Global instability in the elliptic restricted three body problem*

Amadeu Delshams[1], Vadim Kaloshin[2], Abraham de la Rosa[3]
and Tere M. Seara[1]

[1] Universitat Politècnica de Catalunya, Amadeu.Delshams@upc.edu, Tere.M-Seara@upc.edu

[2] University of Maryland at College Park, vadim.kaloshin@gmail.com

[3] Universidad Autónoma de México, abraham.delarosa@gmail.com

January 7, 2015

Abstract

The (planar) ERTBP describes the motion of a massless particle (a comet) under the gravitational field of two massive bodies (the primaries, say the Sun and Jupiter) revolving around their center of mass on elliptic orbits with some positive eccentricity. The aim of this paper is to show that there exist trajectories of motion such that their angular momentum performs arbitrary excursions in a large region. In particular, there exist diffusive trajectories, that is, with a large variation of angular momentum.

The framework for proving this result consists on considering the motion close to the parabolic orbits of the Kepler problem between the comet and the Sun that takes place when the mass of Jupiter is zero. In other words, studying the so-called infinity manifold. Close to this manifold, it is possible to define a scattering map, which contains the map structure of the homoclinic trajectories to it. Since the inner dynamics inside the infinity manifold is trivial, two different scattering maps are used. The combination of these two scattering maps permits the design of the desired diffusive pseudo-orbits, which eventually give rise to true trajectories of the system with the help of shadowing techniques.

Keywords: Elliptic Restricted Three Body problem, Arnold diffusion, splitting of separatrices, Melnikov integral.

1 Main result and methodology

The (planar) ERTBP describes the motion q of a massless particle (a *comet*) under the gravitational field of two massive bodies (the *primaries*, say the *Sun* and *Jupiter*) with mass ratio μ revolving around their center of mass on elliptic orbits with eccentricity e. In this paper we search for trajectories of motion which show a large variation of the angular momentum $G = q \times \dot{q}$. In other words, we search for global instability ("diffusion" is the term usually used) in the angular momentum of this problem.

If the eccentricity vanishes, the primaries revolve along circular orbits, and such diffusion is not possible, since the (planar and circular) RTBP is governed by an autonomous Hamiltonian with two degrees of freedom. This is not the case for the ERTBP, which is a 2+1/2 degree-of-freedom Hamiltonian system with time-periodic Hamiltonian. Our main result is the following

Theorem 1. *There exist two constants $C > 0$, $c > 0$ and $\mu^* = \mu^*(C, c) > 0$ such that for any $0 < e < c/C$ and $0 < \mu < \mu^*$, and for any two values of the angular momentum in the region $C \leq G_1^* < G_2^* \leq c/e$, there exists a trajectory of the ERTBP such that $G(0) < G_1$, $G(T) > G_2$ for some $T > 0$.*

*AD, AR, and TMS were partially supported by the Spanish MINECO-FEDER Grant MTM2012-31714 and the Catalan Grant 2014SGR504.

This result will be a consequence of Theorem 10, where the large C and the small constant c are explicitly computed ($C = 32$, $c = 1/8$), and where it is also shown the existence of trajectories of motion such that their angular momentum performs arbitrary excursions along the region $C \leq G_1^* < G_2^* \leq c/e$.

Let us recall related results about oscillatory motions and diffusion for the RTBP or the ERTBP. They hold close to a region when there is some kind of hyperbolicity in the Three Body Problem, like the Euler libration points [LMS85, CZ11, DGR13], collisions [Bol06], the infinity [LS80, Xia93, Xia92, Moe07, Rob84, MP94, MS14] or near mean motion resonances [FGKR14]. Among these papers, two were very influential for our computations, namely [LS80], where the Laplace's method was used along special complex paths to compute several integrals, and [MP94], which contains asymptotic formulas for a scattering map on the infinity manifold for large values of eG. Another one, [GMS12], is very important for futute related work, since the proof of transversal manifolds of the infinity manifold is established for the RTBP for any $\mu \in (0, 1/2]$.

Concerning the proof of our main result, let us first notice [LS80, GMS12] that, for a non-zero mass parameter small enough and zero eccentricity, the RTBP is not integrable, although for large G its chaotic zones have a size which is exponentially small in G. This phenomenon adds a first difficulty in proving the global instability of the angular momentum G in the ERTBP for large values of G.

The framework for proving our result consists on considering the motion close to the parabolic orbits of the Kepler problem that takes place when the mass parameter is zero. To this end we study the *infinity manifold*, which turns out to be an invariant object topologically equivalent to a normally hyperbolic invariant manifold (TNHIM). On this TNHIM, it is possible to define a *scattering map*, which contains the map structure of the homoclinic trajectories to the TNHIM. Unfortunately, the inner dynamics within the TNHIM is trivial, so it cannot be used combined with the scattering map to produce pseudo-orbits adequate for diffusion, and adds a second difficulty. Because of this, in this paper we introduce the use of *two* different scattering maps whose combination produces the desired diffusive pseudo-orbits, which eventually give rise to true trajectories of the system with the help of the shadowing results given in [GMS15].

The main issue to compute the two scattering maps consists on computing the *Melnikov potential* (51) associated to the TNHIM. The main difficulty for its computation comes from the fact that its size is exponentially small in the momentum G, so it is necessary to perform very accurate estimates for its Fourier coefficients. Such computations are performed in Theorem 7, and they involve a careful treatment of several Fourier expansions, as well as the computation of several integrals using Laplace's method along adequate complex paths, playing both with the eccentric and the true anomaly. To guarantee the convergence of the Fourier series, we have to assume that G is large enough ($G \geq C$), and e small enough ($Ge \leq c$). Under these two assumptions, the dominant part of the Melnikov potential consists on four harmonics, from which it is possible to compute the existence of two functionally independent scattering maps.

The combination of these two scattering maps permits the design of the desired diffusive pseudo-orbits, under the assumption of a mass parameter very small compared to eccentricity ($0 < \mu < \mu^*$), see (80)), which eventually give rise to true trajectories of the system with the help of shadowing techniques.

It is worth noticing that since all the diffusive trajectories found in this paper shadow ellipses close to parabolas of the Kepler problem, that is, with a very large semi-major axis, their energy is close to zero, and the orientation of their semi-major axis only changes slightly at each revolution.

The case of arbitrary eccentricity $0 < e < 1$ and arbitrary parameter mass parameter $0 < \mu < 1$ remains open in this paper. Indeed, the case $eG \approx 1$ involves the analysis of an infinite number of dominant Fourier coefficients of the Melnikov potential, whereas for the case $eG > 1$, the qualitative properties of the Melnikov function should be known without using its Fourier expansion. Larger values of the mass paratemer μ than those considered in this paper involve improving the estimates of the error terms of the splitting of separatrices in complex domains, as is usual when the splitting of separatrices is exponentially small. The computation of the explicit trajectories from the pseudo-orbits found in this paper needs a suitable shadowing result given in [GMS15], which involves the translation to TNHIM of the usual shadowing techniques for NHIM.

The plan of this paper is as follows. In Section 2 we introduce the equations of the ERTBP, as well as the McGehee coordinates to be used to study the motion close to infinity. In Section 3 we recall the geometry of the Kepler problem, when the mass parameter vanishes, close to the *infinity manifold* and its associated separatrix. Next, in Section 4, we study the transversal intersection of the invariant manifolds for the ERTBP, as well as the *scattering map* associated, which depend on the *Melnikov potential* of the problem, whose concrete computation is deferred to Section 6. The global instability is proven in Section 5, using the computation of the Melnikov potential, and is based on the computation of *two different* scattering maps, whose combination gives rise to the diffusive trajectories in the angular momentum.

2 Setting of the problem

If we fix a coordinate reference system with the origin at the center of mass and call q_{S} and q_{J} the position of the primaries, then under the classical assumptions regarding time units, distance and masses normalization, the motion q of a massless particle under Newton's law of universal gravitation is given by

$$\frac{d^2q}{dt^2} = (1-\mu)\frac{q_{\mathrm{S}}-q}{|q_{\mathrm{S}}-q|^3} + \mu\frac{q_{\mathrm{J}}-q}{|q_{\mathrm{J}}-q|^3} \tag{1}$$

where $1-\mu$ is the mass of the particle at q_{S} and μ the mass of the particle at q_{J}. Introducing the conjugate momentum $p = dq/dt$ and the self-potential function

$$U_\mu(q,t;e) = \frac{1-\mu}{|q-q_{\mathrm{S}}|} + \frac{\mu}{|q-q_{\mathrm{J}}|}, \tag{2}$$

equation (1) can be rewritten as a 2+1/2 degree-of-freedom Hamiltonian system with time-periodic Hamiltonian

$$H_\mu(q,p,t;e) = \frac{p^2}{2} - U_\mu(q,t;e). \tag{3}$$

In the (planar) ERTBP, the two primaries are assumed to be revolving around their center of mass on elliptic orbits with eccentricity e, unaffected by the motion q of the comet. In polar coordinates $q = \rho(\cos\alpha, \sin\alpha)$, the equations of motion of the primaries are

$$q_{\mathrm{S}} = \quad \mu r(\cos f, \sin f) \qquad q_{\mathrm{J}} = -(1-\mu)r(\cos f, \sin f). \tag{4}$$

By the first Kepler's law the distance r between the primaries [Win41, p. 195] can be written as a function $r = r(f,e)$

$$r = \frac{1-e^2}{1+e\cos f} \tag{5}$$

where $f = f(t,e)$ is the so called *true anomaly*, which satisfies [Win41, p. 203]

$$\frac{df}{dt} = \frac{(1+e\cos f)^2}{(1-e^2)^{3/2}}. \tag{6}$$

Taking into account the expression (4) for the motion of the primaries, we can write explicitly the denominators of the self-potential function (2)

$$|q-q_{\mathrm{S}}|^2 = \rho^2 - 2\mu r\rho\cos(\alpha - f) + \mu^2 r^2, \tag{7}$$

$$|q-q_{\mathrm{J}}|^2 = \rho^2 + 2(1-\mu)r\rho\cos(\alpha - f) + (1-\mu)^2 r^2. \tag{8}$$

We now perform a standard polar-canonical change of variables $(q,p) \longmapsto (\rho, \alpha, P_\rho, P_\alpha)$

$$q = (\rho\cos\alpha, \rho\sin\alpha), \quad p = \left(P_\rho\cos\alpha - \frac{P_\alpha}{r}\sin\alpha, P_\rho\sin\alpha - \frac{P_\alpha}{r}\cos\alpha\right) \tag{9}$$

to Hamiltonian (3). The equations of motion in the new coordinates are the associated to the Hamiltonian

$$H^*_\mu(\rho,\alpha,P_\rho,P_\alpha,t;e) = \frac{P_\rho^2}{2} + \frac{P_\alpha^2}{2\rho^2} - U^*_\mu(\rho,\alpha,t;e) \tag{10}$$

with a self-potential U_μ^*

$$U_\mu^*(\rho,\alpha,t;e) = U_\mu(\rho\cos\alpha, \rho\sin\alpha, t; e). \tag{11}$$

From now on we will write

$$G = P_\alpha, \qquad y = P_\rho,$$

so that Hamiltonian (10) becomes

$$H_\mu^*(\rho,\alpha,y,G,t;e) = \frac{y^2}{2} + \frac{G^2}{2\rho^2} - U_\mu^*(\rho,\alpha,t;e). \tag{12}$$

Remark 2. In the (planar) circular case $e = 0$ (RTBP), it is clear from equations (5) and (6) that $r = 1$ and $f = t$, and that the expressions for the distances (7) between the primaries depend on the time s and the angle α just through their difference $\alpha - t$. As a consequence, $U_\mu^*(\rho,\alpha,s;0)$ as well as $H_\mu^*(\rho,\alpha,y,G,s;0)$ depend also on s and α just through the same difference $\alpha - t$. This implies that the Jacobi constant $H^* + G$ is a first integral of system.

2.1 McGehee coordinates

To study the behavior of orbits near infinity, we make the McGehee [McG73] non-canonical change of variables:

$$\rho = \frac{2}{x^2} \tag{13}$$

for $x > 0$, which brings the infinity $\rho = \infty$ to the origin $x = 0$. In these McGehee coordinates, the equations associated to Hamiltonian (10) become

$$\frac{dx}{dt} = -\frac{1}{4}x^3 y \qquad \frac{dy}{dt} = \frac{1}{8}G^2x^6 - \frac{x^3}{4}\frac{\partial \mathcal{U}_\mu}{\partial x} \tag{14a}$$

$$\frac{d\alpha}{dt} = \frac{1}{4}x^4 G \qquad \frac{dG}{dt} = \frac{\partial \mathcal{U}_\mu}{\partial \alpha} \tag{14b}$$

where the self-potential $\mathcal{U}_\mu$ is given by

$$\mathcal{U}_\mu(x,\alpha,t;e) = U_\mu^*(2/x^2,\alpha,t;e) = \frac{x^2}{2}\left(\frac{1-\mu}{\sigma_S} + \frac{\mu}{\sigma_J}\right) \tag{15}$$

with

$$|q - q_{\mathrm{S}}|^2 = \sigma_{\mathrm{S}}^2 = 1 - \mu r x^2\cos(\alpha - f) + \frac{1}{4}\mu^2 r^2 x^4,$$

$$|q - q_{\mathrm{J}}|^2 = \sigma_{\mathrm{J}}^2 = 1 + (1-\mu) r x^2\cos(\alpha - f) + \frac{1}{4}(1-\mu)^2 r^2 x^4.$$

It is important to notice that the true anomaly f is present in these equations, so that the equation for f given in (6) should be added to have the complete description of the dynamics.

