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Finiteness of central configurations of five bodies in the plane

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Abstract

We prove there are finitely many isometry classes of planar central configurations (also called relative equilibria) in the Newtonian 5-body problem, except perhaps if the 5-tuple of positive masses belongs to a given codimension 2 subvariety of the mass space.

1. Introduction and statements

Let $(x_k, y_k) \in \mathbb{R}^2$, $k = 1, \dots, n$, be the positions of n points in the plane $\mathbb{R}^2$. We call these points the bodies. Body k has a mass $m_k > 0$. We will study the system

$$\begin{aligned}
&\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = m_2 r_{12}^{-3} \begin{pmatrix} x_{21} \\ y_{21} \end{pmatrix} + m_3 r_{13}^{-3} \begin{pmatrix} x_{31} \\ y_{31} \end{pmatrix} + \cdots + m_n r_{1n}^{-3} \begin{pmatrix} x_{n1} \\ y_{n1} \end{pmatrix} \\
&\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = m_1 r_{12}^{-3} \begin{pmatrix} x_{12} \\ y_{12} \end{pmatrix} + m_3 r_{23}^{-3} \begin{pmatrix} x_{32} \\ y_{32} \end{pmatrix} + \cdots + m_n r_{2n}^{-3} \begin{pmatrix} x_{n2} \\ y_{n2} \end{pmatrix} \\
&\qquad\qquad\qquad\cdots \\
&\begin{pmatrix} x_n \\ y_n \end{pmatrix} = m_1 r_{1n}^{-3} \begin{pmatrix} x_{1n} \\ y_{1n} \end{pmatrix} + \cdots + m_{n-1} r_{(n-1)n}^{-3} \begin{pmatrix} x_{(n-1)n} \\ y_{(n-1)n} \end{pmatrix},
\end{aligned} \tag{1}$$

where $x_{kl} = x_l - x_k$, $y_{kl} = y_l - y_k$ and $r_{kl} = \left(x_{kl}^2 + y_{kl}^2\right)^{1/2} > 0$. Some short notation will be useful. We call $f_k \in \mathbb{R}^2$, $k = 1, \dots, n$, the right-hand sides of the equations. System (1) is

$$q_k = f_k, \quad k = 1, \dots, n, \quad \text{with } q_k = \begin{pmatrix} x_k \\ y_k \end{pmatrix} \in \mathbb{R}^2. \tag{2}$$

Let us recall the meaning of this system. Newton's equations of the n-body problem are the 3-dimensional version of $\ddot{q}_k = -f_k$, $k = 1, \dots, n$. Newton's equations possess few "simple" solutions if $n \geq 3$. They are called homographic or self-similar solutions. In these solutions the configuration remains in the same similarity class, and each of the n bodies behaves as a body in a 2-body problem. Laplace [16], [17] remarked that if there is a $\lambda > 0$ such that $\lambda q_k = f_k$ for all k, $k = 1, \dots, n$, and if the configuration is planar, one may choose velocities such that the motion is self-similar. Wintner [33] called *central configuration* a configuration, planar or not, satisfying Laplace's condition. If the motion is self-similar, the configuration is central. If the central configuration is 3-dimensional, the motion is homothetic and leads to or comes from a total collapse. Extending a result by Sundman, Chazy [5] claimed that if the motion leads to or comes from a total collapse, there is an asymptotic configuration which is central. So he called "figure-limite" what Wintner would call a central configuration. Chazy derived his claim, which is also discussed in [33, §365], from a postulate. The question we consider in the present work has its source in these discussions, as we will see in the history part.

By re-scaling a central configuration we obtain another one with a different λ. There is always a re-scaling factor making $\lambda = 1$, i.e., giving a central configuration solution of (1).

System (1) is, for any angle θ, invariant by the transformation $(x_k, y_k) \mapsto (x_k \cos\theta + y_k \sin\theta, -x_k \sin\theta + y_k \cos\theta)$. We remove this rotation freedom by adding the condition $y_{12} = 0$.

Definition 1. A *positive normalized central configuration* of the planar n-body problem is a solution of (1) satisfying $y_{12} = 0$.

The word "positive" is for "real positive." It refers to the reality hypothesis and to the hypothesis $r_{kl} > 0$. The word is omitted in a real context. But we will study *complex* central configurations and establish in Sections 6 and 9 strong statements about their finiteness. We state here the conclusions concerning the real domain.

THEOREM 1 (Hampton, Moeckel). *Let* $n = 4$. *For any choice of positive masses* $m_1, \dots, m_4$, *there are finitely many positive normalized central configurations.*

In contrast to the original proof by Hampton and Moeckel [13], our proof of Theorem 1 does not require any difficult computation. In both proofs one follows a continuum of central configurations in the complex domain until it reaches a singularity. Our analysis of the singularities is different.

The previous methods could not even specify a 5-tuple of positive masses such that the normalized planar central configurations (also called relative equilibria) of the 5-body problem are finitely many. Developing our analysis we prove the following theorem.

THEOREM 2. *For any choice of masses* $(m_1, \dots, m_5) \in (\mathbb{R}_0^+)^5 \setminus \mathcal{A}$, *where* $\mathbb{R}_0^+$ *is the set of positive real numbers and* $\mathcal{A}$ *is a closed algebraic subset of codimension* 2, *there are finitely many positive normalized central configurations of the planar* 5*-body problem.*

To give the polynomials defining each component of $\mathcal{A}$ and to conclude the proof, we compute and factorize resultants of polynomial systems with integer coefficients. We use a standard computer algebra system.

Once the finiteness is proved, an explicit upper bound on the number of central configurations is obtained by direct application of a Bézout theorem. We embed system (1) into the polynomial system (4). The number of isolated solutions is bounded by product of the degrees of the equations. There are actually nonisolated solutions at infinity, but the version of Bézout theorem given in Example 8.4.6 in [10] applies. However, the bound is so bad that we avoid writing it explicitly.

History of the problem. The equations for the 3-body central configurations were motivated, stated and solved by Euler [9], [7], [8], assuming that the three bodies are collinear, by Lagrange [15] without this assumption. Laplace [16], [17] motivated and stated the equations for the central configurations of $n \geq 4$ bodies, but did not solve them.

In a quite famous paper published in December 1918,[1] Chazy postulated that for any choice of positive masses, all the central configurations are *nondegenerate* critical points of the function $U + \lambda I/2$ restricted to the normalized configurations. (See (6) and (7) for the definition of these functions.)

Then, he noticed that this postulate would imply that the number of normalized central configurations is always finite and does not vary with the masses:

> "Ce postulat admis, il résulte en particulier qu'*à tous les chocs possibles de n corps correspondent un nombre fini de figures-limites de formes distinctes*, et chacune de ces formes distinctes ne dépend que des rapports des masses."

The postulate and the second conclusion are wrong. In 1975, Palmore [25] gave a simple example of a degenerate central configuration, an equilateral triangle of bodies with unit mass and a fourth body with mass $(64\sqrt{3}+81)/249$ at the center of the triangle. Simó [28] showed how the number of 4-body central configurations with a given body in the interior of the triangle formed by the other bodies varies when the four masses vary. Xia [34] proved that in the n-body problem, the relative equilibria may be counted exactly for several

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[1] [5] was published a few weeks after the end of first world war. Here is an indirect testimony of the glorious participation of Chazy to this war, by Jean Guillermet: "Le 27 mai 1918, alors que Jean Chazy commandait la section repérage par le son, dont le poste central se trouvait à Moulin-sous-Touvent, il avait donné avec précision la position de la grosse Bertha, près de Beaumont-en-Beine, alors qu'il ignorait que des obus étaient tombés sur Paris. On avait considéré à l'époque cet extraordinaire calcul comme une véritable acrobatie du repérage par le son."

nonexplicit open sets of the mass space, giving different numbers in different open sets.

But Chazy may still be correct about the finiteness. Wintner [33] believed it and conjectured in 1941:

> §360 "the number $q = q(n; m_1, \dots, m_n)$ of all central configurations belonging to n given m_i is likely to be less than a bound q_n which is independent of the m_i; while q_n itself remains bounded as $n \to \infty$. The largest contribution to $q(n; m_1, \dots, m_n)$ seems to be due to the collinear central configurations. Actually, an enumeration of all $q(n; m_1, \dots, m_n)$ central configurations for arbitrary $n; m_1, \dots, m_n$ represents a fascinating unsolved problem which depends on a complete discussion of certain real algebraic equations."
>
> §365: "this possibility cannot occur unless the n given m_i determine infinitely many central configurations which are distinct in the sense defined at the end of §355. In §360, it appeared to be a reasonable conjecture that such is never the case, i.e., that the integer $q(n; m_1, \cdots, m_n)$ defined at the beginning of §360 always exists. But no proof is known for the truth of this hypothesis."

Wintner's words "q_n itself remains bounded as $n \to \infty$" are disproved by Wintner himself, when he recalls at the next page that Moulton's theorem gives $n!/2$ collinear central configurations of n bodies.

Wintner's following claim is disproved in [24] by a topological estimate, in the cases where the masses are such that the central configurations are all nondegenerate. In such cases, there are more than $(n-1)!(n-2)$ SO(2)-classes of 2-dimensional central configurations. As soon as $n \geq 4$, this is more than the number of collinear central configurations. This estimate is obtained from the Poincaré polynomial $(1+2t)(1+3t)\cdots(1+(n-1)t)$ of the configuration space $\mathbb{CP}_{n-2} \setminus \Delta$, where Δ is the collision set (see [4], [6, p. 324]). According to Conley (see [23]), the $n!/2$ collinear central configurations are saddles of index $n-2$. The estimate follows.

Smale and Shub investigated the central configurations in classical works, each time insisting on the finiteness question ([29, p. 47], [27]). Repeated in [30], the conjecture takes in [31] and [32] the form of Smale's 6th question for the 21st century: *Is the number of relative equilibria finite, in the n-body problem of celestial mechanics, for any choice of positive real numbers $m_1, \dots, m_n$ as the masses?*

Hampton and Moeckel [13] answered positively the question in 2005 for $n = 4$ bodies. The reader may consult their excellent review on the question. The main works they cite on the subject are Kuz'mina [14], Moeckel [20], Xia [34], Albouy [2], Roberts [26], Moeckel [21]. More recently Hampton [11] proved the finiteness of symmetric planar central configurations of 5 bodies, except perhaps if the masses satisfy a given polynomial condition. Hampton and Jensen [12] improved Moeckel [21] by proving the finiteness of the 3-dimensional central configurations of 5 bodies, except perhaps if the masses satisfy a given polynomial condition. Lee and Santoprete [18] proposed a new method to find all the isolated central configurations of five equal masses.

2. **Structure of the proof**

A basic property of the system. Let us count the equations and the unknowns in system (2). We have n vector equations and n vector variables. However, we know that before introducing the normalization $y_{12} = 0$, the solutions are not isolated. One of the $2n$ scalar equation has to be a consequence of the other $2n - 1$. The relation

$$\sum_{k=1}^{n} m_k q_k \wedge f_k = 0, \tag{3}$$

where $\wedge$ is the exterior product, shows that if the $n - 1$ first vector equations are satisfied, then $q_n \wedge f_n = 0$, and the two scalar equations corresponding to both coordinates of the last vector equation $q_n = f_n$ are not independent.

Relation (3) is due to cancellations of pairs of similar terms. Dynamically $\sum_{k=1}^{n} m_k q_k \wedge f_k$ is the time derivative of the angular momentum. The angular momentum is constant along the trajectories of the n-body problem. The cancellations of pairs of similar terms correspond to the so-called action and reaction law. These cancellations also imply the center of mass condition $0 = \sum m_k f_k = \sum m_k q_k$. This linear relation may replace one of the vector equations $q_k = f_k$.

A weak hypothesis on the masses. Our main results assume that all the masses are positive. However, many of our intermediate results only need a weaker assumption on the masses, allowing negative masses or even complex masses.

If $m_1 + \cdots + m_n$ vanishes, the condition $\sum m_k q_k = 0$ above is not a "center of mass condition:" there is no center of mass. From Rule 1c, we will deal with centers of mass of clusters. They should exist, so we assume $\sum_{k \in I} m_k \neq 0$ for any nonempty $I \subset \{1, 2, \ldots, n\}$. In words, *we assume from now on that no subset of bodies has total mass zero.*

Complex positions. Inclusion into a polynomial system. The principle of our proofs is to follow a possible continuum of central configurations in the complex domain and to study its possible singularities there. From now on we consider that $(x_k, y_k) \in \mathbb{C}^2$, $k = 1, \ldots, n$. The positivity condition of the distances r_{kl} shall be dropped. The distances are now bi-valued.

To the variables (x_k, y_k) we add the variables $\delta_{kl} \in \mathbb{C}$, $1 \leq k < l \leq n$, inverses of the distances r_{kl}. We consider δ_{lk} as just another notation for δ_{kl}. System (1) together with the condition $y_{12} = 0$ becomes

$$\begin{aligned}
&\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = m_2\delta_{12}^3 \begin{pmatrix} x_{21} \\ y_{21} \end{pmatrix} + m_3\delta_{13}^3 \begin{pmatrix} x_{31} \\ y_{31} \end{pmatrix} + \cdots, \\
&\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = m_1\delta_{12}^3 \begin{pmatrix} x_{12} \\ y_{12} \end{pmatrix} + m_3\delta_{23}^3 \begin{pmatrix} x_{32} \\ y_{32} \end{pmatrix} + \cdots, \\
&\qquad \cdots \\
&\qquad \delta_{12}^2(x_{12}^2 + y_{12}^2) = 1, \\
&\qquad \delta_{13}^2(x_{13}^2 + y_{13}^2) = 1, \\
&\qquad \cdots \\
&\qquad y_{12} = 0.
\end{aligned} \tag{4}$$

This is a polynomial system in $\mathbb{C}^{2n} \times \mathbb{C}^{n(n-1)/2}$. The (x_k, y_k)'s define a "geometrical configuration," and then the δ_{kl}'s are defined up to multiplication by -1. The geometrical configuration together with one of the $\frac{n(n-1)}{2}$ choices of signs forms a "gravitational configuration," i.e., allows the evaluation of the complex gravitational forces.

Definition 2. A *normalized central configuration* is a solution of (4). A *real normalized central configuration* is a normalized central configuration such that $(x_k, y_k) \in \mathbb{R}^2$ for any $k = 1, \ldots, n$. A *positive normalized central configuration* is a real normalized central configuration such that $\delta_{kl} = \pm(x_{kl}^2 + y_{kl}^2)^{-1/2}$ is positive for any k, l, $k \neq l$.

Definition 2 of a positive normalized central configuration coincides with Definition 1 in the introduction.

Elimination theory. Let N be a positive integer. Following [22], we define a *closed algebraic subset* of the affine space $\mathbb{C}^N$ as the set of common zeroes of a system of polynomials on $\mathbb{C}^N$. A *constructible set* in $\mathbb{C}^N$ is a subset "constructed" from the three postulates: (i) a closed algebraic subset is a constructible set, (ii) the complementary of a constructible set is a constructible set, (iii) the union of two constructible sets is a constructible set.

The polynomial system (4) defines a closed algebraic subset $\mathcal{A} \subset \mathbb{C}^{2n} \times \mathbb{C}^{n(n-1)/2}$. For $n \leq 5$, we will prove that *for most masses this subset is finite.* To distinguish the two possibilities, finitely many or infinitely many points, we will only use the following result from elimination theory.

LEMMA 1. *Let X be a closed algebraic subset of $\mathbb{C}^N$ and $f : \mathbb{C}^N \to \mathbb{C}$ be a polynomial. Either the image $f(X) \subset \mathbb{C}$ is a finite set, or it is the complement of a finite set. In the second case one says that f is* dominating.

Proof. Consider the polynomials defining X as polynomials in $(x, y) \in \mathbb{C}^N \times \mathbb{C}$ that do not depend on the variable y. Consider the system formed by these polynomials and the polynomial $f(x) - y$. The zeroes of this system form a closed algebraic subset $\hat{X} \subset \mathbb{C}^N \times \mathbb{C}$. The image $f(X)$ is the projection on $\mathbb{C}$ of the constructible set $\hat{X}$. The projection of a constructible set is a constructible set (see [22, p. 37]). A constructible set in $\mathbb{C}$ is the set of the zeroes of a nonzero polynomial, i.e., a finite set, or the complement of such a set. □

Potential and moment of inertia. Consider the closed algebraic subset $\mathcal{B} \subset \mathbb{C}^{2n} \times \mathbb{C}^{n(n-1)/2}$ defined by the above relations $\delta_{kl}^2(x_{kl}^2 + y_{kl}^2) = 1$. The first $2n$ variables $(x_1, \dots, x_n, y_1, \dots y_n)$ form local coordinates of the neighborhood in $\mathcal{B}$ of any point. System (1) may be written as

$$\frac{\partial I}{\partial x_k} = -2\frac{\partial U}{\partial x_k}, \qquad \frac{\partial I}{\partial y_k} = -2\frac{\partial U}{\partial y_k}, \tag{5}$$

where

$$U = \sum_{k<l} m_k m_l \delta_{kl} \tag{6}$$

and

$$I = \sum_{k=1}^{n} m_k(x_k^2 + y_k^2) = \frac{1}{m_1 + \cdots + m_n} \sum_{k<l} m_k m_l (x_{kl}^2 + y_{kl}^2) \tag{7}$$

are respectively the Newtonian potential and the moment of inertia, which are locally homogeneous functions of $(x_1, \dots, x_n, y_1, \dots y_n)$ of respective degrees -1 and 2. Computing $\sum_k x_k \partial I/\partial x_k + y_k \partial I/\partial y_k$ with (5) we deduce that any solution of (5) satisfies $I = U$.

System (5) expresses a solution of (1) as a critical point of the function $2U + I$ on $\mathcal{B}$. Thus $2U + I = 3U = 3I$ is locally constant along any continuum of solutions of (5). Let us state this result more precisely.

Lemma 2. *Consider the closed algebraic subset $\mathcal{A} \subset \mathbb{C}^{2n} \times \mathbb{C}^{n(n-1)/2}$ defined by system (4) and the polynomial function U on it defined by expression (6). Then $U(\mathcal{A})$ is a finite set.*

Proof. As $U = I$ on $\mathcal{A}$ we may replace U by $f = 2U + I$ in the statement. According to Lemma 1 it is enough to prove that f is not dominating. A simple statement is easily obtained from [22, p. 42]: a dominating polynomial f on a closed algebraic subset possesses smooth points, i.e., points where the dimension of the tangent space is minimal and where $df \neq 0$. We first notice that the normalization relation $y_{12} = 0$ of system (4) is irrelevant in this discussion. We consider $f = 2U + I$ as a polynomial defined on the closed algebraic subset defined by (4) minus this normalization relation. If f was dominating, it would have a smooth point. But (5) shows that $df = 0$ on the tangent space. Contradiction. □

Remark 1. We can think of Lemma 2 as a kind of Sard's lemma for the function $f = 2U + I$ defined on $\mathcal{B}$: the set of the critical points of f may be infinite, but still the set of the critical values would remain finite. A related statement called Sard's lemma may also be found in [22, p. 42].

Factorization of the distances. For convenience we will use again the variables $r_{kl} = 1/\delta_{kl}$ instead of the δ_{kl}'s. We still think of a closed algebraic subset in the variables x_k, y_k and δ_{kl}.

