AN INTEGRABLE DEFORMATION OF AN ELLIPSE OF SMALL ECCENTRICITY IS AN ELLIPSE

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Abstract. The classical Birkhoff conjecture says that the only integrable convex domains are circles and ellipses. In the paper we show that a version of this conjecture is true for small perturbations of ellipses of small eccentricity.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a strictly convex domain. We say that Ω is C^r if its boundary is a C^r-smooth curve. Consider the billiard problem in Ω: a massless billiard ball moves with unit speed and without friction following a rectilinear path inside the domain Ω. When the ball hits the boundary it is reflected elastically according to the standard reflection law, i.e. the angle of reflection equals the angle of incidence: such trajectories are sometimes called *broken geodesics*.

We call a (possibly not connected) curve $\widehat{\Gamma} \subset \Omega$ a *caustic* if any billiard orbit having one segment tangent to $\widehat{\Gamma}$ is so that all its segments are tangent to $\widehat{\Gamma}$. We call a billiard Ω *locally integrable* if the union of all caustics has nonempty interior; likewise, a billiard Ω is said to be *integrable* if the union of all *smooth convex* caustics has nonempty interior. It follows by rather elementary geometry considerations, (but see e.g. [16, Theorem 4.4] for a detailed proof) that a billiard in an ellipse is integrable: its caustics are indeed cofocal ellipses and hyperbolas.

Birkhoff Conjecture (see Birkhoff [3], Poritsky [13]). *If the billiard in Ω is integrable, then $\partial\Omega$ is an ellipse.*

The most notable result related to the Birkhoff Conjecture is due to Bialy [2] (see also Wojtkowski [19]) who proved that, if convex caustics foliate the whole domain Ω, then Ω has to be a disk. On the other hand, it is simple to construct smooth (but not analytic) locally integrable billiards different from ellipses. In fact, it suffices to arbitrarily perturb an ellipse away from a neighborhood of the two endpoints of the minor axis. More interestingly, Treschev [18] gives indication that there are analytic locally integrable billiards such that the dynamics around one elliptic point is conjugate to a rigid rotation.

There is a quite remarkable relation between properties of billiards and the spectrum of the Laplace operator in Ω. Given a domain Ω, the length spectrum of Ω is defined as the collection of perimeters of its periodic orbits, counted with multiplicity:

$$\mathcal{L}_\Omega := \mathbb{N}\{\text{lengths of closed geodesics in } \Omega\} \cup \mathbb{N}\ell(\partial\Omega),$$

where $\ell(\partial\Omega)$ denotes the length of the boundary.

Denote with $\operatorname{Spec}\Delta$ the spectrum of the Laplace operator in Ω with (e.g) Dirichlet boundary condition, i.e. the set of λ so that

$$\Delta u = \lambda u, \qquad\qquad u = 0 \text{ on } \partial\Omega.$$

From the physical point of view, Dirichlet eigenvalues λ are the eigenfrequencies of the membrane Ω with fixed boundary.

Andersson–Melrose (see [1, Theorem (0.5)]) proved that, for strictly convex C^∞ domains, the length spectrum $\mathcal{L}_\Omega$ contains the singular support of the *wave trace* $t \mapsto \sum_{\lambda_j \in \operatorname{Spec}\Delta} \exp(i\sqrt{-\lambda_j}t)$.

This is, of course, related to inverse spectral theory and to the famous question by M. Kac [10]: "Can one hear the shape of a drum?". More formally: does the Laplace spectrum determine a domain? There is a number of counterexamples to this question (see e.g. [7]), but the domains considered in such examples are neither smooth nor convex. In [15], P. Sarnak conjectures that the set of isospectral planar domains is finite.

In the affirmative direction Hezari–Zelditch proved in [9] that given an ellipse $\mathcal{E}$, any one-parameter C^∞-deformation Ω_ε which preserves the Laplace spectrum (with respect to either Dirichlet or Neumann boundary conditions) and the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry group of the ellipse has to be *flat* (i.e., all derivatives have to vanish for $\varepsilon = 0$). Further historical remarks on the inverse spectral problem can also be found in [9].

2. Our main result

Given a strictly convex domain Ω, we define the associated billiard map f_Ω as follows. Let us fix a point $P_0 \in \partial\Omega$ and denote with s the arc-length parametrization of $\partial\Omega$ starting at P_0 in the counter-clockwise direction; let P_s denote the point on $\partial\Omega$ parametrized by s; by scaling Ω we can always assume that its perimeter is 1. We define the billiard map

$$f_\Omega : \mathbb{T} \times [0,\pi] \to \mathbb{T} \times [0,\pi], \qquad (s,\varphi) \mapsto (s',\varphi'), \tag{1}$$

where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, $P_{s'}$ is the reflection point of a ray leaving P_s with angle φ with respect to the counter-clockwise tangent ray to the boundary $\partial\Omega$ and φ' is the angle of incidence of the ray at $P_{s'}$ with the clockwise tangent. If there is no confusion we will drop the subscript Ω and simply refer to the billiard map as f.

In the sequel, we agree that all caustics that we will consider will be smooth and convex; we will refer to such curves simply as *caustics*.

Let $\widehat{\Gamma}$ be a caustic for Ω; for any $s \in \mathbb{T}$ there exist two rays leaving P_s which are tangent to $\widehat{\Gamma}$, one aligned with the counter-clockwise tangent of $\widehat{\Gamma}$ and the other one with the clockwise tangent; let us denote with $\varphi^{\pm}_{\widehat{\Gamma}}(s)$ their corresponding angles of reflection. Observe that, by reversibility of the dynamics, the trajectory associated with φ^- is the time-reversal of the trajectory associated with φ^+, i.e. $\varphi^- = \pi - \varphi^+$. We can thus restrict our analysis to (e.g.) φ^+; in doing so we will drop, for simplicity, the superscript $+$ from our notations.

The graph $\Gamma = \{(s, \varphi_{\widehat{\Gamma}}(s))\}_{s\in\mathbb{T}}$ is, by definition of a caustic, a (non-contractible) f-invariant curve[1]. Therefore, the restriction $f|_\Gamma$ is a homeomorphism of the circle, and, as such, it admits a rotation number, which we denote with ω. In fact (since we have chosen φ^+ over φ^-), we always have $0 < \omega \leq 1/2$.

Definition. *We say $\widehat{\Gamma}$ is an* integrable rational caustic *if the corresponding (non-contractible) invariant curve Γ consists of periodic points; in particular the corresponding rotation number is rational. If Ω admits integrable rational caustics of rotation number $1/q$ for all $q > 2$, we say that Ω is* rationally integrable.

Remark. *A more standard definition of integrability is existence of a "nice" first integral. Existence of a "nice" first integral for a billiard does not imply that any caustic of rational rotation number is integrable. For instance, the invariant curve corresponding to points belonging to the conciding separatrix arcs of a hyperbolic periodic orbit of f is not integrable. On the other hand, if a caustic with rational rotation number belongs to the interior of a foliation by caustics, then it is, indeed, an integrable rational caustic (see e.g.* [16, Corollary 4.5] *for the general statement and* [8, Proposition 2.8] *for the special case of an ellipse).*

Let us denote with $\mathcal{E}_e \subset \mathbb{R}^2$ an ellipse of eccentricity e and perimeter 1.

Main Theorem. *There exists $e_0 > 0$ such that for any $0 \leq e \leq e_0$ and $K > 0$, there exists $\varepsilon > 0$ so that any rationally integrable C^{39}-smooth domain Ω so that $\partial\Omega$ is C^{39}-K-close and C^1-ε-close to $\mathcal{E}_e$ is an ellipse.*

Remark. *Our requirements for smoothness are probably not optimal and follow from the approach used in our proof (see the proof of Lemma 23 and in particular footnote 7). One could possibly improve them using* [5].

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[1] Indeed, by Birkhoff's Theorem, any f-invariant non-contractible curve has to be a Lipshitz graph.

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3. Our strategy and the outline of the paper

Let us start by exploring the simplified setting of integrable deformations of a disk; we then use this insight to explain the main strategy of our proof in the general case. Let Ω_0 be the unit disk and let us denote polar coordinates with (r, ϕ). Let Ω_ε be a one-parameter family of deformations given in polar coordinates by $\partial\Omega_\varepsilon = \{(r, \phi) = (1 + \varepsilon n(\phi) + O(\varepsilon^2), \phi)\}$. Consider the Fourier expansion

$$n(\phi) = n_0 + \sum_{k>0} n'_k \sin(k\phi) + n''_k \cos(k\phi).$$

Theorem (Ramirez-Ros [14]). *If Ω_ε has an integrable rational caustic $\Gamma_{1/q}$ of rotation number $1/q$ for all sufficiently small ε, then $n'_k = n''_k = 0$ if k is divisible by q.*

Let us now assume that the domains Ω_ε are rationally integrable for all sufficiently small ε: then the above theorem implies that

$$\begin{aligned} n(\phi) &= n_0 + n'_1 \cos\phi + n''_1 \sin\phi + n'_2 \cos 2\phi + n''_2 \sin 2\phi \\ &= n_0 + n^*_1 \cos(\phi - \phi_1) + n^*_2 \cos 2(\phi - \phi_2) \end{aligned}$$

for some ϕ_1 and ϕ_2. Notice that

- n_0 corresponds to an homothety.
- n^*_1 corresponds to a translation in the direction of angle ϕ_1 with the x-axis.
- n^*_2 corresponds to a deformation into an ellipse of small eccentricity with the major axis having angle ϕ_2 with the x-axis.

This implies that, infinitesimally, rationally integrable deformations of a circle are tangent to the 5-parameter family of ellipses. However, there is *no uniformity in q* and, as q increases, the size of perturbation ε such that the above observation holds will decrease.

We now proceed to introduce the main concepts in our proof. Let Ω be a strictly convex domain and consider a tubular neighborhood U_Ω of $\partial\Omega$ so that there are well-defined *tubular coordinates* (s, n), where s is the s-coordinate of the orthogonal projection of the point onto the boundary $\partial\Omega$ and n is the oriented distance along the orthogonal direction to $\partial\Omega$ with $n > 0$ being outside of Ω and $n < 0$ being inside.

Given a domain Ω' with $\partial\Omega' \subset U_\Omega$, one can thus identify it with the graph of a function $\mathbf{n}(s)$ in tubular coordinates. To do that one can project points from $\partial\Omega'$ to $\partial\Omega$ and lift points from $\partial\Omega$ to $\partial\Omega'$. In the sequel we will only consider perturbations

Ω' which can be described by a function $\mathbf{n}(s)$ of this form and we introduce the following (slightly abusing, but suggestive) notation

$$\partial\Omega' = \partial\Omega + \mathbf{n}(s).$$

We then need to define a convenient coordinate system, which was first introduced by Lazutkin [11]. Let Ω be a strictly convex domain; recall that s denotes the arc-length parametrization of $\partial\Omega$ and denote with $\rho(s)$ its radius of curvature at s. Observe that if Ω is C^r, then ρ is C^{r-2}. Define the *Lazutkin parametrization* of the boundary:

$$x(s) = C_\Omega \int_0^s \rho(\sigma)^{-2/3}\, d\sigma, \qquad \text{where } C_\Omega = \left[\int_{\partial\Omega} \rho(\sigma)^{-2/3} d\sigma\right]^{-1}. \tag{2}$$

We call *Lazutkin map* the following change of variables map:

$$\Psi_{\mathrm{L}} : (s,\varphi) \mapsto (x = x(s), y(s,\varphi) = 4C_\Omega\, \rho(s)^{1/3} \sin(\varphi/2)). \tag{3}$$

Also introduce the *Lazutkin density*

$$\mu(x) = \frac{1}{2C_\Omega \rho(x)^{1/3}}, \tag{4}$$

where we denote by $\rho(x) = \rho(s(x))$ the radius of curvature in the Lazutkin parametrization, where $s(x)$ can be obtained by inverting (3). Observe that $\mu(x) = \pi$ for a circle and varies analytically with the eccentricity for an ellipse.