2.1.1 Hamiltonian structure

Under McGehee change of variables (13), the canonical form $d\rho\wedge dy + d\alpha\wedge dG$ is transformed to

$$\omega = -\frac{4}{x^3}dx\wedge dy + d\alpha\wedge dG \tag{16}$$

which, on $x > 0$, is a (non-canonical) symplectic form. Therefore, expressing Hamiltonian (12) in McGehee coordinates

$$\mathcal{H}_\mu(x,\alpha,y,G,t;e) = \frac{y^2}{2} + \frac{x^4G^2}{8} - \mathcal{U}_\mu(x,\alpha,t;e), \tag{17}$$

equations (14) can be written as

$$\frac{dx}{dt} = -\frac{x^3}{4}\left(\frac{\partial \mathcal{H}_\mu}{\partial y}\right) \qquad \frac{dy}{dt} = -\frac{x^3}{4}\left(-\frac{\partial \mathcal{H}_\mu}{\partial x}\right) \tag{18a}$$

$$\frac{d\alpha}{dt} = \frac{\partial \mathcal{H}_\mu}{\partial G} \qquad \frac{dG}{dt} = -\frac{\partial \mathcal{H}_\mu}{\partial \alpha}. \tag{18b}$$

Equivalently, we can write equations (18) as $dz/dt = \{z, \mathcal{H}_\mu\}$ in terms of the Poisson bracket

$$\{f, g\} = -\frac{x^3}{4}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right) + \frac{\partial f}{\partial \alpha}\frac{\partial g}{\partial G} - \frac{\partial f}{\partial G}\frac{\partial G}{\partial \alpha}. \tag{19}$$

3 Geometry of the Kepler problem ($\mu = 0$)

3.1 The infinity manifold

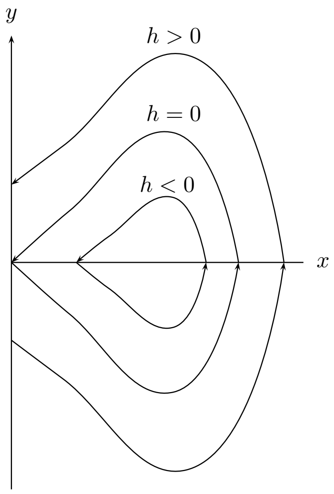

Figure 1: Level curves of $\mathcal{H}_0$ in the $(x \geq 0, y)$ plane, for fixed $G > 0$

For $\mu = 0$ and $G > 0$, Hamiltonian (17) becomes Duffing Hamiltonian

$$\mathcal{H}_0(x, y, G) = \frac{y^2}{2} + \frac{x^4 G^2}{8} - \mathcal{U}_0(x) = \frac{y^2}{2} + \frac{x^4 G^2}{8} - \frac{x^2}{2} \tag{20}$$

and is a first integral, since the system is autonomous. Moreover, $\mathcal{H}_0$ is also independent of e and α. Its associated equations are

$$\frac{dx}{dt} = -\frac{1}{4}x^3 y \qquad \frac{dy}{dt} = \frac{1}{8}G^2 x^6 - \frac{1}{4}x^4 \tag{21a}$$

$$\frac{d\alpha}{dt} = \frac{1}{4}x^4 G \qquad \frac{dG}{dt} = 0 \tag{21b}$$

where it is clear that G is a conserved quantity, which will be restricted to the case $G > 0$ from now on, that is, $G \in \mathbb{R}_+$. The phase space, including the invariant locus $x = 0$ is given by $(x, \alpha, y, G) \in \mathbb{R}_{\geq 0} \times \mathbb{T} \times \mathbb{R} \times \mathbb{R}_+$. From equations (21) it is clear that

$$\mathcal{E}_\infty = \{z = (x = 0, \alpha, y, G) \in \mathbb{R}_{\geq 0} \times \mathbb{T} \times \mathbb{R} \times \mathbb{R}_+\} \tag{22}$$

is the set of equilibrium points of system (21). Moreover, for any fixed $\alpha \in \mathbb{T}, G \in \mathbb{R}$,

$$\Lambda_{\alpha,G} = \{(0, \alpha, 0, G)\}$$

is a parabolic equilibrium point, which is topologically equivalent to a saddle point, since it possesses stable and unstable 1-dimensional invariant manifolds. The union of such points is the 2-dimensional manifold of equilibrium points

$$\Lambda_\infty = \bigcup_{\alpha,G} \Lambda_{\alpha,G}.$$

As we will deal with a time-periodic Hamiltonian, it is natural to work in the extended phase space

$$\tilde{z} = (z, s) = (x, \alpha, y, G, s) \in \mathbb{R}_{\geq 0} \times \mathbb{T} \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{T}$$

just by writing s instead of t in the Hamiltonian and adding the equation

$$\frac{ds}{dt} = 1$$

to systems (18) and (21). We write now the extended version of the invariant sets we have defined so far. For any $\alpha \in \mathbb{T}, G \in \mathbb{R}$, the set

$$\tilde{\Lambda}_{\alpha,G} = \{\tilde{z} = (0, \alpha, 0, G, s), s \in \mathbb{T}\} \tag{23}$$

is a 2π-periodic orbit with motion determined by $ds/dt = 1$. The union of such periodic orbits is the 3-dimensional invariant manifold (the *infinity manifold*)

$$\tilde{\Lambda}_\infty = \bigcup_{\alpha,G} \tilde{\Lambda}_{\alpha,G} = \{(0, \alpha, 0, G, s),\ (\alpha, G, s) \in \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T}\} \simeq \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T}, \tag{24}$$

which is *topologically equivalent to a normally hyperbolic invariant manifold* (TNHIM).

Parameterizing the points in $\tilde{\Lambda}_\infty$ by

$$\tilde{\mathbf{x}}_0 = \tilde{\mathbf{x}}_0(\alpha, G, s) = (\mathbf{x}_0(\alpha, G), s) = (0, \alpha, 0, G, s) \in \tilde{\Lambda}_\infty \simeq \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T}$$

the inner dynamics on $\tilde{\Lambda}_\infty$ is trivial, since it is given by the dynamics on each periodic orbit $\tilde{\Lambda}_{\alpha,G}$:

$$\tilde{\phi}_{t,0}(\tilde{\mathbf{x}}_0) = (0, \alpha, 0, G, s + t) = (\mathbf{x}_0(\alpha, G), s + t) = \tilde{\mathbf{x}}_0(\alpha, G, s + t). \tag{25}$$

3.2 The scattering map

In the region of the phase space with positive angular momentum G, let us now look at the homoclinic orbits to the previously introduced invariant objects.

The equilibrium points $\Lambda_{\alpha,G}$ have stable and unstable 1-dimensional invariant manifolds

$$\begin{aligned} \gamma_{\alpha,G} &= W^{\mathrm{u}}(\Lambda_{\alpha,G}) = W^{\mathrm{s}}(\Lambda_{\alpha,G}) \\ &= \left\{ z = (x, \hat{\alpha}, y, G),\ \mathcal{H}_0(x, y, G) = 0, \hat{\alpha} = \alpha - G \int_{\mathcal{H}_0=0} \frac{x}{y} dx \right\}, \end{aligned}$$

whereas the 2-dimensional manifold of equilibrium points Λ_∞ has stable and unstable 3-dimensional invariant manifolds which coincide and are given by

$$\gamma = W^{\mathrm{u}}(\Lambda_\infty) = W^{\mathrm{s}}(\Lambda_\infty) = \{z = (x, \alpha, y, G),\ \mathcal{H}_0(x, y, G) = 0\}.$$

The surface

$$\tilde{\gamma}_{\alpha,G} = W^{\mathrm{u}}(\tilde{\Lambda}_{\alpha,G}) = W^{\mathrm{s}}(\tilde{\Lambda}_{\alpha,G}) \tag{26}$$

$$= \left\{ \tilde{z} = (x, \hat{\alpha}, y, G, s), s \in \mathbb{T},\ \mathcal{H}_0(x, y, G) = 0, \hat{\alpha} = \alpha - G \int_{\mathcal{H}_0=0} \frac{x}{y} dx \right\} \tag{27}$$

is a 2-dimensional homoclinic manifold to the periodic orbit $\tilde{\Lambda}_{\alpha,G}$ in the extended phase space. The 4-dimensional stable and unstable manifolds of the infinity manifold $\tilde{\Lambda}_\infty$ coincide along the 4-dimensional homoclinic invariant manifold (the *separatrix*), which is just the union of the homoclinic surfaces $\tilde{\gamma}_{\alpha,G}$:

$$\begin{aligned} \tilde{\gamma} &= W^{\mathrm{u}}(\tilde{\Lambda}_\infty) = W^{\mathrm{s}}(\tilde{\Lambda}_\infty) = \bigcup_{\alpha,G} \tilde{\gamma}_{\alpha,G} \\ &= \{\tilde{z} = (x, \alpha, y, G, s), (\alpha, G, s) \in \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T},\ \mathcal{H}_0(x, \alpha, y, G) = 0\} \end{aligned} \tag{28}$$

Due to the presence of the factor $-x^3/4$ in front of equations (21), it is more convenient to parameterize the separatrix $\tilde{\gamma}_{\alpha,G}$ given in (26) by the solutions of the Hamiltonian flow contained in $\mathcal{H}_0 = 0$ in some time τ satisfying (see [MP94])

$$\frac{dt}{d\tau} = \frac{2G}{x^2}. \tag{29}$$

In this way, the homoclinic solution to the periodic orbit $\tilde{\Lambda}_{\alpha,G}$ of system (21) can be written as

$$x_0(t;G) = \frac{2}{G(1+\tau^2)^{1/2}} \tag{30a}$$
$$\alpha_0(t;\alpha,G) = \alpha + \pi + 2\arctan\tau \tag{30b}$$
$$y_0(t;G) = \frac{2\tau}{G(1+\tau^2)} \tag{30c}$$
$$G_0(t;G) = G \tag{30d}$$
$$s_0(t;s) = s + t \tag{30e}$$

where α and G are free parameters and the relation between t and τ is

$$t = \frac{G^3}{2}\Big(\tau + \frac{\tau^3}{3}\Big), \tag{31}$$

which is equivalent to (29) on $\mathcal{H}_0$. From the expressions above, we see that the convergence along the separatrix to the infinity manifold is power-like in τ and t:

$$x_0, y_0, \frac{\alpha - \alpha_0 - \pi}{G} \sim \frac{2}{G\tau} \sim \frac{2}{\sqrt[3]{6t}}, \quad \tau, t \to \pm\infty. \tag{32}$$

We now introduce the notation

$$\begin{aligned}\tilde{\mathbf{z}}_0 &= \tilde{\mathbf{z}}_0(\sigma,\alpha,G,s) = (\mathbf{z}_0(\sigma,\alpha,G),s) \\ &= (x_0(\sigma;G), \alpha_0(\sigma;\alpha,G), y_0(\sigma;G), G, s) \in \tilde{\gamma}\end{aligned} \tag{33}$$

so that we can parameterize any surface $\tilde{\gamma}_{\alpha,G}$ as

$$\tilde{\gamma}_{\alpha,G} = \{\tilde{\mathbf{z}}_0 = \tilde{\mathbf{z}}_0(\sigma,\alpha,G,s) = (\mathbf{z}_0(\sigma,\alpha,G),s),\ \sigma\in\mathbb{R}, s\in\mathbb{T}\}.$$

and we can parameterize the 4-dimensional separatrix as

$$\tilde{\gamma} = W(\tilde{\Lambda}_\infty) = \{\tilde{\mathbf{z}}_0 = \tilde{\mathbf{z}}_0(\sigma,\alpha,G,s) = (\mathbf{z}_0(\sigma,\alpha,G),s), \sigma\in\mathbb{R}, G\in\mathbb{R}_+, (\alpha,s)\in\mathbb{T}^2\}. \tag{34}$$

The motion on $\tilde{\gamma}$ is given by

$$\tilde{\phi}_{t,0}(\tilde{\mathbf{z}}_0) = \tilde{\mathbf{z}}_0(\sigma+t,\alpha,G,s) = (\mathbf{z}_0(\sigma+t,\alpha,G),s+t) \tag{35}$$

and by equations (30), (31) the following asymptotic formula follows:

$$\phi_{t,0}(\tilde{\mathbf{z}}_0) - \phi_{t,0}(\tilde{\mathbf{x}}_0) = (\mathbf{z}_0(\sigma+t,\alpha,G),s+t) - (\mathbf{x}_0(\alpha,G),s+t) \xrightarrow[t\to\pm\infty]{} 0 \tag{36}$$

The *scattering map* S describes the homoclinic orbits to the infinity manifold $\tilde{\Lambda}_\infty$ (defined in (24)) to itself. Given $\tilde{\mathbf{x}}_-, \tilde{\mathbf{x}}_+ \in \tilde{\Lambda}_\infty$, we define

$$S_\mu(\tilde{\mathbf{x}}_-) := \tilde{\mathbf{x}}_+$$

if there exists $\tilde{\mathbf{z}}^* \in W^{\mathrm{u}}_\mu(\tilde{\Lambda}_\infty) \cap W^s_\mu(\tilde{\Lambda}_\infty)$ such that

$$\phi_{t,\mu}(\tilde{\mathbf{z}}^*) - \phi_{t,\mu}(\tilde{\mathbf{x}}_\pm) \to 0 \quad \text{for } t \to \pm\infty. \tag{37}$$

In the case $\mu = 0$ the asymptotic relation (36) implies $S_0(\tilde{\mathbf{x}}_0) = \tilde{\mathbf{x}}_0$ so that that the scattering map $S_0 : \tilde{\Lambda}_\infty \longrightarrow \tilde{\Lambda}_\infty$ is the identity.

