# Constructing many atomic models in $\aleph_1$

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## 1 Introduction

As has been known since at least [**?**] and is carefully spelled out in Chapter 6 of [**?**], for every complete sentence $\psi$ of $L_{\omega_1,\omega}$ (in a countable vocabulary $\tau$) there is a complete, first order theory $T$ (in a countable vocabulary extending $\tau$) such that the models of $\psi$ are exactly the $\tau$-reducts of the atomic models of $T$. This paper is written entirely in terms of the class $\mathbf{At_T}$ of atomic models of a complete first order theory $T$, but applies to $L_{\omega_1,\omega}$ by this translation.

Our main theorem, Theorem 2.8, asserts: Let $T$ be any complete first-order theory in a countable language with an atomic model. If the pseudo-minimal types are not dense, then there are $2^{\aleph_1}$ pairwise non-isomorphic, full[1] atomic models of $T$, each of size $\aleph_1$.

The first section states some old observations about atomic models and develops a notion of 'algebraicity', dubbed pseudo-algebraicity for clarity, that is relevant in this context. We introduce the relevant analogue to strong minimality, pseudo-minimality, and state the pseudo-minimals dense/many models dichotomy. Section 3 expounds a transfer techinique, already used in [**?**] and [**?**] and applied here prove to Theorem 2.8. The gist of the method is to prove a model theoretic property is consistent with ZFC by forcing and then extend the model $M$ of set theory witnessing this result to a model $N$, preserving the property and such that the property is absolute between $V$ and $N$. Section 4 describes a forcing construction, which together with the results of Section 3, yields a proof of Theorem 2.8 in Section 5.

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[1] An atomic model $M$ is *full* if $|\phi(M,\overline{a})| = ||M||$ for every non-pseudo-algebraic formula $\phi(x,\overline{a})$ with $\overline{a}$ from $M$.

## 2 A notion of algebraicity

{1}

Throughout this paper, $T$ will always denote a complete, first-order theory in a countable language that has an atomic model. By definition, a model $M$ of $T$ is atomic if every finite tuple $\overline{a}$ from $M$ realizes a complete formula[2]. The existence of an atomic model is equivalent to the statement that 'every consistent formula $\phi(\overline{x})$ has a complete formula $\psi(\overline{x})$ implies it.' Equivalently, $T$ has an atomic model if and only if, for every $n \geq 1$, the isolated complete $n$-types are dense in the Stone space $S_n(\emptyset)$. We recall some old results of Vaught concerning this context.

{Vaught}

**Fact 2.1.** *Let $T$ be any complete theory in a countable language having an atomic model. Then:*

1. $\mathbf{At_T}$ *is $\aleph_0$-categorical, i.e., every pair of countable atomic models are isomorphic;*
2. $\mathbf{At_T}$ *contains an uncountable model if and only if some/every countable model of* $\mathbf{At_T}$ *has a proper elementary extension.*

The only known arguments for proving amalgamation and thus constructing monster models for $\mathbf{At_T}$ invoke the continuum hypothesis and so are not useful for our purposes. Nevertheless, we argue that many concepts of interest are in fact model independent.

In first-order model theory, if a formula $\phi(x, \overline{a})$ is algebraic, then its solution set cannot be increased in any elementary extension, i.e., if $\overline{a} \subseteq M \preceq N$, then $\phi(M, \overline{a}) = \phi(N, \overline{a})$. However, in the atomic case, the analogous phenomenon can be witnessed by non-algebraic formulas. For example, $(\mathbb{Z}, S)$, the integers with a successor function, is an atomic model of its theory. The formula '$x = x$' is not algebraic, yet $(\mathbb{Z}, S)$ has no proper **atomic** elementary extensions. This inspires the following definition:

{countable}

**Definition 2.2.** Let $M \in \mathbf{At_T}$ be countable[3]. A formula $\phi(x, \overline{a})$ is *pseudo-algebraic in $M$* if $\overline{a}$ is from $M$, and $\phi(N, \overline{a}) = \phi(M, \overline{a})$ for every countable $N \in \mathbf{At_T}$ with $N \succeq M$.

The strong $\aleph_0$-homogeneity (any two finite sequences realizing the same type over the emptyset are automorphic) of the countable atomic model of $T$ yields immediately that pseudo-algebraicity truly depends only on the type of $\overline{a}$ over the emptyset. That is, if

[2] Recall that $\phi(\overline{x})$ is a *complete formula in T* if $\phi(\overline{x})$ is the generator of a principal type, i.e. for every $\psi(\overline{x})$, $T \vdash (\forall \overline{x})[\phi(\overline{x}) \rightarrow \psi(\overline{x})]$ or $T \vdash (\forall \overline{x})[\phi(\overline{x}) \rightarrow \neg\psi(\overline{x})]$ .

[3] In Definition 2.2 it would be equivalent to restrict to countable and $M$ and allow arbitrary cardinality for $N$. It would **not** be equivalent to assert for arbitrary $M$: "$\phi(x, \overline{a})$ is pseudo-algebraic in $M$ if and only if $\phi(M, \overline{a}) = \phi(N, \overline{a})$ for every $N \succeq M$." To see the distinction, consider the extreme case where $M$ is an uncountable atomic model that is maximal, i.e., has no proper atomic elementary extension.

$M, M' \in \mathbf{At_T}$ are each countable and $\mathrm{tp}(\overline{a}, M) = \mathrm{tp}(\overline{a}', M')$, then $\phi(x, \overline{a})$ is pseudo-algebraic in $M$ if and only if $\phi(x, \overline{a}')$ is pseudo-algebraic in $M'$. This observation allows us to extend the notion of pseudo-algebraicity to arbitrary atomic models of $T$.

{arbitrary}

**Definition 2.3.** Let $N \in \mathbf{At_T}$ have arbitrary cardinality.

1. A formula $\phi(x, \overline{a})$ is *pseudo-algebraic in* $N$ if $\overline{a}$ is from $N$, and $\phi(x, \overline{a})$ is pseudo-algebraic in $M$ for some (equivalently, for every) countable $M \preceq N$ containing $\overline{a}$.

2. An element $b \in N$ is *pseudo-algebraic over* $\overline{a}$ *inside* $N$, written $b \in \mathrm{pcl}(\overline{a}, N)$, if $\mathrm{tp}(b/\overline{a}, N)$ contains a formula that is pseudo-algebraic in $N$.

3. Given an infinite subset $A \subseteq N$, $b$ *is pseudo-algebraic over* $A$ *in* $N$, written $b \in \mathrm{pcl}(A, N)$, if and only if $b \in \mathrm{pcl}(\overline{a}, N)$ for some finite $\overline{a} \in A^n$.

As the language of $T$ is countable, for any complete formula $\theta(\overline{y})$, there is a formula $\psi(x, \overline{y})$ of $L_{\omega_1,\omega}$ such that $T \cup \{\psi(x, \overline{y})\} \vdash \theta(\overline{y})$ and for every atomic $M$, every $\overline{a} \in \theta(M)$, and every $b \in M$:

$$b \in \mathrm{pcl}(\overline{a}, M) \qquad \text{if and only if} \qquad M \models \psi(b, \overline{a})$$

Note that this notion allows us to reword Fact 2.1(2): $T$ has an uncountable atomic model if and only if '$x = x$' is not pseudo-algbraic. Here is a second example.

**Example 2.4.** Let $L = \{A, B, \pi, S\}$ and $T$ say that $A$ and $B$ partition the universe with $B$ infinite, $\pi : A \to B$ is a total surjective function and $S$ is a successor function on $A$ such that every $\pi$-fiber is the union of $S$-components. A model $M \models T$ is atomic if every $\pi$-fiber contains exactly one $S$-component. Now choose elements $a, b \in M$ for such an $M$ such that $a \in A$ and $b \in B$ and $\pi(a) = b$. Clearly, $a$ is not algebraic over $b$ in the classical sense, but $a \in \mathrm{pcl}(b, M)$.

Recall that a *t-construction over* $B$ is a sequence $\langle a_i : i < \omega \rangle$ such that, letting $A_i$ denote $B \cup \{a_j : j < i\}$, $\mathrm{tp}(a_i/A_i)$ is generated by a complete formula.

The notion of pseudo-algebraicity has many equivalents. Here are some we use below.

{pcleq}

**Lemma 2.5.** *Suppose* $M \in \mathbf{At_T}$ *and* $b, \overline{a}$ *are from* $M$. *The following are equivalent:*

1. $b \in \mathrm{pcl}(\overline{a}, M)$;

2. *For every* $N \preceq M$, *if* $\overline{a} \in N^n$, *then* $b \in N$;

3. *$b$ is contained inside any maximal $t$-construction sequence $\langle a_\alpha : \alpha < \beta \rangle$ over $\overline{a}$ inside $M$.*

For (3) note that as $T$ has an atomic model, a maximal $t$-construction sequence over a finite set is the universe of a model.

Here is one application of Lemma 2.5.

{outside}

**Lemma 2.6.** *Suppose that $M \in \mathbf{At_T}$, $\overline{a}$ is from $M$, but $\phi(x, \overline{a})$ is not pseudo-algebraic in $M$. Then for every finite $\overline{e}$ from $M$, there is $b \in \phi(M, \overline{a})$ with $b \notin \mathrm{pcl}(\overline{e}, M)$.*

**Proof.** We may assume $\overline{a} \subseteq \overline{e}$. Choose a countable $M^* \preceq M$ containing $\overline{e}$ and, by non-pseudo-algebraicity and Definition 2.2, choose a countable $N^* \in \mathbf{At_T}$ with $N^* \succeq M^*$ and $b^* \in \phi(N^*, \overline{a}) \setminus \phi(M^*, \overline{a})$. As $N^*$ is countable and atomic, choose an elementary embedding $f : N^* \to M$ that fixes $\overline{e}$ pointwise. Then $f(b^*) \in \phi(M, \overline{a})$ and $f(b^*) \notin \mathrm{pcl}(\overline{e}, M)$ as witnessed by $f(M^*)$ and Lemma 2.5(2). $\square_{2.6}$

> I was confused by your preferring to reference 2.3 in place of 2.2 in the Lemma above. I commented out your objection, but I think things are as they should be now. If you agree, you can simply delete this sidebar.

In general, the notion of pseudo-algebraic closure gives rise to a reasonable closure relation. All of the standard van der Waerden axioms for a dependence relation hold in general, with the exception of the Exchange Axiom. Our next definition isolates those formulas on which exchange (and a bit more) hold.

{psmdef}

**Definition 2.7.** Let $M$ be any atomic model and let $\overline{a}$ be from $M$.

- A complete formula $\phi(x, \overline{a})$ is *pseudo-minimal* if it is not pseudo-algebraic, but for every $\overline{a}^* \supseteq \overline{a}$ and $c$ from $M$ and for every $b \in \phi(M, \overline{a})$, if $c \in \mathrm{pcl}(\overline{a}^* b, M)$ but $c \notin \mathrm{pcl}(\overline{a}^*, M)$, then $b \in \mathrm{pcl}(\overline{a}^* c, M)$.
- The class $\mathbf{At_T}$ has *density of pseudo-minimal types* if for some/every $M \in \mathbf{At_T}$, for every non-pseudo-algebraic formula $\phi(x, \overline{a})$, there is $\overline{a}^* \supseteq \overline{a}$ from $M$ and a pseudo-minimal formula $\psi(x, \overline{a}^*)$ such that $\psi(x, \overline{a}^*) \vdash \phi(x, \overline{a})$.

It is immediate that if there is a non-pseudo-algebraic formula then $T$ has an atomic model in $\aleph_1$, so also if pseudo-minimal types are not dense, then $T$ has an atomic model in $\aleph_1$. The main Theorem of this paper is the following:

{big}

**Theorem 2.8.** *Let $T$ be any complete first-order theory in a countable language with an atomic model. If the pseudo-minimal types are not dense, then there are $2^{\aleph_1}$ pairwise non-isomorphic, full, atomic models of $T$, each of size $\aleph_1$.*

# 3 A technique for producing many models of power $\aleph_1$

{2}

The objective of this section is to prove the transfer Theorem 3.3.1 that allows the construction (in ZFC) of many atomic models of a first order theory $T$ in two steps. First force to find a model $(M, E)$ of set theory in which a model of $T$ is coded by stationary sets. Then apply the transfer theorem to code a family of such models in ZFC.