By replacing the arc-length parametrization s with the Lazutkin parametrization x in the definition of the tubular coordinates, we obtain the definition of the *Lazutkin tubular coordinates*. We denote the corresponding perturbation function with $\mathbf{n}(x)$. Observe that if $\partial\Omega = \mathcal{E}$ is an ellipse, ρ is analytic and thus the Lazutkin parametrization is itself an analytic parametrization of $\mathcal{E}$.

Let $\mathbf{n}(x)$ be a C^r deformation of Ω and consider, for $\varepsilon \in (0,1)$, the 1-parameter family of domains

$$\partial\Omega_\varepsilon := \partial\Omega + \varepsilon\mathbf{n}(x).$$

The first step of our proof is to obtain a perturbative version of Ramirez-Ros' Theorem for elliptical domains. In order to do so we first derive a necessary condition for preservation of an integrable rational caustic (in Section 4). We will then define (see Section 5) functions $\{c_q(x), s_q(x)\}_{q>2}$ so that if Ω_ε has an integrable rational caustic $\widehat{\Gamma}^\varepsilon_{1/q}$ of rotation number $1/q$ for some $q > 2$ and all small ε, then

$$\int \mathbf{n}(x)\, \mu(x)\, c_q(x)\, dx = \int \mathbf{n}(x)\, \mu(x)\, s_q(x)\, dx = 0. \tag{5}$$

In fact, in Lemma 9 we derive an perturbative version of the above conditions: more precisely, if a perturbation $\partial\Omega' = \partial\Omega + \varepsilon\mathbf{n}(s)$ has an integrable rational caustic $\Gamma_{1/q}$ for some small $\varepsilon > 0$, then we can replace (5) with an inequality of the absolute value of the integrals being $\mathcal{O}(q^8\varepsilon^2)$-small: observe that, as we hinted above, our estimate is necessarily *non-uniform in* q.

If $\partial\Omega$ is a circle, then $\{c_q, s_q\}$ are given by Fourier Modes (as in Ramirez-Ros' Theorem above); if, on the other hand, $\partial\Omega$ is an ellipse, the functions $\{c_q(x), s_q(x), q > 2\}$ can be explicitly defined using elliptic integrals via action-angle coordinates (see (19)). We then (see Section 6 for definitions) complement these functions with 5 functions

$$\{c_0(x),\ c_q(x),\ s_q(x),\ q = 1, 2\}$$

having the same meaning as the ones described above: four define homothety, translations and rotations, while the fifth one defines hyperbolic rotations. We then show (see Section 7) that for sufficiently small eccentricity, the functions $\{c_q, s_q\}$ also form a basis of L^2.

Remark. *We emphasize that our condition on eccentricity is not an abstract smallness assumption. In fact, we give concrete conditions on the eccentricity for our result to hold. More specifically: one has to check that for some* $N > 1$ *a given* $(2N + 1) \times (2N + 1)$ *correlation matrix* M_N *(defined in* (21)*) is invertible (see Remark 19) and that some explicit condition (given in* (25)*, where* $C^*(e)$ *is defined in Lemma 17) holds true.*

We then conclude the proof (in Section 8) using the following approximation result (Lemma 23): if Ω is rationally integrable and $\partial\Omega$ is an $O(\varepsilon)$-perturbation of an ellipse $\partial\Omega_0 = \mathcal{E}_e$ of small eccentricity e, then there exists an ellipse $\bar{\mathcal{E}}$ such that $\partial\Omega_\varepsilon$ is an $O(\varepsilon^\beta)$-perturbation of $\bar{\mathcal{E}}$ for some $\beta > 1$.

4. A sufficient condition for rational integrability, the Deformation Function, and action-angle variables

Let $\mathcal{E} \subset \mathbb{R}^2$ be an ellipse of perimeter 1; conventionally we let P_0 be one of the end-points of the major axis. Let $\hat{\Gamma}_\omega$ be the caustic of rotation number ω with $0 < \omega < 1/2$. Let $f = f_{\mathcal{E}}$ be the associated billiard map and Γ_ω be the corresponding invariant curve of f of rotation number ω. Then there exists a parametrization $S(\cdot;\omega)$ of the boundary $\mathcal{E}$ in arc-length coordinate s so that f acts as a rigid rotation, i.e. for any $\theta \in \mathbb{T}$

$$f(S(\theta;\omega), \Phi(\theta;\omega)) = (S(\theta + \omega;\omega), \Phi(\theta + \omega;\omega)) \tag{6}$$

where we introduced the shorthand notation $\Phi(\theta;\omega) = \varphi_{\widehat{\Gamma}_\omega}(S(\theta;\omega))$. The functions S and Φ describe *action-angle coordinates*. In other words, (S, Φ) is the change of variables from action-angle coordinates to arc-length and reflection angle. Geometrically, given $S(\theta;\omega)$, consider the ray leaving $P_{S(\theta;\omega)}$ with angle $\Phi(\theta;\omega)$; this ray will be the tangent to $\widehat{\Gamma}_\omega$ and land at the point parametrized by $S(\theta + \omega;\omega)$ with angle $\Phi(\theta + \omega;\omega)$ with respect to the tangent at $S(\theta + \omega;\omega)$.

We can normalize S so that $S(0;\omega) = 0$ is fixed for all $\omega \in (0, 1/2)$. Following Tabanov (see [17]) we can take S and Φ to be analytic in both θ and ω. In particular, for each $\omega \in (0, 1/2)$ the map $\Phi(\cdot;\omega)$ is an (analytic) circle diffeomorphism. Observe additionally that both functions depend analytically on the parameter e and, moreover, for $e = 0$ we have $S(\theta;\omega) = \theta$ and $\Phi(\theta;\omega) = \pi\omega$.

Let now Ω be a deformation of $\mathcal{E}$ identified by a C^{39} function $\mathbf{n}$. Given $p/q \in (0, 1/2) \cap \mathbb{Q}$ with p and q relatively prime, let us define the *Deformation Function* as follows:

$$\mathcal{D}\left(\mathbf{n}, S, \Phi, \frac{p}{q}\right)(\theta) = 2\sum_{k=1}^{q} \mathbf{n}\left(S\left(\theta + k\frac{p}{q}; \frac{p}{q}\right)\right) \sin\Phi\left(\theta + k\frac{p}{q}; \frac{p}{q}\right). \tag{7}$$

In Theorem 1 below we show that the Deformation Function is the leading term of the change of perimeter of the star-shaped polygon inscribed in $\mathcal{E}$ corresponding to an orbit of rotation number p/q starting at $P_{S(\theta)}$. In order to turn the above consideration into a precise statement, we need to introduce some further notation.

First, since in the present article we are interested only in caustics of rotation number $1/q$, we restrict the analysis to this case. Let us thus introduce the convenient shorthand notations $S_q = S(\cdot, 1/q)$ and $\Phi_q = \Phi(\cdot, 1/q)$. Recall that for any ellipse $\mathcal{E}$, every caustic $\widehat{\Gamma}_{1/q}$ of rational rotation number $1/q$ with $q > 2$ is an integrable rational caustic. Recall also that, for any $0 \le s < 1$, we denote by P_s a point whose arc-length to P_0 in the counter-clockwise direction is s. For ease of notation, for any $k = 0, \cdots, q-1$, let $P_k^0(\theta) = P_{S_q(\theta+k/q)}$, so that have that for each $\theta \in \mathbb{T}$ the q-periodic orbit corresponding to θ tangent to the caustic $\widehat{\Gamma}_{1/q}$ is given by the points $P_0^0(\theta), \cdots, P_{q-1}^0(\theta)$. By the variational characterization of periodic orbits (see e.g. [3]), the above points are the vertices of the inscribed q-gon of maximal perimeter with a vertex at $P_{S_q(\theta)}$. Let $L_q^0(\theta)$ be the perimeter of this q-gon, i.e.

$$L_q^0(\theta) = \sum_{k=0}^{q-1} \|P_{k+1}^0(\theta) - P_k^0(\theta)\|,$$

where $\|\cdot\|$ is the Euclidean distance. $\widehat{\Gamma}_{1/q}$ being an integrable rational caustic implies that $L_q^0(\theta)$ is constant in θ. This follows from the fact that every periodic orbit is a critical point for the perimeter: hence, a smooth one parameter family of periodic orbits has a constant perimeter.

Let us denote with $P_0'(\theta)$ the lift of $P_0^0(\theta)$ to $\partial\Omega$. Since Ω is strictly convex, for each $\theta \in \mathbb{T}$, there is a convex q-gon starting at $P_0'(\theta)$ of maximal perimeter. Denote its vertices by $P_k'(\theta),\ k = 0, \cdots, q-1$ and its perimeter by

$$L_q'(\theta) = \sum_{k=0}^{q-1} \|P_{k+1}'(\theta) - P_k'(\theta)\|.$$

Observe that if Ω admits an integrable rational caustic of rotation number $1/q$, then the points $P'_0(\theta), \cdots, P'_{q-1}(\theta)$ are the reflection points of the q-periodic orbit of rotation number $1/q$ starting at $P'_0(\theta)$. Moreover, $L'_q(\theta)$ is also constant.

Theorem 1. *Let $\mathcal{E}_e$ be an ellipse of eccentricity $0 \le e < 1$ and perimeter 1, and let (S, Φ) be the corresponding action-angle coordinates. Then there is $c = c(e) > 0$ such that for any integer q, $q > 2$ and a C^1 deformation $\partial\Omega := \mathcal{E} + \mathbf{n}$ so that Ω has an integrable rational caustic of rotation number $1/q$ and $q^8\|\mathbf{n}\|_{C^1} < c$:*

$$\max_\theta \left| L'_q(\theta) - L^0_q(\theta) - \mathcal{D}(\mathbf{n}, S, \Phi; 1/q)(\theta) \right| \le C\, q^8 \|\mathbf{n}\|^2_{C^1},$$

where $C = C(e, \|\mathbf{n}\|_{C^5})$ depends on the eccentricity e and monotonically on the C^5-norm of $\mathbf{n}$, but is independent of q.