4 Invariant manifolds for the ERTBP ($\mu > 0$)

4.1 The infinity manifold

In order to analyze the structure of system (18), we will write $\mathcal{H}_\mu$ given in (17) as

$$\mathcal{H}_\mu(x,\alpha,y,G,s;e) = \mathcal{H}_0(x,y,G) - \mu\Delta\mathcal{U}_\mu(x,\alpha,s;e) \tag{38}$$

where we have written $\mathcal{U}_\mu$ in (15) as

$$\mathcal{U}_\mu(x,\alpha,s;e) = \mathcal{U}_0(x) + \mu\Delta\mathcal{U}_\mu(x,\alpha,s;e) = \frac{x^2}{2} + \mu\Delta\mathcal{U}_\mu(x,\alpha,s;e), \tag{39}$$

and we proceed to study the dynamics as a perturbation of the limit case $\mu = 0$. From (15),

$$\begin{aligned}\Delta\mathcal{U}_0(x,\alpha,s;e) &= \lim_{\mu\to 0}\Delta\mathcal{U}_\mu(x,\alpha,s;e)\\ &= \frac{x^2}{\left[4 + x^4 r^2 + 4x^2 r\cos(\alpha - f)\right]^{1/2}} + \left(\frac{x^2}{2}\right)^2 r\cos(\alpha - f) - \frac{x^2}{2}\end{aligned} \tag{40}$$

where $r = r(f,e)$ and $f = f(s,e)$ are given, respectively, in (5-6).

For $\mu > 0$, it is clear from equations (18) that the set $\mathcal{E}_\infty$ remains invariant and, therefore, so does the infinity manifold $\tilde{\Lambda}_\infty$, being again a TNHIM, and all the periodic orbits $\tilde{\Lambda}_{\alpha,G}$ also persist. The inner dynamics on $\tilde{\Lambda}_\infty$ is the same that in the case $\mu = 0$, so that the parametrization $\tilde{\mathbf{x}}_0$ as well as its trivial dynamics (25) remain the same.

4.2 The scattering map

From [McG73] we know that $W^{\mathrm{s}}_\mu(\tilde{\Lambda}_\infty)$ and $W^{\mathrm{u}}_\mu(\tilde{\Lambda}_\infty)$ exist for μ small enough and are 4-dimensional in the extended phase space. The existence of a scattering map will depend on the transversal intersection between these two manifolds.

Let us take an arbitrary $\tilde{\mathbf{z}}_0 = (\mathbf{z}_0, s) = (\mathbf{z}_0(\sigma,\alpha,G), s) \in \tilde{\gamma}$ as in (33). Now, we have to construct points in $W^{\mathrm{s}}_\mu(\tilde{\Lambda}_\infty)$ and $W^{\mathrm{u}}_\mu(\tilde{\Lambda}_\infty)$ to measure the distance between them. It is clear from the definition of $\tilde{\gamma}$ that

$$\tilde{\mathbf{v}} = (\nabla\mathcal{H}_0(\mathbf{z}_0), 0)$$

is orthogonal to $\tilde{\gamma} = W^{\mathrm{u}}(\tilde{\Lambda}_\infty) = W^{\mathrm{s}}(\tilde{\Lambda}_\infty)$ at $\tilde{\mathbf{z}}_0$ and then if the normal bundle to $\tilde{\gamma}$ is denoted by

$$N(\tilde{\mathbf{z}}_0) = \{\tilde{\mathbf{z}}_0 + \sigma\,\tilde{\mathbf{v}}, \sigma \in \mathbb{R}\}$$

we have that there exist unique points $\tilde{\mathbf{z}}^{\mathrm{s,u}}_\mu = (z^{\mathrm{s,u}}_\mu, s)$ such that

$$\{\tilde{\mathbf{z}}^{\mathrm{s,u}}_\mu\} = W^{\mathrm{s,u}}_\mu(\tilde{\Lambda}_\infty) \cap N(\tilde{\mathbf{z}}_0). \tag{41}$$

The distance we want to compute between $W^{\mathrm{s}}_\mu(\tilde{\Lambda}_\infty)$ and $W^{\mathrm{u}}_\mu(\tilde{\Lambda}_\infty)$ is the signed magnitude given by

$$d(\tilde{\mathbf{z}}_0,\mu) = \mathcal{H}_0(\tilde{\mathbf{z}}^{\mathrm{u}}_\mu) - \mathcal{H}_0(\tilde{\mathbf{z}}^{s}_\mu). \tag{42}$$

We now introduce the *Melnikov potential* (see [DG00, DLS06])

$$\mathcal{L}(\alpha,G,s;e) = \int_{-\infty}^{\infty}\Delta\mathcal{U}_0(x_0(t;G),\alpha_0(t;\alpha,G),s+t;e)\,dt \tag{43}$$

where $\Delta\mathcal{U}_0$ is defined in (40). Thanks to the asymptotic behavior (32) of the solutions along the separatrix and of the self potential close to the infinity manifold

$$\Delta\mathcal{U}_0(x,\alpha,s;e) = O(x^4) \quad \text{as} \quad x \to \infty$$

this integral is absolutely convergent, and will be computed in detail in Section 6.

Proposition 3. Given $(\alpha, G, s) \in \mathbb{T} \times \mathbb{R}^+ \times \mathbb{T}$, assume that the function

$$\sigma \in \mathbb{R} \longmapsto \mathcal{L}(\alpha, G, s-\sigma; e) \in \mathbb{R} \tag{44}$$

has a non-degenerate critical point $\sigma^* = \sigma^*(\alpha, G, s; e)$. Then for $0 < \mu$ small enough, close to the point $\tilde{\mathbf{z}}_0^* = (\mathbf{z}_0(\sigma^*, \alpha, G), s) \in \tilde{\gamma}$ (see the parameterization in (33)), there exists a locally unique point

$$\tilde{\mathbf{z}}^* = \tilde{\mathbf{z}}^*(\sigma^*, \alpha, G, s; e, \mu) \in W^s_\mu(\tilde{\Lambda}_\infty) \pitchfork W^{\mathrm{u}}_\mu(\tilde{\Lambda}_\infty)$$

of the form

$$\tilde{\mathbf{z}}^* = \tilde{\mathbf{z}}_0^* + O(\mu).$$

Also, there exist unique points $\tilde{\mathbf{x}}_\pm = (0, \alpha_\pm, 0, G_\pm, s) = (0, \alpha, 0, G, s) + O(\mu) \in \tilde{\Lambda}_\infty$ such that

$$\phi_{t,\mu}(\tilde{\mathbf{z}}^*) - \phi_{t,\mu}(\tilde{\mathbf{x}}_\pm) \longrightarrow 0 \quad \text{for } t \to \pm\infty. \tag{45}$$

Moreover, we have

$$G_+ - G_- = \mu \frac{\partial \mathcal{L}}{\partial \alpha}(\alpha, G, s - \sigma^*(\alpha, G, s; e)) + O(\mu^2). \tag{46}$$

Proof. From equation (33) we know that any point $\tilde{\mathbf{z}}_0 \in \tilde{\gamma}$ have the form

$$\tilde{\mathbf{z}}_0 = \tilde{\mathbf{z}}_0(\sigma, \alpha, G, s).$$

As in (41), we consider

$$\tilde{\mathbf{z}}^{\mathrm{s,u}}_\mu = (\mathbf{z}^{\mathrm{s,u}}_\mu, s) \in W^{\mathrm{s,u}}_\mu(\tilde{\Lambda}_\infty) \cap N(\tilde{\mathbf{z}}_0),$$

and we are looking for $\tilde{\mathbf{z}}_0$ such that $\tilde{\mathbf{z}}^{\mathrm{s}}_\mu = \tilde{\mathbf{z}}^{\mathrm{u}}_\mu$. There must exist points $\tilde{\mathbf{x}}_\pm = (\mathbf{z}_\pm, s) \in \tilde{\Lambda}_\infty$ such that

$$\phi_{t,\mu}(\tilde{\mathbf{z}}^{\mathrm{s,u}}_\mu) - \phi_{t,\mu}(\tilde{\mathbf{x}}_\pm) \xrightarrow[t \to \pm\infty]{} 0, \tag{47}$$

moreover $\phi_{t,\mu}(\tilde{\mathbf{z}}^{\mathrm{s,u}}_\mu) - \phi_{t,0}(\tilde{\mathbf{z}}_0) = O(\mu)$ for $\pm t \geq 0$ (see [McG73]). Since $\mathcal{H}_0$ does not depend on time, by (38) and the chain rule we have that

$$\frac{d}{dt}\mathcal{H}_0(\phi_{t,\mu}(\tilde{\mathbf{z}}^{\mathrm{s,u}}_\mu)) = \{\mathcal{H}_0, \mathcal{H}_\mu\}(\phi_{t,\mu}(\tilde{\mathbf{z}}^{\mathrm{s,u}}_\mu)) = -\mu\{\mathcal{H}_0, \Delta\mathcal{U}_\mu\}(\phi_{t,\mu}(\tilde{\mathbf{z}}^{\mathrm{s,u}}_\mu); e).$$

Since $\mathcal{H}_0 = 0$ in $\tilde{\Lambda}_\infty$, using (47) and the trivial dynamics on $\tilde{\Lambda}_\infty$ we obtain

$$\mathcal{H}_0(\tilde{\mathbf{z}}^{\mathrm{s,u}}_\mu) = -\mu \int_0^{\pm\infty} \{\mathcal{H}_0, \Delta\mathcal{U}_\mu\}(\phi_{t,\mu}(\tilde{\mathbf{z}}^{\mathrm{s,u}}_\mu); e)\, dt.$$

Taylor expanding in μ and using the notation (35)

$$\begin{aligned}
\mathcal{H}_0(\tilde{\mathbf{z}}^{\mathrm{u}}_\mu) - \mathcal{H}_0(\tilde{\mathbf{z}}^{\mathrm{s}}_\mu) &= \mu \int_{-\infty}^{\infty} \{\mathcal{H}_0, \Delta\mathcal{U}_0\}(\phi_{t,0}(\tilde{\mathbf{z}}_0); e)\, dt + O(\mu^2) \\
&= \mu \int_{-\infty}^{\infty} \{\mathcal{H}_0, \Delta\mathcal{U}_0\}(\mathbf{z}_0(\sigma + t, \alpha, G), s + t; e)\, dt + O(\mu^2).
\end{aligned} \tag{48}$$

On the other hand, from (43)

$$\mathcal{L}(\alpha, G, s; e) = \int_{-\infty}^{\infty} \Delta\mathcal{U}_0(x_0(\nu - s; G), \alpha_0(\nu - s; \alpha, G), s; e)\, d\nu$$

and then

$$\frac{\partial \mathcal{L}}{\partial s}(\alpha, G, s; e) = -\int_{-\infty}^{\infty} \{\Delta\mathcal{U}_0, \mathcal{H}_0\}(\mathbf{z}_0(\nu - s, \alpha, G), s; e)\, d\nu$$

so that

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial s}(\alpha, G, s - \sigma; e) &= \int_{-\infty}^{\infty} \{\mathcal{H}_0, \Delta\mathcal{U}_0\}(\mathbf{z}_0(\nu - s + \sigma, \alpha, G), s; e)\, d\nu \\
&= \int_{-\infty}^{\infty} \{\mathcal{H}_0, \Delta\mathcal{U}_0\}(\mathbf{z}_0(t, \alpha, G), s + t; e)\, dt
\end{aligned} \tag{49}$$

and therefore, from (48) and (49)

$$d(\tilde{\mathbf{z}}_0,\mu) = \mathcal{H}_0(\tilde{\mathbf{z}}^{\mathrm{u}}_\mu) - \mathcal{H}_0(\tilde{\mathbf{z}}^{\mathrm{s}}_\mu) = \mu\frac{\partial\mathcal{L}}{\partial s}(\alpha, G, s-\sigma; e) + O(\mu^2).$$

For μ small enough, it is clear by the implicit function theorem that a non degenerate critical value σ^* of the function (44) gives rise to a homoclinic point $\tilde{\mathbf{z}}^*$ to $\tilde{\Lambda}_\infty$ where the manifolds $W^{\mathrm{s}}_\mu(\tilde{\Lambda}_\infty)$ and $W^{\mathrm{u}}_\mu(\tilde{\Lambda}_\infty)$ intersect transversally and has the desired form $\tilde{\mathbf{z}}^* = \tilde{\mathbf{z}}^*_0 + O(\mu)$.