The method expounded here has many precursors. Among the earliest are the treatment of Skolem ultrapowers in [**?**] and the study of elementary extensions of models of set theory in [**?**] and [**?**]. Paul Larson introduced the use of iterated generic ultrapowers (used in the different context of Woodin's $\mathbb{P}$-max forcing) in a large cardinal context in [**?**, **?**] and the general method is abstracted in [**?**]. The model theoretic technique used here is described in [**?**] and [**?**]. We formulate a general metatheorem for the construction.

The first subsection describes how to define and maintain satisfaction of formulas in a pre-determined, countable fragment $L_{\mathcal{A}}$ under elementary extensions of $\omega$-models of set theory. Most of this is well-known; we emphasize that only an $\omega$-model and not transitivity is necessary to correctly code sentences of $L_{\omega_1,\omega}$. The second subsection surveys known results about $M$-normal ultrapowers, and Theorem 3.3.1 is proved in the third subsection.

## 3.1 Coding $\tau$-structures into non-transitive models of set theory

In this section, we fix an explicit encoding of a pre-determined countable fragment $L_{\mathcal{A}} = L_{\mathcal{A}}(\tau)$ of $L_{\omega_1,\omega}(\tau)$ for a countable vocabulary $\tau$ into an $\omega$-model $(M, E)$ satisfying $ZFC$. The specific form of this encoding is not important, but it is useful for the reader to see what we assume about $M$ in order that satisfaction is computed 'correctly' for every formula of $L_{\mathcal{A}}$. It will turn out that everything works wonderfully (even when $(M, E)$ is non-transitive) provided $(M, E)$ is an $\omega$-model (that is $\omega^M = \omega^V$), because this guarantees a formula of $L_{\mathcal{A}}$ does not gain additional conjuncts or disjuncts in an elementary extension that is also an $\omega$-model.

Slightly modified the previous sentence. ok?

**Definition 3.1.1.** We say $(M, E)$ is an *$\omega$-model of set theory* if $(M, E) \models ZFC$, $(\omega + 1)^{M,E} = \omega + 1$, and for $n, m \in \omega + 1$, $(M, E) \models nEm$ if and only if $n \in m$.

Fix any countable vocabulary (sometimes called language) $\tau$. In what follows, we will assume that $\tau$ is relational with $\aleph_0$ $n$-ary relation symbols $R^n_m$, but the generalization to other countable languages is obvious.

**Definition 3.1.2.** Fix a particular countable fragment $L_{\mathcal{A}} = L_{\mathcal{A}}(\tau)$ of $L_{\omega_1,\omega}(\tau)$.

- A *Basic Gödel number* has the form $\langle 0, n, m\rangle$, where $n, m \in \omega$. We write this as $\ulcorner R^n_m \urcorner$.
- Let $BG_\tau$ denote the set of Basic Gödel numbers. We now define by induction the set $G_{L_{\mathcal{A}}}$ of Gödel numbers of $L_{\mathcal{A}}$-formulas.

  1. $\ulcorner v_i \urcorner = \langle 1, i\rangle$;
  2. $\ulcorner R^n_m(v_{i_1}, \dots v_{i_n})\urcorner = \langle \ulcorner R^n_m \urcorner, \ulcorner v_{i_1}\urcorner, \dots, \ulcorner v_{i_n}\urcorner\rangle$
  3. $\ulcorner \phi = \psi \urcorner = \langle 2, \ulcorner \phi \urcorner, \ulcorner \psi \urcorner\rangle$;
  4. $\ulcorner \phi \wedge \psi \urcorner = \langle 3, \ulcorner \phi \urcorner, \ulcorner \psi \urcorner\rangle$;
  5. $\ulcorner \exists v_i \phi \urcorner = \langle 4, \ulcorner v_i \urcorner, \ulcorner \phi \urcorner\rangle$;
  6. $\ulcorner \neg\phi \urcorner = \langle 5, \ulcorner \phi \urcorner\rangle$;
  7. If $\psi = \bigwedge_{i\in\omega} \theta_i$ and $\psi \in L_{\mathcal{A}}$, then $\ulcorner \psi \urcorner = \langle 6, f_\psi\rangle$, where $f_\psi$ is the function with domain $\omega$ and $f_\psi(i) = \ulcorner \theta_i \urcorner$.

**Definition 3.1.3.** For a given countable fragment $L_{\mathcal{A}}$, we say an $\omega$-model $(M, E)$ *supports* $L_{\mathcal{A}}$ if $G_{L_{\mathcal{A}}} \in M$ and $G_{L_{\mathcal{A}}} \subseteq M$.

> Added the condition $G_{L_{\mathcal{A}}} \in M$ to the preceding definition. I think this makes the issue of definability go away.

Note that $BG_\tau$ and $G_{L_{\mathcal{A}}}$ are defined in $V$ but they are correctly identified by an $(M, E)$ that supports $L_{\mathcal{A}}$. More precisely, the following lemma is immediate.

**Lemma 3.1.4.** *If $(M, E)$ is an $\omega$-model of set theory supporting $L_{\mathcal{A}}$, then both $BG_\tau$ and $G_{L_{\mathcal{A}}}$ are definable subsets of $M$. Furthermore, if $(N, E) \succeq (M, E)$ is also an $\omega$-model, then $BG^{N,E}_\tau = BG^{M,E}_\tau$, $(N, E)$ supports $L_{\mathcal{A}}$, $G^{N,E}_{L_{\mathcal{A}}} = G^{M,E}_{L_{\mathcal{A}}}$, and $\ulcorner \phi \urcorner^{N,E} = \ulcorner \phi \urcorner^{M,E}$ for every $\phi \in L_{\mathcal{A}}$.*

**Definition 3.1.5.** Suppose $(M, E)$ is an $\omega$-model of set theory, and we have fixed a countable vocabulary $\tau$. A $\tau$-structure $\mathcal{B} = (B, \dots)$ is *inside* $(M, E)$ *via* $g$ if the universe $B \in M$, $g \in M$ is a function with domain $BG_\tau \cup \{\emptyset\}$, $g(\emptyset) = B$ and for each $(n, m) \in \omega^2$, $g(\ulcorner R^n_m \urcorner) = R^n_m(\mathcal{B})$.

**Definition 3.1.6.** If $(M, E)$ is an $\omega$-model of set theory, a $\tau$-structure $\mathcal{B}$ is inside $(M, E)$ via $g$, and $(N, E) \succeq (M, E)$ is an $\omega$-model, then $\mathcal{B}^N$ denotes the -structure with universe $g(\emptyset)^N$ and relations $R^n_m(\mathcal{B}^N) = g(\ulcorner R^n_m \urcorner)^N$. {extension}

Clearly, $\mathcal{B}^N$ is inside $(N, E)$ via $g^N$. Again using the fact that we are working with $\omega$-models, the following is immediate.

{embed}

**Lemma 3.1.7.** *Suppose $(M, E)$ is an $\omega$-model of set theory supporting $L_{\mathcal{A}}$ and a $\tau$-structure $\mathcal{B}$ is inside $(M, E)$ via $g$. Then there is a unique $h \in M$, $h : G_{L_{\mathcal{A}}} \to M$ extending $g$ such that $h(\ulcorner\psi\urcorner) = \psi(\mathcal{B})$ for every $\psi \in L_{\mathcal{A}}$.*

## 3.2 $M$-normal ultrapowers

The idea of using $M$-normal ultrafilters to construct many elementary chains of models of set theory is not new, and the definitions and results of this subsection are presented here for the convenience of the reader.

Fix a countable $\omega$-model $(M, E)$ of set theory. Since $M$ is countable, so is the set $\omega_1^M$. As notation, let

$$\mathfrak{C} = \{B \subseteq \omega_1^M : M \models \text{`}B \text{ is club'}\}$$

In what follows, a function $f$ with domain $\omega_1^M$ is *regressive* if $f(\alpha) < \alpha$ for all $\alpha > 0$.

**Definition 3.2.1.** An *$M$-normal ultrafilter* $\mathcal{U}$ is an ultrafilter on the set $\omega_1^M$ such that

- $\mathfrak{C} \subseteq \mathcal{U}$; and
- For every regressive $f : \omega_1^M \to \omega_1^M$ with $f \in M$, $f^{-1}(\beta) \in \mathcal{U}$ for some $\beta \in \omega_1^M$.

We record an Existence Lemma for $M$-normal ultrafilters.

{existence}

**Lemma 3.2.2.** *Suppose $A \subseteq \omega_1^M$ and $A \in M$. Then there is an $M$-normal ultrafilter $\mathcal{U}$ with $A \in \mathcal{U}$ if and only if $M \models$ '$A$ is stationary'.*

**Proof.** Clearly, if $M \models$ '$A$ is non-stationary', then there is some $B \in \mathfrak{C}$ such that $A \cap B = \emptyset$, so no $M$-normal ultrafilter can contain $A$. For the converse, enumerate the regressive functions in $M$ by $\langle f_n : n \in \omega\rangle$. We construct a nested, decreasing sequence $\langle A_n : n \in \omega\rangle$ of subsets of $\omega_1^M$ such that each $A_n \in M$ and $M \models$ '$A_n$ is stationary' as follows: Put $A_0 := A$ and given $A_n$, by Fodor's Lemma (in $M$!) choose a stationary $A_{n+1} \subseteq A_n$ and $\beta_n$ such that $f_n[A_{n+1}] = \{\beta_n\}$.

As $\mathfrak{C} \cup \{A_n : n \in \omega\}$ has f.i.p., (now working in $V$) it follows that there is an ultrafilter $\mathcal{U}$ containing these sets. Any such $\mathcal{U}$ must be $M$-normal. $\Box_{3.2.2}$

We record three consequences of $M$-normality.

{normal}

**Lemma 3.2.3.** *Suppose that $\mathcal{U}$ is an $M$-normal ultrafilter on $\omega_1^M$. Then:*

1. *If $A \in \mathcal{U} \cap M$, then $M \models$ '$A$ is stationary';*

2. *If $A \in \mathcal{U} \cap M$, $f \in M$, and $f : A \to \omega_1^M$ is regressive, then $f^{-1}(\beta) \in \mathcal{U}$ for some $\beta \in \omega_1^M$; and*

3. *If $\langle A_n : n \in \omega \rangle \in M$ and every $A_n \in \mathcal{U} \cap M$, then $A = \bigcap_{n\in\omega} A_n \in \mathcal{U} \cap M$.*

**Proof.** (1) Choose $A \in \mathcal{U} \cap M$. To see that $A$ is stationary in $M$, choose any $B \in M$ such that $M \models$ '$B$ is club'. Then $B \in \mathfrak{C} \subseteq \mathcal{U}$. As $\mathcal{U}$ is a proper filter, $A \cap B$ is non-empty.

(2) This is 'completely obvious' but rather cumbersome to prove precisely.

Given $f : A \to \omega_1^M$, by intersecting with the club $D := \omega_1^M \setminus \omega$, we may assume $A \subseteq D$. Define $g : \omega_1^M \to \omega_1^M$ by

$$g(\delta) = \begin{cases} f(\delta) & \text{if } \delta \in A \text{ and } f(\delta) \geq \omega \\ f(\delta) + 1 & \text{if } \delta \in A \text{ and } f(\delta) < \omega \\ 0 & \text{if } \delta \notin A \end{cases}$$

Then $g \in M$ and $g$ is regressive, hence $g^{-1}(\beta) \in \mathcal{U}$ for some $\beta$. As $g^{-1}(0)$ is disjoint from $A$ and $A \in \mathcal{U}$, $\beta \neq 0$. Thus, $g^{-1}(\beta) \subseteq A$. It follows that either $f^{-1}(\beta) \in \mathcal{U}$ (when $\beta \geq \omega$) or $f^{-1}(\beta - 1) \in \mathcal{U}$ (when $\beta < \omega$).