Remark. *Notice that in* [12, Proposition 11] *a different (weaker, but cleaner) version of this statement is given, where it suffices to know only $S(\theta, \omega)$. We also point out that $c(e) \to 0$ as $e \to 1$.*

Proof of Theorem 1. Let $\alpha_k(\theta)$ be the angle between $P'_k(\theta) - P^0_k(\theta)$ and the positive tangent to $\mathcal{E}$ at $P^0_k(\theta)$ (see Figure 1). We assume $\alpha_k(\theta)$ to be positive towards the exterior of $\mathcal{E}$, i.e. if $P'_k(\theta)$ is outside of $\mathcal{E}$, then $\alpha_k(\theta) \in (0, \pi)$. Introduce the displacements

$$v_k(\theta) = \|P'_k(\theta) - P^0_k(\theta)\|$$

and let $\varphi_k(\theta) = \Phi_q(\theta + k/q)$. By definition of action-angle coordinates, the edge $P^0_{k+1}(\theta) - P^0_k(\theta)$ has reflection angle $\varphi_k(\theta)$ at $P^0_k(\theta)$ and $\varphi_{k+1}(\theta)$ at $P^0_{k+1}(\theta)$ respectively. Finally, let us introduce the notation $l^0_k(\theta) = \|P^0_{k+1}(\theta) - P^0_k(\theta)\|$ and $l'_k(\theta) = \|P'_{k+1}(\theta) - P'_k(\theta)\|$. Observe that by Corollary 6, for each $k = 0, \cdots, q-1$ we have

$$\frac{1}{\Xi q} \le l'_k(\theta) \le \frac{\Xi}{q} \quad \text{for some} \quad \Xi = \Xi(e, \|\mathbf{n}\|_{C^5}) > 1, \tag{8}$$

and Ξ depends monotonically on $\|\mathbf{n}\|_{C^5}$. For $k = 0, \cdots, q-1$, project $P'_k(\theta)$ onto $\mathcal{E}$ by the orthogonal projection and denote the projected point by $\bar{P}'_k(\theta)$. Observe that, by construction, $\bar{P}'_0(\theta) = P^0_0(\theta)$. Denote, moreover, with $\bar{\varphi}^+_k$ (resp. $\bar{\varphi}^-_k$) the angle between $\bar{P}'_{k+1}(\theta) - \bar{P}'_k(\theta)$ (resp. $\bar{P}'_k(\theta) - \bar{P}'_{k-1}(\theta)$) and the positive (resp. negative) tangent to $\mathcal{E}$ at $\bar{P}'_k(\theta)$ (see Figure 2).

Lemma 2. *Let Ξ be the constant appearing in (8); for any $k = 0, \cdots, q-1$:*

$$|\bar{\varphi}^+_k - \bar{\varphi}^-_k| \le 5\Xi\, q\, \|\mathbf{n}\|_{C^1}.$$

Proof. Since $\|P'_k - \bar{P}'_k\| \le \|\mathbf{n}\|_{C^0}$ for any $k = 0, \cdots, q-1$, the angle between the k-th perturbed edge and the k-th projected edge satisfies

$$\sphericalangle\{P'_k(\theta) - P'_{k+1}(\theta), \bar{P}'_k(\theta) - \bar{P}'_{k+1}(\theta)\} \le \frac{2\|\mathbf{n}\|_{C^0}}{l'_k(\theta) - 2\|\mathbf{n}\|_{C^0}} \le 4\Xi\, q\, \|\mathbf{n}\|_{C^0}$$

where in the last inequality we have used (8): in fact, we know $l'_k(\theta) > \Xi/q$ and by our assumptions on $\mathbf{n}$ we have $\|\mathbf{n}\|_{C^0} \le \|\mathbf{n}\|_{C^1} < c/q^8$, thus, if $c < 1/\Xi$, since $q > 2$:

$$l'_k(\theta) - 2\|\mathbf{n}\|_{C^0} > l'_k(\theta)/2 > 1/(2\Xi q).$$

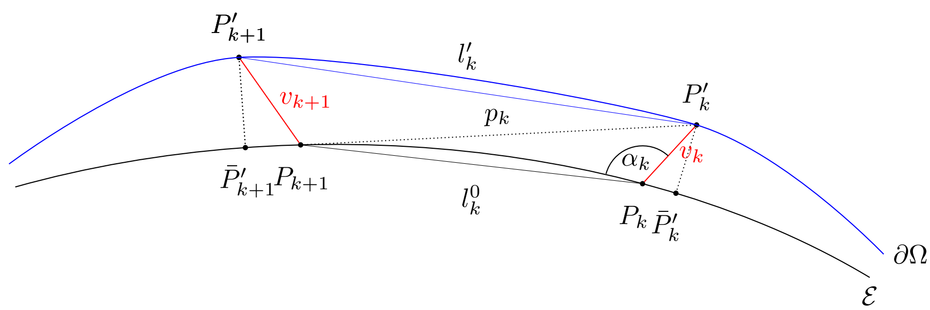

FIGURE 1. Two orbits: unperturbed (in black) and perturbed (in blue)

Since Ω has an integrable rational caustic of rotation number $1/q$, the collection $P'_k(\theta)$, $k = 0, \cdots, q-1$ corresponds to a q-periodic orbit, thus, the angle of incidence at $P'_k(\theta)$ of $P'_k(\theta) - P'_{k+1}(\theta)$ equals the angle of reflection of $P'_{k-1}(\theta) - P'_k(\theta)$. See Figure 2: the angle between the tangent to $\partial\Omega$ at $P'_k(\theta)$ and the tangent to $\mathcal{E}$ at the projected point $\bar{P}'_k(\theta)$ is bounded above by $\mathbf{n}'(S_q(\theta + k/q))$, hence by $\|\mathbf{n}\|_{C^1}$. Therefore, adding the two deviations coming from discrepancy of the tangents to $\partial\Omega$ (resp. $\mathcal{E}$) and discrepancy of end points $P'_i(\theta)$ (resp. $\bar{P}'_i(\theta)$) with $i = k \pm 1, k$ we get that

$$|\bar{\varphi}^+_k - \bar{\varphi}^-_k| \le 4\Xi\, q\, \|\mathbf{n}\|_{C^0} + 2\|\mathbf{n}\|_{C^1},$$

from which we conclude our proof. □

Lemma 3. *For each $k = 0, \cdots, q-1$ let $\bar{\theta}_k$ be so that $\bar{P}'_k(\theta) = P_{S_q(\bar{\theta}_k)}$. Then there exists $C = C(e, \|\mathbf{n}\|_{C^5})$ so that, in the above notations, for any $k = 0, \cdots, q-1$:*

$$|\bar{\theta}_k - \theta_k| \le Cq^3\|\mathbf{n}\|_{C^1} \qquad v_k(\theta) \le Cq^3\|\mathbf{n}\|_{C^1}. \tag{9}$$

Proof. The basic idea of the proof is to consider the worst case scenario of deviation of reflection angles $\bar{\varphi}^\pm_k(\theta)$ from $\varphi_k(\theta)$. Since, unless $\mathcal{E}$ is a circle, the reflection angles

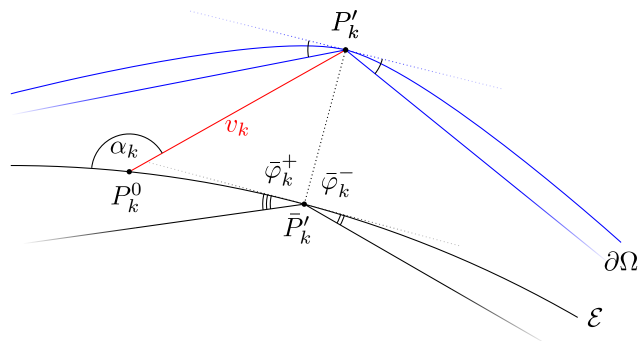

FIGURE 2. Reflection angles: in blue (above) the trajectory of the periodic orbit given by $P'_0, \cdots, P'_{q-1}$; in black (below) the pseudo-orbit given by $\bar P'_0, \cdots, \bar P'_{q-1}$.

φ_k vary depending on the reflection point[2], it is more convenient to keep track of a *first integral*, which is constant along any orbit in the ellipse $\mathcal{E}$. Therefore, it cannot change too rapidly for the perturbed domain Ω. Here is quantification of this phenomenon. Recall that for the ellipse one can explicitly define a conserved quantity (a first integral), as follows. For simplicity, assume $\mathcal{E}$ is centered at the origin and that the major axis is horizontal; let

$$\mathcal{E} = \{x^2/a^2 + y^2/b^2 = 1\},\ 0 < b^2 < a^2.$$

where a and b are chosen so that the ellipse has perimeter 1. Let us introduce so-called *elliptical coordinates* (μ, ψ) on $\mathbb{R}^2$ as follows:

$$x = h \cdot \cosh\mu \cdot \cos\psi, \qquad\qquad y = h \cdot \sinh\mu \cdot \sin\psi$$

where $h^2 = a^2 - b^2$, $0 \le \mu < \infty$, $0 \le \psi < 2\pi$. The family of cofocal ellipses $\mu =$const and hyperbolas $\psi =$const form an orthogonal net of curves[3]. The ellipse $\mathcal{E}$ has the equation $\mu = \mu_0$, where $\cosh^2\mu_0 = a^2/h^2 > 1$. Thus, the length parametrization s of the ellipse can be given as a function of ψ, (see e.g. [17] for an explicit formula): Then, the billiard map has a first integral given by

$$I(\psi, \varphi) = \cos^2\varphi + \frac{\cos^2\psi}{\cosh^2\mu_0}\sin^2\varphi;$$

observe that $I(\psi, \varphi) = I(\psi, \pi - \varphi)$. Recall that $S_q(\cdot)$ denotes the angle parametrization of $\mathcal{E}$ with rotation number $1/q$. Since the elliptic angle ψ is an analytic function of the arc-length parametrization s and S, in turn, is an analytic function of

[2] i.e. reflection angles are smaller close to the basis of the minor axis and larger close to the basis of the major axis

[3] Observe that as $a \to b$, we have $h \to 0$ and $\mu \to \infty$ so that $h\cosh\mu \to a$ and $h\sinh\mu \to a$.

θ, we can define the first integral $I(\theta,\varphi)$ in the (θ,φ) coordinates. Notice that $\cosh^2\mu_0 > 1 \ge \cos^2\psi$; hence

$$\partial_\varphi I(\psi,\varphi) = \left(\frac{\cos^2\psi}{\cosh^2\mu_0} - 1\right)\sin 2\varphi;$$

observe that for any ψ, the function $I(\psi,\cdot)$ is strictly decreasing on $(0,\pi/2)$; moreover $|\partial_\varphi I| < 1$ and

$$|\partial_\varphi I| \in [1-\cosh^{-2}\mu_0, 2]\,\varphi \quad \text{for } \varphi\in[0,\pi/6]. \tag{10}$$

Moreover, this holds in both (ψ,φ) and (θ,φ) coordinates.

Then we claim that there exists k_* so that $\bar\varphi^-_{k_*} \le \Phi_q(\bar\theta_{k_*}) \le \bar\varphi^+_{k_*}$. Observe that by definition

$$f(S_q(\bar\theta_k),\bar\varphi^+_k) = (S_q(\bar\theta_{k+1}),\bar\varphi^-_k);$$

by well-known properties of monotone twist maps, no orbit can cross the invariant curve $\Gamma_{1/q}$, thus we obtain that if $\bar\varphi^+_k < \Phi_q(\bar\theta_k)$ (resp. $\bar\varphi^+_k > \Phi_q(\bar\theta_k)$), then $\bar\varphi^-_{k+1} < \Phi_q(\bar\theta_{k+1})$ (resp. $\bar\varphi^-_{k+1} > \Phi_q(\bar\theta_{k+1})$). We conclude that if our claim does not hold, necessarily, either $\bar\varphi^+_k < \Phi_q(\bar\theta_k)$ or $\bar\varphi^+_k > \Phi_q(\bar\theta_k)$ for all $k = 0,\cdots,q-1$. In the first case, the twist condition implies that $\bar\theta_{k+1}-\bar\theta_k < 1/q$; but this is a contradiction, since $\bar\theta_q = \bar\theta_0+1$ (passing to the covering space $\mathbb{R}$). Similar arguments in the second case also lead to a contradiction; this in turn implies our claim. Moreover, Lemma 2 implies that

$$\bar\varphi^+_{k_*} - \Phi_q(\bar\theta_{k_*}) \le 5\Xi\, q\,\|\mathbf{n}\|_{C^1} < 5q^{-7}.$$