We have $x_k^2 + y_k^2 = (x_k + iy_k)(x_k - iy_k)$. We set $z_k = x_k + iy_k$ and $w_k = x_k - iy_k$. In the case of a real configuration the z_k's form this configuration in the complex plane, while the w_k's form its conjugate. We have $x_k^2 + y_k^2 = z_k w_k$ and $x_{kl}^2 + y_{kl}^2 = r_{kl}^2 = z_{kl}w_{kl}$. System (1) becomes

$$(8) \qquad \begin{pmatrix} z_1 \\ w_1 \end{pmatrix} = m_2 r_{12}^{-1} z_{12}^{-1} w_{12}^{-1} \begin{pmatrix} z_{21} \\ w_{21} \end{pmatrix} + m_3 r_{13}^{-1} z_{13}^{-1} w_{13}^{-1} \begin{pmatrix} z_{31} \\ w_{31} \end{pmatrix} + \cdots$$
$$\cdots$$

which simplifies to

$$(9) \qquad \begin{aligned} z_1 &= m_2 z_{21}^{-1/2} w_{21}^{-3/2} + m_3 z_{31}^{-1/2} w_{31}^{-3/2} + \cdots \\ w_1 &= m_2 z_{21}^{-3/2} w_{21}^{-1/2} + m_3 z_{31}^{-3/2} w_{31}^{-1/2} + \cdots \\ &\cdots . \end{aligned}$$

Here, e.g., $z_{21}^{-1/2}$ is an abuse of notation. This quantity cannot be deduced unambiguously from the variables z_{12}, w_{12} and r_{12}. But as $r_{12}^2 = z_{12}w_{12}$, such a product as $z_{21}^{p/2} w_{21}^{q/2}$ is expressed rationally in these variables if $p \in \mathbb{Z}$, $q \in \mathbb{Z}$ and $p + q$ is even.

The rotation freedom is expressed in (z, w) variables as the invariance of (8) by the map $R_a : (z_k, w_k, r_{kl}) \mapsto (az_k, a^{-1}w_k, r_{kl})$ for any $a \in \mathbb{C}_0$ and any k, l, $k \neq l$. The condition $y_{12} = 0$ we proposed to remove this rotation freedom becomes $z_{12} = w_{12}$.

A discrete symmetry of (8) appears: $(z_k, w_k, r_{kl}) \mapsto (z_k, jw_k, j^2 r_{kl})$ sends solution on solution. Here j is a cubic root of unity. The solutions of (8) come in triples.

Distances and separations. We will use the name "distance" for the $r_{kl} = \sqrt{z_{kl}w_{kl}}$. We will use the name z-separation (respectively w-separation) for the z_{kl}'s (respectively the w_{kl}'s) in the complex plane.

Roberts' continuum. The following continuum of central configurations was published in [26]. It is not considered as disproving Chazy's conjecture because the masses are not all positive.

Here $n = 5$ and $(m_1, m_2, m_3, m_4, m_5) = (1/2, 1/2, 1/2, 1/2, -1/8)$. The first four bodies form a rhombus: $z_1 = -z_3 = w_1 = -w_3 = a$, $z_2 = -z_4 = b$, $w_2 = -w_4 = -b$, where $a \in \mathbb{R}$ and $ib \in \mathbb{R}$ for the real configurations.

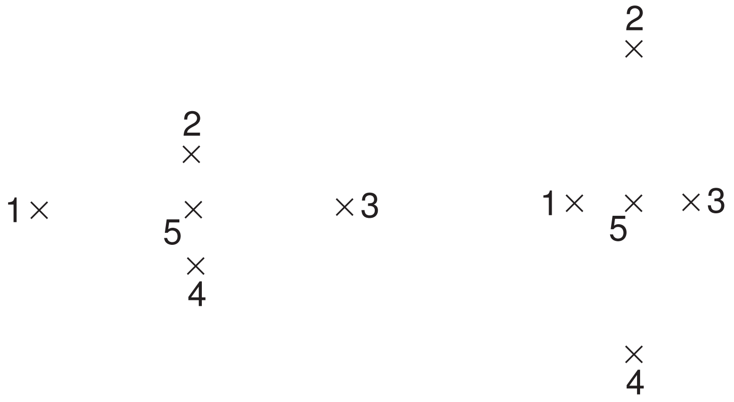

Figure 1. Two of Roberts' central configurations.

The last body is at the center of mass: $z_5 = w_5 = 0$. The algebraic curve defined by the coordinates above and the condition $r_{12} = r_{23} = r_{34} = r_{14} = 1$ satisfies (8). This condition also fixes a relation between a and b: $r_{12}^2 = z_{12}w_{12} = (b-a)(-b-a) = a^2 - b^2 = 1$.

When $a \to 0$, bodies 1, 3 and 5 collide. When $b \to 0$ bodies 2, 4 and 5 collide. We get a "singularity" at both limits, which we will refer to as Roberts' real singularity and describe as a triple contact singularity.

Let now $a \to \infty$ along the real axis. At $a = 1$, we meet the singularity above: $b = \pm\sqrt{a^2 - 1}$ vanishes. This point turns out to be a branch point.

We turn around this branch point and continue along the real axis. There is a choice of leaf. Drawing a cut on the Riemann surface between $a = -1$ and $a = 1$ we visualize the two main choices when going to infinity: $b \sim a$ or $b \sim -a$, corresponding to the two leaves. We get two other singularities, which we will refer to as Roberts' singularities at infinity.

Modified Roberts' continuum. We will consider a central configuration of Roberts' continuum. We change m_5 in $-m_5$, r_{k5} in $-r_{k5}$, $k = 1, \dots, 4$. We get a real normalized central configuration with positive masses (here we normalized with $y_{13} = 0$ instead of our usual $y_{12} = 0$). Still this is not a positive normalized central configuration, as we choose for r_{k5} the negative square root of $z_{k5}w_{k5}$. There is a repulsive Newtonian force instead of attractive for the pairs $(k, 5)$.

However, we get a continuum of solutions of (4) with positive masses. It is similar to Roberts' continuum. The singularities are the same and $U = I = 1$ along both continua.

Singular sequences of normalized central configurations. Set $Z_{kl} = r_{kl}^{-1} w_{kl}^{-1}$ and $W_{kl} = r_{kl}^{-1} z_{kl}^{-1}$. Then $r_{kl} = r_{lk}$, $Z_{kl} = -Z_{lk}$, $W_{kl} = -W_{lk}$. System (4) becomes

$$\begin{aligned}
z_1 &= \quad m_2 Z_{21} + m_3 Z_{31} + \cdots + m_n Z_{n1},\\
z_2 &= m_1 Z_{12} + m_3 Z_{32} + \cdots + m_n Z_{n2},\\
&\cdots\\
z_n &= m_1 Z_{1n} + m_2 Z_{2n} + m_3 Z_{3n} + \cdots + m_{n-1} Z_{(n-1)n},\\
w_1 &= \quad m_2 W_{21} + m_3 W_{31} + \cdots + m_n W_{n1},\\
w_2 &= m_1 W_{12} + m_3 W_{32} + \cdots + m_n W_{n2},\\
&\cdots\\
w_n &= m_1 W_{1n} + m_2 W_{2n} + m_3 W_{3n} + \cdots + m_{n-1} W_{(n-1)n},\\
z_{12} &= w_{12}.
\end{aligned} \tag{10}$$

Let $\mathcal{N} = n(n+1)/2$. To a gravitational configuration

$$\mathcal{Q} = (z_1, z_2, \ldots, z_n, w_1, w_2, \ldots, w_n, \delta_{12}, \ldots, \delta_{(n-1)n})$$

we associate two vectors in $\mathbb{C}^{\mathcal{N}}$

$$\begin{aligned}
\mathcal{Z} &= (\mathcal{Z}_1, \mathcal{Z}_2, \ldots, \mathcal{Z}_{\mathcal{N}}) = (z_1, z_2, \ldots, z_n, Z_{12}, Z_{13}, \ldots, Z_{(n-1)n}),\\
\mathcal{W} &= (\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_{\mathcal{N}}) = (w_1, w_2, \ldots, w_n, W_{12}, W_{13}, \ldots, W_{(n-1)n}).
\end{aligned}$$

The coordinates of $\mathcal{Z}$ are, up to the mass factor, the terms of equations 1 to n in (10). The coordinates of $\mathcal{W}$ are, up to the mass factor, the terms of equations $n+1$ to $2n$ in (10).

A solution $\mathcal{Q}$ of (10) is a normalized central configuration. Consider a sequence $\mathcal{Q}^{(m)}$, $m = 1, 2, \ldots$, of normalized central configurations. Let $Z^{(m)} = \max_{p=1,\ldots,\mathcal{N}} |\mathcal{Z}_p^{(m)}|$ be the modulus of the maximal component of the vector $\mathcal{Z} \in \mathbb{C}^{\mathcal{N}}$. Extract a sub-sequence such that the maximal component is always the same, i.e., $Z^{(m)} = |\mathcal{Z}_p^{(m)}|$ for a p that does not depend on m. Extract again in such a way that the vector $Z^{-1}\mathcal{Z}$ converges. Define $W^{(m)} = \max_{q=1,\ldots,\mathcal{N}} |\mathcal{W}_q^{(m)}|$. Extract again in such a way that there is similarly an integer q such that $W^{(m)} = |\mathcal{W}_q^{(m)}|$ for all m. Extract a last time in such a way that the vector $W^{-1}\mathcal{W}$ converges.

If the initial sequence is such that $\mathcal{Z}$ or $\mathcal{W}$ is unbounded, so is the extracted sequence. Note that $\max_{p=1,\ldots,\mathcal{N}} |\mathcal{Z}_p|$ is bounded away from zero: if the first n components of the vector $\mathcal{Z}$ all go to zero, then the denominators in the other components go to zero and $\mathcal{Z}$ is unbounded. There are two possibilities for the extracted sub-sequences above:

1) $\mathcal{Z}$ and $\mathcal{W}$ are bounded,

2) at least one of these two vectors is unbounded.

Let us show that the first case corresponds to an extracted sequence converging to a normalized central configuration. This is due to the inhomogeneity of the vectors $\mathcal{Z}$ and $\mathcal{W}$. The extracted sequence is such that $Z^{-1}\mathcal{Z}$ and $W^{-1}\mathcal{W}$ converge, so if $(\mathcal{Z},\mathcal{W})$ have two limit points, they are of the form $(\mathcal{Z},\mathcal{W})$ and $(\mathcal{Z}',\mathcal{W}')=(\lambda\mathcal{Z},\mu\mathcal{W})$, with $\lambda>0$ and $\mu>0$. Due to the equation $z_{12}=w_{12}$ we have $\lambda=\mu$. Then $(Z'_{kl})^{-1}=r'_{kl}w'_{kl}=\lambda r'_{kl}w_{kl}$, and on the other hand, $(Z'_{kl})^{-1}=(\lambda Z_{kl})^{-1}=\lambda^{-1}r_{kl}w_{kl}$. Finally $r'_{kl}=\lambda^{-2}r_{kl}$, which is incompatible with $z'_{kl}=\lambda z_{kl}$, $w'_{kl}=\mu w_{kl}$ and the homogeneous equations $r_{kl}^2=z_{kl}w_{kl}$, ${r'_{kl}}^2=z'_{kl}w'_{kl}$, except if $\lambda=1$. The two limit points coincide and the sequence converges.

Definition 3. Consider a sequence of normalized central configurations. A sub-sequence extracted by the above process, in the unbounded case, is called a *singular sequence.*

Our method to prove the finiteness of the central configurations consists essentially of two steps. First, we study *all* possibilities for a singular sequence. *We show that such an unbounded sequence is impossible for most masses.* Second, we use Lemma 1 to prove that if there are infinitely many normalized central configurations, there exist singular sequences, and even singular sequences where some distance goes to zero or to infinity.

Consequently there are finitely many normalized central configurations for most masses.

3. **Tools to classify the singular sequences**

Notation of asymptotic estimates. We already used $a\sim b$ which means, as usual, $a/b\to 1$. We will also use $a\prec b$, $a\preceq b$ and $a\approx b$. The first means $a/b\to 0$, the second a/b is bounded, and the third $a\preceq b$ and $a\succeq b$.

Strokes and circles. One color rules. We pick a singular sequence. We write the indices of the bodies in a figure and use two colors for edges and vertices.

The first color, the z-color, is used to mark the maximal order components of $\mathcal{Z}=(z_1,\dots,z_n,Z_{12},\dots,Z_{(n-1)n})$. They correspond to the components of the converging vector $Z^{-1}\mathcal{Z}$ that do not tend to zero. We draw a circle around the name of body k if the term z_k is of maximal order among all the components of $\mathcal{Z}$. We draw a stroke between the names k and l if the term $Z_{kl}=z_{kl}^{-1/2}w_{kl}^{-3/2}$ is of maximal order among all the components of $\mathcal{Z}$. If there is a maximal order term in an equation, there should be another one. This gives immediately Rule 1a.

Rule 1a. There is something at each end of any z-stroke: another z-stroke or/and a z-circle drawn around the name of the body. A z-circle cannot be isolated; there must be a z-stroke emanating from it. There is at least one z-stroke in the z-diagram.

Definition 4. Consider a singular sequence. We say that bodies k and l are *close* in z-coordinate, or z-close, or that z_k and z_l are close, if $z_{kl} \prec Z$.

We extend this convenient terminology to centers of mass instead of bodies. We can say, e.g., that the center of mass of k and l is close to the origin. The following statement is obvious.

Rule 1b. If bodies k and l are z-close, they are both z-circled or both not z-circled.

Definition 5. An *isolated component of the z-diagram* is a subset of vertices such that no z-stroke is joining a vertex of this subset to a vertex of the complement.

Rule 1c. The center of mass of a set of bodies forming an isolated component of the z-diagram is z-close to the origin.

Proof. Let the bodies of this isolated component be numbered $1, 2, \ldots, p$. Compute $m_1 z_1 + m_2 z_2 + \cdots + m_p z_p$ from (10). All the terms in this expression have the form $m_k m_l Z_{kl}$. Only the terms $m_k m_l Z_{kl}$, $1 \le k \ne l \le p$ may be of maximal order Z. But $m_k m_l Z_{kl}$ and $m_l m_k Z_{lk}$ cancel out. There only remain lower order terms. □

Rule 1d. Consider the z-diagram or an isolated component of it. If there is a z-circled body, there is another one. The z-circled bodies cannot all be z-close together.

Proof. These are easy consequences of the center of mass equation $m_1 z_1 + \cdots + m_n z_n = 0$, or of the equation $m_1 z_1 + \cdots + m_p z_p \prec Z$ obtained from Rule 1c for an isolated component. □

Definition 6. Consider a z-stroke from vertex k to vertex l. We say it is a *maximal z-stroke* if k and l are not z-close.

Rule 1e. An isolated component of the z-diagram has no z-circled vertex if and only if it has no maximal z-stroke.

Proof. If the z-stroke kl is maximal, then $z_{kl} \approx Z$, and so either $z_k \approx Z$ or $z_l \approx Z$. At least one of the ends is circled. If there is no maximal z-stroke, we decompose the isolated component into connected isolated components. All the z_k's of a component are close together, and they are also close to their center of mass, which is close to the origin by Rule 1c. There is no circle. □

Remark 2. No circle in a connected isolated component means no free ends by Rule 1a.

Superposition of two colored diagrams. On the same diagram we also draw w-strokes and w-circles. Graphically we use another color. The previous rules and definitions apply to w-strokes and w-circles. What we will call simply the *diagram* is the superposition of the z-diagram and the w-diagram. We will, for example, adapt Definition 5 of an isolated component: a subset of bodies forms an isolated component of the diagram if and only if it forms an isolated component of the z-diagram and an isolated component of the w-diagram.

Edges and strokes. There is an *edge* between vertex k and vertex l if there is either a z-stroke, or a w-stroke, or both. There are three types of edges, z-edges, w-edges and zw-edges, and only two types of strokes, represented with two different colors. Vertices may also be circled in three different ways, by combining circles of the two colors.

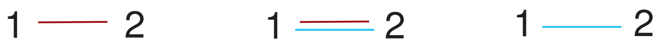

Figure 2. A z-stroke, a z-stroke plus a w-stroke, a w-stroke, forming respectively a z-edge, a zw-edge, a w-edge.

The Roberts' continuum example produces the following collection of diagrams from left to right: $a \to 0$ and 1, 3, 5 collide; $b \to 0$ and 2, 4, 5 collide; $a \to \infty$ and $a \sim b$; $a \to \infty$ and $a \sim -b$ (see Figure 3).

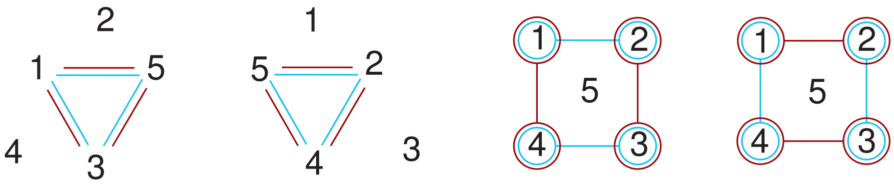

Figure 3. Roberts' continuum at real triple contact and imaginary infinity.

New normalization. Main estimates. One does not change a central configuration by multiplying the z coordinates by $a \in \mathbb{C}_0$ and the w coordinates by a^{-1}. Our diagram is invariant by such an operation, as it considers the z-coordinates and the w-coordinates separately.

We used the normalization $z_{12} = w_{12}$ in the previous considerations. In the following we will normalize instead with $Z = W$. We start with a central configuration normalized with the condition $z_{12} = w_{12}$, then multiply the z-coordinates by $a > 0$, the w-coordinates by a^{-1}, in such a way that the maximal component of $\mathcal{Z}$ and the maximal component of $\mathcal{W}$ have the same modulus, i.e., $Z = W$.

A singular sequence was defined by the condition either $Z\to\infty$ or $W\to\infty$. We also remarked that both Z and W were bounded away from zero. With the new normalization, a singular sequence is simply characterized by $Z=W\to\infty$. In contrast, $Z=W$ tends to a positive constant if a sequence tends to a central configuration.

We set $Z=W=\epsilon^{-2}$. For a singular sequence $\epsilon\to 0$. From now on we only discuss singular sequences. A justification for this normalization and this notation is the simplicity of the following estimates.

Estimate 1. For any (k,l), $1\leq k<l\leq n$, we have $\epsilon\preceq r_{kl}\preceq\epsilon^{-2}$ and $\epsilon^2\preceq z_{kl}\preceq\epsilon^{-2}$. There is a zw-edge between k and l if and only if $r_{kl}\approx\epsilon$. There is a maximal w-edge between k and l if and only if $z_{kl}\approx\epsilon^2$.

Proof. The right-hand side estimates for both r_{kl} and z_{kl} follow from z_{kl}, $w_{kl}\preceq W$. For the left-hand side estimates, we write $Z_{kl}=r_{kl}^{-1}w_{kl}^{-1}\preceq\epsilon^{-2}$ and $W_{kl}=r_{kl}^{-1}z_{kl}^{-1}\preceq\epsilon^{-2}$. Multiplying both inequalities we get $\epsilon\preceq r_{kl}$.

The "equality case" $\epsilon\approx r_{kl}$ requires $Z_{kl}\approx\epsilon^{-2}$, which means a z-stroke, and $W_{kl}\approx\epsilon^{-2}$, which means a w-stroke. Both strokes form a zw-edge.

We have $w_{kl}\preceq\epsilon^{-2}$. Rewrite $w_{kl}=r_{kl}^2z_{kl}^{-1}$, so $W_{kl}^2w_{kl}=z_{kl}^{-3}\preceq\epsilon^{-6}$, which gives $\epsilon^2\preceq z_{kl}$. The "equality case" requires $w_{kl}\approx\epsilon^{-2}$, which means k and l not w-close, and $W_{kl}\approx\epsilon^{-2}$, which means a w-stroke. ☐

Remark 3. By the estimates above, the strokes in a zw-edge are not maximal. A maximal z-stroke never forms a zw-edge. It always forms a z-edge. Definition 6 tells us what is a maximal z-stroke. A maximal z-edge is just the same thing.