(3) Assume not. Let $B := \omega_1^M \setminus A \in \mathcal{U} \cap M$. As in (2) we may assume $B \subseteq (\omega_1^M \setminus \omega)$. Define $f : B \to \omega$ by

$$f(\delta) = \text{least } n \text{ such that } \delta \notin A_n$$

As $f$ is regressive, we get a contradiction from (2). $\square_{3.2.3}$

Given $M$ and an $M$-normal ultrafilter $\mathcal{U}$, we form the ultraproduct $Ult(M,\mathcal{U})$ as follows:

First, consider the (countable!) set of functions $f : \omega_1^M \to M$ with $f \in M$. There is a natural equivalence relation $\sim_{\mathcal{U}}$ defined by

$$f \sim_{\mathcal{U}} g \quad \Leftrightarrow \quad \{\delta \in \omega_1^M : f(\delta) = g(\delta)\} \in \mathcal{U}$$

The objects of $Ult(M,\mathcal{U})$ are the equivalence classes $[f]_{\mathcal{U}}$, and we put

$$Ult(M,\mathcal{U}) \models [f]_{\mathcal{U}} E [g]_{\mathcal{U}} \quad \Leftrightarrow \quad \{\delta \in \omega_1^M : f(\delta) E g(\delta)\} \in \mathcal{U}.$$

For each $a \in M$, we have the constant function $f_a : \omega_1^M \to M$ defined by $f_a(\delta) = a$ for every $\delta \in \omega_1^M$. Every such function $f_a \in M$, hence we get an embedding

$$j : M \to Ult(M,\mathcal{U})$$

defined by $j(a) = [f_a]_{\mathcal{U}}$.

The following Lemmas summarize the results we need:

{basicnormal}

**Lemma 3.2.4.** *Suppose that $(M, E)$ is a countable $\omega$-model of set theory and $\mathcal{U}$ is any $M$-normal ultrafilter on $\omega_1^M$. Then:*

1. $N := Ult(M,\mathcal{U})$ *is a countable $\omega$-model with* $(N, E) \succeq (M, E)$*;*
2. *If $a \in M$ and $M \models$ '$a$ is countable' then* $j(a) = j[a] =_{df} \{j(x) : x \in a\}$*;*
3. $\omega_1^M$ *is a proper initial segment of* $\omega_1^N$ *with* $[id]_\mathcal{U}$ *the least element of* $\omega_1^N \setminus \omega_1^M$.

If we are being pedantic, then the statements of (1) and (3) in the Lemma above should be adjusted (and the proofs should match this. Namely, a more precise statement of (1) would be to replace (N,E) being an elementary extension by the more precise $j : (M, E) \to (N, E)$ is elementary. But, if we went that route on (1) we really ought to change (3) to match, i.e., The image $j[\omega_1^M] =_{df} \{j(a) : a \in \omega_1^M\}$ is a proper initial segment of $\omega_1^N$ with $[id]_\mathcal{U}$ the least element of $\omega_1^N \setminus j[\omega_1^M]$.
Should we make these changes?

**Proof.** We begin with (2). Fix $a \in M$ with $M \models$ '$a$ is countable' and abbreviate $M \models aEb$ by $aEb$. First, for every $bEa$, $f_b(\delta) E f_a(\delta)$ for every $\delta \in \omega_1^M$, so $j(b)Ej(a)$ by Łoś's theorem. Conversely, choose any $g : \omega_1^M \to M$ with $g \in M$ such that $[g]_\mathcal{U} \neq [f_b]_\mathcal{U}$ for every $bEa$. Towards showing that $[g]_\mathcal{U} \neg E j(a)$, using the countability of $a$ in $M$, choose a surjection $\Phi : \omega \to a$ with $\Phi \in M$. In $M$, let

$$A_n = \{\delta \in \omega_1^M : g(\delta) \neq \Phi(n)\}.$$

By separation, each $A_n \in M$ and recursion, since $M$ is an $\omega$-model, $\langle A_n : n \in \omega\rangle \in M$ and each $A_n \in \mathcal{U} \cap M$. Thus, by Lemma 3.2.3(3), $A := \bigcap_{n\in\omega} A_n \in \mathcal{U} \cap M$. Since $g(\delta) \neg Ea$ for every $\delta \in A$, the fact that $A \in \mathcal{U}$ implies that $[g]_\mathcal{U} \neg E j(a)$.

Commented out your final sentence above as I don't think it is needed.

As for (1), that $N \succeq M$ is the Łoś theorem. $N$ is clearly countable, as there are only countably many functions in $M$, and it is an $\omega$-model by (2). As for (3), that $\omega_1^M$ is an initial segment of $\omega_1^N$ follows from (2), and the minimality of $[id]_\mathcal{U}$ in the difference follows from Fodor's Lemma in $M$. $\square_{3.2.4}$

Changed the order of the phrases in the hypotheses of the Lemma below. If we do adopt the more pedantic choice of $j$ being elementary, we should probably interpose some statement before this Lemma about how we now think of $N$ as being an elementary extension...

{Lnormal}

**Lemma 3.2.5.** *Suppose that $(M, E)$ is a countable $\omega$-model of set theory that supports $L_{\mathcal{A}}$ and let $\mathcal{B} = (B, \dots)$ be an $L$-structure inside $(M, E)$ via $g$. Given any $M$-normal ultrafilter $\mathcal{U}$ on $\omega_1^M$, let $N = Ult(M,\mathcal{U})$ and let $\mathcal{B}^N$ be the $L$-structure formed as in Definition 3.1.6 with $h$ as in Lemma 3.1.7. Then:*

1. *For every $L_{\mathcal{A}}$-formula $\psi(x_1, \dots, x_n)$ and all $[f_1]_{\mathcal{U}}, \dots, [f_n]_{\mathcal{U}}$ with each $f_i : \omega_1^M \to B$,*

$$\mathcal{B}^N \models \psi([f_1], \dots, [f_n]) \Longleftrightarrow \{\alpha \in \omega_1^M : (f_1(\alpha), \dots, f_n(\alpha)) \in h(\ulcorner \psi \urcorner)\} \in \mathcal{U}$$

2. *The induced embedding $j : \mathcal{B} \to \mathcal{B}^N$ is $L_{\mathcal{A}}$-elementary; and*

3. *If $\omega_1^M \subseteq B$ and $\theta(x) \in L_{\mathcal{A}}$ has one free variable, then $\mathcal{B}^N \models \theta([id]_{\mathcal{U}})$ if and only if $\{\alpha \in \omega_1^M : \alpha \in h(\ulcorner \theta \urcorner)\} \in \mathcal{U}$.*

## 3.3 A transfer theorem

We bring together the methods of the previous subsections into a general transfer theorem. Recall that we are using Roman letters (M) for models of set theory, Gothic ($\mathcal{B}$) for $\tau$-structures and $\mathcal{B}^M$ denotes a structure supported in $M$, and for a $\tau$-relation $P$, $P^{\mathcal{B}}$ denotes the elements of $\mathcal{B}$ satisfying $P$.

{transfer1}

**Theorem 3.3.1.** *Fix a vocabulary $\tau$ with a distinguished unary predicate $P$ and fix a countable fragment $L_{\mathcal{A}} = L_{\mathcal{A}}(\tau) \subset L_{\omega_1,\omega}(\tau)$. SUPPOSE there is a countable, $\omega$-model $(M, E)$ of set theory supporting $L_{\mathcal{A}}$ and there is a $\tau$-structure $\mathcal{B} = (B, \dots)$ inside $M$ via $g$ satisfying:*

- *$P^{\mathcal{B}} \subseteq \omega_1^M \subseteq B$;*
- *$M \models$ '$P^{\mathcal{B}}$ is stationary/costationary'.*

*THEN for every $X \subseteq \omega_1$ (in V!) there is an $\omega$-model $(N_X, E) \succeq (M, E)$ and a continuous, strictly increasing*[4] *$t_X : \omega_1 \to \omega_1^{N_X}$ satisfying:*

- *$|N_X| = \aleph_1$ and $(\omega_1^{N_X}, E)$ is an $\aleph_1$-like linear order;*
- *for all $\alpha \in \omega_1$, $\mathcal{B}^{N_X} \models P(t_X(\alpha))$ if and only if $\alpha \in X$.*

[4]The function $t_X$ need not be an element of $N_X$.

**Proof.** Fix any $X \subseteq \omega_1$. We construct a continuous chain $\langle M_\alpha : \alpha \in \omega_1 \rangle$ of $\omega$-models of set theory as follows: Put $M_0 := (M, E)$ and at countable limit ordinals, take unions. Now suppose $M_\alpha$ is given. Choose an $M_\alpha$-normal ultrafilter $\mathcal{U}_\alpha$ such that $P^{M_\alpha} \in \mathcal{U}_\alpha$ if and only if $\alpha \in X$. The existence of such a $\mathcal{U}$ follows from Lemma 3.2.2, since by elementarity, letting $\mathcal{B}_\alpha$ denote $\mathcal{B}^{M_\alpha}$, we have that

$$M_\alpha \models \text{`}P^{\mathcal{B}_\alpha} \text{ is a stationary/costationary subset of } \omega_1\text{'}$$

Given such a chain, put $N_X := \bigcup\{M_\alpha : \alpha \in \omega_1\}$ and define $t_X : \omega_1 \to \omega_1^{N_X}$ by $t_X(\alpha) = [id]_{\mathcal{U}_\alpha}$. $\square_{3.3.1}$

This result extends easily to $L(Q)$ and the somewhat more complicated version for $L(aa)$ is treated in section 2 of [**?**].

# 4 The relevant forcing

{3}

Throughout this section, we have a fixed atomic class $\mathbf{At_T}$ that contains uncountable models, for which the pseudo-minimal types are not dense. The objective of this section is introduce a class of $I^*$ of expansions of linear orders, develop the notion of a model $N \in \mathbf{At_T}$ being *striated* by such an order, and prove Theorem 4.2.4, which uses the failure of density of pseudo-minimal types to force the existence of a striated model capable of encoding a nearly arbitrary subset of $\omega_1$.

## 4.1 A class of linear orders

Recall that a linear order is $\aleph_1$-like if every initial segment is countable. It is well-known that there are $2^{\aleph_1}$ $\aleph_1$-like linear orders of cardinality $\aleph_1$. An accessible account of this proof, which underlies this entire paper, appears on page 203 of [**?**]. The key idea of that argument is to code a stationary set of cuts which have a least upper bound. In the current paper, the coding is not so sharp. Instead, we force an atomic model of $T$ that codes a stationary set by infinitary formulas defined using $\mathrm{pcl}$.

We begin by describing a class of $\aleph_1$-like linear orders, colored by a unary predicate $P$ and an equivalence relation $E$ with convex classes. This subsection makes no reference to the class $\mathbf{At_T}$.

**Definition 4.1.1.** Let $\tau_{\mathrm{ord}} = \{<, P, E\}$ and let $\mathbf{I}^*$ denote the collection of $\tau_{\mathrm{ord}}$-structures $(I, <, P, E)$ satisfying:

1. $(I, <)$ is an $\aleph_1$-like dense linear order with minimum element $\min(I)$ (i.e., $|I| = \aleph_1$, but $pred_I(a)$ is countable for every $a \in I$);

2. $P$ is a unary predicate and $\neg P(\min(I))$;

3. $E$ is an equivalence relation on $I$ with convex classes such that

   (a) If $t = \min(I)$ or if $P(t)$ holds, then $t/E = \{t\}$;

   (b) Otherwise, $t/E$ is a (countable) dense linear order without endpoints.

4. The quotient $I/E$ is a dense linear order with minimum element, no maximum element, such that both sets $\{t/E : P(t)\}$ and $\{t/E : \neg P(t)\}$ are dense in it.

Note that for $s \in I$, we denote the equivalence class of $s$ by $s/E$ and the predecessors of the class by $< s/E$. We are interested in well-behaved proper initial segments $J$ of orders $I$ in $\mathbf{I}^*$.

{suitdef}

**Definition 4.1.2.** Fix $(I, <, P, E) \in \mathbf{I}^*$. A proper initial segment $J \subseteq I$ is *suitable* if, for every $s \in J$ there is $t \in J$, $t > s$, with $\neg E(s, t)$.

Note that if $J \subseteq I$ is suitable, then $J$ is a union of $E$-classes and that there is no largest $E$-class in $J$. Accordingly, there are three possibilities for $I \setminus J$:

- $I \setminus J$ has a minimum element $t$. In this case, it must be that $t/E = \{t\}$.
- $I \setminus J$ has no minimum $E$-class. In this case, we call $J$ *seamless*.
- $I \setminus J$ has a minimum $E$-class that is infinite. This will be our least interesting case.