Consider now the solution of system (18) represented by $\phi_{t,\mu}(\tilde{\mathbf{z}}^*)$. By the fundamental theorem of calculus and (38) we have

$$\begin{aligned} G_+ - G_- &= -\int_{-\infty}^{\infty}\frac{\partial\mathcal{H}_\mu}{\partial\alpha}(\phi_{t,\mu}(\tilde{\mathbf{z}}^*))\,dt = \int_{-\infty}^{\infty}\frac{\partial\Delta\mathcal{U}_\mu}{\partial\alpha}(\phi_{t,\mu}(\tilde{\mathbf{z}}^*); e)\,dt \\ &= \mu\int_{-\infty}^{\infty}\frac{\partial\Delta\mathcal{U}_0}{\partial\alpha}(\phi_{t,0}(\tilde{\mathbf{z}}^*_0); e)\,dt + O(\mu^2) \\ &= \mu\int_{-\infty}^{\infty}\frac{\partial\Delta\mathcal{U}_0}{\partial\alpha}(\mathbf{z}_0(\sigma^*+t,\alpha,G), s+t; e)\,dt + O(\mu^2) \\ &= \mu\frac{\partial\mathcal{L}}{\partial\alpha}(\alpha, G, s-\sigma^*; e) + O(\mu^2). \end{aligned}$$

□

Once we have found a critical point $\sigma^* = \sigma^*(\alpha, G, s; e)$ of (44) on a domain of (α, G, s), we can define the *reduced Poincaré function* (see [DLS06])

$$\mathcal{L}^*(\alpha, G; e) := \mathcal{L}(\alpha, G, s-\sigma^*; e) = \mathcal{L}(\alpha, G, s^*; e) \tag{50}$$

with $s^* = s - \sigma^*$. Note that the reduced Poincaré function does not depend on the s chosen, since by Proposition 3

$$\frac{\partial}{\partial s}\left(\mathcal{L}\left(\alpha, G, s-\sigma^*(\alpha, G, s; e); e\right)\right) \equiv 0.$$

Note also that if the function (44) in Proposition 3 has different non degenerate critical points there will exist different scattering maps.

The next proposition gives an approximation of the scattering map in the general case $\mu > 0$

Proposition 4. The associated scattering map $(\alpha_+, G_+, s_+) = S_\mu(\alpha, G, s)$ for any non degenerate critical point σ^* of the function defined in (44) is given by

$$(\alpha, G, s) \longmapsto \left(\alpha - \mu\frac{\partial\mathcal{L}^*}{\partial G}(\alpha, G; e) + O(\mu^2), G + \mu\frac{\partial\mathcal{L}^*}{\partial\alpha}(\alpha, G; e) + O(\mu^2), s\right)$$

where $\mathcal{L}^*$ is the Poincaré reduced function introduced in (50).

Proof. By hypothesis we have a non degenerate critical point σ^* of (44). By definition (50), Proposition 3 gives

$$G_+ - G = \mu\frac{\partial\mathcal{L}^*}{\partial\alpha}(\alpha, G) + O(\mu^2).$$

as well as $G_- = G + O(\mu)$ to get the correspondence between G_+ and G_- that were looking for.

The companion equation to (46)

$$\alpha_+ - \alpha = -\mu\frac{\partial\mathcal{L}^*}{\partial G}(\alpha, G) + O(\mu^2)$$

is a direct consequence of the fact that the scattering map S_μ is symplectic.

Indeed, this is a standard result for a scattering map associated to a NHIM, and is proven in [DLS08, Theorem 8]. For what concerns our scattering map defined on a TNHIM, the only difference is that the stable contraction (expansion) along $W^{\mathrm{s,u}}_\mu(\tilde{\Lambda}_\infty)$ is power-like (32) instead of exponential with respect to time. Therefore we only have to check that Proposition 10 in[DLS08] still holds, namely that Area $(\phi_{t,\mu}(\mathcal{R})) \to 0$ when $t \to 0$ for every 2-cell $\mathcal{R}$ in $W^{\rm s}_\mu(\tilde\Lambda_\infty)$ parameterized by $R : [0,1] \times [0,1] \to W^{\rm s}_\mu(\tilde\Lambda_\infty)$ in such a way that $R(t_1,t_2) \in W^{\rm s}_\mu(\tilde\Lambda_\infty)$, $R(0,t_2) \in \tilde\Lambda_\infty$. But this is a direct consequence of the fact that the stable coordinates contract at least by $C/\sqrt[3]{t}$ (see (32)) and the coordinates along Λ_∞ do not expand at all. □

Remark 5. In the (planar) circular case $e = 0$ (RTBP), $\Delta\mathcal{U}_\mu(x,\alpha,s;e)$ depends on the time s and the angle α just through their difference $\alpha - s$, see Remark 2. From

$$\frac{\partial \Delta\mathcal{U}_\mu}{\partial \alpha}(x,\alpha,s;0) = -\frac{\partial \Delta\mathcal{U}_\mu}{\partial s}(x,\alpha,s;0)$$

one readily obtains

$$\frac{\partial \mathcal{L}}{\partial s}(\alpha,G,s;0) = -\frac{\partial \mathcal{L}}{\partial \alpha}(\alpha,G,s;0)$$

and therefore

$$\frac{\partial \mathcal{L}}{\partial \alpha}(\alpha,G,s-\sigma^*;e) = -\frac{\partial \mathcal{L}}{\partial s}(\alpha,G,s-\sigma^*;0) = 0$$

and consequently the reduced Poincaré function $\mathcal{L}^*$ does not depend on α, and $G_+ = G_- + O(\mu^2)$.

But indeed $G_+ \equiv G_-$ in the circular case, since there exists the first integral provided by the Jacobi constant $C_J = \mathcal{H}_\mu + G$ and as $\mathcal{H}_\mu = 0$ on $\tilde\Lambda_\infty$, $G_+ = G_-$. Therefore in the circular case there is no possibility to find diffusive orbits studying the intersection of $W^{\rm s}_\mu(\tilde\Lambda_\infty)$ and $W^{\rm u}_\mu(\tilde\Lambda_\infty)$ since any scattering map preserves the angular momentum.

5 Global diffusion in the ERTBP

We have already the tools to derive the scattering maps to the infinity manifold $\tilde\Lambda_\infty$, namely Proposition 3 to find transversal homoclinic orbits to $\tilde\Lambda_\infty$ and Proposition 4 to give their expressions. Both of them rely on computations on the Melnikov potential $\mathcal{L}$. Inserting in the Melnikov potential introduced in (43) the expression for $\Delta\mathcal{U}_0$ in (40) we get

$$\mathcal{L}(\alpha,G,s;e) = \int_{-\infty}^{\infty} \left[\frac{x_0^2}{\left[4 + x_0^4 r^2 + 4x_0^2 r \cos(\alpha_0 - f)\right]^{1/2}} + \left(\frac{x_0^2}{2}\right)^2 r\cos(\alpha_0 - f) - \frac{x_0^2}{2} \right] dt \quad (51)$$

where x_0 and α_0, coordinates of the homoclinic orbit defined in (30), are evaluated at t, whereas r and f, defined in (5) and (6), are evaluated at $s+t$.

To evaluate the above Melnikov potential, we will compute its Fourier coefficients with respect to the angular variables α, s. Since x_0 and r are even functions of t and f and α_0 are odd, $\mathcal{L}$ is an even function of the angular variables α, s: $\mathcal{L}(-\alpha,G,-s;e) = \mathcal{L}(\alpha,G,s;e)$, and therefore $\mathcal{L}$ has a Fourier Cosine series with real coefficients $L_{q,k}$:

$$\mathcal{L} = \sum_{q\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} L_{q,k} e^{i(qs+k\alpha)} = L_{0,0} + 2\sum_{k\geq 1} L_{0,k}\cos k\alpha + 2\sum_{q\geq 1}\sum_{k\in\mathbb{Z}} L_{q,k}\cos(qs+k\alpha). \quad (52)$$

The concrete computation of the Fourier coefficients of the Melnikov potential (51) will be carried out in section 6. First, some accurate bounds will be obtained:

Lemma 6. Let $G \geq 32$, $q \geq 1$, $k \geq 2$ and $\ell \geq 0$. Then $|L_{q,\ell}| \leq B_{q,\ell}$ and $|L_{0,\ell}| \leq B_{0,\ell}$, where

$$\begin{aligned} B_{q,0} &= 2^9 2^q \mathrm{e}^{2q} e^q G^{-3/2} \mathrm{e}^{-qG^3/3} \\ B_{q,1} &= 2^7 \mathrm{e}^q (1+e)^4 G^{-7/2} \mathrm{e}^{-qG^3/3} \\ B_{q,-1} &= 2^9 2^q \mathrm{e}^{2q} e^{|1-q|} G^{-1/2} \mathrm{e}^{-qG^3/3} \\ B_{q,k} &= 2^5 2^k \mathrm{e}^q (1+e)^k G^{-2k-1/2} \mathrm{e}^{-qG^3/3} \\ B_{q,-k} &= 2^5 2^{q+2k} \mathrm{e}^{2q} e^{|k-q|} G^{k-1/2} \mathrm{e}^{-qG^3/3} \\ B_{0,\ell} &= 2^8 2^{2\ell} e^\ell G^{-2\ell-3} \end{aligned} \tag{53}$$

Directly from this lemma, we first see that the harmonics $L_{q,\ell}$ are exponentially small in G for $q \geq 1$, so it will be convenient to split the Fourier expansion (52) as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \cdots = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_{\geq 2} \tag{54}$$

where

$$\begin{aligned} \mathcal{L}_0(\alpha, G; e) &= L_{0,0} + 2\sum_{k \geq 1} L_{0,k} \cos k\alpha, \\ \mathcal{L}_q(\alpha, G, s; e) &= 2\sum_{k \in \mathbb{Z}} L_{q,k} \cos(qs + k\alpha), \qquad q \geq 1. \end{aligned} \tag{55}$$

The function $\mathcal{L}_0$ does not depend on the angle s and it contains the harmonics of $\mathcal{L}$ of order 0 in s, which are of finite order in terms of G, $\mathcal{L}_1$ the harmonics of first order, which are of order $\mathrm{e}^{-qG^3/3}$, and all the harmonics of $\mathcal{L}_q$ for $q \geq 2$ are much exponentially smaller in G than those of $\mathcal{L}_1$, so we will estimate $\mathcal{L}_0$ and $\mathcal{L}_1$ and bound $\mathcal{L}_{\geq 2}$.

To this end, it will be necessary to sum the series in (55). From the bounds $B_{q,k}$ in (53) for the harmonics $L_{q,k}$ we get the quotients

$$\frac{B_{q,k+1}}{B_{q,k}} = \frac{2(1+e)}{G^2} \text{ for } k \geq 2, \quad \frac{B_{q,-(k+1)}}{B_{q,-k}} = 4eG \text{ for } k \geq q, \quad \frac{B_{0,\ell+1}}{B_{q,\ell}} = \frac{4e}{G^2} \text{ for } \ell \geq 0, \tag{56}$$

which indicate that, for fixed q, we will need at least the conditions $G > \sqrt{2(1+e)}$ and $eG < 1/4$ to ensure the convergence of the Fourier series. This is the main reason why we are going to restrict ourselves to the region $G \geq C$ large enough and $eG \leq c$ small enough along this paper to get the diffusive orbits.