Define now the *instant first integral* $I^\pm_k = I(\bar\theta_k,\bar\varphi^\pm_k)$; then $I^+_k = I^-_{k+1}$ and since

$$|I^+_k - I^-_k| \le \left|\int_{\bar\varphi^-_k}^{\bar\varphi^+_k} \partial_\varphi I(\bar\theta_k,\varphi)d\varphi\right|.$$

and $\Phi_q(\bar\theta_{k_*}) < C(e)/q$ (applying Lemma 5 to $\mathcal{E}$), by Lemma 2 and (10) we thus conclude (choosing a larger C)

$$|I^+_{k_*} - I_*| < C\,\|\mathbf{n}\|_{C^1}. \tag{11}$$

where $I_* = I(\theta,\varphi_0(\theta))$ and $C = C(e,\|\mathbf{n}\|_{C^5})$; inducing at most q times and applying repeatedly the same argument we conclude $|I^\pm_0 - I_*| < Cq\|\mathbf{n}\|_{C^1}$, that implies

$$|\bar\varphi^\pm_0(\theta) - \varphi_0(\theta)| < Cq^2\|\mathbf{n}\|_{C^1}$$

and inducing on k and using again Lemma 2 we conclude (choosing a larger C)

$$|\bar\theta_k - \theta_k| < Cq^3\|\mathbf{n}\|_{C^1}.$$

The second bound of (9) follows immediately by applying the triangle inequality. □

Lemma 4. *In the notations introduced above we have*

$$\begin{aligned}(12)\qquad \Big| l'_k(\theta) - l^0_k(\theta) - v_k(\theta)\cos\left(\varphi_k(\theta)+\alpha_k(\theta)\right) \\ + v_{k+1}(\theta)\cos\left(\varphi_{k+1}(\theta)-\alpha_{k+1}(\theta)\right)\Big| \le 10\frac{v_k(\theta)^2+v_{k+1}(\theta)^2}{l^0_k(\theta)}.\end{aligned}$$

Proof. Let $p_k(\theta) = \|P'_k(\theta) - P^0_{k+1}(\theta)\|$; applying the Cosine Theorem to the triangle $P^0_k(\theta)P^0_{k+1}(\theta)P'_k(\theta)$ we have

$$p_k(\theta)^2 = v_k(\theta)^2 + l^0_k(\theta)^2 - 2v_k(\theta)l^0_k(\theta)\cos(\varphi_k(\theta)+\alpha_k(\theta)).$$

Likewise, applying it to the triangle $P^0_{k+1}(\theta)P'_{k+1}(\theta)P'_k(\theta)$ we have

$$l'_k(\theta)^2 = v_{k+1}(\theta)^2 + p_k(\theta)^2 + 2v_{k+1}(\theta)p_k(\theta)\cos(\varphi_{k+1}(\theta)-\alpha_{k+1}(\theta)-\delta_{k+1}(\theta)),$$

where $\delta_{k+1}(\theta)$ is the oriented angle $\sphericalangle(P^0_k(\theta)P^0_{k+1}(\theta)P'_k(\theta))$. Combining the above expressions we get

$$\begin{aligned}(13)\qquad l'_k(\theta)^2 - l^0_k(\theta)^2 = v_k(\theta)^2 + v_{k+1}(\theta)^2 - 2v_k(\theta)l^0_k(\theta)\cos(\varphi_k(\theta)+\alpha_k(\theta)) \\ + 2v_{k+1}(\theta)p_k(\theta)\cos(\varphi_{k+1}(\theta)-\alpha_{k+1}(\theta)-\delta_{k+1}(\theta)).\end{aligned}$$

Observe that by the triangle inequality:

$$l^0_k(\theta) - v_k(\theta) - v_{k+1}(\theta) \le l'_k(\theta), p_k(\theta) \le l^0_k(\theta) + v_k(\theta) + v_{k+1}(\theta).$$

Moreover, elementary geometry implies $|\sin\delta_{k+1}(\theta)| \le v_k(\theta)/l^0_k(\theta)$. Now (12) immediately follows dividing both sides of (13) by $l'_k(\theta) + l^0_k(\theta)$ and using the above estimates. □

We can now conclude the proof of Theorem 1; observe that by definition $L^0_q(\theta) = \sum_{k=0}^{q-1} l^0_k(\theta)$ and likewise $L'_q(\theta) = \sum_{k=0}^{q-1} l'_k(\theta)$. By Lemma 4 we thus gather:

$$\begin{aligned}\Big| L'_q(\theta) - L^0_q(\theta) - \sum_{k=0}^{q-1} v_k(\theta)\cos\left(\varphi_k(\theta)+\alpha_k(\theta)\right) \\ + \sum_{k=0}^{q-1} v_{k+1}(\theta)\cos\left(\varphi_{k+1}(\theta)-\alpha_{k+1}(\theta)\right)\Big| \le 20\sum_{k=0}^{q-1}\frac{v_k(\theta)^2}{l^0_k(\theta)}.\end{aligned}$$

Observe that

$$\begin{aligned}\sum_{k=0}^{q-1}\Big[-v_k(\theta)(\cos\varphi_k(\theta)\cos\alpha_k(\theta) - \sin\varphi_k(\theta)\sin\alpha_k(\theta)) \\ + v_{k+1}(\theta)(\cos\varphi_{k+1}(\theta)\cos\alpha_{k+1}(\theta) + \sin\varphi_{k+1}(\theta)\sin\alpha_{k+1}(\theta))\Big] \\ = 2\sum_{k=0}^{q-1} v_k(\theta)\sin\varphi_k(\theta)\sin\alpha_k(\theta).\end{aligned}$$

Notice that, by (9), we have $v_k(\theta)\sin\alpha_k(\theta) = \mathbf{n}(S_q(\theta+k/q)) + O(q^6\|\mathbf{n}\|^2_{C^1})$. Therefore,

$$|L_q'(\theta) - L_q^0(\theta) - \sum_{k=0}^{q-1}\mathbf{n}(S_q(\theta+k/q))\sin\Phi_q(\theta+k/q)| \le Cq^8\|\mathbf{n}\|^2_{C^1}.$$

This completes the proof of Theorem 1. □

5. Lazutkin parametrization and Deformed Fourier Modes

It turns out that for nearly glancing orbits, i.e. orbits having small reflection angle, it is more convenient to study the billiard map f, defined in (1), in Lazutkin coordinates. Recall that Ψ_{L} denotes the Lazutkin change of coordinates defined in (3) and consider the billiard map in Lazutkin coordinates $f_{\mathrm{L}} = \Psi_{\mathrm{L}} \circ f \circ \Psi_{\mathrm{L}}^{-1}$; then f_{L} has the following form (see e.g. [11, (1.4)]):

$$f_{\mathrm{L}} : (x, y) \to (x + y + y^3 g(x, y), y + y^4 h(x, y)), \tag{14}$$

where g and h can be expressed analytically in terms of derivatives of the curvature radius ρ up to order 3: hence if Ω is C^r, g, h are C^{r-5}. Recall that $\widehat{\Gamma}_{1/q} \subset \Omega$ denotes a caustic of rotation number $1/q$, while $\Gamma_{1/q}$ denotes the associated non-contractible invariant curve for the billiard map f. We denote by $\Gamma_{\mathrm{L},1/q}$ the corresponding invariant curve for the billiard map f_{L} in Lazutkin coordinates, i.e. $\Gamma_{\mathrm{L},1/q} = \Psi_{\mathrm{L}}\,\Gamma_{1/q}$. Moreover, let us introduce action-angle coordinates in the Lazutkin parametrization, i.e $(X(\theta,\omega), Y(\theta,\omega)) = \Psi_{\mathrm{L}}(S(\theta,\omega), \Phi(\theta,\omega))$; as before, we define $X_q(\theta) = X(\theta, 1/q)$ and $Y_q = Y(\theta, 1/q)$.

Lemma 5. *Let Ω be a C^5 strictly convex domain and let $\Gamma_{L,1/q}$ be the invariant curve corresponding to an integrable rational caustic of rotation number $1/q$ with $q > 2$, given by*

$$\Gamma_{L,1/q} = \{(x, y_q(x)) : \ x \in \mathbb{T}\}.$$

Then there exists C depending on $\|\rho\|_{C^3}$, such that

$$\left|y_q(x) - \frac{1}{q}\right| < \frac{C}{q^3} \qquad \textit{for any } \ x \in \mathbb{T}. \tag{15}$$

For $k \in \mathbb{Z}$ let $(x_k, y_q(x_k)) = f_L^k(x, y_q(x))$ be an orbit on the invariant curve $\Gamma_{L,1/q}$, and let $\tilde{x}_k$ be a lift of x_k to $\mathbb{R}$; then

$$\left|\tilde{x}_k - \tilde{x}_0 - \frac{k}{q}\right| < \frac{C}{q^2}, \qquad \textit{for } \ 0 \le k \le q. \tag{16}$$

Moreover, if $\Omega = \mathcal{E}_e$ an ellipse of eccentricity e and perimeter 1. the constant C depends on e only and it is such that $C(e) \to 0$ as $e \to 0$.

Corollary 6. *Let Ω be a C^5 strictly convex domain and $q > 2$. Let $(s_k, \varphi_k),\ k = 0, \cdots, q-1$ be a q-periodic orbit of rotation number $1/q$ and $P_k,\ k = 0, \cdots, q-1$ be the corresponding collision points on $\partial\Omega$. Then there is $\Xi = \Xi(\Omega) > 1$, depending on $\|\rho\|_{C^3}$ such that the Euclidean length of each edge $\|P_{k+1} - P_k\|$ satisfies*

$$\frac{1}{\Xi q} \le \|P_{k+1} - P_k\| \le \frac{\Xi}{q}.$$

Moreover, if Ω is a perturbation $\mathbf{n}$ of an ellipse $\mathcal{E}_e$ (i.e. $\partial\Omega = \mathcal{E}_e + \mathbf{n}$), then Ξ depends continuously on the eccentricity e and $\|\mathbf{n}\|_{C^5}$.