Estimate 2. We assume that there is a z-stroke between k and l. Then

$$\epsilon\preceq r_{kl}\preceq 1,\quad \epsilon\preceq z_{kl}\preceq\epsilon^{-2},\quad \epsilon\succeq w_{kl}\succeq\epsilon^2.$$

Under the same hypothesis the "equality cases" are characterized as follows:

$$\begin{aligned}&\text{Left: } r_{kl}\approx\epsilon\Leftrightarrow z_{kl}\approx\epsilon\Leftrightarrow w_{kl}\approx\epsilon\Leftrightarrow zw\text{-edge between } k \text{ and } l,\\ &\text{Right: } r_{kl}\approx 1\Leftrightarrow z_{kl}\approx\epsilon^{-2}\Leftrightarrow w_{kl}\approx\epsilon^2\Leftrightarrow \text{maximal } z\text{-edge between } k \text{ and } l.\end{aligned}$$

Proof. The z-stroke means $Z_{kl}=r_{kl}^{-3}z_{kl}\approx\epsilon^{-2}$. Moreover, we know that $z_{kl}\preceq\epsilon^{-2}$ and $W_{kl}=r_{kl}^{-1}z_{kl}^{-1}\preceq\epsilon^{-2}$. Substituting $z_{kl}\approx r_{kl}^3\epsilon^{-2}$ successively in these inequalities gives $\epsilon\preceq r_{kl}\preceq 1$. Substituting $r_{kl}^{-1}\approx\epsilon^{-2/3}z_{kl}^{-1/3}$ in the second gives $\epsilon\preceq z_{kl}$. Writing instead $Z_{kl}=r_{kl}^{-1}w_{kl}^{-1}\approx\epsilon^{-2}$, $z_{kl}=r_{kl}^2w_{kl}^{-1}\preceq\epsilon^{-2}$, $W_{kl}=r_{kl}^{-3}w_{kl}\preceq\epsilon^{-2}$ and eliminating r_{kl} gives the remaining estimate. Looking at these proofs, we see that the left-hand side inequalities all become "equalities" when $W_{kl}\approx\epsilon^{-2}$, which means there is a w-stroke. The right-hand side inequalities become "equalities" when $z_{kl}\approx\epsilon^{-2}$, which means the bodies are not z-close. ☐

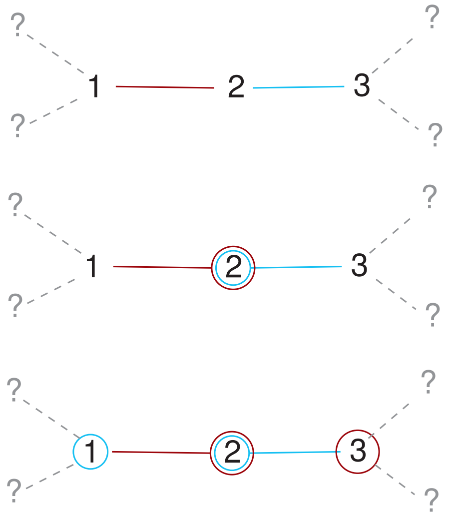

Figure 4. Around a z-edge-w-edge sequence.

Circling method. Estimate 2 shows that in all the cases where there is a z-stroke between k and l, these bodies are close in w-coordinate. Then Rule 1b applies to the w-diagram. Vertices k and l are either both w-circled or both not w-circled.

Given the edges of a diagram we first z-circle the ends of the lines formed by a succession of consecutive z-strokes, and we w-circle the ends of the lines formed by a succession of consecutive w-strokes (Rule 1a). Then, in a second step, we z-circle the vertices which are attached to the previous z-circles by w-strokes (or even by z-edges that we know to be nonmaximal). We do the same with w-circles. The three examples below detail this in common situations. In all the cases, we deduce that the diagram cannot stop there, Rule 1a implying the existence of other circles or other strokes.

Two colors rules. Consequences of a zw-edge. Rules 1a to 1e concern a "one color diagram." They are stated for the z-diagram, but apply as well to the w-diagram. The following rules are numbered 2a to 2h. They concern the diagram obtained by superposition of the z-diagram and the w-diagram.

Rule 2a. There is at least another z-stroke and at least another w-stroke emanating from any zw-edge.

Proof. By Estimate 1 the z-stroke of the zw-edge is not maximal. By Rule 1e and Remark 2 it is not isolated in the z-diagram. So there is another z-stroke from it. Same for the w-stroke. ☐

Rule 2b. *Two consecutive zw-edges.* If there are two consecutive zw-edges, there is a third zw-edge closing the triangle.

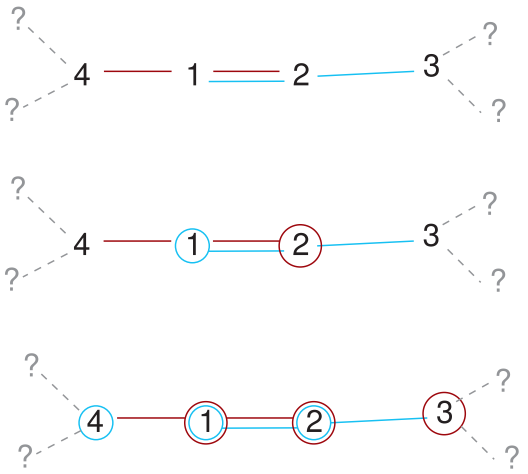

Figure 5. Around a zw-edge with two connected edges.

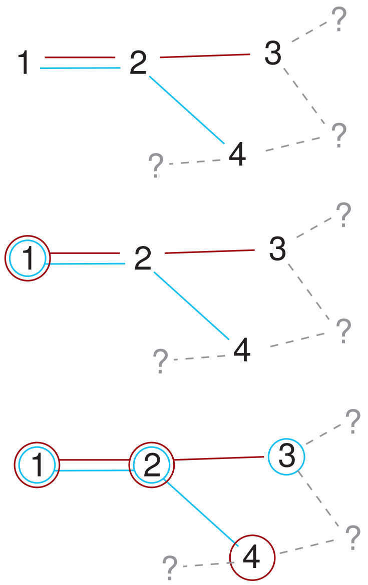

Figure 6. Around a zw-edge with two connected edges in another way.

Proof. Let the consecutive zw-edges be $(1,2)$ and $(2,3)$. By Estimate 2, $z_{13}=z_{12}+z_{23}$ is of order ϵ or less, w_{13} is of order ϵ or less. But "less" is impossible, because, e.g., $Z_{13}=z_{13}^{-1/2}w_{13}^{-3/2}$ would be of greater order than Z. We conclude the proof by using the first equality case of Estimate 1. ☐

In Figure 7 we show the simplest patterns around a zw-edge. In the preceding figures we have shown how to circle the first two. About the third,

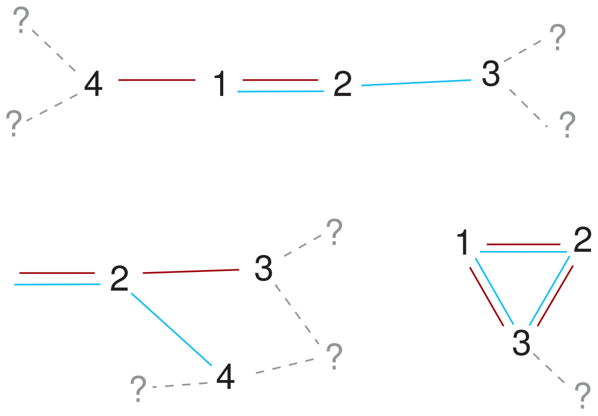

Figure 7. The sub-diagrams with 2 edges (or less) connected to a zw-edge.

bodies 1, 2, and 3 are close, so they are all z-circled or all not z-circled, and they are all w-circled or all not w-circled.

Clusters. Cycles. At the limit when following a singular sequence, the z_k's form clusters. If, for example, bodies 1, 2 and 3 are such that $z_{12} \prec z_{13}$, we say that 1 clusters with 2 in z-coordinate, relatively to the subset of bodies 1, 2, 3. We may then consider a fourth body, which may form a sub-cluster, e.g., together with body 2. Altogether this means $z_{24} \prec z_{12} \prec z_{13}$.

We will often write a *clustering scheme* in each coordinate. In the latter situation we would write simply $z : 24.1 \dots 3$, three dots being the largest separation within this group, one dot the intermediate separation, no dot the smallest separation. (Three different orders of separation appear to be enough in our considerations.)

In the rule below we discuss clustering relations inside a sub-system of three bodies. Nothing forbids that these three bodies form, e.g., in z-coordinate, a cluster relatively to the whole configuration.

Rule 2c. *Skew clustering.* Consider two consecutive edges that are not part of a triangle, e.g., an edge from vertex 1 to vertex 2, an edge from vertex 2 to vertex 3. Then the clustering schemes are $z : 1.2 \dots 3$, $w : 1 \dots 2.3$, or $z : 1 \dots 2.3$, $w : 1.2 \dots 3$. We say there is "skewsymmetric clustering" or simply "skew clustering."

Proof. Suppose the two consecutive edges are z-edges. If we had the same order in z-separation, we would have the same order in w-separation, as $Z_{12} = z_{12}^{-1/2} w_{12}^{-3/2} \approx z_{23}^{-1/2} w_{23}^{-3/2} = Z_{23}$, and the triangle would be closed, by the same argument as for Rule 2b. Excluded. The "skewsymmetric clustering" follows from $Z_{12} \approx Z_{23}$.

The same applies for two consecutive w-edges. Two consecutive zw-edges are forbidden by Rule 2b. In the remaining cases the two consecutive edges have different types. But we know from Estimate 2 that the z-separation corresponding to a w-edge is strictly smaller than the z-separation corresponding to a zw-edge, which is in turn strictly smaller than the z-separation corresponding to a z-edge. The w-separations corresponding respectively to these types of edges follow the inverse order. So there is always skew clustering. ☐

Recall that a z-edge between k and l is called maximal if $z_{kl} \approx \epsilon^{-2}$.

Corollary. *Two consecutive z-edges cannot be maximal if they are not part of a triangle of edges.*

Proof. If the edges are not part of a triangle, Rule 2c applies and gives $z_{12} \prec z_{23}$ or $z_{12} \succ z_{23}$, which contradicts $z_{12} \approx \epsilon^{-2}$ and $z_{23} \approx \epsilon^{-2}$. ☐

There may appear some contradiction if there are cycles of edges. If there is a cycle, one can join two vertices following two different paths of edges. The cumulated separations should be equal in both paths. This gives the following rule.

Rule 2d. *Cycles.* Consider a cycle of edges, the list of z-separations corresponding to these edges, and the maximal order for the z-separations within this list. Two or more of the z-separations are of this order. The corresponding edges have the same type. If there are only two, the corresponding separations are not only of the same order, but equivalent.

Rule 2e. *Triangles.* Consider a triangle of edges in the diagram. Then the edges have the same type (all z-edges or all w-edges or all zw-edges), all the z-separations are of the same order, all the w-separations are of the same order.

Proof. Let the triangle have three edges of different type. Then we need two z-edges or two zw-edges by Rule 2d, corresponding to the greatest z-separation. We need also two edges corresponding to greatest w-separation. Impossible. So the edges have the same type. Suppose the z-separations are not of the same order. So one is of lower order, the other two are equivalent. By, e.g., $z_{12}^{-1/2}w_{12}^{-3/2} \approx z_{23}^{-1/2}w_{23}^{-3/2} \approx z_{31}^{-1/2}w_{31}^{-3/2}$, only one w-separation is of greatest order, contradicting the cycle rule 2d. So the separations are of the same order. ☐

Rules 2b and 2e together give the following easily remembered statement.

Corollary. *Consider three vertices. There are* 6, 3, 2, 1 *or* 0 *strokes joining them. If there are three forming a triangle, they are of the same color.*

Rule 2f. *Fully edged sub-diagrams.* Consider in the diagram: a triangle of edges, plus a fourth vertex attached to the triangle by at least two edges, plus a fifth vertex attached to the four previous vertices by at least two edges, and so on up to a p-th vertex, $p \geq 3$. Then there is indeed an edge between any pair of the p vertices, the edges have the same type, all the z-separations are of the same order, all the w-separations are of the same order.

Proof. Rule 2e is Rule 2f for $p = 3$. Suppose Rule 2f is true for $p-1$ bodies. We add body p, attached with two edges to, let us say, bodies 1 and 2. We apply Rule 2e to the triangle $12p$. Thus $1p$ and $2p$ correspond to the same type of edge, the same z-separations and the same w-separations as 12, i.e., as every kl, $k < l < p$. Consider the edges kp with $3 \leq k \leq p-1$. As, e.g., $z_{kp} = z_{k1} + z_{1p}$, we have $z_{kp} \preceq z_{12}$ and similarly $w_{kp} \preceq w_{12}$. If $z_{kp} \prec z_{12}$, then $Z_{kp} = z_{kp}^{-1/2} w_{kp}^{-3/2}$ and $W_{kp} = z_{kp}^{-3/2} w_{kp}^{-1/2}$ would be respectively larger than $Z_{12} = z_{12}^{-1/2} w_{12}^{-3/2}$ and $W_{12} = z_{12}^{-3/2} w_{12}^{-1/2}$. This would contradict the maximality implied by the z_{12}-stroke or the w_{12}-stroke. So $z_{kp} \approx z_{12}$, $w_{kp} \approx w_{12}$, and there is the same type of edge between k and p as between 1 and 2. ☐

Rule 2g. If four edges form a quadrilateral, then the opposite edges have the same type.

Proof. Rule 2d provides us with two edges of the same type corresponding to the maximal z-separation within the four edges, and two edges of the same type corresponding to the maximal w-separation within the four edges. Suppose there is a pair of adjacent edges of one type, a pair of adjacent edges of another different type. By Rule 2f, the diagonals are not edges. Rule 2c applies to any of these pairs and gives skew clustering, i.e., different order of z-separation. But this contradicts Rule 2d, which gives the same order of z-separation. Finally the two types are the same, or they are different but they alternate along the quadrilateral. ☐

Lemma 2 states that the potential $U = \sum m_k m_l / r_{kl}$ takes finitely many values on the set of normalized central configurations. Here we use this property for the first time. We get a new rule, which we will use in Section 7. The rule is immediately deduced from the fact that U is bounded.

Rule 2h. *Bounded potential.* Consider a singular sequence. Pick bodies k_0 and l_0 such that $r_{k_0 l_0} \preceq r_{kl}$ for any k, l, $1 \leq k < l \leq n$. If $r_{k_0 l_0} \to 0$, then there is another pair of bodies (k_1, l_1) such that $r_{k_1 l_1} \approx r_{k_0 l_0}$.

Corollary. *If there is a zw-edge in the diagram, there is another one.*

Proof. A zw-edge between bodies k_0 and l_0 means $r_{k_0 l_0} \approx \epsilon$, which is the minimal order for a distance according to Estimate 1. Rule 2h applies. ☐

4. Systematic exclusion of 4-body diagrams

We call *a bicolored vertex* of the diagram a vertex which connects at least a stroke of z-color with at least a stroke of w-color. The number of edges from a bicolored vertex is at least 1 and at most $n-1$. The number of strokes from a bicolored vertex is at least 2 and at most $2(n-1)$. Given a diagram, we define C as the maximal number of strokes from a bicolored vertex. We use this number to classify all possible diagrams.

Recall that the z-diagram indicates the maximal terms among a finite set of terms. It is nonempty. If there is a circle, there is an edge of the same color emanating from it. So there is at least a z-edge, and similarly, at least a w-edge.

4.1. *Four bodies. No bicolored vertex.* If there is no bicolored vertex, then C is not defined, there are at most two strokes and they are "parallel." Thus the only possible diagram is the first one in Figure 8.

4.2. *Four bodies.* $C = 2$. There are two cases: a zw-edge exists or not.

If it is present, it should be isolated. This is impossible by Rule 2a.

If it is not present, there are adjacent z-edges and w-edges. From any such adjacency there is no other edge. By trying to continue Figure 4, we see that the only diagram is the second in Figure 8.

4.3. *Four bodies.* $C = 3$. Consider a bicolored vertex with 3 strokes. It is Y-shaped or connects a single stroke to a zw-edge.

Suppose it is Y-shaped, let us say with two z-edges and a w-edge. Then it is w-circled by Rule 1a. By the circling method, the other ends of the z-edges are w-circled. Each of these ends should have a w-edge. This produces a triangle and contradicts Rule 2e.

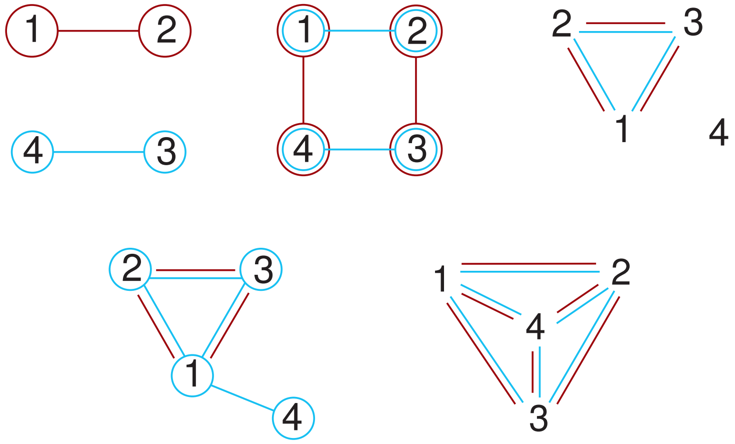

Figure 8. The five remaining 4-body diagrams.

Suppose it is a zw-edge connected with, let us say, a z-edge. By Rule 2a, there is a w-edge on the other side of the zw-edge that cannot close the triangle by Rule 2e and cannot make a quadrilateral by Rule 2g. The diagram cannot just be these three edges as shown in Figure 5. Contradiction.

4.4. *Four bodies.* $C = 4$. Consider a bicolored vertex with 4 strokes. In a first case, it has a zw-edge and two z-edges. Rule 2a requires another w-edge from the zw-edge that closes a triangle that contradicts Rule 2e.

In a second case, the bicolored vertex is as vertex 2 in the first diagram in Figure 6. Any other edge in this diagram would close a triangle, which would contradict Rule 2e. The circling method then gives a contradiction, as shown in Figure 6.

In the last case, the bicolored vertex has two adjacent zw-edges. A third zw-edge closes the triangles by Rule 2b. As $C = 4$ there is one triangle of zw-edges and no other edge in the diagram. This is the third diagram in Figure 8. There is no maximal z-stroke thus no z-circle by Rule 1e. For the same reason there is no w-circle.

4.5. *Four bodies.* $C = 5$. The maximal bicolored vertex should be as vertex 1 in the fourth diagram in Figure 8, which forces the rest of the diagram by Rule 2b and the circling method. Rule 1e shows there is no z-circle.

4.6. *Four bodies.* $C = 6$. This is a fully edged diagram by Rule 2b. There is no circle by Rule 1e. This is the fifth diagram in Figure 8.

The conclusion of this section is that *any singular sequence should converge to one of the five diagrams in Figure 8*.

Remark 4. In the 3-body problem, the finiteness is easy to get, even in the complex domain. The computations in the appendix constitute a proof. It is however a valuable exercise to apply the above method to the 3-body case. Only one diagram, the fully edged diagram without circle, appears to be possible.

As the list of diagrams is not empty, further discussion is needed to prove the finiteness. Several ideas may be used. They mostly use the fact that on a fully edged diagram, any edge is a zw-edge, so any $r_{kl} \approx \epsilon \to 0$ by Estimate 1.

A first idea is to discuss the diagram as we will do in 5.3 and deduce that a singular sequence may approach the diagram only if the masses satisfy (16). If the masses do not satisfy this relation, there are no singular sequences and, consequently, no continuum of normalized central configurations.