Among the harmonics $L_{0,k}$ of 0 order in s, by (56), the harmonic $L_{0,0}$ appears to be the dominant one, but we will also estimate $L_{0,1}$ to get information about the variable α, and bound the rest of harmonics $L_{0,k}$ for $k \geq 2$. Among the harmonics of first order $L_{1,k}$, again by (56), the five harmonics $L_{1,k}$ for $|k| \leq 2$ are the only candidates to be the dominant ones, but the quotients from (53)

$$\frac{B_{1,2}}{B_{1,-1}} = \frac{(1+e)^2}{8\mathrm{e}G^4}, \qquad \frac{B_{1,1}}{B_{1,-1}} = \frac{(1+e)^4}{8\mathrm{e}G^3}, \qquad \frac{B_{1,0}}{B_{1,-1}} = \frac{e}{G} = \frac{eG}{G^2}, \tag{57}$$

indicate that $L_{1,-1}$ and $L_{1,-2}$ appear to be the two dominant harmonics of order 1. Summarizing, to compute the series (52) we estimate only the four harmonics $L_{0,0}$, $L_{0,1}$, $L_{1,-1}$ and $L_{1,-2}$, and bound all the rest, providing the following result, whose proof will also be carried out in section 6.

Theorem 7. *For $G \geq 32$, $eG \leq 1/8$, the Melnikov potential* (51) *is given by*

$$\mathcal{L}(\alpha, G, s; e) = \mathcal{L}_0(\alpha, G; e) + \mathcal{L}_1(\alpha, G, s; e) + \mathcal{L}_{\geq 2}(\alpha, G, s; e) \tag{58}$$

with

$$\mathcal{L}_0(\alpha, G; e) = L_{0,0} + L_{0,1} \cos\alpha + \mathcal{E}_0(\alpha, G; e) \tag{59}$$

$$\mathcal{L}_1(\alpha, G, s; e) = 2L_{1,-1} \cos(s - \alpha) + 2L_{1,-2} \cos(s - 2\alpha) + \mathcal{E}_1(\alpha, G, s; e) \tag{60}$$

where the four harmonics above are given by

$$\begin{aligned}
L_{0,0} &= \frac{\pi}{2G^3}(1+E_{0,0}) \\
L_{0,1} &= -\frac{15\pi e}{8G^5}(1+E_{0,1}) \\
L_{1,-1} &= \sqrt{\frac{\pi}{8G}}\mathrm{e}^{-G^3/3}(1+E_{1,-1}) && (61) \\
L_{1,-2} &= -3\sqrt{2\pi}eG^{3/2}\mathrm{e}^{-G^3/3}(1+E_{1,-2}) && (62)
\end{aligned}$$

and the error functions satisfy

$$\begin{aligned}
|E_{0,0}| &\le 2^{12}G^{-4} + 2^2\,49\,e^2 \\
|E_{0,1}| &\le 2^{13}G^{-4} + e^2 \\
|E_{1-1}| &\le 2^{21}G^{-1} + 2\,49\,e^2 \\
|E_{1,-2}| &\le 2^{17}G^{-1} + \frac{49}{3}e \\
|\mathcal{E}_0| &\le 2^{14}\,e^2G^{-7} \\
|\mathcal{E}_1| &\le 2^{19}\mathrm{e}^{-G^3/3}\left[G^{-7/2} + e^2G^{5/2} + eG^{-3/2}\right] && (63) \\
|\mathcal{L}_{\ge 2}| &\le 2^{28}G^{3/2}\mathrm{e}^{-2G^3/3} && (64)
\end{aligned}$$

The function $\mathcal{L}_1$ contains only harmonics of first order in s, so we can write it as a cosine function in s. Introducing

$$p := -\frac{L_{1,-2}}{L_{1,-1}} = 12eG^2\frac{1+E_{1,-1}}{1+E_{1,-2}} =: 12eG^2(1+E_p) \tag{65}$$

in the definition (60) of $\mathcal{L}_1$ we can write

$$\begin{aligned}
\mathcal{L}_1 &= 2L_{1,-1}\left(\sum_{k\in\mathbb{Z}}\frac{L_{1,k}}{L_{1,-1}}\cos(qs+k\alpha)\right) \\
&= 2L_{1,-1}\left(\cos(s-\alpha) - p\cos(s-2\alpha) + \sum_{k\ne -1,-2}\frac{L_{1,k}}{L_{1,-1}}\cos(qs+k\alpha)\right) \\
&= 2L_{1,-1}\Re\left(\mathrm{e}^{i(s-\alpha)}\left(1 - p\mathrm{e}^{-i\alpha} + \sum_{k\ne -1,-2}\frac{L_{1,k}}{L_{1,-1}}\mathrm{e}^{i(k+1)\alpha}\right)\right) \\
&= 2L_{1,-1}\Re\left(\mathrm{e}^{i(s-\alpha)}B\mathrm{e}^{-i\theta}\right) = 2L_{1,-1}B\cos(s-\alpha-\theta) && (66)
\end{aligned}$$

where $B = B(\alpha, G; e) \ge 0$ and $-\theta = -\theta(\alpha, G; e) \in [-\pi, \pi)$ are the modulus and the argument of the complex expression

$$1 - p\mathrm{e}^{-i\alpha} + \sum_{k\ne -1,-2}\frac{L_{1,k}}{L_{1,-1}}\mathrm{e}^{i(k+1)\alpha} =: B\mathrm{e}^{-i\theta}. \tag{67}$$

Writing also in polar form the quotient of the sum in (67) by the parameter p introduced in (65)

$$E\mathrm{e}^{-i\phi} := \sum_{k\ne -1,-2}\frac{L_{1,k}}{pL_{1,-1}}\mathrm{e}^{i(k+1)\alpha} = -\sum_{k\ne -1,-2}\frac{L_{1,k}}{L_{1,-2}}\mathrm{e}^{i(k+1)\alpha},$$

with $E = E(\alpha, G; e) \ge 0$ and $-\phi = -\phi(\alpha, G; e) \in [-\pi, \pi)$, equation (67) for B and θ reads now as

$$B\mathrm{e}^{-i\theta} = 1 - p\mathrm{e}^{-i\alpha} + pE\mathrm{e}^{-i\phi} \tag{68}$$

or, equivalently, as the couple of real equations

$$B\cos\theta = 1 - p\cos\alpha + pE\cos\phi \tag{69}$$

$$-B\sin\theta = p\sin\alpha - pE\sin\phi. \tag{70}$$

The function $E = E(\alpha, G; e)$ is small, since, by (63), (62) and (65),

$$\begin{aligned}|E| \le \frac{|\mathcal{E}_1|}{|L_{1,-2}|} = \frac{|\mathcal{E}_1|}{|pL_{1,-1}|} &\le \frac{(1+e)^4 G^{-7/2}}{\sqrt{8G/\pi}} + \frac{e^2G^{5/2} + eG^{-3/2}}{3\sqrt{2\pi}eG^{3/2}} \\ &= \frac{1}{3\sqrt{2\pi}}\left(\left(1 + \frac{3\pi(1+e)^4}{2G}\right)\frac{1}{G^3} + eG\right) = O\left(G^{-3}, eG\right),\end{aligned} \tag{71}$$

with an analogous bound for its derivative with respect to α. Writing equation (68) as

$$B\mathrm{e}^{-i\theta} = \widehat{B}\mathrm{e}^{-i\widehat{\theta}} + pE\mathrm{e}^{-i\phi}$$

one gets the explicit formulae for $\widehat{B}$ and $\widehat{\theta}$

$$\begin{aligned}\widehat{B} &= \sqrt{1 - 2p\cos\alpha + p^2} = \sqrt{(1-p)^2 + 4p\sin^2\alpha} \ge 0, \\ \widehat{\theta} &= -2\arctan\left(\frac{p\sin\alpha}{\widehat{B} + 1 - p\cos\alpha}\right) \in (-\pi, \pi]\end{aligned} \tag{72}$$

from which we see that $\widehat{B}$ behaves like a distance to the point $p = 1$ and $\alpha = 0$. The angle $\widehat{\theta}$ is not well defined when $\widehat{B} = 0$, but this happens only for $\alpha = 0$ and $p = 1$, that is, for $G \simeq (12e)^{-1/2}$. A totally analogous property holds for B:

Lemma 8. $B(\alpha, G; e) > 0$ except for $\alpha = 0$ and $p = 1 + \sum_{k\neq -1,-2} L_{1,k}/L_{1,-1}$.

Proof. For $B = 0$, equation (70) reads as

$$\sin\alpha = f(\alpha). \tag{73}$$

where $f(\alpha) = f(\alpha, G; e) := E\sin\phi = E\sin\phi(\alpha, G; e)$. Since $f^2 + (\partial f/\partial\alpha)^2 < 1$ due to (71), there are exactly two simple solutions of equation (73) in the interval $[-\pi/2, 3\pi/2]$; one is $\alpha^*_{0,+} \in (-\pi/2, \pi/2)$ obtained as a fixed point of the contraction $\alpha = \arcsin(f(\alpha, G; e))$, and a second $\alpha^*_{0,-} \in (\pi/2, 3\pi/2)$ fixed point of the contraction $\alpha = \pi - \arcsin(f(\alpha, G; e))$. Taking a closer look at equation (68), we see that if α changes to $-\alpha$, then $-\phi, -\theta, B$ are solutions of (68) or, in other words, ϕ, θ are odd functions of α and B even. Therefore $\alpha = 0, \pi$ are the unique solutions of equation (70) for $B = 0$. Substituting $\alpha = 0, \pi$ in (69) for $B = 0$, only $\alpha = 0$ provides a positive p, which is then given by $p = 1 + pE = 1 + \sum_{k\neq -1,-2} L_{1,k}/L_{1,-1}$. □

We are now in position to find critical points of the function $s \mapsto \mathcal{L}(\alpha, G, s; e)$. To this end we will check that $s \mapsto \mathcal{L}(\alpha, G, s; e)$ is indeed a *cosine-like* function, that is, with a non-degenerate maximum (minimum) and no other critical points. By Theorem 7, The dominant part of the Melnikov potential $\mathcal{L}$ is given by $\mathcal{L}_0 + \mathcal{L}_1$. By equation (58) and the bounds for the error term, for G large enough, the critical points in the variable s are well approximated by the critical points of the function $\mathcal{L}_0 + \mathcal{L}_1$ and therefore will be close to $s - \alpha - \theta = 0, \pi \pmod{2\pi}$ thanks to expression (66). For this purpose, we introduce

$$\mathcal{L}_1^* = \mathcal{L}_1^*(\alpha, G; e) = 2L_{1,-1}B \tag{74}$$

where $B = B(\alpha, G; e)$ is given in (67) and $L_{1,-1}$ is the harmonic computed in (61). With this notation the function $\mathcal{L}_1$ can be written as a cosine function in s

$$\mathcal{L}_1(\alpha, G, s; e) = \mathcal{L}_1^*(\alpha, G; e)\cos(s - \alpha - \theta), \tag{75}$$

and differentiating the Melnikov potential (58) with respecto to s we get

$$\frac{\partial \mathcal{L}}{\partial s} = -\mathcal{L}_1^* \sin(s-\alpha-\theta) + \frac{\partial \mathcal{L}_{\geq 2}}{\partial s} = 0 \Longleftrightarrow \sin(s-\alpha-\theta) = \frac{1}{\mathcal{L}_1^*}\frac{\partial \mathcal{L}_{\geq 2}}{\partial s} \tag{76}$$

which is a equation of the form (73) for $s-\alpha-\theta$ instead of α and $f = \left(\partial \mathcal{L}_{\geq 2}/\partial s\right)/\mathcal{L}_1^*$. Therefore, as long as $B > \left(\partial \mathcal{L}_{\geq 2}/\partial s\right)/(2L_{1,-1})$, which by the estimate (61) for $L_{1,-1}$ and the bound (64) for $\mathcal{L}_{\geq 2}$ happens outside of a neighborhood of size $O\left(G^{3/2}\mathrm{e}^{-G^3/3}\right)$ of the point

$$(\alpha = 0, G = G^*) \text{ where } G^* \approx (12e)^{-1/2} \text{ is such that } p = 1 + \sum_{k\neq -1,-2} L_{1,k}/L_{1,-1}, \tag{77}$$

there exist exactly two no-degenerate critical points $s_{0,\pm}^*$ of the function $s \mapsto \mathcal{L}(\alpha, G, s; e)$.

Let us recall now that the Melnikov function $\mathcal{L}$, as well as its terms $\mathcal{L}_q$ are all expressed as Fourier Cosine series in the angles α and s, or equivalently, they are even functions of (α, s). Consequently, $\partial \mathcal{L}_q/\partial s$ is an odd function of (α, s), and it is easy to check that each critical point s^* is an odd function of α. Moreover, using the Fourier Sine expansion of $\partial \mathcal{L}_q/\partial s$, one sees that it s is a critical point of $s \mapsto \mathcal{L}(\alpha, G, s; e)$, $s+\pi$ too, so $s_{0,-} = s_{0,+} + \pi$. We state all this in the following proposition.