Proof. Recall that by definition $y(s,\varphi) = 4\,C_\Omega\,\rho^{1/3}(s)\sin(\varphi/2)$. By Lemma 5 we have $y \in [1/q - C/q^3, 1/q + C/q^3]$ for some C depending on ρ only. Therefore, $\sin(\varphi/2) \in [1/Cq - 1/q^3, C/q + C^2/q^3]$. Since the angle of reflection is $\sim 1/q$ and curvature is uniformly bounded, we get the required bound on the distance $\|P_{k+1} - P_k\|$. □

Proof of Lemma 5. Choose q_0 (sufficiently large depending on $\|\rho\|_{C^3}$) to be specified in due course and assume $q \ge q_0$. First of all, we claim that we have the preliminary bound

$$y_q(x_k) \le \frac{C_1}{q}, \qquad \text{for any } k = 0, \cdots, q-1,$$

where C_1 is a large constant depending on the maximal and minimal value of the curvature ρ. In fact, recall $\Psi_{\mathrm{L}}^{-1}\Gamma_{\mathrm{L},1/q} = \Gamma_{1/q}$ can be parametrized as the graph $\{(s, \varphi_q(s))\}_{s\in\mathbb{T}}$. Let $(s_k, \varphi_q(s_k)) = \Psi_{\mathrm{L}}^{-1}(x_k, y_q(x_k))$, so that

$$(s_{k+1}, \varphi_q(s_{k+1})) = f(s_k, \varphi_q(s_k))$$

and $\tilde{s}_k$ be a lift to $\mathbb{R}$. Since $\tilde{s}_q = \tilde{s}_0 + 1$, there exists $0 \le k_* < q$ so that $0 < \tilde{s}_{k_*+1} - \tilde{s}_{k_*} \le 1/q$. For fixed s_k, we can find a function $\varphi(s_{k+1})$ so that the ray leaving s_k with angle $\varphi(s_{k+1})$ will collide with $\partial\Omega$ at s_{k+1}; if q_0 is sufficiently large, we can use expansion of the billiard map for small φ in terms of curvature (see e.g. [11, (1.1)]) and conclude that $\varphi_q(s_{k_*}) < C/q$, where $C = C(\|\rho\|_{C^1})$ and thus, by definition of the Lazutkin coordinate map (3) we conclude that

$$y_q(x_{k_*}) \le \frac{C_1}{q},$$

where $C_1 = C_1(\|\rho\|_{C^1})$. By iterating (14), starting from k_*, we conclude by (finite) induction that for any $0 \le k < q$:

$$|y_q(x_{j+1}) - y_q(x_j)| \le \frac{C_0}{q^4}, \qquad y_q(x_j) < \frac{C_1}{q},$$

where $C_0 = \max\{\|g\|, \|h\|\}C_1^4$ and we have possibly chosen a larger C_1. Observe that since $\|g\|$ and $\|h\|$ depend $\|\rho\|_{C^3}$, so does C_0. Moreover, by iterating the first inequality q times we also have

$$|y_q(x_0) - y_q(x_k)| \leq \frac{C_0}{q^3} \qquad \text{for any } k = 0, \cdots, q-1. \tag{17}$$

We now claim that $|y_q(x) - 1/q| \leq 4C_0/q^3$. In fact assume by contradiction that $y_q(x) - 1/q > 4C_0/q^3$; then by (17) we gather that $y_q(x_k) - 1/q > 3C_0/q^3$ for any $0 \leq k < q$. Hence, by (14) and the above estimates, for any $0 \leq k < q$ we have

$$\tilde{x}_{k+1} - \tilde{x}_k \geq \frac{1}{q} + \frac{C_0}{q^3};$$

iterating q times we conclude that

$$\tilde{x}_q - \tilde{x}_0 \geq 1 + \frac{C_0}{q^2},$$

which is a contradiction, since $\tilde{x}_q = \tilde{x}_0 + 1$. A similar argument implies that

$$y_q(x) - \frac{1}{q} < -\frac{4C_0}{q^3}$$

also leads to a contradiction. This implies our claim, which in turn implies (15) and (16). Notice that in order to have C_0/q^3 to be small compared to $1/q$ we need q_0 (and thus q) to be sufficiently large (with respect to $\|\rho\|_{C^3}$).

Observe now that if $\partial\Omega$ is an ellipse of eccentricity e, $\Gamma_{\mathrm{L},1/q} = \{(X_q(\theta), Y_q(\theta))\}_{\theta\in\mathbb{T}}$. Since both X_q and Y_q vary analytically with e and if $\partial\Omega$ is a circle, $Y_q(\theta)$ is the constant function equal to $1/q$. We conclude that we can choose $C(e)$ so that $\lim_{e\to 0} C(e) = 0$. This concludes the proof. □

Lemma 7. *Let $\mathcal{E}_e$ be an ellipse of eccentricity e and perimeter 1; then there exists $C(e)$ with $C(e) \to 0$ as $e \to 0$ so that*

$$\|X_q - \mathrm{Id}\|_{C^1} \leq \frac{C(e)}{q^2}.$$

Proof. In the proof of this statement, to simplify the notation, $C(e)$ denotes a generic constant which depends on e only; its actual value might change from an instance to the next. Recall that $X(0,\omega)$ parametrizes a fixed point P_0 for all $\omega \in [0, 1/3]$, (i.e. one of end points of the major axis). Now consider the q-periodic orbit leaving the point P_0: in angle coordinates the orbit is given by $\{\theta_k = k/q \mod 1\}$. Then by (14) and Lemma 5 we conclude that

$$\left|\frac{X_q(\theta_{k+1}) - X_q(\theta_k)}{\theta_{k+1} - \theta_k} - 1\right| \leq \frac{C(e)}{q^2};$$

by the Mean Value Theorem we conclude that there exists some $\bar{\theta}_k \in (\theta_k, \theta_{k+1})$ so that $|X_q'(\bar{\theta}_k) - 1| < C(e)/q^2$. Likewise, we can find $\bar{\bar{\theta}}_k \in (\bar{\theta}_k, \bar{\theta}_{k+1})$ so that

$|X_q''(\bar{\bar{\theta}}_k)| < C(e)/q$. Hence, for each $\theta \in [\bar{\theta}_k, \bar{\theta}_{k+1}]$ we can write

$$X_q'(\theta) = X_q'(\bar{\bar{\theta}}) + \int_{\bar{\bar{\theta}}_k}^{\theta} \left[X_q''(\bar{\bar{\theta}}_k) + \int_{\bar{\bar{\theta}}_k}^{\theta'} X_q'''(\theta'')d\theta'' \right] d\theta'.$$

Now recall that $X_q(\theta) = S(\theta, 1/q)$, where S is analytic in both arguments; in particular all derivatives of X_q are bounded uniformly in q. Moreover $\|X_q'''\| < C(e)$ such that $C(e) \to 0$ as $e \to 0$, since, as noted before, X_q depends analytically on e and for $e = 0$ the function X_q is the identity.

We conclude that $|X_q'(\theta) - X_q'(\bar{\bar{\theta}})| < C(e)/q^2$ for any $\theta \in [\bar{\theta}_k, \bar{\theta}_{k+1}]$, which implies that $\|X_q' - 1\|_{C^0} < C(e)/q^2$. Our estimate then holds integrating in θ. □

Let $s(x)$ be the length parametrization as a function of the Lazutkin parametrization, which can by obtained by inverting (2). Since $y = 4\,C_\Omega\, \rho(s)^{1/3} \sin(\varphi/2)$, for any $(s,\varphi) \in \Gamma_{1/q}$, (15) implies that:

$$\left| \sin \Phi_q \left(X_q^{-1}(x) \right) - \frac{w_q}{2C_\Omega q \rho(x)^{1/3}} \right| \le \frac{2C}{q^3}.$$

where $w_q = q\sin(\pi/q)/\pi \in [1/2, 1]$. Notice that Lemma 5 implies that in the above expression $C = C(e) \to 0$ as $e \to 0$. To simplify notations let $\eta_q(x) = \sin \Phi_q \left(X_q^{-1}(x) \right)$. Notice, moreover, that $q\eta_q(x)$ has a well defined limit as $q \to \infty$. Recall that in (4) we defined the Lazutkin Density $\mu(x) := 1/(2C_\Omega \rho(x)^{1/3})$. Recall that the density function $\mu(x)$ given above, depends only on the domain (i.e. on the eccentricity e if $\partial\Omega = \mathcal{E}$); in particular, it does not depend on q. Using the previous bound we have

$$\left| \frac{q\eta_q(x)}{w_q \mu(x)} - 1 \right| \le \frac{C}{q^2} \tag{18}$$

for some C depending on C_Ω and ρ. For any $q > 2$ define[4]

$$c_q(x) = \frac{q\eta_q(x)}{w_q\mu(x)} \, \frac{1}{X_q'(X_q^{-1}(x))} \, \sin 2\pi q X_q^{-1}(x), \tag{19a}$$

$$s_q(x) = \frac{q\eta_q(x)}{w_q\mu(x)} \, \frac{1}{X_q'(X_q^{-1}(x))} \, \cos 2\pi q X_q^{-1}(x). \tag{19b}$$

Observe that by Lemma 7, the above functions approximate the corresponding Fourier Modes as $q \to \infty$; we will refer to them as the *Deformed Fourier Modes*. The next lemma gives a bound on the speed of this approximation.

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[4] We will define the first five functions $c_i(x)$, $i = 0, 1, 2$, $s_i(x)$, $i = 1, 2$ respectively in the next section.

Lemma 8. *Let $\mathcal{E}$ be an ellipse of eccentricity e; there exists $C^*(e)$ with $C^*(e) \to 0$ as $e \to 0$ so that for any $q > 2$,*

$$\|s_q - \sin(2\pi q\,\cdot)\|_{C^0} < \frac{C^*(e)}{q^2}, \qquad \|c_q - \cos(2\pi q\,\cdot)\|_{C^0} < \frac{C^*(e)}{q^2}.$$

Proof. By (18) we have that $\frac{q\eta_q(x)}{w_q\mu(x)} - 1$ has variation at most $C(e)q^{-2}$. By Lemma 7 we have $\|X_q' - 1\| < C(e)q^{-2}$. Combining the above two estimates we obtain the required bounds. □

Lemma 9. *In notations of Theorem 1, let $\mathcal{E}_e$ be an ellipse of perimeter 1 and eccentricity e and $\partial\Omega$ be a perturbation of $\mathcal{E}_e$ identified by a C^5-smooth function[5] $\mathbf{n}(x)$; assume that Ω has an integrable rational caustic of rotation number $1/q$ for some $2 < q < c(e)\|\mathbf{n}\|_{C^1}^{-1/8}$. Then there exists $C = C(e, \|\mathbf{n}\|_{C^5})$ so that:*

$$\left|\int \mathbf{n}(x)\mu(x)a_q(x)dx\right| \le Cq^8\|\mathbf{n}\|_{C^1}^2,$$

where $a_q = c_q$ or s_q.

Proof. Denote $\mathcal{D}(\theta) = [\mathcal{D}(\mathbf{n}, S, \Phi; 1/q)](\theta)$ the Deformation Function given by (7); then by definition we have

$$\begin{aligned}\int_0^1 \mathcal{D}(\theta)\sin(2\pi q\theta)\,d\theta &= q\int_0^1 \mathbf{n}\left(X_q(\theta)\right)\sin\Phi_q\left(\theta\right)\,\sin(2\pi q\theta)\,d\theta \\ &= \int_0^1 \mathbf{n}\left(X_q(\theta)\right)\,\left[q\eta_q(X_q(\theta))\right]\,\sin(2\pi q\theta)\,d\theta.\end{aligned}$$

Notice that if Ω has an integrable rational caustic of a rotation number $1/q$ for some $q > 2$, then, using the notation introduced in Theorem 1, perimeters $L_q^0(\theta)$ and $L_q'(\theta)$ of the q-gons inscribed in $\mathcal{E}$ and $\partial\Omega$, respectively, are constant. Therefore, Theorem 1 implies that the Deformation Function $\mathcal{D}(\theta)$ is $Cq^8\|\mathbf{n}\|_{C^1}^2$ close to a constant. Since, for any k, $\int_{k/q}^{(k+1)/q}\sin(2\pi q\theta)\,d\theta = 0$, we conclude that

$$\left|\int_0^1 \mathcal{D}(\theta)\sin(2\pi q\theta)\,d\theta\right| \le Cq^8\|\mathbf{n}\|_{C^1}^2$$

[5] We abuse notation and denote with $\mathbf{n}$ the perturbation as a function of the Lazutkin coordinate x; observe that since the change of variable is analytic, norms in arc-length and Lazutkin parametrization differ by some constant depending on e.