We record one easy Lemma.

{seamless}

**Lemma 4.1.3.** *If $(I, <, P, E) \in \mathbf{I}^*$ and $J \subseteq I$ is a seamless proper initial segment, then for every finite $\mathcal{S} \subseteq I$ and $w \in J$ such that $w > \mathcal{S} \cap J$, there is an automorphism $\pi$ of $(I, <, P, E)$ that fixes $\mathcal{S}$ pointwise, and $\pi(w) \notin J$.*

**Proof.** Fix $I, J, \mathcal{S}$ as above. As $J$ is seamless, we can find $t, t' \in I \setminus \mathcal{S}$ satisfying:

- $t/E$ and $t'/E$ are both singletons;
- $t, t'$ satisfy the same $\mathcal{S}$-cut, i.e., for each $s \in \mathcal{S}$, $s < t$ iff $s < t'$;
- $t < w < t'$;
- $t \in J$, but $t' \notin J$.

We will produce an automorphism $\pi$ of $(I,<,E,P)$ that fixes $\mathcal{S}$ pointwise and $\pi(t)=t'$. This suffices, as necessarily $\pi(w)\notin J$ for any such $\pi$. To produce such a $\pi$, first choose a suitable proper initial segment $K\subseteq I$ containing $\mathcal{S}\cup\{t,t'\}$. Note that $K$ is countable, and is a union of $E$-classes. Consider the structure $(K/E,<,P)$ formed from the quotient $K/E$, where $<$ is the inherited linear order and $P(r/E)$ if and only if $P(r)$ held in $(I,<,E,P)$. Now $Th(K/E,<,P)$ is known to be $\aleph_0$-categorical and eliminate quantifiers. [The theory is axiomatized by asserting that $<$ is dense linear order with a least element but no greatest element, and $P$ is a dense/codense subset.] Thus, there is an automorphism $\pi_0$ of $(K/E,<,P)$ fixing $\mathcal{S}/E$ pointwise and $\pi(t/E)=t'/E$. As every $E$-class of $K$ is either a singleton or a countable, dense linear order, there is an automorphism $\pi_1$ of $(K,<,E,P)$ fixing $\mathcal{S}$ pointwise and $\pi_1(t)=t'$ and such that $\pi_1(x)/E=\pi_0(x/E)$. Now the automorphism $\pi$ of $(I,<,E,P)$ defined by $\pi(u)=\pi_1(u)$ if $u\in K$, and $\pi(u)=u$ for each $u\in I\setminus K$ is as desired. $\square_{4.1.3}$

The following construction codes a nearly arbitrary subset $S\subseteq\omega_1$ into an $I^S\in\mathbf{I}^*$. We construct orderings that avoid the third case of Definition 4.1.2.

{IS}

**Construction 4.1.4.** *Let $S\subseteq\omega_1$ with $0\notin S$. There is $I^S=(I^S,<,P,E)\in\mathbf{I}^*$ that has a continuous, increasing sequence $\langle J_\alpha:\alpha\in\omega_1\rangle$ of proper initial segments such that:*

1. *If $\alpha\in S$, then $I^S\setminus J_\alpha$ has a minimum element $a_\alpha$ satisfying $P(a_\alpha)$; and*
2. *If $\alpha\notin S$ and $\alpha>0$, then $J_\alpha$ is seamless.*

**Proof.** Let $\tau_{\text{ord}}=\{<,P,E\}$ and $\mathcal{A}$ be the $\tau_{\text{ord}}$-structure with universe singleton $\{a\}$ with both $P(a)$ and $E(a,a)$ holding. Let $\mathcal{B}=(\mathbb{Q},<,P,E)$, where $(\mathbb{Q},<)$ is a countable dense linear order with no endpoints, $P$ fails everywhere, and all elements are $E$-equivalent. Combine these to get a (countable) $\tau_{\text{ord}}$-structure $\mathcal{C}$ formed by the dense/codense (with no endpoints) concatenation of countably many copies of both $\mathcal{A}$ and $\mathcal{B}$. Finally, take $\mathcal{D}$ to be the concatenation $\mathcal{A}\hat{}\mathcal{C}$.

Using these $\tau_{\text{ord}}$-structures as building blocks, form a continuous sequence of $\tau_{\text{ord}}$-structures $J_\alpha$, where $J_\alpha$ is an $\tau_{\text{ord}}$-substructure and an initial segment of $J_\beta$ whenever $\alpha<\beta$ by: $J_0$ is the one-element structure $\{\min(I)\}$ with $\neg P(\min(I))$. For $\alpha<\omega_1$ a non-zero limit ordinal, take $J_\alpha$ to be the increasing union of $\langle J_\beta:\beta<\alpha\rangle$. Given $J_\alpha$, form $J_{\alpha+1}$ by

$$J_{\alpha+1}=\begin{cases} J_\alpha\hat{}\mathcal{D} & \text{if } \alpha\in S\\ J_\alpha\hat{}\mathcal{C} & \text{if } \alpha\notin S\end{cases}$$

Finally, take $I^S$ to be the increasing union of $\langle J_\alpha:\alpha<\omega_1\rangle$. $\square_{4.1.4}$

## 4.2 Striated models and forcing

In this section we introduce the notion of a striation of a model - a decomposition of a model $N$ of $T$ into uncountably many countable pieces satisfying certain constraints on $\mathrm{pcl}$. We will show later how to code stationary sets by specially constructed (forced) striated models.

### 4.2.1 Striated Models

Fix an atomic $N \in \mathbf{At_T}$ and some $I = (I, <, E, P) \in \mathbf{I}^*$.

{striate}

**Definition 4.2.1.** We say $N$ is *striated by $I$* if there are $\omega$-sequences $\langle \overline{a}_t : t \in I \rangle$ satisfying:

- $N = \bigcup\{\overline{a}_t : t \in I\}$; (As notation, for $t \in I$, $N_{<t} = \bigcup\{\overline{a}_j : j < t\}$.)
- If $t = \min(I)$, then $\overline{a}_t \subseteq \mathrm{pcl}(\emptyset, N)$;
- For $t > \min(I)$, $a_{t,0} \notin \mathrm{pcl}(N_{<t}, N)$;
- For each $t$ and $n \in \omega$, $a_{t,n} \in \mathrm{pcl}(N_{<t} \cup \{a_{t,0}\}, N)$.

Note: In the definition above, we allow $a_{s,m} = a_{t,n}$ in some cases when $(s, m) \neq (t, n)$. However, if $s < t$, then the element $a_{t,0} \neq a_{s,m}$ for any $m$. Also, if $\mathrm{pcl}(\emptyset, N) = \emptyset$, we do not define $\overline{a}_{\min(I)}$. Although $E$ and $P$ don't appear explicitly in either Definition 4.2.1 or Definition 4.2.2, $E$ is needed for the following notations and $P$ plays a major role later.

The idea of our forcing will be to force the existence of a striated atomic model $N_I$ indexed by a linear order $I \in \mathbf{I}^*$ with universe $X = \{x_{t,n} : t \in I, n \in \omega\}$. Such an $N_I$ will have a 'built in' continuous sequence $\langle N_\alpha : \alpha \in \omega_1 \rangle$ of countable, elementary substructures, where the universe of $N_\alpha$ will be $X_\alpha = \{x_{t,n} : t \in J_\alpha, n \in \omega\}$ for some initial segment $J_\alpha$ of $I$. We start with the assumption that pseudo-minimal types are not dense so some formula $\delta(x, \overline{f})$ has 'no pseudo-minimal extension'. We absorb the constants $\overline{f}$ into the language and use the assumption of 'no pseudo-minimal extension' to make the set

$$\{\alpha \in \omega_1 : I \setminus J_\alpha \text{ has a least element}\}$$

(infinitarily) definable. To make this precise, we introduce some notation.

Suppose that $(I, <, P, E) \in \mathbf{I}^*$ and $N = \{a_{t,n} : t \in I, n \in \omega\}$ is striated by $I$. For any suitable $J \subseteq I$, let $N_J$ denote the substructure with universe $\{a_{t,n} : t \in J, n \in \omega\}$. Abusing notation slightly, given any $s \in I \setminus \{\min(I)\}$, let

$$J_{<s} = \{s' \in I : s' < s \text{ and } \neg E(s', s)\}$$

Thus, $J_{<s}$ is a suitable proper initial segment of $I$, and we denote its associated $L$-structure, $\{a_{t,n} : t \in J_{<s}, n < \omega\}$, by $N_{<s}$. With this notation, we now describe three relationships between an element and a substructure of this sort.

{catchetal}

**Definition 4.2.2.** Suppose $N$ is striated by $(I, <, P, E)$, $J \subseteq I$ suitable, and $b \in N \setminus N_J$.

- $b$ *catches* $N_J$ if, for every $e \in N$, $e \in \mathrm{pcl}(N_J \cup \{b\}, N) \setminus N_J$ implies $b \in \mathrm{pcl}(N_J \cup \{e\}, N)$.
- $b$ *has unbounded reach in* $N_J$ if there exists $s^* \in J$ such that, letting $A$ denote $\mathrm{pcl}(N_{<s^*} \cup \{b\}, N) \cap N_J$, for every $s \in J$ with $s > s^*$ there is a $c \in A - N_{<s}$.
- $b$ *has bounded effect in* $N_J$ if there exists $s^* \in J$ such that $\mathrm{pcl}(N_{<s} \cup \{b\}, N) \cap N_J = N_{<s}$ for every $s > s^*$ with $s \in J$.

Clearly, an element $b$ cannot have both unbounded reach and bounded effect in $N_J$, but the properties are not complementary.

{full}

**Definition 4.2.3.** A model $M$ with uncountable cardinality is said to be *full* if for every $\overline{a} \in M$ every non-algebraic $p \in S_{at}(\overline{a})$ is realized $|M|$-times in $M$.

The remainder of this section is devoted to the proof of the following Theorem.

{forcing}

**Theorem 4.2.4.** *Suppose $\delta(x)$ is a complete, non-pseudo algebraic formula with no pseudo-minimal extension. For every $(I, <, P, E) \in \mathbf{I}^*$ there is a c.c.c. forcing $\mathbb{Q}_I$ such that in $V[G]$, there is a full, atomic $N_I \models T$ striated by $(I, <)$ such that:*

1. *For every suitable initial segment $J \subseteq I$, $N_J \preceq N_I$;*
2. *If $t \in I$ and $P(t)$ holds, then $a_{t,0}$ catches and has unbounded reach in $N_{<t}$;*
3. *If $J \subseteq I$ is seamless, then for every $b \in N_I \setminus N_J$, if $b$ catches $N_J$, then $b$ has bounded effect in $N_J$.*

**Proof.** The hypothesis that $\delta(x)$ has no pseudo-minimal extension means for every $\phi(x, \overline{a})$ which implies $\delta(x)$ and is not pseudoalgebraic there do not exist $\overline{a}^*, c, b$ satisfying the Definition 2.7 of pseudominimality. Replacing $\overline{a}^*, c, b$ by $\overline{b}, e, c$, our hypothesis on $\delta(x)$ translates into the following statement:

{good}

**Fact 4.2.5.** *Assume $\delta(x)$ has no pseudo-minimal extension. For any $M \in \mathbf{At_T}$, for any $\overline{a}$ from $M$ and any $c \in \delta(M)$ for which $c \notin \mathrm{pcl}(\overline{a}, M)$, there are $\overline{b}$ and $e$ from $M$ such that*

1. $e \in \mathrm{pcl}(\overline{a}\overline{b}c, M) \setminus \mathrm{pcl}(\overline{a}\overline{b}, M)$*; but*

2. $c \notin \mathrm{pcl}(\overline{a}\overline{b}e, M)$.