Proposition 9. Let $\mathcal{L}$ be the Melnikov potential given in (58), $G \geq 32$ and $eG \leq 1/8$. Then, except for a neighborhood of size $O\left(G^{3/2}\mathrm{e}^{-G^3/3}\right)$ of the point $(\alpha = 0, G = G^*)$ given in (77), $s \mapsto \mathcal{L}(\alpha, G, s; e)$ is a *cosine-like* function, and its critical points are given by

$$s_{0,+}^* = s_{0,+}^*(\alpha, G; e) = \alpha + \theta + \varphi_+^*, \qquad s_{0,-}^* = s_{0,+}^* + \pi = \alpha + \theta + \pi + \varphi_\pm^*$$

where $\theta = \theta(\alpha, G; e)$ is given in (67) and $\varphi_+^* = O\left(G^{3/2}\mathrm{e}^{-G^3/3}\right)$.

From the proposition above we know that there exist $s_{0,-}^*$ and $s_{0,+}^*$, non degenerate critical points of $s \mapsto \mathcal{L}(\alpha, G, s; e)$. Therefore, we can define two different reduced Poincaré functions (50)

$$\begin{aligned}\mathcal{L}_\pm^*(\alpha, G; e) &= \mathcal{L}(\alpha, G, s_{0,\pm}^*; e)\\ &= \mathcal{L}_0(\alpha, G; e) \pm \mathcal{L}_1^*(\alpha, G; e) + \mathcal{E}_\pm(\alpha, G; e).\end{aligned}$$

By the symmetry properties of $\mathcal{L}_q$, it turns out that each $\mathcal{L}_q^*(\alpha, G; e) = \mathcal{L}(\alpha, G, t_{0,}^*; e)$ is an even function of α. Moreover, since $s_{0,-}^* = s_{0,+}^* + \pi$, one has that $\mathcal{L}_q^* = (-1)^q \mathcal{L}_q^*$, so we can write the reduced Poincaré map as

$$\mathcal{L}_\pm^* = \mathcal{L}_0 \pm \mathcal{L}_1^* + \mathcal{L}_2^* \pm \mathcal{L}_3^* + \mathcal{L}_4^* \pm \cdots \tag{78}$$

From the expression for the scattering map given in Proposition 4 we can define two different scattering maps, namely

$$S_\pm(\alpha, G, s) = \left(\alpha + \mu \frac{\partial \mathcal{L}_\pm^*}{\partial G}(\alpha, G; e) + O(\mu^2), G - \mu \frac{\partial \mathcal{L}_\pm^*}{\partial \alpha}(\alpha, G; e) + O(\mu^2), s\right). \tag{79}$$

These two scattering maps are different since they depend on the two reduced Poincaré-Melnikov potentials $\mathcal{L}_\pm^*$. From their expression (79), the scattering maps $S_\pm$ follow closely the level curves of the Hamiltonians $\mathcal{L}_\pm^*$. More precisely, up to $O(\mu^2)$ terms, $S_\pm$ is given by the time $-\mu$ map of the Hamiltonian flow of Hamiltonian $\mathcal{L}_\pm^*$. The $O(\mu^2)$ remainder will be negligible as long as

$$|\mu| < \left|\frac{\partial \mathcal{L}_\pm^*}{\partial G}\right|, \left|\frac{\partial \mathcal{L}_\pm^*}{\partial \alpha}\right|.$$

Nevertheless, since we want to switch scattering maps, we will need to impose

$$|\mu| < |\mathcal{L}_1^*| = 2|L_{1,-1}B| = O\left(G^{-1/2}\mathrm{e}^{-G^3/3}\right),$$

that is, μ exponentially small with respect to G in the region $C \le G \le c/e$ which a fortiori is satisfied for

$$0 < \mu < \mu^* = \mathrm{e}^{-(c/e)^3/3}. \tag{80}$$

This is the relation between the eccentricity and the mass parameter that we need to guarantee that our main result holds. This kind of relation is typical in problems with exponential splitting, when the bound of the remainder, here $O(\mu^2)$, is obtained through a direct application of the Melnikov method for the real system. To get better estimates for this remainder, one needs to bound this remainder for complex values of the parameter t or τ of the parameterization (30) of the unperturbed separatrix. Such approach has recently been used for in the RTBP in [GMS12] and it is likely to work in the ERTBP, allowing us to consider any $\mu \in (0, 1/2]$, that is, imposing no restrictions on the mass parameter.

We want to show now that the foliations of $\mathcal{L}^*_\pm$ = constant are different, since this will imply that the scattering maps $S_\pm$ are different. Even more, we will design a mechanism in which we will determine the places in the plane (α, G) where we will change from one scattering map to the other, obtaining trajectories with increasing angular momentum G. To check that the level curves of $\mathcal{L}^*_+$ and $\mathcal{L}^*_-$ are different, and indeed transversal, we only need to check that their Poisson bracket is zero. Since $\mathcal{L}^*_+$ and $\mathcal{L}^*_-$ are even functions of α, their Poisson bracket $\{\mathcal{L}^*_+, \mathcal{L}^*_+\}$ will be an odd function of α, so we already know that it will have a factor $\sin\alpha$. Using equation (78) we can write

$$\{\mathcal{L}^*_+, \mathcal{L}^*_+\} = \{\mathcal{L}_0 + \mathcal{L}^*_1 + \mathcal{L}^*_2 + \cdots, \mathcal{L}_0 - \mathcal{L}^*_1 + \mathcal{L}^*_2 - \cdots\} = -2\{\mathcal{L}_0, \mathcal{L}^*_1\} + \mathcal{E}_3$$

where $\mathcal{E}_3$ contains only Poisson brackets of odd order

$$\mathcal{E}_3 = -2\left(\{\mathcal{L}_0, \mathcal{L}^*_3\} + \{\mathcal{L}^*_1, \mathcal{L}^*_2\}\right) - 2 \sum_{q\ge 0, q \text{ odd} \ge 5} \sum_{q=0}^{[q/2]} \{\mathcal{L}^*_{q'}, \mathcal{L}^*_{q-q'}\}.$$

Therefore, by the bounds (53) for the harmonics $L_{q,k}$, the error term $\mathcal{E}_3 = O\left(\mathrm{e}^{-G^3}\right)$ is much exponentially smaller for large G than $\{\mathcal{L}_0, \mathcal{L}^*_1\}$, which is $O\left(\mathrm{e}^{-G^3/3}\right)$ and we now compute.

By splitting $\mathcal{L}_0$, using (59), and $\mathcal{L}^*_1 = 2L_{1,-1}B$, using (61), (72) and (71), in their dominant and non-dominant parts

$$\mathcal{L}_0 = \widehat{\mathcal{L}}_0 + \mathcal{E}_0, \qquad L_{1,-1} = \widehat{L}_{1,-1}(1 + E_{0,1}), \qquad B = \widehat{B} + E_B,$$

after a straightforward computation, we arrive at

$$-2\{\mathcal{L}_0, \mathcal{L}^*_1\} = -2\{\widehat{\mathcal{L}}_0, \widehat{\mathcal{L}}^*_1\} + \mathcal{E}_J$$

where

$$-2\{\mathcal{L}_0, \mathcal{L}^*_1\} = \frac{-\mathcal{L}^*_1}{B^2} \frac{3\pi p \sin\alpha}{G^4} d$$

with

$$d = \left[1 - \frac{25}{4}\frac{eG}{G^3}\cos\alpha - \frac{5}{48}\frac{B^2}{G}\left[1 + \frac{1}{2G^3} - \frac{-\cos\alpha + p}{B^2} \cdot \frac{24eG}{G^2}\right]\right].$$

and a small error term

$$E_J = O\Big(G^{-5} + eG^{-3} + e^2G^3 + pe^2G^4\big(1 + p(eG + G^{-6})\big)\Big)G^{-1/2}\mathrm{e}^{-G^3/3}$$

5.1 Strategy for diffusion

The previous lemma tells us that the level curves of $\mathcal{L}^*_+$ and $\mathcal{L}^*_-$ are transversal in the region $G \ge 32$ and $eG \le 1/8$, except for the three curves $\alpha = 0$, $\alpha = \pi$ and $d = 0$ (which, by the way, are also transversal to any of these level curves, see figure 2).

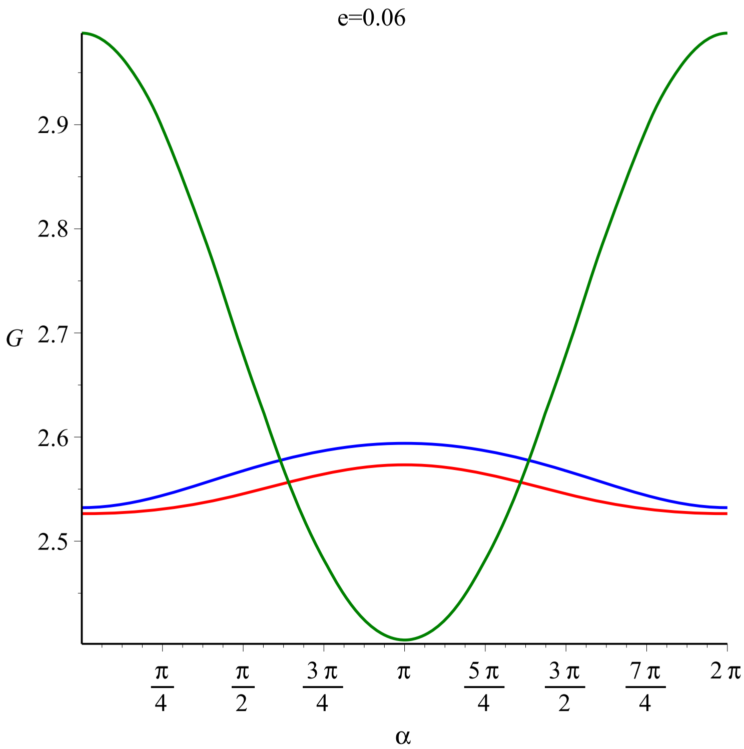

Figure 2: Level Sets of $\mathcal{L}_+^*$ ($\mathcal{L}_-^*$) in Blue (Red) and d=0 in Green

Thus, apart from these curves, at any point in the plane (α, G) the slopes $dG/d\alpha$ of the level curves of $\mathcal{L}_+^*$ and $\mathcal{L}_-^*$ are different, and we are able to choose which level curve increases more the value of G, when both slopes are positive, or alternatively, to choose the level curve which decreases less the value of G, when both slopes are negative (see Figure 3). In the same way, we can find trajectories along which the angular momentum performs arbitrary excursions. More precisely, given an arbitrary finite sequence of values G_i, $i = 1, \ldots, n$ we can find trajectories which satisfy $G(T_i) = G_i$, $i = 1, \ldots, n$.

Strictly speaking, This mechanism given by the application of scattering maps produce indeed pseudo-orbits, that is, heteroclinic connections between different periodic orbits in the infinity manifold which are commonly known as transition chains after Arnold's pioneering work [Arn64]. The existence of true orbits of the system which follow closely these transition chains relies on shadowing methods, which are standard for partially hyperbolic periodic orbits (the so-called whiskered tori in the literature) lying on a normally hyperbolic invariant manifold (NHIM). Such shadowing methods are equally applicable in our case, where we have an infinity manifold $\tilde{\Lambda}_\infty$ which is only topologically equivalent to a NHIM (see [Rob88, Rob84, Moe02, Moe07, GL06, GLS14]) and [GMS15].