A second idea is to deduce from expression (7) that the moment of inertia I tends to zero while approaching the fully edged diagram. By Lemma 2, the moment of inertia is constant on a continuum of central configurations. So it is constantly equal to zero. But if the masses are positive, I is positive on real configurations. Under this hypothesis on the masses, there could still exist a continuum of normalized central configurations, but it would not contain any real central configuration. This reality argument will be used in the proof of Theorem 5.

A third idea is to prove that a continuum of normalized central configurations should approach *several* diagrams. As at least one of the three distances should be dominating, there exist singular sequences such that this distance goes to zero, and singular sequences such that it goes to infinity. A singular sequence of the latter type cannot exist, as it cannot go to the only diagram that has no distance going to infinity. We will often use this idea in the proofs of our theorems.

5. **Five remaining 4-body diagrams. First finiteness result**

In the previous process of eliminating diagrams, our only hypothesis on the masses is that no subset of bodies has zero mass. We could not eliminate the diagrams in Figure 8. Some singular sequence could still exist and approach any of these diagrams. Here we restrict to *real positive masses*. Still this is not enough: each diagram will be excluded *except if the masses satisfy a polynomial relation*. In Sections 5.1 to 5.5 we obtain the constraints on the masses corresponding to each of the five diagrams from Figure 8, numbered horizontally from top left to bottom right.

5.0. *On a pair of disconnected fully edged subdiagrams*. We define a class of diagrams consisting of a fully edged isolated component of z-color and a fully edged isolated component of w-color. In particular, this class contains the first diagram in Figure 8. Some *diagrams from the* 5*-body case* also fall into the framework described below.

To construct a diagram in this class we start with two normalized central configurations. We stretch one along the z-axis and the other one along the w-axis. Each stretching is done in such a way that the stretched coordinate is of order ϵ^{-2} while the other coordinate is of order ϵ^2. The result looks like a singular sequence. The diagram has a fully edged isolated component of z-color and a fully edged isolated component of w-color. The first diagram in Figure 8 is the simplest example.

Reciprocally let us assume that bodies 1 to p form a diagram of z-color, fully z-edged and z-circled, and bodies $p+1$ to n form a similar diagram of w-color.

According to Rule 2f, $z_{kl} \approx z_{12}$ for any k, l, $k < l \leq p$. As the vertices $1, \dots, p$ are z-circled while the other vertices are not, we also have $z_{kl} \approx z_{12} \approx \epsilon^{-2}$ for all k, l, $k \leq p$, $p+1 \leq l$. By Rule 2f again, $w_{(p+1)(p+2)} \approx w_{kl}$ for any k, l, $p+1 \leq k < l \leq n$. All this information is condensed in the clustering scheme:

$z:1\dots 2\dots 3\dots(p+1)(p+2)n\dots 4\dots p$. The corresponding estimates in w give $w:p+1\dots p+2\dots 123p\dots p+3\dots n$.

According to these estimates, in each of the first $2p$ equations of system (9), the first p terms in the right-hand side dominate: the w_{kl}'s are smaller in these terms, while the z_{kl}'s are of the same order as in the remaining terms of the right-hand side.

Consequently, the system is decoupled in the limit: bodies 1 to p form a central configuration, as do bodies $p+1$ to n.

We got a description of a singular sequence corresponding to the considered disconnected diagrams. Is this description complete? At a first look it seems hopeless to look for further equations involving the crossed terms. These small terms correspond to interactions between a central configuration and the other. By a perturbation of the positions of the bodies 1 to p, one should be able to balance the contribution of these crossed terms.

But a closer look shows that these contributions cannot be balanced in general, and there is a constraint corresponding to the crossed terms. Consider the equations of central configurations (2) in vector form $q_k=f_k$, suppose $1\le k\le p$ and set $f_k=f_k^i+f_k^e$, where f_k^i is the contribution of the first p bodies or "internal" bodies and f_k^e are the "crossed terms," i.e., the contribution of the other bodies, or "exterior" bodies. The scalar quantity $m_1q_1\wedge f_1^i+m_2q_2\wedge f_2^i+\cdots+m_pq_p\wedge f_p^i$ vanishes in the limit for any configuration of the first p bodies. Finally,

$$0=m_1q_1\wedge f_1+\cdots+m_pq_p\wedge f_p=m_1q_1\wedge f_1^e+m_2q_2\wedge f_2^e+\cdots+m_pq_p\wedge f_p^e$$

is an interesting constraint on the f_k^e's, i.e., on the crossed terms.

5.1. *The disconnected diagram.* In the case of the first diagram in Figure 8 the clustering scheme is $z:1\dots 34\dots 2$, $w:3\dots 12\dots 4$. We write

$$\begin{aligned}0&=f_1\wedge q_1=m_2r_{12}^{-3}q_1\wedge q_2+m_3r_{13}^{-3}q_1\wedge q_3+m_4r_{14}^{-3}q_1\wedge q_4,\\0&=f_2\wedge q_2=m_1r_{12}^{-3}q_2\wedge q_1+m_3r_{23}^{-3}q_2\wedge q_3+m_4r_{24}^{-3}q_2\wedge q_4\end{aligned}\tag{11}$$

and combine

$$0=m_1m_3r_{13}^{-3}q_1\wedge q_3+m_1m_4r_{14}^{-3}q_1\wedge q_4+m_2m_3r_{23}^{-3}q_2\wedge q_3+m_2m_4r_{24}^{-3}q_2\wedge q_4.$$

Remarkably the four terms are quite similar. Here $r_{kl}^2=z_{kl}^{-1}w_{kl}^{-1}\sim z_k^{-1}w_l^{-1}$, because each factor is the separation between something near the center of mass and something far away. The r_{kl}'s are of the same order. Again by the clustering scheme, $q_k\wedge q_l\sim z_kw_l$. Finally our equation tells that

$$m_1m_3(z_1w_3)^{-1/2}\pm m_2m_3(z_2w_3)^{-1/2}\pm m_1m_4(z_1w_4)^{-1/2}\pm m_2m_4(z_2w_4)^{-1/2}$$

is small compared to one of the four terms. By the center of mass $z_1\sim -m_2z_0$, $z_2\sim m_1z_0$, $w_3\sim -m_4w_0$, $w_4\sim m_3w_0$ for some nonvanishing complex numbers

z_0 and w_0. We get

$$\frac{m_1m_3}{\sqrt{m_2m_4}} \pm \frac{m_2m_3}{\sqrt{-m_1m_4}} \pm \frac{m_1m_4}{\sqrt{-m_2m_3}} \pm \frac{m_2m_4}{\sqrt{m_1m_3}} = 0, \tag{12}$$

or by setting $m_k = \mu_k^2$,

$$\mu_1^3\mu_3^3 \pm i\mu_2^3\mu_3^3 \pm i\mu_1^3\mu_4^3 \pm \mu_2^3\mu_4^3 = 0.$$

Half of the choices of signs enter the factorization $(\mu_1^3 \pm i\mu_2^3)(\mu_3^3 \pm i\mu_4^3) = 0$, which has no solution with $(\mu_1, \mu_2, \mu_3, \mu_4) \in \mathbb{R}^4$. The real solutions correspond to $(\mu_1\mu_3)^3 = (\mu_2\mu_4)^3$ and $(\mu_2\mu_3)^3 = (\mu_1\mu_4)^3$. Removing the cubes and dividing one by the other gives $\mu_1^2 = \mu_2^2$. We get two conditions on the positive masses:

$$m_1 = m_2 \quad \text{and} \quad m_3 = m_4. \tag{13}$$

5.2. *The quadrilateral diagram.* Here we study the second diagram on Figure 8. Rule 1c shows that $m_1z_1 + m_4z_4$ and $m_2z_2 + m_3z_3$ are close to the origin, i.e., are not z-maximal. In particular, they are close together. But the clustering scheme $z : 12 \dots 34$, $w : 14 \dots 23$ gives $z_1 \sim z_2$ and $z_3 \sim z_4$. So finally,

$$m_1m_3 = m_2m_4. \tag{14}$$

The same analysis in w-coordinate gives the same constraint.

5.3. *The isolated triangle.* Here we study the third diagram in Figure 8. The diagram with a zw-edged triangle without circles shows that the dominant terms of (10) correspond to an equilibrium of the three body problem. It is an "absolute" equilibrium, which is stronger than a relative equilibrium.

The configuration in a real relative equilibrium is a central configuration. Instead of (2), it satisfies the system $\lambda q_k = f_k$, with $\lambda > 0$. For an equilibrium, $\lambda = 0$. As homogeneity gives $\lambda I = U$, the potential U of an equilibrium is zero.

There is an absolute equilibrium of three bodies in Roberts' example, with masses 1, $-1/4$, 1. To any collinear 3-body configuration we can associate masses making it an equilibrium (see [3]).

The constraint on the masses for a 3-body equilibrium gives a constraint on the masses for the third diagram in Figure 8. System (10) reduces to its main terms:

$$m_2Z_{12}+m_3Z_{13} \prec \epsilon^{-2}, \quad m_2W_{12}+m_3W_{13} \prec \epsilon^{-2}, \quad m_1Z_{21}+m_3Z_{23} \prec \epsilon^{-2}, \quad \cdots$$

or

$$\frac{Z_{12}}{m_3} \sim \frac{Z_{23}}{m_1} \sim \frac{Z_{31}}{m_2}, \qquad \frac{W_{12}}{m_3} \sim \frac{W_{23}}{m_1} \sim \frac{W_{31}}{m_2}. \tag{15}$$

As $Z_{kl} = w_{kl}^{-1} r_{kl}^{-1}$, $W_{kl} = z_{kl}^{-1} r_{kl}^{-1}$, there is a number $\rho \in \mathbb{C}_0$ such that $w_{kl} \sim \rho z_{kl}$. The configuration is collinear in the limit and

$$m_1 z_{23}^2 \sim \pm m_2 z_{31}^2 \sim \pm m_3 z_{12}^2.$$

Setting $m_k = \mu_k^2$, this relation becomes

$$\mu_1 z_{23} \sim \varepsilon_2 \mu_2 z_{31} \sim \varepsilon_3 \mu_3 z_{12}, \quad \text{with } \varepsilon_2^4 = \varepsilon_3^4 = 1.$$

The relation $z_{12} + z_{23} + z_{31} = 0$ gives $\mu_1^{-1} + (\varepsilon_2 \mu_2)^{-1} + (\varepsilon_3 \mu_3)^{-1} = 0$, thus only three choices with $\sqrt{m_k} = \mu_k > 0$:

$$\frac{1}{\sqrt{m_1}} = \frac{1}{\sqrt{m_2}} + \frac{1}{\sqrt{m_3}}, \ \frac{1}{\sqrt{m_2}} = \frac{1}{\sqrt{m_1}} + \frac{1}{\sqrt{m_3}}, \ \frac{1}{\sqrt{m_3}} = \frac{1}{\sqrt{m_1}} + \frac{1}{\sqrt{m_2}}. \tag{16}$$

5.4. *The kite diagram.* Here we study the fourth diagram in Figure 8. We first prove an interesting result about the position of the origin relative to the z-coordinates of the bodies. Recall that the origin is also the z-coordinate of the center of mass of the configuration.

PROPOSITION 1. *If in a singular sequence body n and body 1 are such that $z_{1n} \prec z_{kn}$ and $w_{1n} \approx w_{kn}$ for all k, $1 < k < n$, then z_1, z_n and the origin form a cluster of size $\approx z_{1n}$, i.e., $z_1, z_n \preceq z_{1n}$.*

Proof. We neglect the terms with $1 < k < n$ in the last two equations of (9). We have $z_n \sim m_1 r_{1n}^{-3} z_{1n}$, $w_n \sim m_1 r_{1n}^{-3} w_{1n}$. But, by the w-center of mass, $w_n \preceq w_{1n}$, which is the order of the w-size of the configuration. Thus $r_{1n} \succeq 1$ and $z_n \sim m_1 r_{1n}^{-3} z_{1n} \preceq z_{1n}$. □

An *underlined z-clustering scheme* is the z-clustering scheme where a cluster is underlined. Each body j of this cluster is such that $z_j \preceq R$, where $R = \max z_{kl}$, k and l being bodies in the cluster. In words, the origin "belongs" to the underlined cluster.

The fourth diagram in Figure 8 corresponds to the clustering schemes $z : 2 \dots 41 \dots 3$, $w : 123 \dots 4$. Proposition 1 with $n = 4$ gives the position of the origin in the z-scheme. Here the underlined clustering scheme is $z : 2 \dots \underline{41} \dots 3$. The main information we extract from this scheme is that z_1 is in the limit at the center of mass of z_2 and z_3.

Let us consider system (9). The first pair of equations is consequence of the last three and the center of mass, thus we forget it. The second and third pairs are reduced to their dominant terms. The diagram and the w-cluster 123 give

$$m_1 Z_{21} \sim -m_3 Z_{23}, \quad w_1 \sim m_1 W_{21} + m_3 W_{23},$$

$$m_1 Z_{31} \sim -m_2 Z_{32}, \quad w_1 \sim m_1 W_{31} + m_2 W_{32},$$

or

$$\frac{Z_{12}}{m_3} \sim \frac{Z_{23}}{m_1} \sim \frac{Z_{31}}{m_2}, \quad m_1W_{21} + m_3W_{23} \sim m_1W_{31} + m_2W_{32}. \tag{17}$$

Comparing to the similar computation in the previous diagram, we have an equation less, but we know that z_1 is at the z-center of mass of the triangle z_1, z_2, z_3, which gives the relation $m_2z_{12} \sim m_3z_{31}$. Our equations are now homogeneous. We simply set $z_{12} = m_3$, $z_{31} = m_2$. Using $Z_{kl} = r_{kl}^{-3}z_{kl}$, the identities above among the Z_{kl}'s become

$$\frac{m_1}{r_{12}^3} \sim -\frac{m_2 + m_3}{r_{23}^3} \sim \frac{m_1}{r_{13}^3}. \tag{18}$$

Consider the identity $w_{12} + w_{23} + w_{31} = 0$. Multiplying it by (18) gives the second identity (17). Multiplying it by directly by the first identity (17) gives

$$-\frac{1}{m_3r_{12}} - \frac{1}{m_2r_{13}} \sim \frac{1}{m_1r_{23}}, \tag{19}$$

which basically tells us that the three infinite contributions to the potential U cancel out. The system being homogeneous, (18) may be written as $r_{23} = \sqrt[3]{m_2 + m_3}$, $r_{12} = -\varepsilon_2\sqrt[3]{m_1}$, $r_{13} = -\varepsilon_3\sqrt[3]{m_1}$, with $\varepsilon_2^3 = \varepsilon_3^3 = 1$. The masses are positive, and we use the cubic root symbol for the positive cubic root. If one of the ε_k's was nonreal, the other one should be its conjugate according to (19). The left-hand side of (19) would be negative, while the right-hand side is positive. This is a contradiction. Finally,

$$\varepsilon_2 = \varepsilon_3 = 1,\ r_{12} = r_{13},\ (m_2+m_3)r_{12}^3 = -m_1r_{23}^3,\ m_2m_3r_{12} = -m_1(m_2+m_3)r_{23},$$

giving the relation

$$m_1^2(m_2 + m_3)^4 = m_2^3m_3^3. \tag{20}$$

5.5. *Fully edged diagram.* Here we study the fifth diagram in Figure 8. We met a 3-body equilibrium in Section 5.3. Here we have a 4-body equilibrium. Such an equilibrium satisfies a complicated mass condition that we will not discuss here. We will show in Theorem 3 and Theorem 5 how to reach our main conclusions without discussing this condition.

5.6. *First finiteness result.* We collect the result of this part in a finiteness statement that we will improve later.

THEOREM 3. *Suppose $n = 4$ and $m_k > 0$, $k = 1, \dots, 4$. System (4), which defines the normalized central configurations in the complex domain, possesses finitely many solutions, except perhaps if after some renumbering, the masses satisfy either condition (14), or condition (16), or condition (20).*

Proof. Recall that the δ_{kl}'s, $1 \le k < l \le 4$, are the six inverses of the mutual distances in the configuration. Elementary geometry shows that giving five of the δ_{kl}^2's determines finitely many geometrical configurations up to rotation. If there are infinitely many solutions of (4), at least two of the δ_{kl}'s should take infinitely many values. We suppose δ_{34} does, and we take it as the polynomial function in Lemma 1. There is a sequence of normalized central configurations such that $\delta_{34} \to 0$, i.e., $|z_{34}w_{34}| \to \infty$. Whatever the renormalization is, Z or W is unbounded on this sequence. We extract a singular sequence. It corresponds to one of the diagrams in Figure 8. It cannot be the fifth diagram, where all the edges are zw-edges, which according to Estimate 1 means that all the distances $r_{kl} \approx \epsilon \to 0$. The codimension 2 mass condition (13) of the first diagram is included in condition (14). So if there are infinitely many solutions, the masses should satisfy one of the stated conditions. ☐

6. **More finiteness results in the 4-body case**

6.1. *On the distances in the* 3*rd and* 4*th diagrams in Figure* 8. Consider n bodies and a diagram with zw-edges between any pair of the first $n-1$ vertices. By Estimate 1 we know that $z_{kl}, w_{kl} \approx \epsilon$ for $k \neq l < n$. If $z_n \preceq \epsilon$ and $w_n \preceq \epsilon$, then we have the fully edged diagram. We exclude this case. Either z_n, or w_n, or both escape from the cluster, which has size ϵ. Let us assume $w_n \succeq z_n$. This implies $\epsilon \prec w_n$ and $w_{1n} \sim \cdots \sim w_{(n-1)n}$.

First case. We assume that $\epsilon \prec z_n$ or that z_n stays with the other z_k's but does not cluster, i.e., $z_{kn} \approx \epsilon$. We have $z_{1n} \sim z_{2n} \sim \cdots \sim z_{(n-1)n}$ in the first subcase and only $z_{1n} \approx z_{2n} \approx \cdots \approx z_{(n-1)n}$ in the second subcase. This gives $r_{1n} \sim \pm r_{2n} \sim \cdots \sim \pm r_{(n-1)n}$ and $r_{1n} \approx r_{2n} \approx \cdots \approx r_{(n-1)n}$ respectively. According to Proposition 2 below, there is a contradiction with the position of w_n except if these distances are bounded.

Recall that "body n is w-close to the center of mass" (Definition 4) is equivalent to "vertex n is not w-circled" but is weaker than "body n clusters with the center of mass in w-coordinate," which means that there exists an $l \neq n$ such that $w_n \prec w_l$.

Proposition 2. *If in a singular sequence all the distances from a given body to the other bodies are unbounded, then this body clusters with the center of mass in z-coordinate and in w-coordinate.*

Proof. If all the distances from body n go to infinity, the equation $w_n = \sum_{k<n} m_k r_{kn}^{-3} w_{kn}$ shows that $w_n \prec w_{ln}$ for some $l < n$. Then $w_n \prec w_l$. Same for z. ☐

Remark 5. An example is the singular sequence at infinity contained in the Roberts' example in the regime when the fifth body is at the center of mass and at infinite distance from all the other bodies.

Second case. We assume that z_n clusters with a body, e.g., $z_{1n} \prec \epsilon$. The w-configuration is made of a cluster and an isolated body. By the center of mass, $(M - m_n)w_1 \sim -m_n w_n$, where M is the total mass. The equation $w_n \sim m_1 r_{1n}^{-3} w_{1n}$ gives more precisely $r_{1n}^3 \sim M^{-1} m_1^{-1}(M - m_n)$. The other distances go to infinity: $r_{kn}^2 = z_{kn} w_{kn} \approx (z_{kn}/z_{1n})(z_{1n} w_{1n}) \approx \epsilon/z_{1n}$.