On the other hand, let us rewrite $x = X_q(\theta)$, $\theta = X_q^{-1}(x)$: we obtain:

$$\begin{aligned}&\int_0^1 \mathbf{n}(x)\ [q\eta_q(x)]\ \sin(2\pi q X_q^{-1}(x))\ dX_q^{-1}(x)\\ &= w_q\int_0^1 \mathbf{n}(x)\ \mu(x)\ \frac{q\eta_q(x)}{w_q\mu(x)}\ \frac{1}{X_q'(X_q^{-1}(x))}\ \sin(2\pi q X_q^{-1}(x))\,dx\\ &= w_q\int_0^1 \mathbf{n}(x)\mu(x)s_q(x)dx,\end{aligned}$$

which gives the required inequality for s_q. Repeating the argument verbatim, replacing $\sin(2\pi q\theta)$ with $\cos(2\pi q\theta)$ gives the corresponding inequality for c_q; this concludes the proof. □

Lemma 10. *Let $\mathbf{n}(x)$ be a C^1 function, $\mathcal{E}_e$ be an ellipse of eccentricity e and perimeter 1. Then there is $C = C(e) > 0$ such that for each $q > 2$ we have*

$$\left|\int \mathbf{n}(x)\mu(x)c_q(x)dx\right| \le \frac{C\|\mathbf{n}\|_{C^1}}{q}, \qquad \left|\int \mathbf{n}(x)\mu(x)s_q(x)dx\right| \le \frac{C\|\mathbf{n}\|_{C^1}}{q}.$$

Remark 11. *In the above lemma, $C(e)$ does not tend to 0 together with e.*

Proof. Since $\mu(x)$ is analytic, the function $\mathbf{n}(x)\mu(x)$ is C^1-smooth; hence, its q-th Fourier coefficients

$$\int \mathbf{n}(x)\,\mu(x)\ \sin(2\pi qx)\,dx, \qquad \int \mathbf{n}(x)\,\mu(x)\ \cos(2\pi qx)\,dx$$

are, in absolute value, bounded above by $c\|\mathbf{n}\|_{C^1}q^{-1}$ for some $c = c(e)$. Using Lemma 8 we have that the maximal difference $|c_q(x) - \cos(2\pi qx)|$ and $|s_q(x) - \sin(2\pi qx)|$ is $C^*(e)q^{-2}$. This implies the required estimate. □

6. Selection of translational, rotational, and deformational functional directions

In this section we introduce the "missing" 5 Deformed Fourier Modes, denoted with $c_0, s_q, c_q,\ q = 1, 2$. These five functions correspond to homothety, a pair of translation functions, a rotation, and a deformation of an ellipse to another of nearly identical eccentricity.

In principle, we can define the first four of these functions for an arbitrary smooth convex domain Ω_0; we refrain to do so since all remaining Deformed Fourier Modes have been defined only for ellipses. The reader could easily modify our construction and apply it to the more general case.

In order to define $s_q, c_q,\ q = 0, 1, 2$ we need to use the geometry of the ellipse $\mathcal{E}$. To fix ideas, assume that the origin $O \in \mathbb{R}^2$ is in the interior of $\mathcal{E}$. We construct c_0 to define homotheties, the first pair s_1, c_1 to define translations and the second pair s_2, c_2 to define rotations of $\mathcal{E}$ around the origin and deformations changing eccentricity.

Let (r,ϕ) denote polar coordinates and let $r(\phi)$ be the polar equation for $\mathcal{E}$, i.e. $\mathcal{E} = \{(r(\phi),\phi)\}_{\phi\in\mathbb{T}}$. Let s be length parametrization of $\mathcal{E}$ starting at $(0,r(0))$ and $s(\phi)$ be the corresponding function, which is invertible and let $\phi(s)$ denote its inverse. Recall that we can assume without loss of generality that the perimeter of $\mathcal{E}$ is 1. Let $\psi(\phi)$ be angle between the normal to $\mathcal{E}$ at $(\phi, r(\phi))$ and the radial direction, measured in the counterclockwise direction. Naturally, $\psi(s) := \psi(\phi(s)),\ r(s) := r(\phi(s))$ and all functions on $\mathcal{E}$ can be given with respect to either the ϕ-parametrization or the s-parametrization and differ via an analytic change of variable.

Consider the ellipse $\mathcal{E}^{\mathrm{h}}$ obtained by replacing the radial component $r(\phi)$ with $(1+\varepsilon)r(\phi)$ and denote with $\mathbf{n}^{\mathrm{h}}$ the corresponding perturbation function so that $\mathcal{E}^{\mathrm{h}} = \mathcal{E} + \mathbf{n}^{\mathrm{h}}$. Let $\mathbf{n}^{\mathrm{h}}_*(s) = \varepsilon r(s)\cos\psi(s)$.

Lemma 12. *For C depending on the eccentricity e we have*

$$\|\mathbf{n}^h - \mathbf{n}^h_*\|_{C^{39}} \le C\|\mathbf{n}^h_*\|^2_{C^{39}}.$$

Likewise, for any unit vector (a_1,b_1), consider the ellipse $\mathcal{E}^{\mathrm{t}}$ obtained by translating $\mathcal{E}$ by a vector $\varepsilon(a_1,b_1)$ and denote with $\mathbf{n}^{\mathrm{t}}$ the corresponding perturbation function. Choose $\alpha\in[0,2\pi)$ and such that $\tan\alpha = b_1/a_1$. Let $\mathbf{n}^{\mathrm{t}}_*(s) = \varepsilon\cos(\phi(s) - \alpha + \psi(s))$.

Lemma 13. *For C depending on the eccentricity e we have:*

$$\|\mathbf{n}^t - \mathbf{n}^t_*\|_{C^{39}} \le C\|\mathbf{n}^t_*\|^2_{C^{39}}.$$

Define

$$c_0(s) := r(\phi(s))\cos\psi(s),\quad c_1(s) := \cos(\phi(s)+\psi(s)),\quad s_1(s) := \sin(\phi(s)+\psi(s)).$$

Then using the last two functions we can realize any translation $\varepsilon(a_1,b_1)$ up to ε^2 by choosing $\mathbf{n} = \varepsilon(a_1c_1 + b_1s_1)$.

Consider now an ellipse $\mathcal{E}^{\mathrm{r}}$ obtained by rotating $\mathcal{E}$ by angle ε around the origin and denote with $\mathbf{n}^{\mathrm{r}}$ the corresponding perturbation function. Let $\mathbf{n}^{\mathrm{r}}_*(s) = \varepsilon r(s)\sin\psi(s)$.

Lemma 14. *For C depending on the eccentricity e we have*

$$\|\mathbf{n}^r - \mathbf{n}^r_*\|_{C^{39}} \le C\|\mathbf{n}^r_*\|^2_{C^{39}}.$$

Now, let $\mathcal{E}$ an ellipse of eccentricity e and semimajor axis 1, i.e.

$$\mathcal{E} = \left\{x^2 + \frac{y^2}{1-e^2} = 1\right\}.$$

Denote by $r_e(\phi)$ and $\psi_e(\phi)$ parameters associated to this domain. For small ε consider an ε-deformation of $\mathcal{E}_\varepsilon$ into the ellipse

$$\mathcal{E}^{\mathrm{e}} = \left\{\frac{x^2}{(1+\varepsilon)^2} + \frac{(1+\varepsilon)^2y^2}{1-e^2} = 1\right\}$$

obtained by hyperbolic rotation

$$L_\varepsilon : (x, y) \to ((1+\varepsilon)x, (1+\varepsilon)^{-1}y). \tag{20}$$

The eccentricity of this ellipse is $e' = e + c\varepsilon + O(\varepsilon^2)$, where $c = c(e)$. Let $\mathbf{n}^{\mathrm{e}}$ be the corresponding perturbation function. Define the function

$$\mathbf{n}^{\mathrm{e}}_*(s) = \varepsilon r(s) \cos(2\phi(s) + \psi(s)).$$

Lemma 15. *For C depending on the eccentricity e we have*

$$\|\mathbf{n}^e - \mathbf{n}^e_*\|_{C^{39}} \le C\|\mathbf{n}^e_*\|^2_{C^{39}}.$$

Proofs of Lemmata 12-15. The proofs follow from elementary geometry and are left to the reader. □

Remark 16. *The number 39 in the statements of Lemmata 12-15 could in fact be replaced with r for any $r \ge 0$, since all perturbations involved are analytic functions.*

Suppose $0 < e_0 < 1$ and $0 \le e < e_0$. Notice that $\psi_e(k\pi/2) = 0,\ k = 0, 1, 2, 3$, while $r_e(\phi)$ is analytic and strictly positive. Define $\max_\phi |\sin\psi_e(\phi)| = \varrho(e)$. Naturally, $\varrho(e) \in [0, 1)$ and $\psi_e(\phi) \in (-\pi/2, \pi/2)$ for all ϕ. Then there is a function $\theta_e(\phi)$ such that $\theta_e(k\pi/2) = k\pi,\ k = 0, 1, 2, 3$, and

$$\sin\psi_e(\phi) = \varrho(e)\sin 2\theta_e(\phi).$$

There is a function $\theta^*_e(\phi)$ such that such that $\theta_e(k\pi/2) = k\pi/2,\ k = 0, 1, 2, 3$, and

$$\cos(2\phi + \psi_e(\phi)) = \cos 2\theta^*_e(\phi).$$

Notice that as $e \to 0$ we have that

$$\max_\phi\{|\theta^*_e(\phi) - \phi|, |\theta_e(\phi) - \phi|\} \to 0.$$

Finally, we define

$$c_2(s) := \cos 2\theta^*_e(\phi(s)), \qquad s_2(s) := \sin 2\theta_e(\phi(s))$$

We can now extend Lemma 8:

Lemma 17. *In the notation of Lemma 8 and possibly increasing $C^*(e)$, for any positive integer q we have*

$$\|c_0 - 1\|_{C^0} \le C^*(e), \quad \|c_q - \cos(2\pi q\cdot)\|_{C^0} \le \frac{C^*(e)}{q^2}, \quad \|s_q - \sin(2\pi q\cdot)\|_{C^0} \le \frac{C^*(e)}{q^2}.$$

Proof. The case $q > 2$ is covered by Lemma 8. The case $q = 0, 1, 2$ can be done by direct inspection of the definition of these functions. □

7. The Deformed Fourier basis

In the previous section we completed the definition of the Deformed Fourier modes by introducing the first 5 modes; in this section, for convenience of notation let us rename the functions c_k and s_k as follows: for $j \geq 0$ define e_j so that $e_{2j} = c_j$ and $e_{2j+1} = s_{j+1}$; let us also introduce the corresponding Fourier Modes $e^{\mathbb{F}}_j$ so that $e^{\mathbb{F}}_{2j} = \cos(2\pi j\cdot)$ and $e^{\mathbb{F}}_{2j+1} = \sin(2\pi(j+1)\cdot)$.