With all these elements, we can finally state our main result

Theorem 10. *Let $G_1^* < G_2^*$ large enough and $e > 0$, $\mu > 0$ small enough. More precisely $32 \leq G_1^* < G_2^* \leq 1/(8e)$ and $0 < \mu < \mu^* = e^{-(8e)^{-3}/3}$. Then, for any finite sequence of values $G_i \in (G_1^*, G_2^*)$, $i = 1, \ldots, n$, there exists a trajectory of the ERTBP such that $G(T_i) = G_i$, $i = 1, \ldots, n$ for some $0 < T_i < T_{i+1}$. In particular, for any two values $G_1 < G_2 \in (G_1^*, G_2^*)$, there exists a trajectory such that $G(0) < G_1$, and $G(T) > G_2$ for some time $T > 0$.*

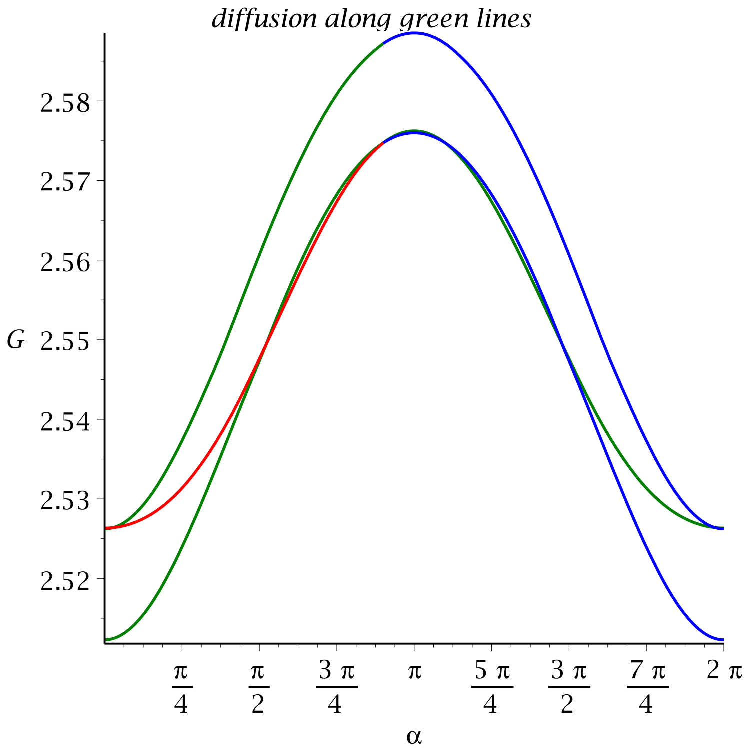

Figure 3: Zone of diffusion: Level curves of $\mathcal{L}_+^*$ ($\mathcal{L}_-^*$) in blue (red) and diffusion trajectories in green.

6 Computation of the Melnikov potential: Proof of Theorem 7

The main difficulty to compute the Melnikov potential is that it is given by an integral (51) where the coordinates of the separatrix x_0 and α_0 are given implicitly (30) in terms of the time t through the variable τ (31), whereas r and f are given in terms of $s+t$ through the differential equation (6) defining the true anomaly f. To evaluate the above Melnikov potential, we will compute its Fourier Cosine series (52) in the angles s, α.

The next proposition gives formulas for its Fourier coefficients. To this we will consider the Fourier expansion of the functions:

$$r(f(t))^n \ \mathrm{e}^{imf(t)} = \sum_{q\in\mathbb{Z}} c_q^{n,m} \mathrm{e}^{iqt} \tag{81}$$

which can be found in [MP94] and [Win41, p. 204]. Using that r is an even function and that f is and odd function, one readily sees that the above coefficients are real and indeed they satisfy

$$c_{-q}^{n,-m} = c_q^{n,m} = \bar{c}_q^{n,m}. \tag{82}$$

Once these coefficients $c_q^{n,m}$ are introduced we can give explicit formulas for the Fourier coefficients of the Melnikov potential $\mathcal{L}$.

Proposition 11. The Melnikov potential given in (51) or in (52) can be written as

$$\mathcal{L} = \sum_{q\in\mathbb{Z}} L_q \mathrm{e}^{iqs} \quad \text{with} \quad L_q = \sum_{k\in\mathbb{Z}} L_{q,k} \mathrm{e}^{ik\alpha}. \tag{83}$$

with

$$L_{q,0} = \sum_{l\geq 1} c_q^{2l,0} N(q,l,l) \tag{84a}$$

$$L_{q,1} = \sum_{l\geq 2} c_q^{2l-1,-1} N(q,l-1,l) \tag{84b}$$

$$L_{q,-1} = \sum_{l\geq 2} c_q^{2l-1,1} N(q,l,l-1) \tag{84c}$$

$$L_{q,k} = \sum_{l\geq k} c_q^{2l-k,-k} N(q,l-k,l) \quad \text{for } k\geq 2 \tag{84d}$$

$$L_{q,-k} = \sum_{l\geq k} c_q^{2l-k,k} N(q,l,l-k) \quad \text{for } k\geq 2 \tag{84e}$$

and

$$N(q,m,n) = \frac{2^{m+n}}{G^{2m+2n-1}} \binom{-1/2}{m} \binom{-1/2}{n} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{iq(\tau+\tau^3/3)\,G^3/2}}{(\tau-i)^{2m}(\tau+i)^{2n}} d\tau \tag{85}$$

Proof. We write Melnikov potential (51) as:

$$\mathcal{L} = \tilde{\mathcal{L}}_1 + \int_{-\infty}^{\infty} \left[\left(\frac{x_0^2}{2}\right)^2 r\cos(\alpha_0 - f) - \frac{x_0^2}{2} \right] dt, \tag{86}$$

where

$$\tilde{\mathcal{L}}_1 = \int_{-\infty}^{\infty} \frac{x_0^2}{\left[4 + x_0^4 r^2 + 4x_0^2 r\cos(\alpha_0 - f)\right]^{1/2}}\, dt$$

can be written as

$$\tilde{\mathcal{L}}_1 = \int_{-\infty}^{\infty} \frac{x_0^2}{2} \left(1 + \frac{x_0^2}{2} r(f(t+s))\mathrm{e}^{i(\alpha_0 - f(t+s))}\right)^{-1/2} \left(1 + \frac{x_0^2}{2} r(f(t+s))\mathrm{e}^{-i(\alpha_0 - f(t+s))}\right)^{-1/2} dt. \tag{87}$$

Using

$$(1+z)^{-\frac{1}{2}} = \sum_{l=0}^{\infty} \binom{-1/2}{l} z^l$$

we get

$$\tilde{\mathcal{L}}_1 = \sum_{k\geq 0}\sum_{l\geq k} \tilde{L}_k^l + \sum_{k<0}\sum_{l\leq k} \tilde{S}_k^l$$

where

$$\tilde{L}_k^l = \frac{1}{2^{2l-k+1}} \binom{-1/2}{l-k} \binom{-1/2}{l} \int_{-\infty}^{\infty} x_0^{4l-2k+2}\, [r(f(t+s))]^{2l-k}\, \mathrm{e}^{ik\alpha_0} \mathrm{e}^{-ikf(t+s)}\, dt; \quad 0\leq k\leq l$$

$$\tilde{S}_k^l = \frac{1}{2^{-2l+k+1}} \binom{-1/2}{k-l} \binom{-1/2}{-l} \int_{-\infty}^{\infty} x_0^{-4l+2k+2} [r(f(t+s))]^{-2l+k} \mathrm{e}^{ik\alpha_0} \mathrm{e}^{-ikf(t+s)} dt; \quad l\leq k\leq -1.$$

With these expressions is easy to see that $\tilde{L}_0^0$ cancels out the last term in the integral (86) and that $\tilde{L}_1^1 + \tilde{S}_{-1}^{-1}$ cancels the cosine term, and so

$$\mathcal{L} = \sum_{l\geq 1} \tilde{L}_0^l + \sum_{l\geq 2} \tilde{L}_1^l + \sum_{l\leq -2} \tilde{S}_{-1}^l + \sum_{k>1}\sum_{l\geq k} \tilde{L}_k^l + \sum_{k<-1}\sum_{l\leq k} \tilde{S}_k^l. \tag{88}$$

Now we perform the change of variable

$$t = \frac{G^3}{2}\Big(\tau + \frac{\tau^3}{3}\Big), \qquad dt = \frac{G^3}{2}(1+\tau^2)d\tau$$

introduced in (31), and we use the formulas for x_0 and α_0 given in (30a) and (30b). In particular we will use that:

$$x_0^2 = \frac{4}{G^2(1+\tau^2)} = 2Gd\tau, \qquad \mathrm{e}^{i\alpha_0} = \frac{\tau - i}{\tau + i}\mathrm{e}^{i\alpha},$$

and the expansion in Fourier series given in (81), obtaining

$$\begin{aligned}
\tilde{L}_k^l &= \mathrm{e}^{ik\alpha}\frac{2^{2l-k}}{G^{4l-2k-1}}\binom{-1/2}{l}\binom{-1/2}{k-l}\sum_{q\in\mathbb{Z}}\mathrm{e}^{iq\,s}c_q^{2l-k,-k}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{iq(\tau+\tau^3/3)\,G^3/2}}{(\tau-i)^{2(l-k)}(\tau+i)^{2l}}d\tau; \qquad 0\le k\le l \\
&= \mathrm{e}^{ik\alpha}\sum_{q\in\mathbb{Z}}\mathrm{e}^{iq\,s}c_q^{2l-k,-k}N(q,l-k,l) \qquad (89a)
\end{aligned}$$

$$\begin{aligned}
\tilde{S}_k^l &= \mathrm{e}^{ik\alpha}\frac{2^{-2l+k}}{G^{-4l+2k-1}}\binom{-1/2}{-l}\binom{-1/2}{k-l}\sum_{q\in\mathbb{Z}}\mathrm{e}^{iq\,s}c_q^{-2l+k,-k}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{iq(\tau+\tau^3/3)\,G^3/2}}{(\tau-i)^{-2l}(\tau+i)^{2(k-l)}}d\tau; \quad l\le k\le -1 \\
&= \mathrm{e}^{ik\alpha}\sum_{q\in\mathbb{Z}}\mathrm{e}^{iq\,s}c_q^{-2l+k,-k}N(q,-l,k-l) \qquad (89b)
\end{aligned}$$

substituting now equations (89a) and (89b) in the expansion (88) we get

$$\begin{aligned}
\mathcal{L} = &\sum_{q\in\mathbb{Z}}\mathrm{e}^{iqs}\sum_{l\ge1}c_q^{2l,0}N(q,l,l) \\
&+\sum_{q\in\mathbb{Z}}\mathrm{e}^{i(qs+\alpha)}\sum_{l\ge2}c_q^{2l-1,-1}N(q,l-1,l)+\sum_{q\in\mathbb{Z}}\mathrm{e}^{i(qs-\alpha)}\sum_{l\le-2}c_q^{-2l-1,1}N(q,-l,-l-1) \\
&+\sum_{q\in\mathbb{Z}}\sum_{k\ge2}\mathrm{e}^{i(qs+k\alpha)}\sum_{l\ge k}c_q^{2l-k,-k}N(q,l-k,l)+\sum_{q\in\mathbb{Z}}\sum_{k\le-2}\mathrm{e}^{i(qs+k\alpha)}\sum_{l\le k}c_q^{-2l+k,-k}N(q,-l,k-l)
\end{aligned} \qquad (90)$$

Now changing the indexes $l \to -l$ and $k \to -k$ in the third and fifth terms one obtain the desired formulas for the Fourier coefficients $L_{q,k}$. □

In view of proposition (11) and formulas (84), to compute the dominant part of the Melnikov potential and obtain effective bounds of the errors we will need to estimate the constants $c_q^{n,m}$ defined in (81) and the integrals $N(q,m,n)$ defined in (85) for $q \ge 0$ and only for indices m, n satisfying $n \ge 0$, $m \le n+1$. Alternatively to (5), it will be very convenient to express the distance r between the primaries as

$$r = 1 - e\cos E \qquad (91)$$

in terms of the *eccentric anomaly* E, given by the Kepler equation [Win41, p. 194]

$$t = E - e\sin E. \qquad (92)$$

This is done in the next three propositions.