Conclusion. The r_{kn}^2's are at most of order ϵ/z_{1n}. This is at most ϵ^{-1}. The order ϵ^{-1} is possible if and only if $z_{1n} \approx \epsilon^2$, which corresponds to the second case above, and, according to Estimate 1, to a maximal w-edge between 1 and n. This is the situation of the fourth diagram in Figure 8. The first and the second cases cover all the possibilities for the third diagram in Figure 8. We combine these results in an easy estimate on the product of two nonadjacent distances; i.e., $r_{ij} r_{kl}$, where i, j, k and l are distinct indices.

PROPOSITION 3. *Consider n bodies and a diagram with zw-edges between any pair of the first $n-1$ vertices. Then the product of any two nonadjacent distances tends to zero.*

THEOREM 4. *Suppose $n = 4$ and $m_k > 0$, $k = 1, \dots, 4$. System (4), which defines the normalized central configurations in the complex domain, possesses finitely many solutions except perhaps if the masses are equal.*

Proof. We repeat the argument in the proof of Theorem 3. If there were infinitely many solutions of (4), at least two of the δ_{kl}'s should take infinitely many values. We suppose δ_{12} does, and we take it as the polynomial function in Lemma 1. There is a sequence of normalized central configurations such that $|\delta_{12}| \to \infty$, i.e., $z_{12} w_{12} \to 0$. As $Z_{12} W_{12} = \delta_{12}^4$, Z or W is unbounded on this sequence. We extract a singular sequence.

In the first and the second diagrams in Figure 8, no distance is going to zero: the edges correspond to distances ≈ 1, while the pairs of vertices without edges correspond to distances going to infinity. This is immediately deduced from the clustering scheme and the usual estimates.

So we are in the third, the fourth or the fifth diagram, and other distances, say r_{23} and r_{31}, should also go to zero. They also take infinitely many values.

Consider $r_{12} r_{34}$. First suppose it takes finitely many values. Push again r_{12} to zero. According to Proposition 3, no renumbered diagram has a finite nonzero $r_{12} r_{34}$ with $r_{12} \to 0$. This is a contradiction.

According to Lemma 1, $r_{12}^2 r_{34}^2 = z_{12} w_{12} z_{34} w_{34}$ is a dominating polynomial. Push it to infinity. In the third, the fourth and the fifth diagram, it goes to zero, according to Proposition 3. We are in the first or the second diagram. The constraint on the masses in the first diagram are $m_1 = m_3$ and $m_2 = m_4$, or $m_1 = m_4$ and $m_2 = m_3$. In the second diagram the constraint is $m_1 m_2 = m_3 m_4$, which happens to include both cases of the first diagram.

We repeat successively with r_{23} and r_{31} the argument with r_{12}, giving $m_1m_3 = m_2m_4$ and $m_1m_4 = m_2m_3$. For positive masses, the remaining case is equal masses. □

THEOREM 5. *Suppose $n = 4$ and $m_k > 0$, $k = 1, \dots, 4$. There exists finitely many real normalized central configurations.*

Proof. If there are infinitely many solutions of (4), infinitely many being real, then Lemma 2 shows that there are infinitely many solutions of (4) for which $U = I = \sum_k m_k(x_k^2 + y_k^2)$ takes the same real positive value I_0. We repeat the proof of Theorem 4, replacing system (4) by the same system together with the polynomial equation $I = I_0$. We exclude all the singular sequences, except in the equal mass case. In this case, we push a distance, say r_{12}, to zero, which again excludes the first two diagrams. The fifth diagram is such that I tends to zero (and thus $I = 0$ by Lemma 2). It is excluded by the condition $I_0 > 0$. One checks that conditions (16) and (20) corresponding to the remaining two diagrams are impossible with equal positive masses. □

Theorem 5 implies Theorem 1 as stated in the introduction. Theorem 5 also excludes the continua where $x_1, y_1, x_2, \dots, y_4$ are real but the mutual distances $r_{kl} = \pm\sqrt{x_{kl}^2 + y_{kl}^2}$ are not supposed to be positive.

To get Theorem 1 from Theorem 4 we could also remark that in the equal mass case, the central configurations are known (see [1], [2]) and are finitely many. Compared to Theorem 5 this would have two disadvantages. First, [2] uses a computer algebra system, while our proof of Theorem 5 does not require any difficult computation. Second, as [1] only concerns the positive central configurations, we would not get Theorem 5, but just Theorem 1.

Note that Hampton and Moeckel also deduced Theorem 1 from a stronger statement concerning the complex central configurations.

We close here our list of results on the 4-body problem. *From now on we study the planar 5-body problem.*

7. Systematic construction of the 5-body diagrams

In Sections 7 to 9 we shall prove Theorem 2. The proof consists of two main parts.

First, in Section 7, we derive a list of problematic diagrams. This is similar to the study of Section 4 in the 4-body case. This list of sixteen diagrams is in Figure 11. It is analogous to the list of five diagrams in Figure 8.

Second, in Section 8, we present calculations showing that except if some explicit relations on the five masses are satisfied, thirteen of these diagrams cannot be approached by a singular sequence. Similar relations probably also exist for the three other diagrams, but we found them too complicated and avoided their discussion.

We conclude in Section 9 by showing that a continuum of central configuration should approach at least one of the thirteen diagrams. For generic masses it is impossible. It remains possible for positive masses satisfying some explicit polynomial relations.

The first part (Section 7) also has substructure. As in the case of the 4-body problem, we divide possible diagrams into groups according to the maximal number of strokes from a bicolored vertex. During the analysis of all the possibilities we rule some of them out immediately, some with further arguments, and some, the ones we cannot exclude without further hypotheses on the masses, are incorporated into the list of sixteen in Figure 11. The second group consists of six diagrams in Figures 9 and 10.

7.1. *Five bodies. No bicolored vertex.* We start by analyzing diagrams with at least two connected components. We state a new proposition and then eliminate a triple of certain diagrams.

PROPOSITION 4. *Suppose a diagram has one and only one maximal z-edge and that this edge forms an isolated component. Call 1 and 2 the ends of this edge and suppose $m_1^3 + m_2^3 \neq 0$. Then a body k, $3 \leq k \leq n$, cannot be such that $w_{1k} \prec w_{1l}$ for all the other bodies $l \neq k$, $3 \leq l \leq n$.*

Proof. Using Rule 1a, we first deduce that 1 and 2 are z-circled. By Rule 1e, no other body is z-circled: if another body were z-circled, then there would be another maximal z-stroke. The clustering scheme is $z : 1 \dots c \dots 2$, where c is a cluster close to the origin formed by $z_3, \dots, z_n$.

Estimate 1 applied to maximal z-edges is $w_{12} \approx \epsilon^2$. This is as small as possible: as there is no other maximal z-edge, $w_{12} \prec w_{1k}$ for any $k \geq 3$. We apply Proposition 1 to bodies 1 and 2, switching the roles of coordinates z and w. In w-coordinate, the origin forms with w_1 and w_2 a small cluster of size $w_{12} \approx \epsilon^2$.

We take, for example, $k = n$ in the proposition; i.e., assume that body n is such that $w_{1n} \prec w_{1l}$, $3 \leq l < n$. The w-clustering scheme contains the small cluster $w : \dots \underline{12}.n \dots$.

We write $0 = m_1 f_1 \wedge q_1 + m_2 f_2 \wedge q_2 = \sum m_k m_l r_{kl}^{-3} q_k \wedge q_l$, $k = 1, 2$, $l = 3, \dots, n$. Here $q_k \wedge q_l = z_k w_l - w_k z_l \sim z_k w_l$ by the above considerations. We write $r_{kl}^3 = r_{kl} z_{kl} w_{kl}$, $z_{kl} \sim -z_k$, $w_{kl} \sim w_l$. Finally $r_{kl}^{-3} q_k \wedge q_l \sim -r_{kl}^{-1}$. The terms with $l = n$ dominate the above sum, and we have

$$m_1 r_{1n}^{-1} \sim -m_2 r_{2n}^{-1}.$$

Squaring this identity gives $m_1^2 z_{1n}^{-1} w_{1n}^{-1} \sim m_2^2 z_{2n}^{-1} w_{2n}^{-1}$. By clustering, $w_{2n} \sim w_{1n}$. By homogeneity and center of mass, we substitute m_2 for z_{1n} and $-m_1$ for z_{2n}, giving $m_1^2/m_2 = -m_2^2/m_1$, or $m_1^3 + m_2^3 = 0$, which is excluded by hypothesis. □

Let us now consider the disconnected diagrams. They have an isolated edge, which cannot be a zw-edge according to Rule 2a. Let us say it is a z-edge. The complement has 3 bodies. These three can have one, two or three w-edges.

For one w-edge, the attached bodies have to be w-circled by Rule 1a. This is the first diagram in Figure 9.

For two w-edges, take the same diagram and draw the second w-edge from 4 to 5. We show that vertex 4 with two w-edges is w-circled. As vertices 3 and 5 are w-circled, Rule 1e shows that at least one of w_{34} and w_{45} is maximal. If both, then it contradicts the skew-clustering Rule 2c. Thus, only one is maximal and, therefore, 4 is w-circled.

For three w-edges, there are three possibilities: the number of w-circled vertices is either zero, or two or three (one is not possible by w-center of mass).

The remaining five diagrams are the three in Figure 9 and the first two in Figure 11. Proposition 4 shows that the first and third diagrams in Figure 9 are impossible.

Consider the second diagram, with a z_{12}-edge, a w_{34}-edge, a w_{45}-edge, and nothing else. As we said, one of the w-edges is not maximal. The corresponding distance goes to zero. The inverse of this distance cannot be the only infinite contribution to the potential that is bounded (Rule 2h). We check one by one the nine remaining distances r_{kl}: none is going to zero. This is a contradiction.

The two remaining disconnected diagrams are shown in Figure 11, first line. They will be discarded under further conditions on the masses in Section 8.

7.2. *Five bodies.* $C = 2$. As in the 4-body case, there is no zw-edge, and we start with Figure 4. The color of each exterior circle forces the color of the edge (supposed unique by the case hypothesis) from this circle. If the two edges go to the same vertex, we get the diagram corresponding to Roberts' continuum at infinity, shown as the third in Figure 11.

If the two edges go to different vertices, the circling method demands a cycle with alternating colors, incompatible with the odd number of edges.

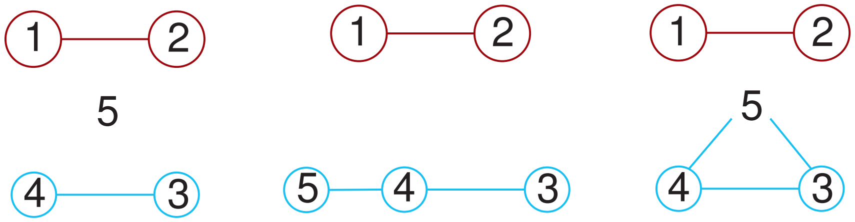

Figure 9. Three impossible disconnected diagrams.

7.3. *Five bodies.* $C = 3$. Suppose there is a Y-shaped vertex with, let us say, two z-edges. By Rule 1a we w-circle the contact and the other ends of the two drawn z-edges. By Rule 2e we should now draw from these ends two w-edges to the fifth vertex. We see a quadrilateral with two adjacent z-edges and two adjacent w-strokes, excluded by Rule 2g. Contradiction.

So there is no Y-shaped bicolored vertex. Then there is a zw-edge that we should continue as in Figure 5. If we had a one color Y-shaped vertex in this figure, it should be at vertex 3 or at vertex 4, as vertex 5 cannot be connected to vertices 1 and 2. This would imply a triangular or a quadrilateral cycle among vertices 1 to 4, of such a type excluded by Rule 2e or Rule 2g. So there is no branching at all.

The diagram is a simple line, open or closed. The following two propositions study these interesting diagrams.

Proposition 5. *If an isolated component of the diagram is a closed line connecting successively $p \geq 4$ of the bodies, then this component is of one color or there are no adjacent edges of the same type. Moreover, Rule* 2*a* (*the edges at both sides of a zw-edge should have different types*) *and Rule* 2*d* (*cycle condition*) *should be respected.*

Proof. Adjacent zw-edges are forbidden by Rule 2b. If there is a sequence of at least two z-edges, and if this sequence is not all the component, it connects to a w-stroke. Call k the connecting vertex and j the vertex connected to k by the z-edge. Vertex k is the end of a w-stroke, so it is w-circled by Rule 1a. By Rule 1b, j has to be w-circled too. But there is no w-stroke coming from j, which contradicts Rule 1a. ☐

Proposition 6. *If an isolated component of the diagram is an open line connecting successively p of the bodies* (*just one line without branching, with edges of any type*), *it has only one color.*

Proof. By the argument of the previous proof, if the line has two colors, there are no adjacent edges of the same type. If the end was a z-edge, we would have a w-stroke before, and then w-clustering with the final body, thus w-circled and isolated in the w-diagram, contradicting Rule 1a. So the end is a zw-edge, connected to a z-edge or a w-edge. But by Rule 2a, a z-edge *and* a w-edge should emanate from a zw-edge. Contradiction. ☐

Here we have a zw-edge. The open line is excluded. We consider the closed line. The zw-edge is connected to a z-edge on one side, to a w-edge on the other side. A quadrilateral is impossible according to Rule 2g. If the closed line is a pentagon, we cannot alternate the types and have another zw-edge (which would have different types on both sides). There remains the pentagon with exactly one zw-edge, which is excluded by Rule 2h.

7.4. *Five bodies.* $C = 4$. Consider a w-edge and three z-edges from the same vertex. These edges connect the five vertices. By Rule 1a, we w-circle the contact and the other ends of the three drawn z-edges. We should now draw three w-edges from these three ends. Each would contradict Rule 2e.

Suppose there are two w-edges and two z-edges. Suppose the numeration is as around vertex 5 in diagram 7, Figure 11. The only possible other edges according to Rule 2e are the two horizontal edges in this diagram. We will show that the z_{12} and w_{34} edges should exist. The color of these edges is forced by Rule 2e. We will have two attached triangles, one of each color.

To show the existence of the z_{12} and w_{34} edges, suppose that, for example, the z_{12}-edge is missing. Then by Rule 1a, vertices 1 and 2 have to be z-circled. By Rule 2c applied to z-edges 15 and 52, there is a skew clustering and 5 has to be z-circled too, being z-close to a circled vertex. However, z-circle around 5 implies in turn z-circles around 3 and 4. This is a contradiction as no further z-edges can be drawn neither from 3 nor from 4. The z_{12}-edge is there, and we have two attached triangles.

This “butterfly diagram” may be circled in three different ways, represented as diagrams 6 to 8 in Figure 11. The only constraint is that vertex number 5, connecting both attached triangles, cannot be circled. If it was, e.g., z-circled, then by Rule 1b, vertices 3 and 4 would also be z-circled, but there are no z-edges from them.

Suppose there are two zw-edges; then Rule 2b enforces a third zw-edge. We have a triangle of zw-edges that is isolated due to $C = 4$. As an isolated zw-edge is excluded by Rule 2a, the two possibilities are diagrams 4 and 5 in Figure 11. As the triangle has no maximal edge, the vertices are not circled (see Estimate 1 and Rule 1e).

Suppose there is one zw-edge, one z-edge, and one w-edge emanating from, e.g., vertex 2 as on the first diagram in Figure 6.

One possibility is not to have edges emanating from 1. Then we are forced to z and w circle vertices and get the third diagram in Figure 6. Vertices 3 and 4 require further edges. If these edges go among 134, it violates Rule 2e. There should be edges from bodies 3 and 4 to body 5. According to Rule 2g,

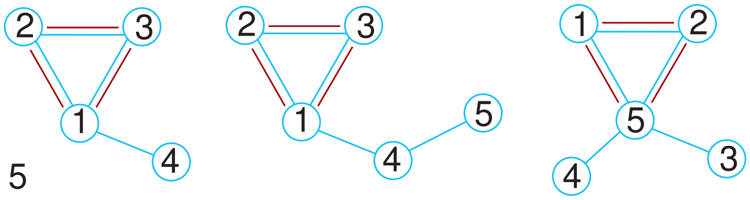

Figure 10. Three impossible diagrams with $C = 5$, 5 and 6 respectively.

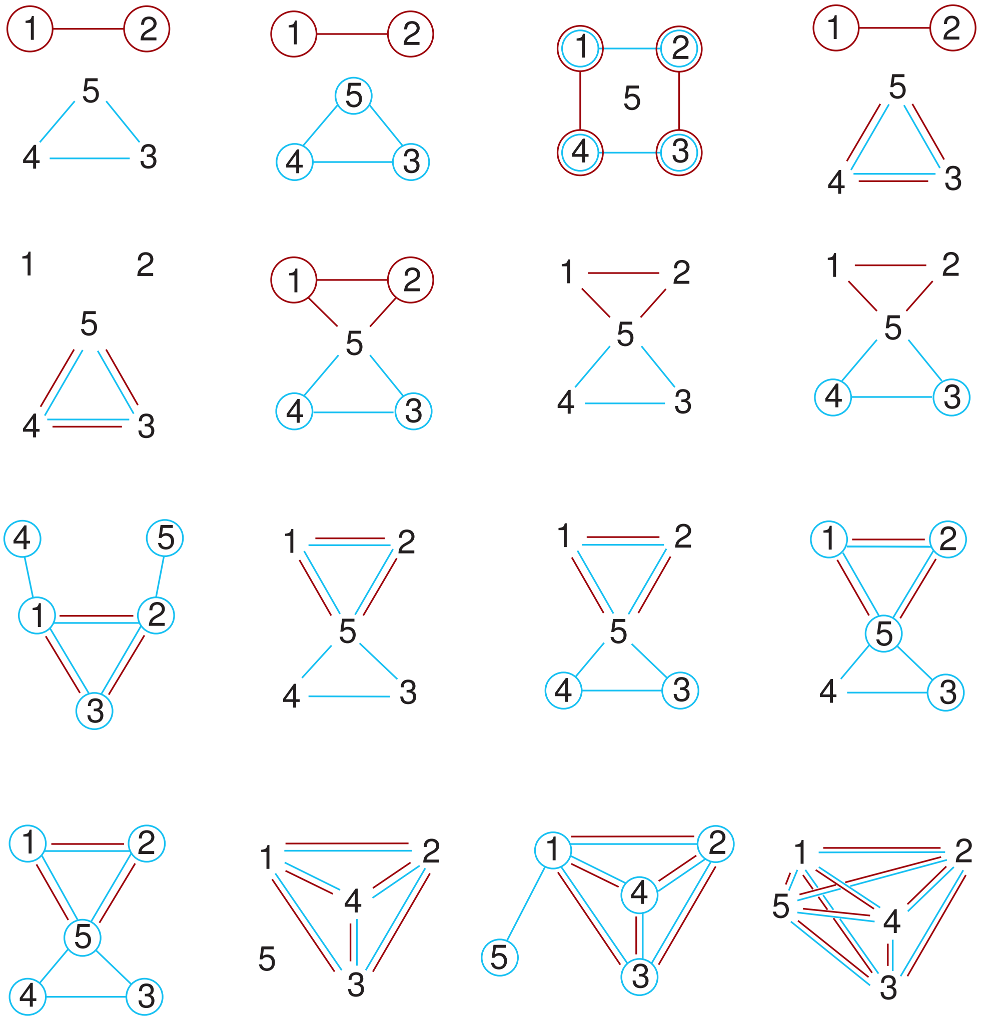

Figure 11. The sixteen 5-body diagrams.

none of these edges is a zw-edge. There is only one zw-edge in the diagram, which contradicts Rule 2h.

The other possibility is to have edges emanating from 1. Edges 13 and 14 are prohibited by Rule 2e. A zw-edge is prohibited by Rule 2b and $C = 4$. Thus, we should have a 15 stroke, e.g., a z-edge. Then Rule 1a forces a w-circle around vertex 1 and consequently, by Estimate 2, also w-circles around 2, 3 and 5. A w-stroke should emanate from vertex 3 and may only go to vertex 5, again by Rule 2e. We have a quadrilateral 1235. By Rule 2g it has a zw-edge between 3 and 5. By Rule 2a another w-stroke should emanate from this zw-edge. Only one does not contradict the triangle rule 2e, a w_{45}-stroke. But this makes a quadrilateral 2345 that contradicts Rule 2g.