Fix, for the whole of the proof, some $(I, <, E, P) \in \mathbf{I}^*$. We wish to construct an atomic model $N_I \models T$, whose complete diagram contains variables $\{x_{t,n} : t \in I, n \in \omega\}$, that is striated by $(I, <)$, and includes $\delta(x_{t,0})$, whenever $I \models P(t)$. We begin by defining a forcing notion $\mathbb{Q}_I$ and prove that it satisfies the c.c.c. Then, we exhibit several collections of subsets of $\mathbb{Q}_I$ and prove that each is dense and open. Fact 4.2.5 will only be used in showing the sets witnessing 'unbounded reach' (i.e., Group F of the constraints) are dense. Finally in Section 4.4, we argue that if $G \subseteq \mathbb{Q}_I$ is a generic filter meeting each of these dense open sets, then $V[G]$ will contain an atomic model $N_I$ of $T$ satisfying the conclusions of Theorem 4.2.4.

## 4.3 The forcing

{forcedetail}

Our forcing $\mathbb{Q}_I$ consists of 'finite approximations' of this complete diagram. The conditions will be complete types in variable with a specific kind of indexing that we now describe.

{finiteseq}

**Notation 4.3.1.** A *finite sequence* $\overline{x}$ *from* $\langle x_{t,n} : t \in I, n \in \omega\rangle$ is *indexed by* $u$ if it has the form $\overline{x} = \langle x_{t,m} : t \in u, m < n_t\rangle$, where $u \subseteq I$ is finite and $1 \le n_t < \omega$ for every $t \in u$.

Given a finite sequence $\overline{x}$ indexed by $u$ and $\langle n_t : t \in u\rangle$ and given a proper initial segment $J \subseteq I$, let $u\restriction_J = u \cap J$ and $\overline{x}\restriction_J = \langle x_{t,m} : t \in u\restriction_J, m < n_t\rangle$.

As well, if $p(\overline{x})$ is a complete type in the variables $\overline{x}$, then $p\restriction_J$ denotes the restriction of $p$ to $\overline{x}\restriction_J$, which is necessarily a complete type. For $s \in I$, the symbols $u\restriction_{<s}$ and $\overline{x}\restriction_{\le s}$ are defined analogously, setting $J = I\restriction_{<s}$ and $I\restriction_{\le s}$, respectively. If $\overline{x}$ arises from a type $p$ that we are keeping track of, we write $n_{p,t}$ for $n_t$. These various notations may be combined to yield, for example, $p\restriction_{\le s/E}$.

The forcing $\mathbb{Q}_I$ will consist of finite approximations of a complete diagram of an $L$-structure in the variables $\{x_{t,\ell} : t \in I, \ell \in \omega\}$. Recall that the property, '$a \in \mathrm{pcl}(\overline{b})$' is enforced by a first order formula; this justifies 'say' in the next definition.

{forcedef}

**Definition 4.3.2** ( $(\mathbb{Q}_I, \le_{\mathbb{Q}})$). $p \in \mathbb{Q}_I$ if and only if the following conditions hold:

1. $p$ is a complete (principal) type with respect to $T$ in the variables $\overline{x}_p$, which are a finite sequence indexed by $u_p$ and $n_{p,t}$ (when $p$ is understood we sometimes write $n_t$);
2. If $t \in u_p$ and $P(t)$ holds, then $p \vdash \delta(x_{t,0})$;
3. If $t = \min(I)$, then $p$ 'says' $\{x_{t,n} : n < n_t\} \subseteq \mathrm{pcl}(\emptyset)$;

4. If $p$ 'says' $x_{t,0} \in \mathrm{pcl}(\emptyset)$, then $t = \min(I)$;

5. For all $t \in u_p$, $t \neq \min(I)$, $p$ 'says' $x_{t,0} \notin \mathrm{pcl}(\overline{x}_p\restriction_{<t})$; and

6. For all $t \in u_p$ and $m < n_t$, $p$ 'says' $x_{t,m} \in \mathrm{pcl}(\overline{x}_p\restriction_{<t} \cup \{x_{t,0}\})$.

For $p, q \in \mathbb{Q}_I$, we define $p \leq_{\mathbb{Q}_I} q$ if and only if $\overline{x}_p \subseteq \overline{x}_q$ and the complete type $p(\overline{x}_p)$ is the restriction of $q(\overline{x}_q)$ to $\overline{x}_p$.

We begin with some easy observations.

{truncate}

**Lemma 4.3.3.** *For every $p \in \mathbb{Q}_I$ and every proper initial segment $J \subseteq I$, $p\restriction_J \in \mathbb{Q}_I$ and $p\restriction_J \leq_{\mathbb{Q}_I} p$.*

{extendauto}

**Lemma 4.3.4.** *Every automorphism $\pi$ of $(I, <, E, P)$ naturally extends to an automorphism $\pi'$ of $\mathbb{Q}_I$ via the mapping $x_{t,n} \mapsto x_{\pi(t),n}$.*

{chaincondit

**Lemma 4.3.5.** *Suppose $p \in \mathbb{Q}_I$ and $u_p \neq \emptyset$. Enumerate $u_p = \{s_i : i < d\}$ with $s_i <_I s_{i+1}$ for each $i$. For any $M \in \mathbf{At_T}$ and any $\overline{b}$ from $M$ realizing $p(\overline{x}_p)$, there is a sequence $M_0 \preceq M_1 \preceq \cdots \preceq M_{d-1} = M$ of elementary substructures of $M$ satisfying:*

- *For each $i < d$, $\overline{b}\restriction_{<s_i} \subseteq M_i$; and*
- *For $0 < i < d$, $b_{s_i,0} \in M_i \setminus M_{i-1}$.*

**Proof.** By induction on $d = |u_p|$. For $d = 0, 1$ there is nothing to prove, so assume $d \geq 2$ and the Lemma holds for $d - 1$. Fix any $M \in \mathbf{At_T}$ and choose any realization $\overline{b}$ of $p(\overline{x}_p)$ in $M$. Clearly, the subsequence $\overline{a} := \overline{b}\restriction_{<s_{d-1}}$ realizes the restriction $q := p\restriction_{<s_{d-1}}$. As $b_{s_{d-1},0} \notin \mathrm{pcl}(\overline{a}, M)$, there is $M_{d-2} \preceq M$ such that $\overline{a}$ is from $M_{d-2}$, but $b_{s_{d-1},0} \notin M_{d-2}$. Then complete the chain by applying the inductive hypothesis to $M_{d-2}$ and $q$. $\Box_{4.3.5}$

The 'moreover' in the following lemma emphasizes that in proving density we are showing how to assign levels to a elements of a finite sequence in a model which need not be striated.

{consistent}

**Lemma 4.3.6.** *Suppose $J \subseteq I$ is an initial segment and $p, q \in \mathbb{Q}_I$ satisfy $p\restriction_J \leq_{\mathbb{Q}} q$ and $u_q \subseteq J$. Then there is $r \in \mathbb{Q}_I$ with $\overline{x}_r = \overline{x}_p \cup \overline{x}_q$, $r \geq_{\mathbb{Q}} p$ and $r \geq_{\mathbb{Q}} q$. Moreover, if $M \in \mathbf{At_T}$, $\overline{a}$ realizes $p\restriction_J$, $\overline{a}\overline{b}$ realizes $p$, and $\overline{ac}$ realizes $q$, then $\overline{a}\overline{b}\overline{c}$ realizes $r$.*

**Proof.** If $u_p = \emptyset$, then take $r = q$, so assume otherwise. Choose any $M \in \mathbf{At_T}$ and fix a realization $\overline{b}$ of $p(\overline{x}_p)$ in $M$. Let $\overline{a} = \overline{b}\restriction_J$. Write $u_p = \{s_i : i < d\}$ with $s_i <_I s_{i+1}$ for each $i$. Apply Lemma 4.3.5 to $M$ and $\overline{b}$ and choose $\ell < d$ least such that $\overline{a} \subseteq M_\ell$. As $q(\overline{x}_q)$ is generated by a complete formula and $\overline{a} \subseteq M_\ell$, there is $\overline{c} \subseteq M_\ell$ such that $\overline{ac}$ (when properly indexed) realizes $q$. Now define $r(\overline{x}_r)$ to be the complete type of $\overline{b}\overline{c} = \overline{a}\overline{b}\overline{c}$ in $M$ in the variables $\overline{x}_r = \overline{x}_p \cup \overline{x}_q$. $\Box_{4.3.6}$

{ccc}

**Claim 4.3.7.** $(\mathbb{Q}_I, \leq_{\mathbb{Q}})$ *has the c.c.c.*

Proof. Let $\{p_i : i < \aleph_1\} \subseteq \mathbb{Q}_I$ be a collection of conditions. We will find $i \neq j$ for which $p_i$ and $p_j$ are compatible. We successively reduce this set maintaining its uncountability. By the $\Delta$-system lemma we may assume that there is a single $u^*$ such that for all $i, j$, $u_{p_i} \cap u_{p_j} = u^*$. Further, by the pigeonhole principle we can assume that for each $t \in u^*$, $n_{p_i,t} = n_{p_j,t}$. We can use pigeon-hole again to guarantee that all the $p_i$ and $p_j$ agree on the finite set of shared variables. And finally, since $I$ is $\aleph_1$-like we can choose an uncountable set $X$ of conditions such that for $i < j$ and $p_i, p_j \in X$ all elements of $u^*$ precede anything in any $u_{p_i} \setminus u^*$ or $u_{p_j} \setminus u^*$ and that all elements of $u_{p_i} \setminus u^*$ are less that all elements of $u_{p_j} \setminus u^*$.

Finally, choose any $i < j$ from $X$. Let $J = \{s \in I : s \leq \max(u_{p_i})\}$. By Lemma 4.3.6 applied to $p_i$ and $p_j$ for this choice of $J$, we conclude that $p_i$ and $p_j$ are compatible. $\Box_{4.3.7}$

Recall that a set $X \subseteq \mathbb{Q}_I$ is dense if for every $p \in \mathbb{Q}_I$ there is a $q \in X$ with $q \geq p$ and $X \subseteq \mathbb{Q}_I$ is open if for every $p \in X$ and $q \geq p$, then $q \in X$.

In the remainder of Section 4.3 we list the crucial 'constraints', which are sets of conditions, and we prove each of them to be dense and open in $\mathbb{Q}_I$.

**A. Surjectivity** Our first group of constraints ensure that for any generic $G \subseteq \mathbb{Q}_I$, for every $(t, n) \in I \times \omega$, there is $p \in G$ such that $x_{t,n} \in \overline{x}_p$. To enforce this, for any $(t, n) \in I \times \omega$, let

$$\mathcal{A}_{t,n} = \{p \in \mathbb{Q}_I : x_{t,n} \in \overline{x}_p\}$$

{surjective}

**Claim 4.3.8.** *1. For every $t \in I \setminus \{\min(I)\}$ and every $n \in \omega$, $\mathcal{A}_{t,n}$ is dense and open;*

*2. If $\mathrm{pcl}(\emptyset) \neq \emptyset$, then $\mathcal{A}_{\min(I),n}$ is dense and open for every $n \in \omega$.*

*Moreover, in either case, given $(t, n) \in I \times \omega$ and any $p \in \mathbb{Q}_I$, there is $q \in \mathcal{A}_{t,n}$ with $q \geq_{\mathbb{Q}} p$ and $u_q = u_p \cup \{t\}$.*

**Proof.** Each of these sets are trivially open. We first establish density for (1) and (2) when $n = 0$. For $t = \min(I)$, (1) is vacuous. For (2), choose any $p \in \mathbb{Q}_I$. If $x_{\min(I),0} \in \overline{x}_p$, there is nothing to prove, so assume it is not. Pick any $M \in \mathbf{At_T}$. Choose $\overline{b}$ from $M$ realizing $p$ and choose $a \in \mathrm{pcl}(\emptyset, M)$. Then define $q$ by $\overline{x}_q = \overline{x}_p \cup \{x_{\min(I),0}\}$ and $q(\overline{x}_q) = \mathrm{tp}(\overline{b}a, M)$. Next, we show that $\mathcal{A}_{t,0}$ is dense for every $t > \min(I)$. To see this, choose any $p \in \mathbb{Q}_I$. If $t \in u_p$, then necessarily $x_{t,0} \in \overline{x}_p$, so there is nothing to prove. Thus, assume $t \notin u_p$. Take $J = \{s \in I : s < t\}$. Pick $M \in \mathbf{At_T}$ and choose a realization $\overline{a}$ of $p\restriction_J$ in $M$.

