $k \leq n$.

Finally, for each $i \in \omega$ take $X_i = \bigcup\{F^n_i : n \in \omega\}$. ∎

5.4 Proof of Theorem 5.1

By the remarks following Definition 1.6, it suffices to show that T has ω_1-NDOP. Choose a family $\{M_i : i < \omega\}$ of a-saturated models that contain and are independent over common a-saturated model $M_\emptyset$, and let M be a-prime over $\bigcup\{M_i : i < \omega\}$. Fix a nonalgebraic type $p \in S(M)$. We will eventually show that p is nonorthogonal to some M_i, which suffices by Theorem 1.3.

Let $\overline{M}_K$ denote the stable system indexed by K, where $M_{\{i\}} = M_i$ for each $i \in \omega$. Choose a stable system $\overline{M}_J = \{M_u : u \in J\}$ extending $\overline{M}_K$ satisfying Clauses (1)–(4) of Proposition 5.5.

We adopt the notation prior to Lemma 5.15 for the whole of this section, not only for the stable system $\overline{M}$ in the claim below, but also for the related systems $\overline{A}$ and $\overline{B}$ that follow. For each $X \subseteq \omega$ let

$$\mathcal{N}_X = \{N \preceq M : N \text{ is a-prime over } M_X \text{ and } M \text{ is a-prime over } N \cup M_\omega\}$$

For each finite $\Delta \subseteq L$, let $W_\Delta = \{X \subseteq \omega : p \not\perp^\Delta N \text{ for some } N \in \mathcal{N}_X\}$ and let $W = \bigcup\{W_\Delta : \Delta \subseteq L \text{ finite}\}$.

Claim 5.17 *For all $X \subseteq \omega$, at least one of $X, (\omega - X) \in W$.*

Proof. Fix $X \subseteq \omega$ and let $Y = \omega \setminus X$. Let N_X, N_Y be a-prime models over M_X, M_Y respectively. Let M^* be a-prime over $N_X \cup \mathcal{N}_Y \cup M_\omega$. We argue that M^* is also a-prime over *each of the four* sets M_ω, $N_X \cup M_\omega$, $N_Y \cup M_\omega$, and $N_X \cup N_Y$.

To see this, first note that by applying Proposition 5.5(4) with $I = (\mathcal{P}(X) \cup \mathcal{P}(Y)) \cap J$ and $I^* = J$, M_ω is a-constructible over $M_X \cup M_Y$. By Lemma 5.15(3b) M_ω is a-constructible over $N_X \cup N_Y$. Thus, M^* is a-constructible (hence a-prime) over $N_X \cup N_Y$. As well, by Lemma 5.15(2) N_X is a-constructible over M_ω. By Lemma 5.15(3a) N_Y is a-constructible over $M_\omega \cup N_X$. Hence M^* is a-prime over M_ω as well as $M_\omega \cup N_X$. That M^* is a-prime over $M_\omega \cup N_Y$ is symmetric.

But now, recall that M is also a-prime over M_ω. So there is an isomorphism $h : M^* \to M$ fixing M_ω pointwise. Since $\overline{M}$ is a stable system $M_X \underset{M_\emptyset}{\perp} M_Y$. Also, $h(N_X)$ is a-prime over M_X, hence dominated by M_X over $M_\emptyset$ and dually, $h(N_Y)$ is dominated by M_Y over $M_\emptyset$. Thus $h(N_X) \underset{M_\emptyset}{\perp} h(N_Y)$. But $p \in S(M)$ and M is a-prime over $h(N_X) \cup h(N_Y)$. By NDOP, either $p \not\perp h(N_X)$ or $p \not\perp h(N_Y)$. As the cases are symmetric, assume $p \not\perp h(N_X)$, Finally, since M is a-prime over $h(N_X) \cup M_\omega$, $h(N_X) \in \mathcal{N}_X$, so $X \in W$. ∎

Claim 5.18 *Each W_Δ is a $\mathbf{\Sigma^1_1}$-subset of $\mathcal{P}(\omega)$.*

Proof. Fix $\Delta \subseteq L$ finite. Choose $C \subseteq M$ countable such that p is based and stationary over C. Since M is a-atomic over M_ω we can choose a countable set $Z \subseteq M_\omega$ such that $\mathrm{tp}(C/Z) \vdash \mathrm{tp}(C/M_\omega)$. Using Lemma 5.7 find a stable system $\overline{A}$ indexed by J in which every A_u is countable, $Z \subseteq A_\omega$, and $(\overline{A}, \overline{M})$ is a linked pair of stable systems. Note that $\mathrm{tp}(C/M_\omega)$ does not fork over A_ω by transitivity.