Suppose finally there is a zw-edge and two z-edges. The junction vertex is w-circled with the same contradiction as in the case $C = 3$, first paragraph.

7.5. *Five bodies.* $C = 5$. The case of a zw-edge and three z-edges is excluded exactly as the case of a w-edge and three z-edges. See the first paragraph of 7.4.

Consider a zw-edge, two z-edges and a w-edge. They connect the five vertices. By Rule 2e, no edge from the end of the w-edge can exist, as it would close an impossible triangle. Rule 1a gives a w-circle at the end of this w-edge. Due to the center of mass, another vertex is w-circled. But all the other bodies are w-close so they are all w-circled. Ends of z-edges need further w-edges, which are impossible by Rule 2e. So we get the following conclusion.

PROPOSITION 7. *In a* 5*-body diagram with* $C = 5$*, any bicolored vertex with five strokes has two* zw*-edges and another stroke.*

Suppose vertex 1 is such a bicolored vertex. By Rule 2b two consecutive zw-edges need to be completed into a triangle of zw-edges. The triangle of zw-edges is formed by 123 and a w-edge connects 14 (see the first two diagrams in Figure 10). Notice that by Estimate 1, vertices 1, 2 and 3 all either have w-circles or all have no w-circles.

7.5.1. Suppose there is no edge from vertex 4, which is then w-circled.

7.5.1.1. Suppose there is no other edge at all. By center of mass the w_{14}-edge is maximal and 1, 2 and 3 are w-circled. This is the first diagram in Figure 10. By Rule 1a, vertex 5 is not circled. The w-clustering scheme is $w : 123\ldots5\ldots4$. We claim that the z-clustering scheme should be $z : 2\ldots5.\underline{14}\ldots3$. To prove this claim, first observe that $z_{14} \approx \epsilon^2$ by Estimate 1. We see that vertex 5 cannot be too close from the small cluster z_1, z_4. If we had $z_{54} \approx \epsilon^2$, there should be an edge between 4 and 5 by Estimate 1. But there is no edge there. So we can apply Proposition 1 to bodies 4 and 1. This gives the small cluster with the origin, vertex 1 and vertex 4 in the z-scheme.

As there is no edge between k and 5, $k = 1, \ldots, 4$, and $w_{k5} \approx \epsilon^{-2}$, we have $r_{k5}^2 = z_{k5} w_{k5} \approx z_{k5}\epsilon^{-2} \succ 1$. According to Proposition 2, body 5 clusters with the origin in z coordinate. We deduce the clustering scheme as announced.

More precisely, $z_5 \sim m_1 z_{15}^{-1/2} w_{15}^{-3/2} + m_4 z_{45}^{-1/2} w_{45}^{-3/2}$, as the omitted terms have larger z_{k5}. This is $z_5 \sim m_1 z_{15} r_{15}^{-3} + m_4 z_{45} r_{45}^{-3}$, but the right-hand side is smaller than the left-hand side, according to the clustering scheme and $r_{k5} \succ 1$. This is a contradiction.

7.5.1.2. Suppose there are other edges, but not from vertex 4. As $C = 5$, such an edge does not reach vertex 1. So it should join vertex 5 to vertices 2 or 3. Two such edges would fully edge 1235 by Rule 2f, giving $C > 5$. So we are left with diagrams similar to the first in Figure 10, but with one more edge, joining vertex 5 to 2, or to 3, which is similar. If there is the z_{25}-edge, the circling method puts double circles everywhere, contradicting Rule 1a. If there is the w_{25}-edge, this is the ninth diagram in Figure 11. The same diagram without w-circles around vertices 1, 2 and 3 is impossible. Actually, by the center of mass, this would force the clustering scheme $w:4\dots123\dots5$. We will exclude this clustering scheme while further discussing this diagram.

7.5.2. There are edges from vertex 4. By Rule 2e, they should go to vertex 5, so there is just one edge from 4. Suppose there is another edge from 5. As $C=5$, it cannot go to 1, so it closes a quadrilateral and there is no further edge. By Rule 2g, the edge 45 is a zw-edge. Rule 2a applied to this edge and Rule 2g applied to the quadrilateral contradict each other.

Thus we have a triangular kite with a long string (long of two edges), without further edge. If the edge between 4 and 5 is a zw-edge, Rule 2a is violated. If it is a z-edge, 5 is w-circled without w-edges. Let us consider the only case left, a one color long string, the second diagram in Figure 10.

Let us show that all the vertices are w-circled, as shown in the diagram. By Rule 1a, vertex 5 is w-circled. By Rule 1e, at least one of w_{14}, w_{45} is maximal. Both being maximal contradicts the skewsymmetry Rule 2c. Only one is w-maximal. The clustering scheme is either $w:123.4\dots5$ or $w:123\dots4.5$. In both cases there are two main clusters and the center of mass puts w-circles everywhere.

7.5.2.1. We study the skew-clustering of first type. Here $w_{14}\prec w_{45}$, $z_{45}\prec z_{14}$ and $r_{14}\to 0$. The clustering scheme is $z:2\dots3\dots1.45$, $w:123.4\dots5$. By Proposition 1 applied to bodies 5 and 4, the origin forms with these bodies a small cluster: $z:2\dots3\dots1.\underline{45}$. We write $z_4=m_1r_{14}^{-3}z_{14}+\cdots+m_5r_{45}^{-3}z_{54}$. We claim that the first term in the right-hand side dominates all the other terms, which is a contradiction. As $z_4\preceq z_{45}\prec z_{14}$ and $r_{14}\to 0$, it dominates the left-hand side and the term $r_{45}^{-3}z_{45}$. Comparing the terms written in the form $z_{k4}^{-1/2}w_{k4}^{-3/2}$, we see it also dominates the remaining terms.

7.5.2.2. We study the skew-clustering of second type. Here $r_{45}\to 0$ and $w_{45}\prec w_{14}$. Let us try to emphasize the main characteristics of this case in a general remark.

Further remark on Rule 2*c and skew clustering.* In the situation of Rule 2c, let us say w_{14}-edge, some edge from 4 to 5 with $w_{45}\prec w_{14}$, no edge from 1 to 5, we have not only $z_{14}\prec z_{45}$ as stated by Rule 2c, but also $Z_{14}\prec Z_{45}$ and $Z_{15}\prec Z_{45}$. These estimates are obvious if there is a z_{45}-stroke, but still valid if there is a w_{45}-edge, being then simple consequences of the hypotheses $W_{14}\approx W_{45}$ and $w_{45}\prec w_{14}$.

In this situation, if we subtract equations $z_4=m_1Z_{14}+m_5Z_{54}+\cdots$, $z_5=m_1Z_{15}+m_4Z_{45}+\cdots$, we find $z_{45}=(m_4+m_5)r_{45}^{-3}z_{45}+m_1Z_{15}-m_1Z_{14}+\cdots$.

Here $m_4+m_5\neq 0$, and by the nonmaximality of the corresponding edge, $r_{45}\to 0$. The first term in the right-hand side dominates all the displayed terms. One should find elsewhere a term that cancels this dominant term.

We claim that in the case hypothesis such term does not exist. The clustering scheme is $z:2\ldots3\ldots14.5$, $w:123\ldots4.5$. We have $z_5=m_1z_{15}^{-1/2}w_{15}^{-3/2}+\cdots+m_4z_{45}^{-1/2}w_{45}^{-3/2}$. The omitted terms are smaller than the first: same w but larger z. We have $z_4=m_1z_{14}^{-1/2}w_{14}^{-3/2}+\cdots+m_5z_{54}^{-1/2}w_{54}^{-3/2}$. The first term is larger than the omitted terms: same w, smaller z. Subtracting, no term balances the dominating term $(m_4+m_5)Z_{45}$. This is a contradiction.

7.6. *Five bodies.* $C=6$. There are six strokes and at most four edges, so there should be at least two zw-edges from the considered vertex, which will be conventionally number 5. There is the triangle 125 of zw-edges.

Suppose there are also the w_{35} and the w_{45} edges. There are four possibilities in Figure 11, all being "butterfly diagrams" made of two attached triangles. If there is nothing between 3 and 4, this is the last diagram in Figure 10. There are w-circles at 3 and 4 by Rule 1a, at 5 by skew clustering Rule 2c, then everywhere by Rule 1b. The two skew clustering options are similar. We choose $z:1\ldots35.4\ldots2$, $w:3\ldots152.4$. Application of Proposition 1 to bodies 3 and 5 gives that z_3, z_5 and the origin form a cluster of size z_{35}. The underlined clustering scheme is $z:1\ldots\underline{35}.4\ldots2$.

We estimate $z_4=\cdots+m_3z_{34}^{-1/2}w_{34}^{-3/2}+m_5z_{54}^{-1/2}w_{54}^{-3/2}$. The last term dominates the omitted two (same w but smaller z) and the displayed one (same z but smaller w). However, by skew clustering, $r_{45}\to 0$. Thus, $z_{54}\prec z_4$, which contradicts clustering of 5 near the origin.

If there are the w_{35} and the z_{45} edges, there is a w-circle around 3, and then around the four other bodies, which form a cluster in w-coordinate. But a w-circle around 4 is isolated; this is impossible.

If we are not in the above cases, there are Y-shape contacts with three zw-edges. By Rule 2b, this is the fourteenth diagram in Figure 11, which has no circle by Rule 1e together with Estimate 1, which tells that zw-edges are not maximal edges.

7.7. *Five bodies.* $C=7$. There are three zw-edges and another stroke. By Rule 2b, the only possibility is the fifteenth diagram in Figure 11, a "big kite." The circling method applies.

7.8. *Five bodies.* $C=8$. Rule 2b implies the fully zw-edged diagram, the last in Figure 11. There is no maximal edge, thus no circle, by Rule 1e.

The conclusion of this section is that *any singular sequence should converge to one of the sixteen diagrams in Figure 11.* Diagrams in Figure 11 are numbered left to right within each row and rows ordered top to bottom.

8. The sixteen remaining five-body diagrams

In the previous process of eliminating 5-body diagrams, we supposed that the mass of any cluster is nonzero and that $m_k^3+m_l^3\neq 0$ for any k,l. We could not eliminate the diagrams in Figure 11. Some singular sequence could still exist and approach any of these diagrams. Here we restrict to *real positive masses*. Still this is not enough: the first thirteen diagrams will be excluded *except if the masses satisfy a polynomial relation*. We number 8.1 to 8.16 the discussions of the constraints on the masses corresponding to each of the sixteen diagrams from Figure 11, ordered horizontally from top left to bottom right.

8.1. The clustering scheme is $z:1\dots345\dots2$, $w:3\dots12\dots5\dots4$ according to Rule 2e, Estimate 2 and Proposition 4. The w-triangle without circle indicates that the expressions for w_3, w_4 and w_5 in (9) have only two dominant terms, giving

$$\frac{W_{34}}{m_5}\sim\frac{W_{45}}{m_3}\sim\frac{W_{53}}{m_4}. \tag{21}$$

The relation

$$\frac{Z_{34}}{m_5}\sim\frac{Z_{45}}{m_3}\sim\frac{Z_{53}}{m_4} \tag{22}$$

follows from relations such as $z_3=m_4Z_{43}+m_5Z_{53}+\cdots$, where the omitted terms are among the "crossed terms" Z_{kl}, $k=1,2$, $l=3,4,5$. But $Z_{kl}=z_{kl}^{-1/2}w_{kl}^{-3/2}$, where both z_{kl} and w_{kl} are separations of maximal order in this diagram. So the crossed terms may be neglected in front of any other term. We must still estimate the left-hand side term z_3. It belongs to the cluster z_3, z_4, z_5. The center of mass of this cluster is of the same order as $m_3z_3+m_4z_4+m_5z_5$, which is $\prec Z_{43}$ as sum of crossed terms. On the other hand, this cluster has size of order $z_{45}=r_{45}^3Z_{45}\prec Z_{45}$, the corresponding edge being nonmaximal. Combining, $z_3\prec Z_{45}$ may be neglected also. We get one of the relation (22), the others being similar. Together with (21) this gives a constraint as in 5.3. This constraint is one of the following:

$$\frac{1}{\sqrt{m_3}}=\frac{1}{\sqrt{m_4}}+\frac{1}{\sqrt{m_5}},\quad\frac{1}{\sqrt{m_4}}=\frac{1}{\sqrt{m_5}}+\frac{1}{\sqrt{m_3}},\quad\frac{1}{\sqrt{m_5}}=\frac{1}{\sqrt{m_3}}+\frac{1}{\sqrt{m_4}}. \tag{23}$$

8.2. This is the second five-body diagram. There is the w-color triangle 345 and the z-color segment 12. Every vertex has a circle of the same color as the edges from it. The computations are similar to those in 5.1. We first compute $m_1q_1\wedge q_1+m_2q_2\wedge q_2$. In the homogeneous expression of the leading term we can substitute a finite value for w_3, w_4 and w_5, and set similarly

$z_1 = -m_2$, $z_2 = m_1$. Thus

$$\frac{m_1m_3}{\sqrt{-m_2w_3}} \pm \frac{m_2m_3}{\sqrt{m_1w_3}} \pm \frac{m_1m_4}{\sqrt{-m_2w_4}} \pm \frac{m_2m_4}{\sqrt{m_1w_4}} \pm \frac{m_1m_5}{\sqrt{-m_2w_5}} \pm \frac{m_2m_5}{\sqrt{m_1w_5}} = 0.$$

We set $m_1 = \mu_1^2$ and $m_2 = \mu_2^2$ and multiply the previous equation by $\mu_1\mu_2$:

$$\frac{\mu_1^3m_3}{\sqrt{-w_3}} \pm \frac{\mu_2^3m_3}{\sqrt{w_3}} \pm \frac{\mu_1^3m_4}{\sqrt{-w_4}} \pm \frac{\mu_2^3m_4}{\sqrt{w_4}} \pm \frac{\mu_1^3m_5}{\sqrt{-w_5}} \pm \frac{\mu_2^3m_5}{\sqrt{w_5}} = 0.$$

Setting

$$x = i\frac{\mu_2^3}{\mu_1^3}$$

and dividing by $i\mu_1^3$ gives

$$\pm\frac{m_3}{\sqrt{w_3}}(1 \pm x) \pm \frac{m_4}{\sqrt{w_4}}(1 \pm x) \pm \frac{m_5}{\sqrt{w_5}}(1 \pm x) = 0.$$

The discussion of cases is mainly about the ± sign in the factors $1 \pm x$. There are two cases: the three signs are the same, or one is different.

8.2.1. *Same three signs.* The condition does not involve m_1 and m_2:

$$\frac{m_3}{\sqrt{w_3}} \pm \frac{m_4}{\sqrt{w_4}} \pm \frac{m_5}{\sqrt{w_5}} = 0. \tag{24}$$

Eliminating the $\sqrt{w_k}$'s gives the condition $A = 0$, where

$$A = \frac{m_3^4}{w_3^2} + \frac{m_4^4}{w_4^2} + \frac{m_5^4}{w_5^2} - \frac{2m_4^2m_5^2}{w_4w_5} - \frac{2m_5^2m_3^2}{w_5w_3} - \frac{2m_3^2m_4^2}{w_3w_4}. \tag{25}$$

A solution to our problem consists of a three-body central configuration with masses m_3, m_4, m_5, constrained by $A = 0$. Here the two masses $m_1 = \mu_1^2$ and $m_2 = \mu_2^2$ are not constrained. Finally, there is the center of mass constraint $m_3w_3 + m_4w_4 + m_5w_5 = 0$.

The resulting condition on the masses is obtained by straightforward elimination (resultant) with the equations given in the appendix on three-body central configurations. There are eight factors, corresponding respectively to the three usual Euler cases, both factors of the fourth Euler case and the three remaining complex Lagrange case.

Elimination gives a homogeneous symmetric polynomial in m_3, m_4, m_5 with integer coefficients. In its factorization we erase the powers of the linear factor $m_3 + m_4 + m_5$. There remain nine factors. Each of the 3-body factors generates one factor, except the quadratic one that gives two factors, $(m_4 - m_5)^2 + (m_5 - m_3)^2 + (m_3 - m_4)^2$ and an irreducible factor of degree 12 having only + signs. The other seven factors are irreducible and have respective degrees 36, 36, 36, 22, 28, 28 and 28. So we get nine irreducible polynomial conditions on the masses, at least one of them being nonzero when the masses are positive. Here are these polynomial factors:

$$\begin{aligned}
L_1 &= 6522m_3^{15}m_4^{12}m_5^9 + 3528m_3^{17}m_5^2m_4^{17} + 563 \text{ other terms, symmetric in } m_4, m_5,\\
L_2 &= 2414m_3^{15}m_4^{12}m_5^9 + 3528m_3^{17}m_5^2m_4^{17} + 563 \text{ other terms, symmetric in } m_3, m_5,\\
L_3 &= 495m_4^{24}m_5^{12} - 452m_3^{17}m_5^2m_4^{17} + 563 \text{ other terms, symmetric in } m_3, m_4,\\
L_4 &= -110m_3^7m_4^7m_5^8 - 30m_3^7m_4^{10}m_5^5 + 262 \text{ other terms, symmetric in } m_3, m_4, m_5,\\
L_5 &= m_3^2 + m_4^2 + m_5^2 - m_4m_5 - m_5m_3 - m_3m_4,\\
L_5' &= 2m_5^2m_4^{10} + m_3^{12} + 89 \text{ other terms, symmetric in } m_3, m_4, m_5,\\
L_6 &= 866m_4^{11}m_5^{17} - 1456m_3^6m_4^{15}m_5^7 + 431 \text{ other terms, symmetric in } m_4, m_5,\\
L_7 &= -208m_4^{11}m_5^{17} - 25528m_3^6m_4^{15}m_5^7 + 431 \text{ other terms, symmetric in } m_3, m_5,\\
L_8 &= -184m_4^{11}m_5^{17} + 9440m_3^6m_4^{15}m_5^7 + 431 \text{ other terms, symmetric in } m_3, m_4.
\end{aligned}$$

8.2.2. *One different sign.* We choose, e.g.,

$$\Big(\pm\frac{m_3}{\sqrt{w_3}} \pm \frac{m_4}{\sqrt{w_4}}\Big)(1+x) \pm \frac{m_5}{\sqrt{w_5}}(1-x) = 0. \tag{26}$$

Notice that $x = 0$ or, as well, $x = \infty$ gives again the case 8.2.1. Eliminating the $\sqrt{w_k}$'s gives the condition

$$A(1+x^4) + 4B(x+x^3) + 2Cx^2 = 0, \tag{27}$$

where A is defined by (25) and

$$\begin{aligned}
B &= \Big(\frac{m_3^2}{w_3} - \frac{m_4^2}{w_4} + \frac{m_5^2}{w_5}\Big)\Big(\frac{m_3^2}{w_3} - \frac{m_4^2}{w_4} - \frac{m_5^2}{w_5}\Big),\\
C &= \frac{3m_3^4}{w_3^2} + \frac{3m_4^4}{w_4^2} + \frac{3m_5^4}{w_5^2} + \frac{2m_4^2m_5^2}{w_4w_5} + \frac{2m_5^2m_3^2}{w_5w_3} - \frac{6m_3^2m_4^2}{w_3w_4}.
\end{aligned}$$

We express (27) in the new variable $y = (x + x^{-1})/2 = i(\mu_2^3\mu_1^{-3} - \mu_1^3\mu_2^{-3})/2$

$$Ay^2 + 2By + (C-A)/2 = 0. \tag{28}$$

Again, the resulting condition on the masses is obtained by straightforward elimination with the equation computed in the appendix. We get eight conditions, corresponding to the eight types of 3-body central configurations. They are polynomials in y, m_3, m_4, m_5 with integer coefficients. We denote them as polynomials $P_k(y)$ in y with coefficients depending on m_3, m_4, m_5. The respective degrees in y are 10, 10, 10 for the three usual Euler cases, 6 and 4 for the two factors of the fourth Euler case, 8, 8, 8 for the remaining complex

Lagrange cases:

$$\begin{aligned}
P_1(y) &= (m_3+m_4+m_5)^4 L_1 y^{10} + \cdots + K_1 = 0,\\
P_2(y) &= (m_3+m_4+m_5)^4 L_2 y^{10} + \cdots + K_2 = 0,\\
P_3(y) &= (m_3+m_4+m_5)^4 L_3 y^{10} + \cdots + K_3 = 0,\\
P_4(y) &= (m_3+m_4+m_5)^2 L_4 y^{6} + \cdots + K_4 = 0,\\
P_5(y) &= (m_3+m_4+m_5)^2 L_5 L_5' y^{4} + \cdots + K_5 = 0,\\
P_6(y) &= (m_3+m_4+m_5)^4 L_6 y^{8} + \cdots + K_6 = 0,\\
P_7(y) &= (m_3+m_4+m_5)^4 L_7 y^{8} + \cdots + K_7 = 0,\\
P_8(y) &= (m_3+m_4+m_5)^4 L_8 y^{8} + \cdots + K_8 = 0.
\end{aligned}$$

The L_i's are expressed in Section 8.2.1. The K_i's will be presented in Section 8.2.2.1. The coefficients of P_3, P_4, P_5 and P_8 are symmetric in (m_3, m_4).