As $\delta$ is not pseudo-algebraic, by Lemma 2.6 there is $b \in M$ realizing $\delta$ with $b \notin \mathrm{pcl}(\overline{a}, M)$. Let $q \in \mathbb{Q}_I$ be defined by $\overline{x}_q = \overline{x}_p\restriction_J \cup \{x_{t,0}\}$ and the complete type $q(\overline{x}_q) =$

$\mathrm{tp}(\overline{a}b, M)$. Then $q \geq_{\mathbb{Q}} p\restriction_J$ and by Lemma 4.3.6, there is $r \in \mathbb{Q}$ with $r \geq_{\mathbb{Q}} q$ and $r \geq_{\mathbb{Q}} p$. Visibly, $r \in \mathcal{A}_{t,0}$.

Next, we prove by induction on $n$ that if $\mathcal{A}_{t,n}$ is dense, then so is $\mathcal{A}_{t,n+1}$. But this is trivial. Fix $t$ and choose $p \in \mathbb{Q}_I$ arbitrarily. By our inductive hypothesis, there is $q \geq p$ with $x_{t,n} \in \overline{x}_q$. If $x_{t,n+1} \in \overline{x}_q$, there is nothing to prove, so assume otherwise. Then, necessarily, $n_{q,t} = n + 1$. Let $r$ be the extension of $q$ with $\overline{x}_r = \overline{x}_q \cup \{x_{t,n+1}\}$ and $r(\overline{x}_r)$ the complete type generated by $q(\overline{x}_q) \cup \{x_{t,n+1} = x_{t,n}\}$.

The final sentence holds by inspection of the proof above. $\square_{4.3.8}$

**B. Henkin witnesses**

For every $t \in I$, for every finite sequence $\overline{x}$ (indexed as in Notation 4.3.1) from $I\restriction_{<t} \times \omega$, and for every $L$-formula $\phi(y, \overline{x})$, $\mathcal{B}_{\phi,t}$ is the set of $p \in \mathbb{Q}$ such that:

1. $\overline{x} \subseteq \overline{x}_p$; and
2. Some $s \in u_p$ and $m < n_{p,s}$ satisfy $s < t$ and $p(\overline{x}_p) \vdash (\exists y)\phi(y, \overline{x}) \to \phi(x_{s,m}, \overline{x})$.

{Henkin}

**Claim 4.3.9.** *For each $t \in I$, finite sequence $\overline{x}$ from $I\restriction_{<t} \times \omega$, and $\phi(y, \overline{x})$, $\mathcal{B}_{\phi,t}$ is dense and open.*

**Proof.** Fix $t \in I$ and $\phi(y, \overline{x})$ as above. Choose any $p \in \mathbb{Q}_I$. By using Claim 4.3.8 and extending $p$ as needed, we may assume $\overline{x} \subseteq \overline{x}_p$. Let $q$ denote $p\restriction_{<t}$. Then $q \in \mathbb{Q}_I$ and $q \leq_{\mathbb{Q}} p$ by Lemma 4.3.3. As $\overline{x} \subseteq I_{<t} \times \omega$, $\overline{x} \subseteq \overline{x}_q$, so by adding dummy variables to $\phi$ we may assume $\overline{x} = \overline{x}_q$. Choose any $M \in \mathbf{At_T}$ and any realization $\overline{b}$ of $q$. There are now a number of cases.

Case 1: $M \models \neg\exists y\phi(y, \overline{b})$. Then as $q(\overline{x})$ generates a complete type, $q \vdash \neg\exists y\phi(y, \overline{x}_q)$, hence $p \in \mathcal{B}_{\phi,t}$.

So, we assume this is not the case. Fix a witness $c \in M$ such that $M \models \phi(c, \overline{b})$. There are now several cases depending on the complexity of $c$ over $\overline{b}$. In each of them, we will produce $r \geq_{\mathbb{Q}} q$ with $u_r \subseteq I\restriction_{<t}$ and $r(\overline{x}_r) \vdash \exists y\phi(y, \overline{x})$.

Case 2: $c \in \mathrm{pcl}(\emptyset, M)$. If $\min(I) \notin q$, then let $\overline{x}_r = \overline{x}_q \cup \{x_{\min(I),0}\}$ and if $\min(I) \in q$, then let $\overline{x}_r = \overline{x}_q \cup \{x_{\min(I),m}\}$, where $m = n_{q,\min(I)}$. Regardless, put $r(\overline{x}_r) = \mathrm{tp}(\overline{b}c, M)$.

Case 3: $c \notin \mathrm{pcl}(\overline{b}, M)$. Choose $s^* > u_q$ with $s^* < t$. Let $\overline{x}_r = \overline{x}_q \cup \{x_{s^*,0}\}$ and again take $r(\overline{x}_r) = \mathrm{tp}(\overline{b}c, M)$. It is easily checked that $r \in \mathbb{Q}_I$.

Case 4: $c \in \mathrm{pcl}(\overline{b}, M) \setminus \mathrm{pcl}(\emptyset, M)$. For each $s \in u_q$, let $\overline{x}\restriction_{\leq s}$ be the subsequence of $\overline{x}$ consisting of all $x_{t,m} \in \overline{x}$ with $t \leq s$, and let $\overline{b}\restriction_{\leq s}$ be the corresponding subsequence of $\overline{b}$. Using this as notation, choose $t^* \in u_q \setminus \{\min(I)\}$ least such that $c \in \mathrm{pcl}(\overline{b}\restriction_{\leq t^*}, M)$. Again, let $\overline{x}_r = \overline{x}_q \cup \{x_{t^*,m}\}$, where $m = n_{q,t^*}$, and let $r(\overline{x}_r) = \mathrm{tp}(\overline{b}c, M)$. As in the case above, it is easily verified that $r \in \mathbb{Q}_I$.

Now, in any of Cases 2,3,4, by Lemma 4.3.6 we can find $p^* \geq_{\mathbb{Q}} p$ and $p^* \geq_{\mathbb{Q}} r$. $\square_{4.3.9}$

**C. Fullness** Suppose $\overline{x}$ is a finite sequence (indexed as in Notation 4.3.1), $t \in I$, and $\phi(y,\overline{x})$ is an $L$-formula such that $\phi(y,\overline{x})$ 'says' '$y$ is not pseudo-algebraic over $\overline{x}$.'

$$\mathcal{C}_{\phi,t} = \{p \in \mathbb{Q}_I : \text{ there is } s > t, s \in u_p, \overline{x} \subseteq \overline{x}_p, p \vdash \phi(x_{s,0},\overline{x})\}$$

{fullness}

**Claim 4.3.10.** *Each is $\mathcal{C}_{\phi,t}$ is dense and open.*

**Proof.** Fix $\phi(y,\overline{x})$ and $t$, and choose any $p \in \mathbb{Q}_I$. By extending $p$ as needed, by Claim 4.3.8 we may assume $\overline{x} \subseteq \overline{x}_p$. Choose any countable $M \in \mathbf{At_T}$ and choose any realization $\overline{b}$ of $p(\overline{x}_p)$ in $M$. As $\phi(y,\overline{b})$ is not pseudo-algebraic, there is $N \in \mathbf{At_T}$, $N \succeq M$, and $c \in N \setminus M$ satisfying $N \models \phi(c,\overline{b})$. Choose any $s \in I$ such that $s > \max(u_p)$ and $s > t$ with $I \models \neg P(s)$. Define $q$ by: $\overline{x}_q = \{x_{s,0}\} \cup \overline{x}_p$ and $q(\overline{x}_q) = \mathrm{tp}(c\overline{b}, N)$. Then $q \geq_{\mathbb{Q}} p$ and $q \in \mathcal{C}_{\phi,t}$. $\square_{4.3.10}$

**D+E. Determining level** The definition of the forcing implies that $x_{t,n}$ is pseudo-algebraic over $\overline{x}_p\restriction_{<t} \cup \{x_{t,0}\}$ for any $p \in \mathbb{Q}_I$ with $x_{t,n} \in \overline{x}_p$, but it might also be algebraic over some smaller finite sequence (at a lower level). If this occurs, we 'adjust the level' by finding some $s < t$ and $m$ and insisting that $x_{t,n} = x_{s,m}$. To make this precise involves defining two families of constraints and showing that each is dense and open. The first family is actually a union of two.

$\mathcal{D}_{t,n} = \mathcal{D}^1_{t,n} \cup \mathcal{D}^2_{t,n}$ where

1. $\mathcal{D}^1_{t,n} = \{p : x_{t,n} \in \overline{x}_p$ and $p$ 'says' $x_{t,0} \in \mathrm{pcl}(\overline{x}_p\restriction_{<t} \cup \{x_{t,n}\})\}$;
2. $\mathcal{D}^2_{t,n} = \{p : x_{t,n} \in \overline{x}_p$, there are $s \in u_p$, $s < t$, and $m < n_{p,s}$ such that $p(\overline{x}_p) \vdash x_{t,n} = x_{s,m}\}$.

The second family is parameterized by $\overline{x}, t, n$. Let $\overline{x}$ be any finite sequence (cf. Notation 4.3.1) indexed by $u$ with $s = \max(u) < t$.

$\mathcal{E}_{t,n,\overline{x}} = \{p \in \mathbb{Q}_I : \overline{x} \cup \{x_{t,n}\} \subseteq \overline{x}_p$ and **either** $p$ 'says' $x_{t,n} \notin \mathrm{pcl}(\overline{x})$ **or** $p$ 'says' $x_{t,n} = x_{s,m}$ for some $m\}$

{level1}

**Claim 4.3.11.** *For all $(t,n) \in I \times \omega$ and for all finite sequences $\overline{x}$ indexed by $u$ with $\max(u) < t$, $\mathcal{E}_{t,n,\overline{x}}$ is dense and open.*

**Proof.** Once more, 'Open' is clear. Let $s = \max(u)$. Given any $p \in \mathbb{Q}_I$, by iterating Claim 4.3.8 we may assume $\overline{x} \cup \{x_{t,n}\} \subseteq \overline{x}_p$. If $p$ 'says' $x_{t,n} \notin \mathrm{pcl}(\overline{x})$, then $p \in \mathcal{E}_{t,n,\overline{x}}$, so assume $p$ 'says' $x_{t,n} \in \mathrm{pcl}(\overline{x})$. From our conditions on $\overline{x}$, this implies $x_{t,n} \in \mathrm{pcl}(\overline{x}_p\restriction_{\leq s})$. So put $m = n_{p,s}$, let $\overline{x}_q = \overline{x}_p \cup \{x_{s,m}\}$ and let $q(\overline{x}_q)$ be the complete type generated by $p(\overline{x}_p) \cup$ '$x_{t,n} = x_{s,m}$'. $\square_{4.3.11}$

{level}

**Claim 4.3.12.** *For every $t \in I \setminus \{\min(I)\}$ and every $n \in \omega$, $\mathcal{D}_{t,n}$ is dense and open.*

**Proof.** Choose any $p \in \mathbb{Q}_I$. By Claim 4.3.8 we may assume $x_{t,n} \in \overline{x}_p$. Choose any $M \in \mathbf{At_T}$ and choose $\overline{b}$ in $M$ realizing $p$. There are now several cases.

Case 1. If $b_{t,0} \in \mathrm{pcl}(\overline{b}\restriction_{<t} \cup \{b_{t,n}\})$, then $p \in \mathcal{D}^1_{t,n}$, so assume this is not the case.

Case 2. If $b_{t,n} \in \mathrm{pcl}(\emptyset, M)$ and $\min(I) \notin u_p$, then define $q$ by $\overline{x}_q = \overline{x}_p \cup \{x_{\min(I),0}\}$ and $q(\overline{x}_q) = \mathrm{tp}(\overline{b}b_{t,n}, M)$.

Case 3. If $b_{t,n} \in \mathrm{pcl}(\overline{b}_{\leq s}, M)$ for some $s \in u_p$, $s < t$, then define $q$ by $\overline{x}_q = \overline{x}_p \cup \{x_{s,m}\}$ (where $m = n_{p,s}$) and $q(\overline{x}_q)$ be the extension of $p(\overline{x}_p)$ by '$x_{t,n} = x_{s,m}$.'