Subclaim 5.19 *$X \in W_\Delta$ if and only if there exist a countable $\overline{B}$ such that $(\overline{A}, \overline{B})$ is a linked pair of stable systems and* $\mathrm{tp}(C/B_\omega)$ *does not fork over A_ω, a countable model N' that is pseudo ℓ-atomic over $\overline{B}_X$, a countable set $D \supseteq N'C$, a tuple d from D, a formula $\varphi(x, yz) \in \Delta$, and a a type $q \in S(D)$ that does not fork over N' such that $R_\Delta((p|D) \cup \{\varphi(x, db)\}) < R_\Delta(p)$ for some (every) b realizing q.*

It is easily verified that

$$\{(N', \overline{B}, X) : N' \text{ is pseudo } \ell\text{-atomic over } \overline{B}_X\}$$

is a Borel subset of a product of Polish spaces projecting onto $\mathcal{P}(\omega)$ so the $\mathbf{\Sigma^1_1}$-ness of W_Δ is an immediate consequence of the Subclaim.

To establish the subclaim (and hence the claim) first suppose that $X \in W_\Delta$. Choose $N \in \mathcal{N}_X$ such that $p \not\perp^\Delta N$. Choose D_0, d, φ, and $q_0 \in S(D_0)$ from Definition 2.15 witnessing this. Choose a countable $N' \preceq N$ such that q_0 is based on N'. Since N', C and the language L are countable, we can find a countable subset $D \subseteq D_0$ containing $N'Cd$ and $q \in S(D)$ parallel to q_0 such that $R_\Delta((p|D) \cup \{\varphi(x, db)\}) < R_\Delta(p)$ for any b realizing q. Since N' is a-atomic over M_X, it is also pseudo ℓ-atomic over $\overline{M}_X$ by Lemma 5.13. Choose $E \subseteq M_X$ countable so that for all finite tuples e from N', all $\varphi(x, y)$ and all $u \in J \cap \mathcal{P}(X)$ there is an $L(E)$-formula $\psi(x) \in tp(e/M_X)$ that decides $\varphi(x, M_u)$.

Now arguing as in Lemma 5.7 there is a stable system $\overline{B}$ indexed by J such that each B_u is countable, $E \subseteq B_X$, $A_u \preceq B_u \preceq M_u$, and $(\overline{A}, \overline{B})$ is a linked pair of stable systems. Since $C \underset{A_\omega}{\perp} M_\omega$, we have $C \underset{A_\omega}{\perp} B_\omega$. By our choice of E, N' is pseudo ℓ-atomic over $\overline{B}_X$.

Conversely, fix $X \in \mathcal{P}(\omega)$ and assume $\overline{B}$, N', D, d, φ, and q are as in the Subclaim. It follows immediately from Definition 2.15 that $p \not\perp^\Delta N'$. By Proposition 5.9 there is an elementary map $f : \overline{B} \to \overline{M}$ such that $f|A_\omega = id$ and $f(B_u) \subseteq M_u$ for each $u \in J$. Since $\mathrm{tp}(C/M_\omega)$ does not fork over A_ω and since A_ω is a model, $\mathrm{tp}(B_\omega/CA_\omega) = \mathrm{tp}(f(B_\omega)/CA_\omega)$. Let σ be an automorphism of $\mathfrak{C}$ extending f that fixes CA_ω pointwise. Since p is based and stationary over C, its parallelism class is invariant under the action of σ. Thus, by replacing the given $\overline{B}$ by $f(\overline{B})$, p by $\sigma(p)$, and N' by $\sigma(N')$, we may assume that $B_u \subseteq M_u$ for all $u \in J$ while preserving $p \not\perp^\Delta N'$.