The imaginary number y should be a root of a P_k. As all the coefficients of P_k are real, this is a *codimension* 2 condition on y, m_3, m_4, m_5. We should have $P_k(y) + P_k(-y) = 0$ and $P_k(y) - P_k(-y) = 0$.

8.2.2.1. *Special codimension* 2 *case.* The odd part of P_k has the factor y. A special case is $y = 0$ and $P_k(0) = 0$ for some k between 1 and 8. For real positive masses, $y = 0$ means $m_1 = m_2$. The conditions $P_k(0) = 0$ are nontrivial polynomial conditions on m_3, m_4, m_5. The $P_k(0)$'s, $k = 1, \ldots, 8$, are respectively

$$\begin{aligned}
K_1 &= 33066 m_4^8 m_5^{12} m_3^{20} + 49220 m_4^{21} m_5^3 m_3^{16} + 859 \text{ other terms},\\
K_2 &= 9806 m_4^8 m_5^{12} m_3^{20} + 946 m_3^{30} m_4^{10} + 859 \text{ other terms},\\
K_3 &= 186 m_3^{30} m_4^{10} + 1206 m_4^6 m_5^7 m_3^{27} + 859 \text{ other terms},\\
K_4 &= 2 m_3^{21} m_4^3 - 10 m_3^{21} m_5^3 + 319 \text{ other terms},\\
K_5 &= 16 m_3^7 m_4^9 + 2 m_5^{15} m_3 + 141 \text{ other terms},\\
K_6 &= 94 m_4^{25} m_3^7 - 36 m_3^{25} m_5^7 + 557 \text{ other terms},\\
K_7 &= -68 m_4^{25} m_3^7 - 774 m_3^{25} m_5^7 + 557 \text{ other terms},\\
K_8 &= 1354 m_4^{25} m_3^7 + 126 m_3^{25} m_5^7 + 555 \text{ other terms}.
\end{aligned}$$

8.2.2.2. *General codimension* 2 *case.* The other possibility is $P_k(y) + P_k(-y) = 0$ and $\big(P_k(y) - P_k(-y)\big)/y = 0$. We have two polynomial conditions in $y^2 = (-m_2^3 m_1^{-3} - m_1^3 m_2^{-3} + 2)/4$. We check they have no common factor. The condition on the five masses is codimension 2.

8.3. Same constraint as 5.2:

$$m_1 m_3 = m_2 m_4. \tag{29}$$

8.4 *and* 8.5. The cyclic relations (21) and (22) are evident on the diagram, giving again the constraint (23).

8.6–8.8 *and* 8.10–8.13. *Seven butterflies.* The clustering scheme is z : $1\dots345\dots2$, $w : 3\dots125\dots4$ for the seven butterfly diagrams. But the relative position of the center of mass is not always the same. In all the cases, we write

$$\begin{aligned} z_5 =\, & A_z + B_z, \quad \text{with } A_z = m_1Z_{15} + m_2Z_{25}, \qquad B_z = m_3Z_{35} + m_4Z_{45},\\ w_5 =\, & A_w + B_w, \quad \text{with } A_w = m_1W_{15} + m_2W_{25}, \quad B_w = m_3W_{35} + m_4W_{45}. \end{aligned}$$

Then

$$m_1z_1 + m_2z_2 = -m_5A_z + \cdots\,, \tag{30}$$

where the omitted terms are of lower order than the two terms in A_z. We have three similar relations with A_w, B_z, B_w. Finally we expand as in 5.1:

$$0 = m_1f_1 \wedge q_1 + m_2f_2 \wedge q_2 = m_5q_5 \wedge A + S, \tag{31}$$

with

$$A = \begin{pmatrix} A_z \\ A_w \end{pmatrix}, \qquad S = \sum m_km_lr_{kl}^{-3}q_k \wedge q_l, \quad k = 1, 2, \quad l = 3, 4.$$

8.6. Here we have $A_w \preceq \epsilon^2$, $B_z \preceq \epsilon^2$ by the usual estimates on maximal edges. Then $m_1w_1 + m_2w_2 \preceq \epsilon^2$ by (30), $w_5 \preceq \epsilon^2$ by the clustering scheme. Finally $B_w = w_5 - A_w \preceq \epsilon^2$ and $A_z = z_5 - B_z \preceq \epsilon^2$. In (31), $A_zw_5 \preceq \epsilon^4$, $A_wz_5 \preceq \epsilon^4$, and thus $S \preceq \epsilon^4$. Each term of S is of order ϵ^2. This conclusion is the same as in 5.1. We deduce in the same way the relation among the masses

$$m_1 = m_2 \quad \text{and} \quad m_3 = m_4. \tag{32}$$

8.7, 8.8, 8.10–8.13. In the other butterfly diagrams we will look for other types of relations. In all these diagrams we have an isolated triangle of z-edges without z-circles, giving

$$\frac{Z_{12}}{m_5} \sim \frac{Z_{25}}{m_1} \sim \frac{Z_{51}}{m_2}. \tag{33}$$

8.7. The lower w-wing is similar to the upper z-wing, so we have also relations (21). Subtracting from the w_1-equation the w_2-equation, neglecting W_{kl} with $k = 1, 2$ and $l = 3, 4$, which are as small as possible for this diagram, and finally, as the corresponding edge is not maximal, neglecting $w_{12} = r_{12}^3W_{12}$ in front of W_{12}, we get

$$(m_1 + m_2)W_{12} \sim m_5W_{51} + m_5W_{25}. \tag{34}$$

Note that the three terms are of the same order according to Rule 2e.

8.7.1. A particular case compatible with this equation is

$$\frac{W_{12}}{m_5} \sim \frac{W_{25}}{m_1} \sim \frac{W_{51}}{m_2}. \tag{35}$$

This together with relation (33) gives the same relation as in 5.3:

$$m_1^{-1/2} \pm m_2^{-1/2} \pm m_5^{-1/2} = 0. \tag{36}$$

8.7.2. In the general case, one of the $\sim$ relations in (35) is not satisfied. Then none of the $\sim$ relations in (35) is satisfied: (34) implies

$$-\frac{m_1 m_2}{m_1+m_2}\Big(\frac{W_{51}}{m_2} - \frac{W_{25}}{m_1}\Big) \sim m_1\Big(\frac{W_{25}}{m_1} - \frac{W_{12}}{m_5}\Big) \sim m_2\Big(\frac{W_{12}}{m_5} - \frac{W_{51}}{m_2}\Big).$$

These three differences are of the same order as W_{12}. We define

$$h = -\frac{m_1 m_2}{m_1+m_2}\Big(\frac{W_{51}}{m_2} - \frac{W_{25}}{m_1}\Big)$$

and get $A_w = (m_1+m_2)h$. We know that $m_1 w_1 + m_2 w_2 = -m_5 A_w + \cdots$, where the two terms forming A_w dominate the omitted terms. As we just assumed that these two terms do not cancel each other, $-m_5 A_w \sim m_1 w_1 + m_2 w_2$, or

$$-m_5 h \sim \frac{m_1 w_1 + m_2 w_2}{m_1 + m_2}. \tag{37}$$

The small cluster w_1, w_2, w_5 has size $w_{12} = r_{12}^3 W_{12}$. As the corresponding edge is not maximal, $r_{12} \to 0$ and $w_{12} \prec W_{12} \approx h$. The center of mass in the right-hand side of (37) is close to w_5, so $w_5 \sim -m_5 h$. As $w_5 = A_w + B_w$, we get $B_w \sim -(m_1+m_2+m_5)h$.

8.7.3. We continue with the corresponding hypothesis and deductions concerning the other wing of the butterfly. We have

$$(m_3+m_4)Z_{34} \sim m_5 Z_{53} + m_5 Z_{45} \tag{38}$$

and $B_z = (m_3+m_4)g$, $g \approx Z_{34}$, $z_5 \sim -m_5 g$, $A_z \sim -(m_3+m_4+m_5)g$.

In (31) there is the term

$$q_5 \wedge A = z_5 A_w - w_5 A_z \sim -m_5 g(m_1+m_2)h - m_5 h(m_3+m_4+m_5)g = -m_5 M g h,$$

with $M = m_1 + \cdots + m_5 \neq 0$. The second term in (31) should be estimated and compared to the first term. We will do that after excluding some cases.

As we saw, the small cluster w_1, w_2, w_5 is located around $-m_5 h$. We have *two cases*: $h \prec w_{34} \approx w_{45} \approx w_{53}$ or $h \approx w_{34}$. Similarly for the other wing, the small cluster z_3, z_4, z_5 is located around $-m_5 g$. We have *again two cases*: $g \prec z_{12} \approx z_{23} \approx z_{31}$ or $g \approx z_{12}$.

In the case $h \prec w_{34}$, w_5 occupies in the limit the center of mass of the triple w_3, w_4, w_5 and we have, together with (21) and (38), the same system as in 5.4. The relation among the masses is

$$m_5^2(m_3+m_4)^4 = m_3^3 m_4^3. \tag{39}$$

Similarly, in the case $g \prec z_{12}$, the relation among the masses is

$$m_5^2(m_1+m_2)^4 = m_1^3 m_2^3. \tag{40}$$

We got a relation among the masses in all the cases except if $h \approx w_{34} \approx W_{12}$ and $g \approx z_{12} \approx Z_{34}$. In this last case, we come back to our discussion of (31). The term in S is of the same or lower order than $(z_{12}w_{34})^{-1/2}$. The product gh is of order $w_{34}Z_{34} \approx (z_{34}w_{34})^{-1/2}$. As $z_{34} \prec z_{12}$, the product gh dominates the other terms in (31). This is a contradiction.

8.8. Here the cluster z_3, z_4, z_5 has size ϵ^2, the lower wing having maximal w-edges. We know that $m_3z_3+m_4z_4 \preceq Z_{35}$, and $Z_{35} \approx \epsilon^2$ by the maximality of the z_{35}-edge. Then the cluster and in particular z_5 are at the center of mass of z_1, z_2. Relation (34) is also valid. We get relation (40) by the same arguments giving this relation in the discussion of diagram 8.7.

8.10 *and* 8.11. Here $m_1w_1 + m_2w_2 = m_1m_5W_{51} + m_2m_5W_{52} + \cdots$. As 1 and 2 are not circled, the left-hand side is small: w_1, w_2, and consequently their center of mass, are close to zero. This gives the last $\sim$ relation (35), the first being also valid, as immediately seen in the diagram. Together with (33), we get (36).

8.11, 8.12, 8.13. The edge 12 is a double edge, so it is not maximal. Relation (34) still holds as well as the other arguments of 8.8. We get relation (40).

8.9.1. Here we consider the ninth diagram in Figure 11 and make a first hypothesis. We assume that w_4 clusters with the cluster w_1, w_2, w_3. It cannot be as close as if the pair 14 had a zw-edge. The w-clustering scheme is $w : 5 \ldots 4.123$. By the estimates, $z_{25} \prec z_{14} \prec z_{12}$ and the z-clustering scheme is $z : 3 \ldots 25 \ldots 1.4$. Proposition 1 applies to 2 and 5, placing the origin in the small cluster z_2, z_5. The underlined clustering scheme is $z : 3 \ldots \underline{25} \ldots 1.4$. We have $z_4 \sim m_1r_{14}^{-3}z_{14}$ and $w_4 \sim m_1r_{14}^{-3}w_{14}$, so $m_1^2r_{14}^{-4} \sim z_4w_4 \approx \epsilon^{-1}$, $r_{14} \approx \epsilon^{1/4}$, $z_{14} \approx \epsilon^{7/4}$, $w_{14} \approx \epsilon^{-5/4}$, $r_{24} \approx r_{34} \approx \epsilon^{-1/8}$, $r_{53} \approx r_{51} \approx r_{54} \approx \epsilon^{-1/2}$. Except r_{14}, only the distances $r_{12} \approx r_{23} \approx r_{31} \approx \epsilon$ tend to zero. We have

$$0 = \frac{Z_{23}}{m_1}(w_{23}+w_{31}+w_{12}) = \frac{Z_{23}w_{23}}{m_1} + \frac{Z_{31}w_{31}}{m_2} + \frac{Z_{12}w_{12}}{m_3} + A + B, \tag{41}$$

where

$$A = w_{31}\Big(\frac{Z_{23}}{m_1} - \frac{Z_{31}}{m_2}\Big) = \frac{w_{31}}{m_1m_2}(z_3 - m_4Z_{43} - m_5Z_{53}),$$

$$B = w_{12}\Big(\frac{Z_{23}}{m_1} - \frac{Z_{12}}{m_3}\Big) = -\frac{w_{12}}{m_1m_3}(z_2 - m_4Z_{42} - m_5Z_{52}).$$

We have $Z_{43} \approx Z_{42} \approx \epsilon^{11/8}$, $Z_{25} \approx \epsilon^2$ (as $r_{25} \approx 1$ for a maximal edge), $Z_{53} \approx \epsilon^{5/2}$. Finally $A + B \sim (m_1m_2)^{-1}w_{31}z_3 \approx \epsilon^2$. Note that the first three terms in the right-hand side of (41) are terms of the potential divided by

$m_1m_2m_3$. Then $-A-B$ estimates this contribution to the potential, which is surprisingly small. There remains one and only one infinite contribution to the potential, $m_1m_4r_{14}^{-1}$. This is a contradiction, the potential is bounded.

8.9.2. The other possibility for the ninth diagram in Figure 11 is $w_{14} \approx w_{25} \approx \epsilon^{-2}$. We have two maximal w-edges on the diagram. We have $r_{14} \approx r_{25} \approx 1$. The z-clustering scheme is $z : 14 \dots 3 \dots 25$. The w-clustering scheme may be $w : 4.5 \dots 123$ or $w : 4 \dots 5 \dots 123$. We get $z_4 = m_1z_{14}r_{14}^{-3} + m_5z_{54}r_{45}^{-3} + \cdots$ and $z_5 = m_2z_{25}r_{25}^{-3} + m_4z_{45}r_{45}^{-3} + \cdots$. As $z_{45} \approx \epsilon$, both z_4 and z_5 cannot be as small as $z_{14}r_{14}^{-3} \approx z_{25}r_{25}^{-3} \approx \epsilon^2$. The second term should dominate the first in one equation and thus also in the other. (It is the same term.) This forces $r_{45}^3 \sim m_4 + m_5$ and $m_4z_4 \sim -m_5z_5$. We have $w_{45} \approx \epsilon^{-1}$. Only $w : 4.5 \dots 123$ is possible. The other clustering scheme is impossible. We also have $m_4z_1 \sim -m_5z_2$.

The identity $m_4z_4 + m_5z_5 \prec z_5$ also fixes $m_1z_1 + m_2z_2 + m_3z_3 \prec z_1$. Substituting $m_4z_1 \sim -m_5z_2$ we get $m_4m_2z_2 + m_4m_3z_3 \sim m_1m_5z_2$. Up to a factor, $z_2 = m_4m_3$, $z_3 = m_1m_5 - m_2m_4$, $z_1 = -m_3m_5$ in the limit.

On the other hand,

$$\frac{Z_{23}}{m_1} \sim \frac{Z_{31}}{m_2} \sim \frac{Z_{12}}{m_3}. \tag{42}$$

Multiplying by the relation $w_{23} + w_{31} + w_{12} = 0$ gives the cancellation of the three infinite contributions to the potential:

$$-\frac{1}{m_1r_{23}} - \frac{1}{m_2r_{31}} \sim \frac{1}{m_3r_{12}}.$$

We combine with this other consequence of (42),

$$\frac{m_1m_5 - m_2m_4 - m_3m_4}{m_1r_{23}^3} \sim \frac{-m_3m_5 - m_1m_5 + m_2m_4}{m_2r_{13}^3} \sim \frac{m_4 + m_5}{r_{12}^3}. \tag{43}$$

We eliminate the distances and find a polynomial relation among the masses, with integer coefficients. After factorizing some powers of the masses, we find the irreducible polynomial

$$132m_1^6m_3^7m_4^3m_2^5m_5^3 + 269m_1^6m_3^6m_4^3m_2^6m_5^3 + 372 \text{ other terms},$$

homogeneous of degree 18 in (m_1,m_2,m_3), homogeneous of degree 6 in (m_4,m_5), symmetric under simultaneous transposition of (m_1,m_2) and (m_4,m_5).

8.14–8.16. The relation between the masses are difficult to compute in these diagrams. But the estimates for the distances obtained in Section 6 apply. Proposition 3 applies: the product of two nonadjacent distances tends to zero.

9. Conclusions on the 5-body case

We first recall some simple tricks to estimate the distances between bodies when a singular sequence approaches one of the diagrams in Figure 11. We only use Estimates 1 and 2 of Section 3, based on the *new normalization* introduced just before.

A distance of order ϵ corresponds to a zw-edge. No edge or a simple edge, i.e., a z-edge or a w-edge, corresponds to a distance of higher order. A maximal simple edge corresponds to a distance of order 1, i.e., bounded and bounded away from zero. In the sixteen diagrams in Figure 11, a simple edge happens to be maximal if and only if at least one of its ends is circled (compare Rule 1e). There are nonmaximal simple edges in diagrams 1, 7, 8 and 10, and they correspond to distances r_{kl} such that $\epsilon \prec r_{kl} \prec 1$.

Estimates on the distances without edge require a case-by-case analysis in diagram 14 (see Section 6.1) and diagram 5. In other diagrams, the clustering scheme gives a simple and precise estimate. We will avoid further case-by-case analysis and use only the simple estimates. For example, Proposition 8 below looks like Proposition 3. We call a 4-*product* a quantity $p_{ij} = r_{ij}^2 r_{kl}^2 r_{lm}^2 r_{mk}^2$, where i, j, k, l, m are all the indices from 1 to 5.

Proposition 8. *In the limit corresponding to diagrams* 9 *to* 16 *in Figure* 11, *any* 4-*product is bounded. In diagrams* 14 *to* 16, *any* 4-*product tends to zero.*

Proof. In all these diagrams except number 10 the w-edges are maximal and correspond to distances ≈ 1. All the distances without edges are $\approx \epsilon^{-1/2}$ with the following exceptions. In diagram 9, $r_{45} \approx 1$. In diagram 10, all the distances are of lower order than in diagram 11, due to the nonmaximality of the w-edges. In diagram 14, the distances are of lower order than in diagram 15 and are estimated in Section 6.1.