Let us define the following operator acting on L^2:

$$\mathcal{L} : v(x) \mapsto \sum_{j=0}^{\infty} \hat{v}_j e_j(x)$$

where $\hat{v}_j$ is the j-th Fourier coefficient of v, i.e. $v = \sum_{j=0}^{\infty} \hat{v}_j e^{\mathbb{F}}_j$. In the sequel we will denote by $\|\cdot\|_{L^2\to L^2}$ the usual operator norm in L^2 given by:

$$\|T\|_{L^2\to L^2} = \sup_{f:\,\|f\|_{L^2}\leq 1} \|T f\|_{L^2}.$$

Proposition 18. *There exists $e_* > 0$ so that if $\mathcal{E}$ is an ellipse of eccentricity $e \in [0, e_*]$, the operator $\mathcal{L}$ is bounded and invertible as an operator from L^2 to L^2.*

Proof. We will proceed in two steps: for some large positive integer $N > 2$ to be specified later, we introduce an auxiliary list of vectors

$$\mathcal{B}_N = (e_0, e_1, \cdots, e_{2N}, e^{\mathbb{F}}_{2N+1}, e^{\mathbb{F}}_{2N+2}, \cdots)$$

We prove that $\mathcal{B}_N$ forms a basis of L^2 provided that the eccentricity e is sufficiently small. Then, using this fact we will prove that $\mathcal{B}$ is indeed a basis of L^2. Let M_N be the $(2N + 1) \times (2N + 1)$ *correlation matrix* whose (i, j)-entry is given by

$$[M_N]_{ij} = \int e^{\mathbb{F}}_i(x) e_j(x) dx. \tag{21}$$

Remark 19. *Observe that if $e = 0$ (i.e. if $\mathcal{E}$ is a circle) for any N, the matrix M_N is a multiple of the identity, because Lemma 17 implies that $e_q = e^{\mathbb{F}}_q$. Since M_N depends analytically on e, we conclude that for any N there exists $e_*(N)$ so that M_N is invertible for every $0 \leq e < e_*(N)$. Moreover, since if $N' > N$, the matrix $M_{N'}$ contains the matrix M_N as a minor; we conclude that if e is so that $M_{N'}$ is invertible, so is M_N.*

Lemma 20. *Let $N > 2$ be an integer so that the matrix M_N, defined by (21), is invertible. Then $\mathcal{B}_N$ is a basis of L^2.*

Proof. Observe that $\mathcal{B}_N$ can be obtained by the Fourier Basis $(e^{\mathbb{F}}_q)_{q\geq 0}$ by replacing the first $2N + 1$ basis elements with $(e_q)_{0\leq q\leq 2N}$. In order to show that $\mathcal{B}_N$ is a basis it thus suffices to check that every element of $(e^{\mathbb{F}}_q)_{0\leq q\leq N}$ can be expressed as a linear combination of elements in $\mathcal{B}_N$. To this end, invertibility of the matrix M_N is enough. □

The above lemma implies that we can write L^2 as the direct sum

$$L^2 = L^2_N \oplus \tilde{L}^2_N \tag{22}$$

where L^2_N is the subspace spanned by $(e_q)_{0\le q\le 2N}$ and $\tilde{L}^2_N$ is its complement spanned by $(e^{\mathbb{F}}_q)_{q>2N}$. Denote by $\tilde{\Pi}_N$ the (non-orthogonal) projection on $\tilde{L}^2_N$ (see Figure 3).

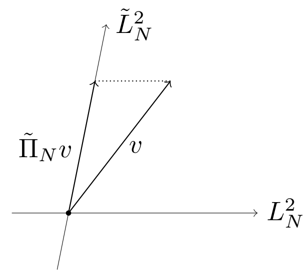

FIGURE 3. Direct sum decomposition of L^2 and definition of $\tilde{\Pi}_N$

Lemma 21. *Let $C^*(e)$ be the constant appearing in the statement of Lemma 17 and assume e to be so small that $C^*(e) < 1/2$. Then, in the above notations, we have:*

$$\|\tilde{\Pi}_N\|_{L^2\to L^2} \le (1 - 2C^*(e))^{-1}.$$

Proof. Observe that by [4, Theorem 2], we have

$$\|\tilde{\Pi}_N\|^2_{L^2\to L^2} = \left[1 - \sup_{v\in L^2_N,\, u\in\tilde{L}^2_N,\, \|v\|_{L^2}=\|u\|_{L^2}=1} \left|\int v(x)u(x)dx\right|^2\right]^{-1}. \tag{23}$$

Moreover, Lemma 17 implies, in particular:

$$\|e_n - e^{\mathbb{F}}_n\|_{L^2} \le C^*(e). \tag{24}$$

Since $v \in L^2_N$, we can write $v(x) = \sum_{k=0}^{2N} a_k e_k(x)$; let

$$\hat{v}(x) = v(x) - \sum_{k=0}^{2N} a_k e^{\mathbb{F}}_n(x).$$

By orthogonality of Fourier modes $\hat{v}(x) - v(x)$ is perpendicular to $\tilde{L}^2_N$. Since $u \in \tilde{L}^2_N$ and $\|u\|_{L^2} = 1$, application of the Schwartz inequality gives

$$\left|\int v(x)u(x)dy\right| = \left|\int \hat{v}(x)u(x)dx\right| \le \|\hat{v}\|_{L^2}.$$

Moreover,

$$\|\hat{v}\|_{L^2}^2 = \int \left| \sum_{k=0}^{2N} a_k (e_k(x) - e_k^{\mathbb{F}}(x)) \right|^2 dx$$

and using again the Schwartz inequality and (24), we conclude:

$$\|\hat{v}\|_{L^2}^2 \le \left[\sum_{k=0}^{2N} a_k^2 \right] C^*(e)^2.$$

On the other hand, $\|v\|_{L^2} = 1$ and we can also write:

$$\|\hat{v}\|_{L^2} = \left\| \sum_{k=0}^{2N} a_k e_k^{\mathbb{F}} - v \right\|_{L^2} \ge \left[\sum_{k=0}^{2N} a_k^2 \right]^{1/2} - 1.$$

Combining the two estimates for $\|\hat{v}\|_{L^2}$ and using that $C^*(e) < 1/2$ yields:

$$\left[\sum_{k=0}^{2N} a_k^2 \right]^{1/2} \le (1 - C^*(e))^{-1},$$

which in turn allows to conclude the proof by plugging this estimate in (23). □

By Lemma 20 we know that $\mathcal{B}_N$ is a basis of L^2, hence we can write any $v \in L^2$ as

$$v = \sum_{k=0}^{2N} a_k e_k + \sum_{k=2N+1}^{\infty} a_k e_k^{\mathbb{F}}$$

for some unique sequence $(a_k)_{k \ge 0}$. Define the operator on L^2 given by:

$$\mathcal{L}_N : v \mapsto \sum_{k=0}^{\infty} a_k e_k.$$

Lemma 22. *If $N > 2$, M_N is invertible, and e so small that*

$$C^*(e) < \frac{1}{2} \left[1 - \frac{1}{1 + N^{3/2}} \right] \tag{25}$$

then $\|\mathcal{L}_N - \mathrm{Id}\|_{L^2 \to L^2} \le \frac{1}{2}$. In particular, $\mathcal{L}_N$ is a bounded and invertible as an operator from L^2 to L^2.

Proposition 18 then immediately follows from the above lemma. □

Proof of Lemma 22. By definition, $L_N^2 \subset \ker[\mathcal{L}_N - \mathrm{Id}]$, thus, for any $v \in L^2$:

$$[\mathcal{L}_N - \mathrm{Id}]\, v = [\mathcal{L}_N - \mathrm{Id}]\, \tilde{\Pi}_N\, v,$$

Hence by Lemma 21:

$$\begin{aligned}\|\mathcal{L}_N - \mathrm{Id}\|_{L^2\to L^2} &= \sup_{v:\|v\|_{L^2}\leq 1} \|[\mathcal{L}_N - \mathrm{Id}]v\|_{L^2}\\ &\leq \sup_{\tilde v\in\tilde L^2_N:\|\tilde v\|_{L^2}\leq(1-2C^*(e))^{-1}} \|[\mathcal{L}_N - \mathrm{Id}]\tilde v\|_{L^2}.\end{aligned}$$

By definition there exists a sequence $(a_q)_{q>2N}$ so that

$$\tilde v = \sum_{q=2N+1}^{\infty} a_q e^{\mathbb{F}}_q,$$

hence, by the Cauchy Inequality

$$\begin{aligned}\|[\mathcal{L}_N - \mathrm{Id}]\tilde v\|_{L^2} &\leq \sum_{q=2N+1}^{\infty} |a_q|\|e_q - e^{\mathbb{F}}_q\|_{L^2}\\ &\leq \left[\sum_{q=2N+1}^{\infty}|a_q|^2\right]^{1/2}\left[\sum_{q=2N+1}^{\infty}\|e_q - e^{\mathbb{F}}_q\|^2_{L^2}\right]^{1/2}.\end{aligned}$$

Thus, using Parseval identity we conclude $\sum_{q=2N+1}^{\infty}|a_q|^2 = \|\tilde v\|^2_{L^2}$ and, therefore, by Lemma 8 we gather

$$\begin{aligned}\|\mathcal{L}_N - \mathrm{Id}\|_{L^2\to L^2} &\leq \frac{C^*(e)}{1-2C^*(e)}\left[\sum_{q=2N+1}^{\infty}\frac{1}{q^4}\right]^{1/2}\\ &\leq \frac{C^*(e)}{2(1-2C^*(e))N^{3/2}} < \frac{1}{2},\end{aligned}$$

where the last inequality follows from (25). □

8. Proof of the Main Theorem

The proof of our Main Theorem relies on the following crucial

Lemma 23. *Let $\mathcal{E}_e$ be an ellipse of perimeter 1 and eccentricity $e\in[0,e_*]$ sufficiently small so that Lemma 22 applies. Let Ω be a rationally integrable C^{39} deformation identified by a C^{39} function $\mathbf{n}(x)$, i.e. $\partial\Omega := \mathcal{E}+\mathbf{n}$. Then there exists an ellipse $\bar{\mathcal{E}}$ and $\bar{\mathbf{n}}$ so that $\partial\Omega = \bar{\mathcal{E}}+\bar{\mathbf{n}}$ and*

$$\|\bar{\mathbf{n}}\|_{C^1} \leq C(e,\|\mathbf{n}\|_{C^{39}})\,\|\mathbf{n}\|^{703/702}_{C^1}.$$

Before giving the proof of Lemma 23, let us use it to give the

Proof of the Main Theorem. Let $\mathcal{E}_e$ be an ellipse of perimeter 1 and eccentricity $e\in[0,e_*)$, where e_* is the constant appearing in the statement of Lemma 23. Let us fix $K>0$ arbitrarily and ε sufficiently small to be specified later. Denote with

$\mathbb{E}_\varepsilon(\mathcal{E})$ the set of ellipses whose C^0-Hausdorff distance from $\mathcal{E}$ is not larger than 2ε, i.e. $\mathbb{E}_\varepsilon(\mathcal{E}) = \{\mathcal{E}' \subset \mathbb{R}^2, \mathrm{dist}_{\mathrm{H}}(\mathcal{E},\mathcal{E}') \le 2\varepsilon\}$. We assume ε so small that every $\mathcal{E}' \in \mathbb{E}_\varepsilon(\mathcal{E})$ has perimeter $\ell' \in [3/4, 5/4]$ and eccentricity $e' \in [0, e_*]$. Observe that any ellipse in $\mathbb{R}^2$ can be parametrized by 5 real quantities (e.g. the coefficients of the corresponding quadratic equation): let $A_\varepsilon(\mathcal{E})$ be the set of parameters $a \in \mathbb{R}^5$ corresponding to ellipses in $\mathbb{E}_\varepsilon(\mathcal{E})$; then $A_\varepsilon(\mathcal{E})$ is compact.