Case 4. If none of the previous cases occur, choose $s^* < t$ with $s^* > u_p \cap I_{<t}$, $I \models \neg P(s^*)$. Define $q$ by $\overline{x}_q = \overline{x}_p \cup \{x_{s^*,0}\}$ and $q(\overline{x}_q) = \mathrm{tp}(\overline{b}b_{t,n}, M)$ (i.e. $x_{s^*,0} = x_{t,n}$). Now since Case 1 fails, $q$ satisfies Condition 5) in the definition of $\mathbb{Q}_I$ at level $t$, and since Case 3 fails, Condition 5) holds at level $s^*$. And in q, Condition 6) holds for $x_{t,n}$ since $b_{t,n} = b_{s^*,0}$. The other conditions are inherited from $p$, so $q \in \mathbb{Q}_I$. $\square_{4.3.12}$

**F. Achieving unbounded reach**

Suppose $s_0/E < s_1 < t$ are from $I$ with $I \models P(t)$, $s_0 \neq \min(I)$, and $I \models \neg P(s_0)$ (so $s_0/E$ is infinite and dense).

$\mathcal{F}_{t,s_0,s_1}$ is the set of $p \in \mathbb{Q}_I$ such that there exists $s_2 \in u_p$ with $s_1 < s_2 < t$ such that (recalling Notation 4.3.1) $p$ 'says'

$$x_{s_2,0} \in \mathrm{pcl}(\{x_{t,0}\} \cup \overline{x}_p\restriction_{\leq s_0/E}).$$

{blocking}

**Claim 4.3.13.** *Each $\mathcal{F}_{t,s_0,s_1}$ is dense and open.*

**Proof.** Open is clear. Choose any $p \in \mathbb{Q}_I$. By Claim 4.3.8 we may assume $x_{t,0} \in \overline{x}_p$. By Lemma 4.3.3 we have the sequence of extensions:

$$p\restriction_{\leq s_0/E} \ \leq_{\mathbb{Q}} \ p\restriction_{<t} \ \leq_{\mathbb{Q}} \ p\restriction_{\leq t} \ \leq_{\mathbb{Q}} \ p.$$

Fix $M \in \mathbf{At_T}$ and choose sequences $\overline{a}$, $\overline{d}$, $\overline{c}$ from $M$ such that $\overline{a}\overline{d}\overline{c}$ realizes $p\restriction_{\leq t}$, with $\overline{a}$ realizing $p\restriction_{\leq s_0/E}$ and $\overline{c}$ realizing $p\restriction_{=t}$. Let $c_0 \in \overline{c}$ be the interpretation of $x_{t,0}$. Thus, $M \models \delta(c_0)$ and $c_0 \notin \mathrm{pcl}(\overline{a}, M)$. Using Fact 4.2.5, choose $\overline{b}$ and $e$ from $M$ such that $e \in \mathrm{pcl}(\overline{a}\overline{b}c_0, M) \setminus \mathrm{pcl}(\overline{a}\overline{b}, M)$, but $c_0 \notin \mathrm{pcl}(\overline{a}\overline{b}e, M)$. We will find conditions in $\mathbb{Q}$ that assign levels to $\overline{b}$ and $e$ to satisfy $\mathcal{F}_{t,s_0,s_1}$.

As the class $s_0/E$ has no last element, by using Claim 4.3.9 (Henkin witnesses) $\lg(\overline{b})$ times, we can construct $q \in \mathbb{Q}_I$, $q \geq_{\mathbb{Q}} p\restriction_{\leq s_0/E}$ satisfying $q(\overline{x}_q) = \mathrm{tp}(\overline{a}\overline{b}, M)$ and $u_q \subseteq I\restriction_{\leq s_0/E}$.

Next, by Lemma 4.3.6 there is $q_1 \geq_{\mathbb{Q}} q$, $q_1 \geq_{\mathbb{Q}} p\restriction_{<t}$, and $u_{q_1} \subseteq I\restriction_{<t}$. By Lemma 4.3.6 again, there is $q_2 \geq_{\mathbb{Q}} q_1$, $q_2 \geq_{\mathbb{Q}} p\restriction_{\leq t}$, and $u_{q_2} \subseteq I\restriction_{\leq t}$. Indeed, by the 'Moreover' clause of Lemma 4.3.6, we may additionally assume that $q_2(\overline{x}_{q_2}) = \mathrm{tp}(\overline{a}\overline{b}\overline{d}\overline{c}, M)$ (and so $q_1(\overline{x}_{q_1}) = \mathrm{tp}(\overline{a}\overline{b}\overline{d}, M)$).

Now, choose $s_2 \in I$ such that $I \models \neg P(s_2)$, $s_1 < s_2 < t$, and $s_2 > s$ for every $s \in u_{q_1}$. Define $r$ by $\overline{x}_r = \overline{x}_{q_2} \cup \{x_{s_2,0}\}$ and $r(\overline{x}_r) = \mathrm{tp}(\overline{a}\overline{bd}\overline{c}e, M)$. It is easily checked that $r \in \mathbb{Q}_I$ and visibly, $r \geq_{\mathbb{Q}} q_2$. As well, $r \in \mathcal{F}_{t,s_0,s_1}$.

Finally, by a final application of Lemma 4.3.6, since $u_r \subseteq I\restriction_{\leq t}$ and $r \geq_{\mathbb{Q}} p\restriction_{\leq t}$, there is $p^* \geq_{\mathbb{Q}} p$ with $p^* \geq_{\mathbb{Q}} r$. As $p^* \in \mathcal{F}_{t,s_0,s_1}$, we conclude that $\mathcal{F}_{t,s_0,s_1}$ is dense. $\square_{4.3.13}$

## 4.4 Proof of Theorem 4.2.4

{finarg}

Given a linear order $I$ we construct a model $N = N_I$ of the theory $T$. That is, we verify that the forcing $(\mathbb{Q}_I, \leq_{\mathbb{Q}})$ satisfies the conclusions of Theorem 4.2.4. Suppose $G \subseteq \mathbb{Q}_I$ is a filter meeting every dense open subset. Let

$$X[G] = \bigcup\{p(\overline{x}_p) : p \in G\}$$

Because of the dense subsets $\mathcal{A}_{t,n}$, $X[G]$ describes a complete type in the variables $\{x_{t,n} : t \in I, n \in \omega\}$.[5] Intuitively, we want to build a with domain given by these variables. But the Level conditions, Claim 4.3.12 introduced a natural equivalence relation $\sim_G$ on $X[G]$ defined by

$$x_{t,n} \sim_G x_{s,m} \qquad \text{if and only if} \qquad X[G] \text{ ‘says’ } x_{t,n} = x_{s,m}$$

Let $N[G]$ be the $\tau$-structure with universe $X[G]/\sim_G$. Each element of $N[G]$ has the form $[x_{t,n}]$, which is the equivalence class of $x_{t,n}$ (mod $\sim_G$). As each $p \in \mathbb{Q}_I$ describes a complete (principal) formula with respect to $T$, $N[G]$ is an atomic set. As well, it follows from Claim 4.3.9 that $N[G] \models T$.

For each $t \in I$ such that $P(t)$ holds, let $N_{<t} = \{[x_{w,n}] : \text{some } x_{s,m} \in [x_{w,n}] \text{ with } s < t\}$. Similarly, for each $s \in I \setminus \{\min(I)\}$ with $\neg P(s)$, let $N_{<s} = \{[x_{w,n}] : w/E < s/E\}$.

By repeated use of Claim 4.3.9, both $N_{<t}$ and $N_{<s}$ are elementary substructures of $N[G]$. Note that $N_{<s'} = N_{<s}$ whenever $E(s', s)$.

For simplicity, let $a_{w,n} \in N[G]$ denote the class $[x_{w,n}]$. Given any $(w, n)$, if there is a least $s \in I$ such that $a_{w,n} = a_{s,m}$ for some $m \in \omega$, then we say $a_{w,n}$ *is on level* $s$. For an arbitrary $(w, n)$, a least $s$ need not exist, but it does in some cases. In particular, Definition 4.3.2.5 and the level constraint ($\mathcal{E}_{w,0,\overline{x}}$) imply that any $a_{w,0}$ is on level $w$ for any $w \in I$. As well, because of the Level constraints (group $D + E$) for any $t$ such that $P(t)$ holds and for any $n > 0$,

$$a_{t,n} \text{ is on level } t \text{ if and only if } a_{t,0} \in \mathrm{pcl}(N_{<t} \cup \{a_{t,n}\}, N[G])$$

[5]If $\mathrm{pcl}(\emptyset) = \emptyset$, then $X[G]$ is in the variables $\{x_{t,n} : n \in \omega, t \in I \setminus \{\min(I)\}\}$. For clarity of exposition, we will assume that $\mathrm{pcl}(\emptyset) \neq \emptyset$.

As $|I| = \aleph_1$ and the fact that each $a_{t,0} \notin \mathrm{pcl}(N_{<t}, N[G])$, $||N[G]|| = \aleph_1$. Finally, it follows from the density of the 'Fullness conditions' that $N[G]$ is full.

It remains to verify that $N[G]$ satisfies the three conditions of Theorem 4.2.4. First, for any initial segment $J \subseteq I$ without a maximum element (in particular, for any suitable $J$) the density of the Henkin conditions offered by Claim 4.3.9 and the Tarski-Vaught criterion imply that $N_J \preceq N[G]$.

Second, suppose $t \in I$ and $P(t)$ holds. We show that $a_{t,0}$ catches and has unbounded reach in $N_{<t}$. Note that since $I{\restriction}_{<t}$ is suitable, $N_{<t} \preceq N[G]$, hence $\mathrm{pcl}(N_{<t}, N[G]) = N_{<t}$. To see that $a_{t,0}$ catches $N_{<t}$, choose any $a_{s,m} \in \mathrm{pcl}(N_{<t} \cup \{a_{t,0}\}, N[G]) \setminus N_{<t}$. By taking an appropriate finite sequence $\overline{x}$ witnessing the pseudo-algebraicity, the density of the constraints $\mathcal{E}_{s,m,\overline{x}}$ allow us to assume $s \leq t$. However, if $s < t$, then we would have $a_{s,m} \in N_{<t}$. Thus, the only possibility is that $(s, m) = (t, n)$ for some $n \in \omega$ and that $a_{t,n}$ is on level $t$. It follows from the displayed remark above that $a_{t_0} \in \mathrm{pcl}(N_{<t} \cup \{a_{t,n}\}, N[G])$. Thus, $a_{t,0}$ catches $N_{<t}$. We also argue that $a_{t,0}$ has unbounded reach in $N_{<t}$. To see this, choose any $s_0 < t$, $s_0 \neq \min(I)$ with $I \models \neg P(s_0)$. For any $s_1$ satisfying $s_0/E < s_1/E < t$, choose $p \in G \cap \mathcal{F}_{t,s_0,s_1}$ and choose $s_2 \in u_p$ from there. Now, the element $a_{s_2,0} \in \mathrm{pcl}(N_{<s_0} \cup \{a_{t,0}\}, N[G])$. As well, since $s_1/E < s_2/E < t/E$, $a_{s_2,0} \notin N_{<s_1}$, so $a_{t,0}$ has unbounded reach in $N_{<t}$.

It remains to verify (3) of Theorem 4.2.4. Choose a seamless $J \subseteq I$ and suppose some $b \in N[G] \setminus N_J$ catches $N_J$. Say $b$ is $a_{t^*,n}$, where necessarily $t^* \in I \setminus J$. We must show $b$ has bounded effect in $N_J$. By the fundamental theorem of forcing, there is $p \in G$ such that

$$p \Vdash a_{t^*,n} \text{ catches } N_J.$$

Thus, among other things, $p \Vdash$ '$a_{t^*,n} \neq a_{s,m}$' for all $s \in J$, $m \in \omega$.

Choose any $s^* \in J$ such that $s^* > s$ for every $s \in u_p \cap J$.

{forceit}

**Claim 4.4.1.** $p \Vdash \mathrm{pcl}(\{b\} \cup N_{<s^*}[\dot{\mathbf{G}}], N[\dot{\mathbf{G}}]) \cap N_J[\dot{\mathbf{G}}] \subseteq N_{<s^*}[\dot{\mathbf{G}}]$.