Now fix an enumeration $\langle a'_n : n \in \omega \rangle$ of N'. Since both N' and B_X are countable and since M realizes every a-isolated type over M_X, the existence theorem for a-isolated types allows us to find $N'' = \langle a''_n : n \in \omega \rangle$ from M such that $\langle a''_n : n \in \omega \rangle$ is an a-construction sequence over M_X with $\text{tp}(N'/B_X) = \text{tp}(N''/B_X)$. Since both N' and N'' are pseudo ℓ-atomic over $\overline{B}_X$, Lemma 5.15(1) implies that $\text{tp}(N'/B_\omega) = \text{tp}(N''/B_\omega)$. Since M is $\aleph_1$-homogeneous there is $C'' \subseteq M$ such that $\text{tp}(N'C/B_\omega) = \text{tp}(N''C''/B_\omega)$. Thus $p'' \not\perp^\Delta N''$, where $p'' \in S(C'')$ is conjugate to p over B_ω. Note that $\text{tp}(C''/M_\omega) = \text{tp}(C/M_\omega)$ since $\text{tp}(C/B_\omega) \vdash \text{tp}(C/M_\omega)$. Since M is $\aleph_1$-homogeneous over M_ω, there is $N_0 = \langle a_n : n \in \omega \rangle$ from M such that

$$C''\langle a''_n : n \in \omega \rangle \equiv_{M_\omega} C\langle a_n : n \in \omega \rangle$$

Summarizing all of this, N_0 is a countable subset of M, $p \not\perp^\Delta N_0$, and N_0 is a-constructible over M_X.

Next, let $\hat{N} = \langle a_n : n < \beta \rangle$ be an a-constructible model over M_X, whose construction sequence end extends $N_0 = \langle a_n : n \in \omega \rangle$. By Lemma 5.15(2) $\hat{N}$ is an a-construction sequence over M_ω. Let $\hat{M}$ be a-prime over $\hat{N}M_\omega$. Note that $\hat{M}$ is also a-prime over N_0M_ω. But recall that M is a-prime over M_ω and N_0 is a countable subset of M. Thus M is also a-prime over N_0M_ω. So, by the uniqueness of a-prime models, there is an isomorphism $h : \hat{M} \to M$ over N_0M_ω. Finally, take $N = h(\hat{N})$. Since $N_0 \subseteq N$, $p \not\perp^\Delta N$ and M is a-prime over NM_ω. Thus, N witnesses that $X \in W_\Delta$, which completes the proof of Claim 5.18. ∎

Claim 5.20 *$p \not\perp M_F$ for some finite $F \subseteq \omega$.*

Proof. We first argue that W is not meagre. If it were, then by Lemma 5.16(1) there would be $X \subseteq \omega$ such that X and $\omega \setminus X \notin W$, which would contradict Claim 5.17.

Since W is not meagre, some W_Δ is not meagre. Fix such a Δ. Since $\mathbf{\Sigma^1_1}$-subsets of a Polish space have the property of Baire (see e.g., Theorem 7 of XII.8 of [3]) it follows from Claim 5.18 that there is a nonempty open subset $U_{F,G}$ of $\mathcal{P}(\omega)$ such that $U_{F,G} \setminus W_\Delta$ is meagre. But $U_{F,G}$ is naturally homeomorphic to $\mathcal{P}(\omega)$, so the translation of Lemma 5.16(2) is that there are sets $\{X_i : i \in \omega\} \subseteq W_\Delta$ such that $X_i \cap X_j = F$ for all $i < j < \omega$. For each $i \in \omega$ choose $N_i \in \mathcal{N}_{X_i}$ such that $p \not\perp^\Delta N_i$. Since $\overline{M}$ is a stable system $\{M_{X_i} : i \in \omega\}$ is independent over M_F. Since each N_i is a-prime over M_{X_i} and since M_F is a-saturated, it follows that $\{N_i : i \in \omega\}$ is independent over M_F. Since $p \not\perp^\Delta N_i$ for each i, Proposition 2.16 entails that $p \not\perp M_F$. ∎

To complete the proof of the theorem, fix a finite $F \subseteq \omega$ such that $p \not\perp M_F$. Taking $I = \{\emptyset\} \cup \{\{i\} : i \in F\}$ and $I^* = \mathcal{P}(F)$ in Clause (4) of Proposition 5.5, M_F is a-prime over $\bigcup\{M_i : i \in F\}$. As F is finite, it follows from NDOP that $p \not\perp M_i$ for some $i \in F$ and we finish. ∎

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