Let now $\mathbf{n}$ be a C^{39} perturbation with $\|\mathbf{n}\|_{C^{39}} < K$ and $\|\mathbf{n}\|_{C^1} < \varepsilon$ and consider the domain Ω given by

$$\partial\Omega = \mathcal{E}_e + \mathbf{n}.$$

For any 5-tuple of parameters $a \in A$ we associate the corresponding ellipse $\mathcal{E}_a$ and perturbation $\mathbf{n}_a$ so that $\partial\Omega = \mathcal{E}_a + \mathbf{n}_a$. Observe that the tubular coordinates (s, n) of Ω change analytically with respect to a, hence $\mathbf{n}_a$ varies analytically with respect to a. In particular, we can assume ε so small that for any $a \in A_\varepsilon(\mathcal{E})$, $\|\mathbf{n}_a\|_{C^{39}} < 2K$. Moreover, the function $a \mapsto \|\mathbf{n}_a\|_{C^1}$ is a continuous function and as such it will have a minimum, which we denote by $a_* \in A_\varepsilon(\mathcal{E})$. Let $\mathcal{E}_*$ and $\mathbf{n}_*$ be the corresponding ellipse and perturbation, respectively; then

$$0 \le \|\mathbf{n}_*\|_{C^1} \le \|\mathbf{n}\|_{C^1} \le \varepsilon.$$

Modulo a possible linear rescaling (which also rescales linearly $\mathbf{n}$, since the Lazutkin perimeter is normalized to be 1) we can assume that $\bar{\mathcal{E}}$ has perimeter 1 and, thus, apply Lemma 23 to $\mathcal{E}_*$ and $\mathbf{n}_*$ and obtain $\bar{\mathcal{E}}_*$ and $\bar{\mathbf{n}}_*$. But if ε is small enough, then there exists $\varrho \in (0, 1)$ so that $\|\bar{\mathbf{n}}_*\|_{C^1} \le \varrho\|\mathbf{n}_*\|_{C^1}$. Hence, by the triangle inequality,

$$\mathrm{dist}_{\mathrm{H}}(\mathcal{E}, \bar{\mathcal{E}}_*) \le \mathrm{dist}_{\mathrm{H}}(\mathcal{E}, \Omega) + \mathrm{dist}_{\mathrm{H}}(\Omega, \bar{\mathcal{E}}_*) \le (1 + \varrho)\varepsilon < 2\varepsilon$$

thus $\bar{\mathcal{E}}_* \in \mathbb{E}_\varepsilon(\mathcal{E})$. Since $\|\mathbf{n}_*\|_{C^1}$ was minimal, we conclude that $\|\mathbf{n}_*\|_{C^1} = \|\bar{\mathbf{n}}_*\|_{C^1} = 0$, i.e. $\Omega = \mathcal{E}_*$ is an ellipse. □

We conclude this article by giving the

Proof of Lemma 23. Let us once again rename the basis vectors c_k and s_k as follows: let $e_j, j \ge 0$ so that $e_{2j} = c_j$ and $e_{2j+1} = s_{j+1}$. First, we claim that the vectors $\{e_j\}_{0\le j\le 4}$ are μ-orthogonal to the subspace generated by $\{e_j\}_{j>4}$. Indeed, for any fixed $0 \le j \le 4$ and $\varepsilon > 0$ small, consider the deformation of the ellipse $\mathcal{E}_e$ into the ellipse[6] $\mathcal{E}_{e'}(\varepsilon) = \mathcal{E}_e + \varepsilon e_j + O(\varepsilon^2)$. Certainly, all caustics $\Gamma_{1/q}$ with $q > 2$ are preserved; therefore, by Lemma 9, for $4 < q \le \varepsilon^{-1/9}$ we can conclude:

$$\left|\varepsilon \int e_j(x)\ \mu(x)\, e_q(x)\, dx\right| \le Cq^8\varepsilon^2 \le C\varepsilon^{10/9}. \tag{26}$$

Since ε can be chosen arbitrarily and the functions $\{e_j\}$ do not depend on the perturbation, but only on $\mathcal{E}$, we proved μ-orthogonality.

[6] Indeed $e = e'$ unless $j = 4$.

Now, let us decompose

$$\mathbf{n}(x) = \mathbf{n}^{(5)}(x) + \mathbf{n}^{\perp}(x) \tag{27}$$

where $\mathbf{n}^{\perp}$ is μ-orthogonal to the subspace spanned by $\{e_j\}_{j\leq 4}$ and $\mathbf{n}^{(5)}$ is its complement; then $\mathbf{n}^{(5)} = \sum_{j=0}^{4} a_j e_j$.

We claim that $|a_j| < C\|\mathbf{n}\|_{C^1}$, where $C = C(e)$ depends on eccentricity only. By μ-orthogonality we have

$$\|\mathbf{n}^{(5)}\|^2_{L^2_\mu} + \|\mathbf{n}^{\perp}\|^2_{L^2_\mu} = \|\mathbf{n}\|^2_{L^2_\mu} \leq \|\mathbf{n}\|^2_{C^1},$$

where $\|\cdot\|_{L^2_\mu}$ denotes the L^2 norm induced by the inner product with the weight μ, i.e. $\|f\|_{L^2_\mu} = \|\sqrt{\mu} f\|_{L^2}$; clearly this norm is equivalent to the standard L^2 norm. In particular , we have $\|\mathbf{n}^{(5)}\|_{L^2} \leq C\|\mathbf{n}\|_{C^1}$. This implies that $|a_j| \leq C\|\mathbf{n}\|_{C^1}$ for a constant depending on e. Since all $e_j, 0 \leq j \leq 4$ are analytic, we also have

$$\|\mathbf{n}^{(5)}\|_{C^{39}} < C\|\mathbf{n}\|_{C^1}. \tag{28}$$

Now let $\bar{\mathcal{E}}(a_0, \cdots, a_4)$ be the ellipse obtained by applying to $\mathcal{E}$ the homothety by $(1+a_0)$, the translation in the direction (a_1, a_2), the rotation by a_3 around the origin and the hyperbolic rotation L_{a_4}, defined in (20) and let $\bar{\mathbf{n}}$ be the corresponding perturbation function so that $\bar{\mathcal{E}} = \mathcal{E} + \bar{\mathbf{n}}$. By Lemmata 12–15 we conclude that

$$\|\bar{\mathbf{n}} - \mathbf{n}^{(5)}\|_{C^{39}} \leq C\|\mathbf{n}\|^2_{C^1}. \tag{29}$$

Next, we show that the component $\mathbf{n}^{\perp}$ of the decomposition (27) is L^2-small and then deduce that it is indeed C^1-small.

Let us define the operator $\mathcal{L}_\mu$ from $L^2 \to L^2$ given by $\mathcal{L}_\mu v(x) = \mu(x)\cdot[\mathcal{L}v](x)$; then by Proposition 18 and since both $\mu(x)$ and $\mu(x)^{-1}$ are both bounded and analytic, we conclude that $\mathcal{L}_\mu : L^2 \to L^2$ is a bounded invertible operator; therefore, so is its adjoint $\mathcal{L}^*_\mu$. Hence, using Parseval's Identity:

$$\begin{aligned}\|\mathbf{n}^{\perp}\|^2_{L^2} = \|(\mathcal{L}^*_\mu)^{-1}\mathcal{L}^*_\mu \mathbf{n}^{\perp}\|^2_{L^2} \leq C\|\mathcal{L}^*_\mu \mathbf{n}^{\perp}\|^2_{L^2} = C\sum_{q=0}^{\infty}\left|\int \mathcal{L}^*_\mu(\mathbf{n}^{\perp})e^{\mathbb{F}}_q\right|^2 \\ = C\sum_{q=0}^{\infty}\left|\int \mathbf{n}^{\perp}\mathcal{L}_\mu(e^{\mathbb{F}}_q)\right|^2 \leq C\sum_{q=0}^{\infty}\left|\int \mathbf{n}^{\perp}\mu e_q\right|^2 \leq C\sum_{q=5}^{\infty}|\tilde{n}_q|^2,\end{aligned}$$

where we used the fact that $\mathcal{L}_\mu e^{\mathbb{F}}_q = \mu\cdot\mathcal{L}e^{\mathbb{F}}_q = \mu\cdot e_q$ and $\tilde{n}_q$ is defined as:

$$\tilde{n}_q := \int_0^1 \mathbf{n}(x)\mu(x)e_q(x)dx.$$

Notice that these numbers are *not* the coefficients of the decomposition of $\mathbf{n}\cdot\mu$ in the basis $\mathcal{B}$, because $\mathcal{B}$ is not an orthonormal basis. Fix $\alpha < 1/8$ to be specified later and let $q_0 = [\|\mathbf{n}\|_{C^1}^{-\alpha}]$, where $[x]$ denotes the integer part of x; by Lemma 9, for any $4 < q \le q_0$, we have

$$|\tilde{n}_q| \le Cq^8\|\mathbf{n}\|_{C^1}^2 \le C\|\mathbf{n}\|_{C^1}^{2-8\alpha}$$

where C depends on e and on $\|\mathbf{n}\|_{C^5}$, but is independent of q. Then

$$\sum_{q=5}^{q_0} |\tilde{n}_q|^2 \le C\|\mathbf{n}\|_{C^1}^{4-17\alpha}.$$

We now apply Lemma 10 to $\tilde{n}_q$ for $q > q_0$: we obtain

$$|\tilde{n}_q|^2 \le C\frac{\|\mathbf{n}\|_{C^1}^2}{q^2}.$$

Therefore, for $q \ge q_0$ we have that

$$\sum_{q=q_0+1}^{\infty} |\tilde{n}_q|^2 \le C\|\mathbf{n}\|_{C^1}^{2+\alpha}.$$

Combining the two above estimates and optimizing for α (i.e. we choose $\alpha = 1/9$), we conclude that $\|\mathbf{n}^\perp\|_{L^2} \le C\|\mathbf{n}\|_{C^1}^{19/18}$: in order to upgrade this L^2 estimate to a C^1 estimate, first, observe that we have:

$$\|\mathbf{n}^\perp\|_{C^1} \le \|D\mathbf{n}^\perp\|_{L^1} + \|D^2\mathbf{n}^\perp\|_{L^1} \le \|D\mathbf{n}^\perp\|_{L^2} + \|D^2\mathbf{n}^\perp\|_{L^2}.$$

We then use standard Sobolev interpolation inequalities (see e.g. [6]): for any $\delta > 0$ and any $1 \le j \le 2$ we have,

$$\|D^j\mathbf{n}^\perp\|_{L^2} \le C\left[\delta\|\mathbf{n}^\perp\|_{C^{39}} + \delta^{-j/(39-j)}\|\mathbf{n}^\perp\|_{L^2}\right].$$

Optimizing the above estimate[7], we choose $\delta = \|\mathbf{n}\|_{C^1}^{703/702}$. Observe that $\|\mathbf{n}^\perp\|_{C^{39}}$ is uniformly bounded using (28). Thus, we conclude that

$$\|\mathbf{n}^\perp\|_{C^1} \le C(e, \|\mathbf{n}\|_{C^{39}})\|\mathbf{n}\|_{C^1}^{703/702}.$$

Observe, moreover, that C above depends monotonically on $\|\mathbf{n}\|_{C^{39}}$.

Hence, we have:

$$\Omega = \mathcal{E} + \bar{\mathbf{n}} + \left[\mathbf{n}^{(5)} - \bar{\mathbf{n}} + \mathbf{n}^\perp\right]$$

where by the above estimate and (29) we gather

$$\|\mathbf{n}^{(5)} - \bar{\mathbf{n}} + \mathbf{n}^\perp\|_{C^1} < C(e, \|\mathbf{n}\|_{C^{39}})\|\mathbf{n}\|_{C^1}^{703/702}.$$

Then $\Omega = \bar{\mathcal{E}} + \bar{\mathbf{n}}$, where $\bar{\mathbf{n}}$ is obtained by $[\mathbf{n}^{(5)} - \bar{\mathbf{n}} + \mathbf{n}^\perp]$ via the analytic transformation which maps Lazutkin tubular coordinates in a neighborhood of $\mathcal{E}$ to Lazutkin tubular

[7] The number 39 has indeed been chosen to be minimal among those for which the above interpolation inequality provides an useful bound.

coordinates in a neighborhood of $\bar{\mathcal{E}}$; since this transformation is $O(\|\mathbf{n}\|_{C^1})$-close to the identity, we conclude our proof. □

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