**Proof.** If not, then there is $q \in \mathbb{Q}_I$ satisfying $q \geq p$ and a finite $A \subseteq N_{<s^*}[\dot{\mathbf{G}}]$ such that

$$q \Vdash \mathrm{pcl}(Ab, N[\dot{\mathbf{G}}]) \cap N_J[\dot{\mathbf{G}}] \not\subseteq N_{<s^*}[\dot{\mathbf{G}}]$$

Without loss, we may assume that if $a_{t,m} \in A$, then $t \in u_q$. As $J$ is seamless, by Lemma 4.1.3, choose an automorphism $\pi$ of $(I, <, E, P)$ such that $\pi{\restriction}_{\geq \min(u_p \setminus J)} = id$; $\pi(t^*) = t^*$; $\pi{\restriction}_{u_p} = id$; $\pi{\restriction}_{u_q \cap I_{<s^*}} = id$, but $\pi(s^*) \notin J$. By Lemma 4.3.4, $\pi$ extends to an automorphism $\pi'$ of $\mathbb{Q}_I$ given by $x_{t,m} \mapsto x_{\pi(t),m}$. By our choice of $\pi$, $\pi'(p) = p$. While $\pi'(q)$ need not equal $q$, we do have $p \leq \pi'(q)$. Now

$$\pi'(q) \Vdash \mathrm{pcl}(Ab, N[\dot{\mathbf{G}}]) \cap N_{\pi(J)}[\dot{\mathbf{G}}] \not\subseteq N_{<\pi(s^*)}[\dot{\mathbf{G}}]$$

But this contradicts $p \Vdash a_{t^*,n}$ catches $N_J$. [To see this, choose $H$ generic with $\pi'(q) \in H$, hence also $p \in H$. Choose $e \in (\mathrm{pcl}(Ab, N[H]) \cap N_{\pi(J)}[H]) \setminus N_{<\pi(s^*)}[H]$. As $A \subseteq N_J[H]$, $e \in \mathrm{pcl}(N_J[H] \cup \{b\}, N[H])$. Moreover, as $N_J[H] \preceq N_{<\pi(s^*)}[H]$, $e \notin N_J[H]$. But, since $N_J[H] \cup \{e\} \subseteq N_{\pi(J)}[H]$ and $b \notin N_{\pi(J)}[H]$, it follows that $b \notin \mathrm{pcl}(N_J[H] \cup \{e\}, N[H])$. That is, $e$ witnesses that $b$ does not catch $N_J[H]$.] $\square_{4.4.1}$

> I cleaned up the justification between the brackets [...] as to why we have a contradiction to catching. Please read and see that I am not confusing myself.

As Claim 4.4.1 holds for any sufficiently large $s^* \in J$, $a_{t,n}$ has bounded effect in $N_J$ This concludes the proof Theorem 4.2.4. $\square_{4.2.4}$

# 5 Proof of Theorem 2.8

{5}

Now we prove the main theorem, Theorem 2.8, by using the transfer lemma, Theorem 3.3.1 to move from coding a model by $S$ in $M[G]$ (Theorem 4.2.4) to $2^{\aleph_1}$ models in $V$.

We prove Theorem 2.8 under the assumption that a countable, transitive model $(M, \epsilon)$ of a suitable finitely axiomatizable subtheory of ZFC exists.[6] As the existence of the latter is provable from ZFC (using the Reflection Theorem) we obtain a proof of Theorem 2.8 in ZFC.

As the pseudo-minimal types are not dense, we can find a complete formula $\delta(x, \overline{a})$ that is not pseudo-algebraic, but has no pseudo-minimal extension. As having $2^{\aleph_1}$ models is invariant under naming finitely many constants, we absorb $\overline{a}$ into the signature and write $\delta(x)$ for this complete formula.

Fix a countable, transitive model $(M, \epsilon)$ of ZFC with $T, \tau \in M$ and we begin working inside it. In particular, choose $S \subseteq \omega_1^M \setminus \{0\}$ such that

$$(M, \epsilon) \models \text{‘}S \text{ is stationary/costationary’}$$

Next, perform Construction 4.1.4 inside $M$ to obtain $I = (I^S, <, P, E) \in \mathbf{I}^*$.

Next, we force with the c.c.c. poset $\mathbb{Q}_{I^S}$ and find $(M[G], \epsilon)$, where $G$ is a generic subset of $\mathbb{Q}_{I^S}$. As the forcing is c.c.c., it follows that all cardinals as well as stationarity, are preserved, Thus, $\omega_1^{M[G]} = \omega_1^M$ and $(M[G], \epsilon) \models$ ‘$S$ is stationary/costationary’.

As Construction 4.1.4 is absolute, $I^{M[G]} = I^M = I^S$. According to Theorem 4.2.4, inside $M[G]$ there is an atomic, full $N_I \models T$ that is striated according to $(I^S, <, P, E)$. Write

[6] Alternatively, one could use the fragment $ZFC^0$ of [**?**].

the universe of $N_I$ as $\{a_{t,n} : t \in I^S, n \in \omega\}$. Inside $M[G]$ we have the mapping $\alpha \mapsto J_\alpha$ given by Construction 4.1.4. For every $\alpha \in \omega_1^{M[G]}$, let $N_\alpha$ be the $\tau$-substructure of $N_I$ with universe $\{a_{t,n} : t \in J_\alpha, n \in \omega\}$. It follows from Theorem 4.2.4 and Construction 4.1.4 that for every non-zero $\alpha \in \omega_1^{M[G]}$:

- $N_\alpha \preceq N_I$;
- If $\alpha \in S$, then $I^S \setminus J_\alpha$ has a least element $t(\alpha)$ and $a_{t(\alpha),0}$ both catches and has unbounded reach in $N_\alpha$;
- If $\alpha \notin S$, then every $b \in N_I \setminus N_\alpha$ that catches $N_\alpha$ has bounded effect in $N_\alpha$.

Now, still working inside $M[G]$, we identify a 3-sorted structure $N^*$ that encodes this information. The vocabulary of $N^*$ will be

$$\tau^* = \tau \cup \{U, V, W, <_U, <_V, P, E, R_1, R_2\}.$$

$N^*$ is the $\tau^*$-structure in which

- $\{U, V, W\}$ are unary predicates that partition the universe;
- $(U^{N^*}, <_U)$ is $(\omega_1^{M[G]}, <)$;
- $(V^{N^*}, <_V, P, E)$ is $(I^S, <, P, E)$;
- $W^{N^*}$ is $N_I$ (the $\tau$-functions and relations only act on the $W$-sort);
- $R_1 \subseteq U \times V$, with $R_1(\alpha, t)$ holding if and only if $t \in J_\alpha$; and
- $R_2 \subseteq U \times W$, with $R_2(\alpha, b)$ holding if and only if $b \in N_\alpha$.

Note that $S \subseteq \omega_1^{M[G]}$ is a $\tau^*$-definable subset of the $U$-sort of $N^*$ ($\alpha \in S$ if and only if $V \setminus R_1(\alpha, V)$ has a $<_V$-minimal element). Also, on the $W$-sort, the relation '$b \in \mathrm{pcl}(\overline{a})$' is definable by an infinitary $\tau^*$-formula. Thus, the relations '$b$ catches $N_\alpha$' , '$b$ has unbounded reach in $N_\alpha$' and '$b$ has bounded effect in $N_\alpha$' are each infinitarily $\tau^*$-definable subsets of $U \times W$.

By construction, $N^* \models \psi$, where the infinitary $\psi$ asserts: 'For every non-zero $\alpha \in U$, **either** every element of $W^{N^*}$ that catches $N_\alpha$ also has unbounded reach in $N_\alpha$ **or** there is an element of $W^{N^*}$ that catches $N_\alpha$ and has bounded effect in $N_\alpha$.'

To distinguish between these two possibilities, there is an infinitary $\tau^*$-formula $\theta(x)$ such that for $x$ from the $U$-sort, $\theta(x)$ holds if and only if there exists $b \in N_I \setminus N_{J_x}$ that catches and has unbounded reach in $N_{J_x}$. Thus, for non-zero $\alpha \in \omega_1^{M[G]}$ we have

$$N^* \models \theta(\alpha) \quad \Longleftrightarrow \quad \alpha \in S$$

Now, identify a countable fragment $L_{\mathcal{A}}$ of $L_{\omega_1,\omega}(\tau^*)$ to include the formulas mentioned in the last three paragraphs, along with infinitary formulas ensuring $\tau$-atomicity.

Now, we switch our attention to $V$, and apply Theorem 3.3.1 to $(M[G], \epsilon)$, $L_{\mathcal{A}}$, and $N^*$. This gives us a family $(M_X, E)$ of elementary extensions of $(M[G], \epsilon)$, each of size $\aleph_1$, indexed by subsets $X \subseteq \omega_1$ $(= \omega_1^V)$. Each of these models of ZFC has an $\tau^*$-structure, which we call $N_X^*$ inside it. As well, for each $X \subseteq \omega_1$, there is a continuous, strictly increasing mapping $t_X : \omega_1 \to U^{N_X^*}$ with the property that

$$N_X^* \models \theta(t_X(\alpha)) \quad \Longleftrightarrow \quad \alpha \in X$$

Let $(I^X, <^X, E^X, P^X)$ be the '$V$-sort' of $N_X^*$. Clearly, each $I^X \in \mathbf{I}^*$.

Finally, the $W$-sort of each $\tau^*$-structure $N_X^*$ is the universe of a $\tau$-structure, striated by $I^X$. We call this 'reduct' $N_X$. Note that by our choice of $L_{\mathcal{A}}$ and the fact that $N_X^* \succeq_{L_{\mathcal{A}}} N^*$, we know that every $\tau$-structure $N_X$ is an atomic model of $T$ and is easily seen to be of cardinality $\aleph_1$. Thus, the proof of Theorem 2.8 reduces to the following:

**Claim.** If $X \setminus Y$ is stationary, then there is no $\tau$-isomorphism $f : N_X \to N_Y$.

**Proof.** Fix $X, Y \subseteq \omega_1$ such that $X \setminus Y$ is stationary and by way of contradiction assume that $f : N_X \to N_Y$ were a $\tau$-isomorphism. Consider the $\tau^*$-structures $N_X^*$ and $N_Y^*$ constructed above. As notation, for each $\alpha \in \omega_1^V$, let $N_\alpha^X$ and $N_\alpha^Y$ denote $\tau$-elementary substructures with universes $R_2(t_X(\alpha), N_X^*)$ and $R_2(t_Y(\alpha), N_Y^*)$, respectively.

Next, choose a club $C_0 \subseteq \omega_1$ such that for every $\alpha \in C_0$:

- $\alpha$ is a limit ordinal;
- The restriction of $f : N_\alpha^X \to N_\alpha^Y$ is a $\tau$-isomorphism.

Denote the set of limit points of $C_0$ by $C$. As $C$ is club and $(X \setminus Y)$ is stationary, choose $\alpha$ in their intersection. Fix a strictly increasing $\omega$-sequence $\langle \alpha_n : n \in \omega \rangle$ of elements from $C_0$ converging to $\alpha$. As $\alpha \in X$, we can choose an element $b \in N_X \setminus N_\alpha^X$ such that $b$ catches $N_\alpha^X$ and has unbounded reach in $N_\alpha^X$. That is, there is $\gamma < \alpha$ such that for every $\beta$ satisfying $\gamma < \beta < \alpha$,

$$\mathrm{pcl}(N_\gamma^X \cup \{b\}, N_X) \cap N_\alpha^X \not\subseteq N_\beta^X.$$

Fix $n \in \omega$ such that $\alpha_n > \gamma$. Then, for every $m \geq n$

$$\mathrm{pcl}(N_{\alpha_n}^X \cup \{b\}, N_Y) \cap N_\alpha^X \not\subseteq N_{\alpha_m}^X.$$

Thus, as '$b \in \mathrm{pcl}(\overline{a})$' is preserved under $\tau$-isomorphisms and $f[N_{\alpha_m}^X] = N_{\alpha_m}^Y$ setwise, we have that $f(b)$ both catches and has unbounded reach in $N_\alpha^Y$. As $\alpha \notin Y$, we obtain a contradiction from $N_Y^* \models \neg\theta(t_Y(\alpha))$ and $N_Y^* \models \psi$.