We estimate the 4-products. In diagram 9, $p_{12} \approx 1$ and the others tend to zero. In diagrams 11, 12, 13, $p_{14} \approx p_{13} \approx p_{24} \approx p_{23} \approx 1$ and the others tend to zero. In diagram 10, the p_{ij}'s are of lower order. In diagram 15, all the p_{ij}'s tend to zero. In diagram 14, the p_{ij}'s are of lower order. ☐

Proposition 9. *If along a singular sequence a distance tends to zero, there are three distinct indices* k, l, m *such that the three distances* r_{kl}, r_{lm} *and* r_{mk} *tend to zero and such that, furthermore, the* 4-*product* $p_{ij} = r_{ij}^2 r_{kl}^2 r_{lm}^2 r_{mk}^2$ *tends to zero, where* i *and* j *are the other two indices in the set* $\{1, \dots, 5\}$.

Proof. We easily check that only diagrams 2, 3, 6 have no distance going to zero. By another inspection of the list of diagrams, we see that all the other diagrams possess a triangle of nonmaximal edges. We take k, l, m as the vertices of such a triangle. The corresponding distances tend to zero. The other distance r_{ij} in the 4-product corresponds to an edge, and is consequently bounded, in all the diagrams except maybe in diagrams 5, 14 and 15. But in these diagrams the triangle klm has zw-edges and the three distances $\approx \epsilon$, while $r_{ij} \preceq \epsilon^{-2}$ by Estimate 1. $\square$

Remark 6. Assume that a single relation between the masses of the form $Q(m_i, m_j, m_k) = 0$, where Q is a polynomial and i, j, k are distinct indices from 1 to 5, allows a singular sequence to approach one of the diagrams from number 1 to number 13 in Figure 11. By inspecting the list of conditions obtained in Section 8, we see that vertices i, j and k are always joined by an isolated triangle of strokes, either in the z-diagram or in the w-diagram. We have $r_{ij} \approx r_{jk} \approx r_{ki}$ by Rule 2e. If these distances ≈ 1, we are in the second diagram, Case 8.2.1. They tend to zero in the other diagrams, and the 4-product with this triangle of distances also tends to zero, by the argument we used to prove Proposition 9.

In the statement of Theorem 2, let us replace the words "positive normalized central configurations" by the words "normalized central configurations" (see Definition 2). Theorem 2 is a corollary of the stronger

THEOREM 6. *For any choice of masses* $(m_1, \dots, m_5) \in (\mathbb{R}_0^+)^5 \setminus \mathcal{A}$, *where* $\mathbb{R}_0^+$ *is the set of positive real numbers and* $\mathcal{A}$ *is a closed algebraic subset of codimension* 2, *there are finitely many normalized central configurations of the planar* 5-*body problem.*

Proof. Given all the distances, only finitely many normalized configurations are possible. Recall that on a continuum of normalized central configurations, a polynomial has only finitely many values or is dominating (see Lemma 1). Thus, on a continuum of normalized configurations, at least one of the r_{ij}^2's is dominating. Push it to zero. By Proposition 9, some 4-product $p_{ij} = r_{ij}^2 r_{kl}^2 r_{lm}^2 r_{km}^2$ also tends to zero, as also do r_{kl}, r_{lm} and r_{mk}. As any nonzero quantity going to zero on the continuum, p_{ij}, r_{kl}^2, r_{lm}^2 and r_{mk}^2 are dominating. We push p_{ij} to infinity, thus forming a singular sequence that approaches a diagram. According to Proposition 8, this is one of the first eight diagrams in Figure 11.

i) Suppose it is diagram 2. We number the vertices as in the figure. There is a polynomial condition on the masses, which defines a codimension 2 set in Case 8.2.2. We put this set, and all the similar sets obtained by renumbering the five bodies, in the exceptional set $\mathcal{A}$. Case 8.2.2 is now excluded.

In the other case 8.2.1, the condition of the masses is codimension one and involves only the three masses m_3, m_4, m_5. The maximality of the 4 edges corresponding to the 4-product p_{12} shows that $p_{12} \approx r_{12} \approx r_{34} \approx r_{45} \approx r_{35} \approx 1$ on a singular sequence approaching our diagram.

If p_{12} is dominating, we push it to infinity, thus forming a new singular sequence that approaches a diagram which cannot be again diagram 2 with the same numbering. By Proposition 8, this diagram is again among the first eight diagrams. A condition on the masses corresponds to the new limiting diagram. By Remark 6, this condition cannot be a single condition on the masses m_3, m_4, m_5, as p_{12} should then be bounded. It is an independent condition. We add the corresponding codimension 2 sets to the exceptional set $\mathcal{A}$.

If p_{12} is not dominating, we push r_{12} to infinity while p_{12} remains constant. We can do that only if r_{12} is dominating. If r_{12} is not dominating, we keep it constant and push another of the four distances to zero or to infinity. Indeed one of the four distances r_{12}, r_{45}, r_{35}, r_{34} is among the three distances r_{kl}, r_{lm}, r_{mk} involved in p_{ij}, which are dominating.

In any of these cases the limiting diagram cannot be number 2 with a similar numbering. By Proposition 9, as $p_{12} \approx 1$, it cannot be diagrams 14 to 16. (Interestingly, diagram 9 is also avoided, as in the only case where p_{12} remains bounded, r_{12} tends to zero.) By Remark 6, the corresponding condition cannot be a single condition on the masses m_3, m_4, m_5. It is an independent condition. We add the corresponding codimension 2 sets to the exceptional set $\mathcal{A}$. Case 8.2.1 is now forbidden, and a singular sequence can no longer approach diagram 2 as $p_{ij} \to \infty$.

ii) Suppose it approaches diagram 6. The condition $m_1 = m_2$, $m_3 = m_4$, or the same condition after renumbering the bodies, should be satisfied. Adding the corresponding codimension 2 sets to $\mathcal{A}$ forbids this possibility.

iii) Suppose the singular sequence approaches diagram 3 as $p_{ij} \to \infty$. We number the vertices as in the figure. We have the condition $m_1m_3 = m_2m_4$. The distance r_{13} goes infinity, so it is dominating. Push $r_{12}r_{34}$ to infinity, or if not dominating, keep it constant and push r_{13} to zero. By Proposition 3, we go to one of the first thirteen diagrams. None of the other polynomial conditions obtained in Section 8 have the factor $m_1m_3 - m_2m_4$. But we could find the same condition again. This happens if the sequence tends to the third diagram again, with bodies 1 and 3 on a diagonal of the square, body 2 and 4 on the other diagonal, and body 5 at the center. This is impossible in the case where r_{13} tends to zero, as no distance tends to zero in diagram 3. This is also impossible in the case where $r_{12}r_{34}$ goes to infinity, as this quantity is bounded on such a diagram. A second independent condition on the masses should be satisfied. We add the corresponding codimension 2 sets to $\mathcal{A}$ to forbid this case.

iv) Each of the remaining diagrams gives a single relation among three masses, let us say m_3, m_4, m_5. By Remark 6, p_{12} tends to zero. We push it to infinity and get another relation that, again by Remark 6, cannot be a single polynomial relation among m_3, m_4, m_5. The two independent relations define a codimension 2 closed algebraic subset, which we add to $\mathcal{A}$. This concludes the construction of $\mathcal{A}$. The last possibility for a singular sequence is now forbidden. There is no continuum of normalized central configurations if the masses do not belong to $\mathcal{A}$. ☐

Example. There are finitely many normalized central configurations with $m_1 = 1$, $m_2 = 2$, $m_3 = 3$, $m_4 = 4$, $m_5 = 5$.

Proof. We repeat the first paragraph of the previous proof, showing that one of the first eight diagrams should be approached. In each diagram, we check if our masses may satisfy the conditions. We compute the polynomials in 8.2.1 on these masses, permuted in all possible ways. They are nonzero. Case 8.2.2.1 requires two equal masses. Ours are not equal. We compute the even parts of the polynomials in 8.2.2.2. They are nonzero. Diagram 2 in Figure 11 cannot be approached. As there is no relation between the masses such as $m_3^2(m_1+m_2)^4 = m_1^3m_2^3$ or $m_1m_3 = m_2m_4$ or $m_1^{-1/2} \pm m_2^{-1/2} \pm m_3^{-1/2} = 0$, the other seven diagrams cannot be approached. ☐

Remark 7. If the matter is to check the finiteness for a given set of five rational masses, one can avoid the computation of the big polynomial conditions in Sections 8.2 and 8.9.2. They were obtained by eliminating variables. We can start the elimination process after substituting the masses.

Remark 8. A 5-tuple of masses with some equal masses easily falls in the exceptional set. This is consistent with Robert's counter-example and with the analogous complex counter-example we presented in Section 2. In these examples, the masses are respectively $(4,4,4,4,-1)$ and $(4,4,4,4,1)$.

Appendix on the complex three-body central configurations

Consider the central configurations of the three-body problem. In view of the application to diagram 8.2, (i) the bodies are numbered 3, 4, 5; (ii) we need the positions of the bodies, the origin being their center of mass; (iii) we only need these positions up to re-scaling; (iv) we need all the complex central configurations; (v) we need the projection of the configuration on one of the complex coordinates axis, namely w as defined in Section 2, *factorization of the distances.* We start with the equations

$$\begin{aligned} z_3 &= m_4Z_{43} + m_5Z_{53}, & w_3 &= m_4W_{43} + m_5W_{53},\\ z_4 &= m_3Z_{34} + m_5Z_{53}, & w_4 &= m_3W_{34} + m_5W_{53},\\ z_5 &= m_3Z_{35} + m_4Z_{45}, & w_5 &= m_3W_{35} + m_4W_{45}, \end{aligned} \tag{44}$$

with the same notation as in (10), i.e., $Z_{kl} = r_{kl}^{-3}z_{kl}$, etc. This implies $m_3z_3 + m_4z_4 + m_5z_5 = 0$, $m_3w_3 + m_4w_4 + m_5w_5 = 0$. We set $M = m_3 + m_4 + m_5$. We assume $M \neq 0$, which is consistent with our general hypothesis on the masses. By writing $Mz_3 = m_4 z_{43} + m_5 z_{53}$ and so on, the system becomes

$$\begin{pmatrix}0\\0\end{pmatrix} = m_4\Big(\frac{1}{r_{34}^3} - \frac{1}{M}\Big)\begin{pmatrix}z_{43}\\w_{43}\end{pmatrix} + m_5\Big(\frac{1}{r_{35}^3} - \frac{1}{M}\Big)\begin{pmatrix}z_{53}\\w_{53}\end{pmatrix}, \tag{45}$$

$$\begin{pmatrix}0\\0\end{pmatrix} = m_3\Big(\frac{1}{r_{34}^3} - \frac{1}{M}\Big)\begin{pmatrix}z_{34}\\w_{34}\end{pmatrix} + m_5\Big(\frac{1}{r_{45}^3} - \frac{1}{M}\Big)\begin{pmatrix}z_{54}\\w_{54}\end{pmatrix},$$

$$\begin{pmatrix}0\\0\end{pmatrix} = m_3\Big(\frac{1}{r_{35}^3} - \frac{1}{M}\Big)\begin{pmatrix}z_{35}\\w_{35}\end{pmatrix} + m_4\Big(\frac{1}{r_{45}^3} - \frac{1}{M}\Big)\begin{pmatrix}z_{45}\\w_{45}\end{pmatrix}.$$

If the three vectors $\binom{z_3}{w_3}$, $\binom{z_4}{w_4}$, $\binom{z_5}{w_5}$ are not on a line, the pairs of vectors in the right-hand sides are independent. Thus $r_{34}^3 = r_{35}^3 = r_{45}^3 = M$.

We call the case where $r_{34}^3 = r_{35}^3 = r_{45}^3 = M$ the Lagrange case and the case where the three vectors are on a line the Euler case. In the complex domain, the intersection of the Lagrange case and the Euler case is not empty. In the Euler case, we have the triangular inequality $\pm r_{45} \pm r_{35} \pm r_{34} = 0$. Fix $M = 1$, which does not restrict the generality. The distances r_{kl} in the Lagrange case are either 1, j or j^2, where j satisfies $1 + j + j^2 = 0$. So we are both in the Lagrange and the Euler case if and only if the three distances are 1, j and j^2.

In the Euler case we multiply the first w-equation in (44) by w_{45}, the second by w_{53}, the third by w_{34} and sum up. By grouping the terms in W_{kl}, we obtain two interesting expressions:

$$0 = \begin{vmatrix} m_3 & m_4 & m_5 \\ w_{45} & w_{53} & w_{34} \\ W_{45} & W_{53} & W_{34} \end{vmatrix} \tag{46}$$

or, expanding along the third line and using $Mw_3 = m_4 w_{43} + m_5 w_{53}$, $Mw_4 = \dots$,

$$w_3 W_{45} + w_4 W_{53} + w_5 W_{34} = 0. \tag{47}$$

These computations show that if (47) and

$$m_3 w_3 + m_4 w_4 + m_5 w_5 = 0 \tag{48}$$

are satisfied, then (46) is satisfied. This homogeneous condition is, in the collinear case, necessary for the existence of a re-scaling such that (44) is satisfied. We will use equations (47) and (48) as equations for Euler configurations. Note that (47) does not depend on the masses, which is related to a remark by Marchal (see [19, p. 44], [3]).

The relevant choices of signs in $r_{kl} = \pm w_{kl}$ fix four classes of Euler's central configurations. Three classes correspond to the three real Euler central configurations. Each comes with two pairs of complex configurations. For example, in the case $r_{34} = w_{34}$, $r_{45} = w_{45}$, $r_{53} = -w_{53}$, (47) becomes

$$0 = \frac{w_3}{w_{45}^2} - \frac{w_4}{w_{53}^2} + \frac{w_5}{w_{34}^2}.$$

The numerator is an irreducible polynomial in (w_3, w_4, w_5) of degree 5. The fourth class, with $w_{34} = r_{34}$, $w_{45} = r_{45}$, $w_{53} = r_{53}$, corresponds to the equation

$$0 = \frac{w_3}{w_{45}^2} + \frac{w_4}{w_{53}^2} + \frac{w_5}{w_{34}^2}.$$

The numerator is the quarter of

$$(w_{45}^2+w_{53}^2+w_{34}^2)\big((-w_3+w_4+w_5)w_{45}^2+(-w_4+w_5+w_3)w_{53}^2+(-w_5+w_3+w_4)w_{34}^2\big).$$

The first factor is also a factor of $w_{34}^3 - w_{53}^3$, so it vanishes if w_{34} and w_{53} are two distinct cubic roots of the same number. Then w_{45} is the third cubic root. We are in the Lagrange case mentioned above.

To compute all the Euler cases, we may take the numerator of

$$0 = \frac{w_3}{w_{45}r_{45}} + \frac{w_4}{w_{53}r_{53}} + \frac{w_5}{w_{34}r_{34}}$$

and eliminate the r_{kl}'s using the polynomial conditions $r_{45} = w_{45}$, $r_{53}^2 = w_{53}^2$, $r_{34}^2 = w_{34}^2$.

In the same way, to avoid a lengthy discussion of the Lagrange cases, one can use the relations $z_{34}^3 w_{34}^3 = z_{45}^3 w_{45}^3 = z_{53}^3 w_{53}^3$, substitute $z_{34} = 1$, $z_{53} = -1 - z_{45}$, and eliminate z_{45} between the first and the second equations. Factorizing, one can observe the common factor with the Euler case.

We may form a polynomial in w_3, w_4, w_5, product of the polynomials in the Euler case and in the Lagrange case. The factors are

$$\begin{aligned}
S_1 = {} & w_4^5 + w_5^2 w_4^3 - 2w_4^4 w_3 + 4w_5^3 w_3 w_4 - 2w_4^4 w_5 + w_3^2 w_4^3 + 4w_5 w_3 w_4^3 + w_3^2 w_5 w_4^2 \\
& - 2w_5^4 w_4 + w_5^3 w_4^2 - 5w_3 w_5^2 w_4^2 - w_3^3 w_4^2 + w_5^5 + 2w_3^4 w_4 + w_3^2 w_5^3 + w_3^2 w_5^2 w_4 \\
& - 2w_3 w_5^4 + 2w_3^4 w_5 - w_3^3 w_5^2 - w_3^5 - 4w_3^3 w_5 w_4, \\
S_2 = {} & w_4^5 + w_5^2 w_4^3 - 2w_4^4 w_3 - 4w_5^3 w_3 w_4 - 2w_4^4 w_5 + w_3^2 w_4^3 + 4w_5 w_3 w_4^3 - w_3^2 w_5 w_4^2 \\
& + 2w_5^4 w_4 - w_5^3 w_4^2 - w_3 w_5^2 w_4^2 - w_3^3 w_4^2 - w_5^5 + 2w_3^4 w_4 - w_3^2 w_5^3 + 5w_3^2 w_5^2 w_4 \\
& + 2w_3 w_5^4 + 2w_3^4 w_5 - w_3^3 w_5^2 - w_3^5 - 4w_3^3 w_5 w_4, \\
S_3 = {} & w_3^5 - 2w_3^4 w_4 - 2w_3^4 w_5 + w_3^3 w_4^2 + w_3^3 w_5^2 + 4w_3^3 w_5 w_4 - 5w_3^2 w_5 w_4^2 + w_3^2 w_5^2 w_4 \\
& - w_3^2 w_5^3 + w_3^2 w_4^3 + 4w_5 w_3 w_4^3 + w_3 w_5^2 w_4^2 - 4w_5^3 w_3 w_4 - 2w_4^4 w_3 + 2w_3 w_5^4 \\
& - w_5^3 w_4^2 - 2w_4^4 w_5 + w_5^2 w_4^3 + w_4^5 - w_5^5 + 2w_5^4 w_4,
\end{aligned}$$

$$\begin{aligned}
S_4 &= w_3^3 - w_3^2 w_4 - w_3 w_4^2 + w_4^3 + 3w_3 w_5 w_4 - w_3^2 w_5 - w_5 w_4^2\\
&\quad - w_5^2 w_4 - w_3 w_5^2 + w_5^3,\\
S_5 &= w_3^2 - w_3 w_4 + w_4^2 - w_5 w_3 - w_5 w_4 + w_5^2,\\
S_6 &= w_3^4 - 2w_3^3 w_4 + w_3 w_4^3 + w_4^4 - 2w_3^3 w_5 + 6w_3^2 w_5 w_4 - 3w_5 w_3 w_4^2 - 5w_5 w_4^3\\
&\quad - 3w_3 w_5^2 w_4 + 9w_5^2 w_4^2 + w_5^3 w_3 - 5w_5^3 w_4 + w_5^4,\\
S_7 &= w_3^4 + w_3^3 w_4 - 2w_3 w_4^3 + w_4^4 - 3w_3^2 w_5 w_4 - 5w_3^3 w_5 + 6w_5 w_3 w_4^2 - 2w_5 w_4^3\\
&\quad + 9w_3^2 w_5^2 - 3w_3 w_5^2 w_4 - 5w_5^3 w_3 + w_5^3 w_4 + w_5^4,\\
S_8 &= w_3^4 - 5w_3^3 w_4 + 9w_3^2 w_4^2 - 5w_3 w_4^3 + w_4^4 + w_3^3 w_5 - 3w_3^2 w_5 w_4 - 3w_5 w_3 w_4^2\\
&\quad + w_5 w_4^3 + 6w_3 w_5^2 w_4 - 2w_5^3 w_3 - 2w_5^3 w_4 + w_5^4.
\end{aligned}$$

The power of the common factor S_5 is irrelevant in our discussion. We may substitute $w_3 = (m_4 w_{34} + m_5 w_{35})/M$, etc., or express w_3 through the relation $m_3 w_3 + m_4 w_4 + m_5 w_5 = 0$.

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