Proposition 12. Let $n, m, q \in \mathbb{Z}$, $n, q \ge 0$, $m \le n+1$. Then the Fourier coefficients $c_q^{n,m}$ defined in (81) satisfy

$$\left|c_q^{n,m}\right| \le \begin{cases} 2^{q+n+1}\mathrm{e}^{q\sqrt{1-e^2}}e^{|m-q|} & m \ge 0 \\ (1+e)^{n+1} & m \le -1 \end{cases}$$

Proof. In the integral formula for the Fourier coefficients

$$c_q^{n,m} = \frac{1}{2\pi}\int_0^{2\pi} r^n \mathrm{e}^{imf}\mathrm{e}^{-iqt}dt \qquad (93)$$

we change the variable of integration to the eccentric anomaly (92) ($dt = rdE$) to get

$$c_q^{n,m} = \frac{1}{2\pi}\int_0^{2\pi}\left(r\mathrm{e}^{if}\right)^m r^{n+1-m}\mathrm{e}^{-iqt}dE. \qquad (94)$$

To compute $c_q^{n,m}$ from (94) we will use the identity (see [Win41, p. 202])

$$\left(r\mathrm{e}^{if}\right)^n = a\mathrm{e}^{iE/2} - \frac{e}{2a}\mathrm{e}^{-iE/2}, \qquad a = \frac{\sqrt{1+e}+\sqrt{1-e}}{2}$$

which readily implies

$$r\mathrm{e}^{if} = a^2\mathrm{e}^{iE} - e + \frac{e^2}{4a^2}\mathrm{e}^{-iE} = (a\mathrm{e}^{iE/2} - \frac{e}{2a}\mathrm{e}^{-iE/2})^2 \tag{95a}$$

$$a^2 + \frac{e^2}{4a^2} = 1, \quad a^2 - \frac{e^2}{4a^2} = \sqrt{1-e^2}, \quad a^4 + \frac{e^2}{16a^4} = 1 - e^2, \qquad a^4 - \frac{e^2}{16a^4} = \sqrt{1-e^2}. \tag{95b}$$

To bound the integral (93) for $m \geq 0$ we will consider two different cases: $0 \leq q \leq m$ and $0 \leq m < q$. Let us first consider the case $0 \leq q \leq m$. By the analyticity and periodicity of the integral we change the path of integration from $\Im(E) = 0$ to $\Im E = \ln(2a^2/e)$:

$$E = u + i\ln\left(\frac{2a^2}{e}\right) \qquad u \in [0, 2\pi]$$

so that

$$\mathrm{e}^{iE} = \mathrm{e}^{iu - \ln(2a^2/e)} = \frac{e}{2a^2}\mathrm{e}^{iu}$$

and then, by (95a), (95b), and (92)

$$\begin{aligned}
r\mathrm{e}^{if} &= \frac{e}{2}\mathrm{e}^{iu} - e + \frac{e}{2}\mathrm{e}^{-iu} = e(\cos u - 1) \\
r &= 1 - \frac{e}{2}\left(\frac{e}{2a^2}\mathrm{e}^{iu} + \frac{2a^2}{e}\mathrm{e}^{-iu}\right) = 1 - \frac{e^2}{4a^2}\mathrm{e}^{iu} - a^2\mathrm{e}^{-iu} \\
&= 1 - \left(\frac{e^2}{4a^2} + a^2\right)\cos u + i\left(\frac{e^2}{4a^2} - a^2\right)\sin u = 1 - \cos u + i\sqrt{1-e^2}\sin u \\
\mathrm{e}^{-it} &= \frac{2a^2\mathrm{e}^{-iu}}{e}\exp\left(\frac{e^2}{4a^2}\mathrm{e}^{iu} - a^2\mathrm{e}^{-iu}\right) = \frac{2a^2\mathrm{e}^{-iu}}{e}\exp\left(-\sqrt{1-e^2}\cos u + i\sin u\right)
\end{aligned}$$

therefore

$$\begin{aligned}
\left|r\mathrm{e}^{if}\right| &= e(1-\cos u) \leq 2e \\
|r| &= \sqrt{(1-\cos u)^2 + (1-e^2)\sin^2 u} = \sqrt{2\,(1-\cos u) - e^2\sin^2 u} \leq 2 \\
\left|\mathrm{e}^{-it}\right| &= \frac{2a^2}{e}\exp\left(-\sqrt{1-e^2}\cos u\right) \leq \frac{2a^2}{e}\mathrm{e}^{\sqrt{1-e^2}}.
\end{aligned}$$

Since $2a^2 \leq 2$, substituting these bounds in (94) we find directly the desired result for $0 \leq q \leq m$.

For the the case $0 \leq m < q$ we now perform the change of the integration variable through

$$E = v - i\ln\left(\frac{2a^2}{e}\right), \qquad v \in [0, 2\pi]$$

so that

$$\mathrm{e}^{iE} = \mathrm{e}^{iv + \ln(2a^2/e)} = \frac{2a^2}{e}\mathrm{e}^{iv}$$

and then, by (95a), (95b), and (92)

$$\begin{aligned}
r\mathrm{e}^{if} &= \frac{2a^4}{e}\mathrm{e}^{iv} - e + \frac{e^3}{8a^4}\mathrm{e}^{-iv} = \frac{2}{e}\left(\left(a^4 + \frac{e^4}{16a^4}\right)\cos v - \frac{e^2}{2} + i\left(a^4 - \frac{e^4}{16a^4}\right)\sin v\right) \\
&= \frac{2}{e}\left(\cos v - \frac{e^2}{2}(1+\cos v) + i\sqrt{1-e^2}\sin v\right) \\
r &= 1 - \frac{e^2}{4a^2}\mathrm{e}^{-iv} - a^2\mathrm{e}^{iv} = 1 - \cos v - i\sqrt{1-e^2}\sin v \\
\mathrm{e}^{-it} &= \frac{e\mathrm{e}^{-iv}}{2a^2}\exp\left(a^2\mathrm{e}^{iv} - \frac{e^2}{4a^2}\mathrm{e}^{-iv}\right) = \frac{e\mathrm{e}^{-iv}}{2a^2}\exp\left(\sqrt{1-e^2}\cos u + i\sin u\right)
\end{aligned}$$

therefore as before

$$\left|r\mathrm{e}^{if}\right| \leq 2, \qquad |r| \leq 2 \qquad \left|\mathrm{e}^{-it}\right| \leq \frac{e}{2a^2}\mathrm{e}^{\sqrt{1-e^2}}.$$

Since $2a^2 \geq 1$, substituting these bounds in (94) we find the desired result for $0 \leq m < q$.

For $m \leq -1$ we bound directly the integral (94) for $E \in [0, 2\pi]$. Since $|\mathrm{e}^{if}| = |\mathrm{e}^{-it}| = 1$ we have

$$|c_q^{n,m}| \leq \frac{1}{2\pi}\int_0^{2\pi} |r|^{n+1} dE$$

by noticing that $|r| \leq (1+e)$ we conclude the proof of the bounds for the $c_q^{n,m}$. □

As we can see from equations (84) the Fourier coefficients of the Melnikov potential $\mathcal{L}$ depend on the function $N(q,m,n)$ defined in (85), so to bound (or to compute) these Fourier coefficients we need to bound (or to compute) $N(q,m,n)$.

Introducing

$$I(q,m,n) = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{iq\frac{G^3}{2}\left(\tau+\frac{\tau^3}{3}\right)}}{(\tau-i)^{2m}(\tau+i)^{2n}} d\tau$$

$N(q,m,n)$ can be written as

$$N(q,m,n) = \frac{2^{m+n}}{G^{2m+2n-1}}\binom{-1/2}{m}\binom{-1/2}{n} I(q,m,n).$$

We will call

$$h(\tau) = i\Big(\frac{\tau^3}{3} + \tau\Big) \tag{96}$$

the variable term in the exponencial of the integral, so that

$$I(q,m,n) = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{q\frac{G^3}{2}h(\tau)}}{(\tau-i)^{2m}(\tau+i)^{2n}} d\tau. \tag{97}$$

Since the integral $I(q,m,n)$ involves an exponential, it will be useful the Laplace's method [Erd56] of integration. In particular on a complex path with $\Im(h(\tau)) = 0$. So, let us define the path

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5 \tag{98}$$

where $0 < \varepsilon < 1$, τ^* is a point such that $\Im(h(\tau^*)) = 0$ that will be fixed later, and

$$\begin{aligned}
\Gamma_1 &= \{\tau \in \mathbb{C} | \Im(h(\tau)) = 0\} \cap \{\tau \in \mathbb{C} | \Re(\tau) \leq \Re(-\bar{\tau}^*)\}\\
\Gamma_5 &= \{\tau \in \mathbb{C} | \Im(h(\tau)) = 0\} \cap \{\tau \in \mathbb{C} | \Re(\tau) \geq \Re(\ \tau^*)\}\\
\Gamma_2 &= \{\tau \in \mathbb{C} | \Im(h(\tau)) = 0\} \cap \{\tau \in \mathbb{C} | \Re(-\bar{\tau}^*) \leq \Re(\tau) \leq 0\} \cap \{\tau \in \mathbb{C} || \tau - i| \geq c\varepsilon\}\\
\Gamma_4 &= \{\tau \in \mathbb{C} | \Im(h(\tau)) = 0\} \cap \{\tau \in \mathbb{C} | 0 \leq \Re(\tau) \leq \Re(\ \tau^*)\} \cap \{\tau \in \mathbb{C} || \tau - i| \geq c\varepsilon\}\\
\Gamma_3 &= \{\tau \in \mathbb{C} | \Im(h(\tau)) \leq 0\} \cap \{\tau \in \mathbb{C} || \tau - i| = c\varepsilon\}.
\end{aligned} \tag{99}$$

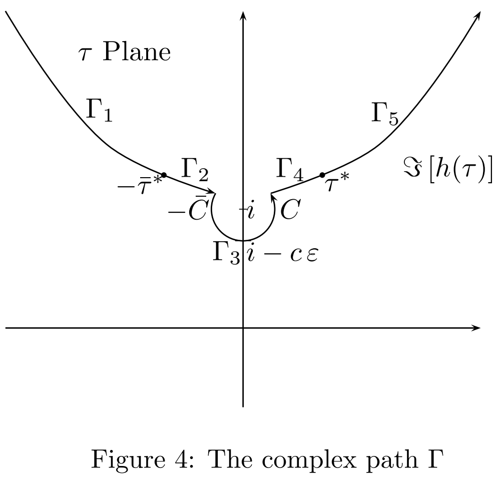

Figure 4: The complex path Γ

By means of the Cauchy-Goursat theorem and a limit argument, the integral $I(q,m,n)$, defined in (97) over the real axis, is equal to the one taken over the path Γ thinking of τ as a complex number (see [LS80]). In fact, by the same argument, its value does not depend on ε.

The positive branch of the hyperbola defined by $\Im(h(\tau)) = 0$ intersects the circumference of radius ε in two points that can be expressed as C_ε and $-\bar{C}_\varepsilon$ defined by:

$$C_\varepsilon = \Gamma_3 \cap \Gamma_4 \qquad -\bar{C}_\varepsilon = \Gamma_3 \cap \Gamma_2 \tag{100}$$

Since the integral over Γ does not depend on ε, we will choose a particular value of ε to bound $I(q,m,n)$ and consequently $N(q,m,n)$ defined in (85). Later on, in proposition 17 we will just compute the ε-independent terms of this integral.

It is not difficult to see that, if we define the function

$$u(\tau)=h(i)-h(\tau)=-\frac{2}{3}-i\Big(\frac{\tau^3}{3}+\tau\Big)=(\tau-i)^2-\frac{i}{3}(\tau-i)^3, \tag{101}$$

then

$$u(\Gamma_1\cup\Gamma_2),\ u(\Gamma_4\cup\Gamma_5)\subset\mathbb{R}_0^+.$$

Moreover, if $\tau^-\in\Gamma_1\cup\Gamma_2$ then $\tau^+=-\bar{\tau}^-\in\Gamma_4\cup\Gamma_5$ and

$$u(\tau^-)=u(\tau^+).$$

On the other hand one can see that u is an increasing function while moving along $\Gamma_1\cup\Gamma_2$ or $\Gamma_4\cup\Gamma_5$ in the direction of increasing imaginary part. Therefore u has two inverses in $\mathbb{R}_0^+$: τ^+ and τ_-. Before writing them down let us notice that the point C_ε defined in (100) can be written as

$$C_\varepsilon=i+\varepsilon\,\mathrm{e}^{i\theta_\varepsilon}\quad\text{with}\quad\theta_\varepsilon\in(0,\pi/2) \tag{102}$$

and has the following expression in the coordinates u defined in (101)

$$u(C_\varepsilon)=|u(C_\varepsilon)|=\varepsilon^2\left|1-\frac{\varepsilon}{3}i\mathrm{e}^{i\theta_\varepsilon}\right|=\varepsilon^2\ k_\varepsilon \tag{103}$$

with

$$k_\varepsilon=\left|1-\frac{\varepsilon}{3}i\mathrm{e}^{i\theta_\varepsilon}\right|=\sqrt{\Big(1+\frac{\varepsilon}{3}\sin\theta_\varepsilon\Big)^2+\Big(\frac{\varepsilon}{3}\cos\theta_\varepsilon\Big)^2}\geq 1,$$

since by construction, $\theta_\varepsilon\in(0,\pi/2)$ and then $0<\sin\theta_\varepsilon$.

Now, we can write the inverses of the function u

$$\begin{aligned}\tau^+:&[u(C_\varepsilon),+\infty)\longrightarrow\Gamma_4\cup\Gamma_5 & \tau^-:&[u(C_\varepsilon),+\infty)\longrightarrow\Gamma_1\cup\Gamma_2\\ &u\longmapsto\xi(u)+i\eta(u), & &u\longmapsto-\xi(u)+i\eta(u).\end{aligned} \tag{104}$$

The change (101) is useful over $\Gamma_1\cup\Gamma_2$ and $\Gamma_4\cup\Gamma_5$, thus performing this change in (85), we have that for any $\varepsilon>0$

$$N(q,m,n)=\frac{d_{m,n}\mathrm{e}^{-q\frac{G^3}{3}}}{G^{2m+2n-1}}\left[\int_{u(C_\varepsilon)}^{\infty}[F^+_{m,n}(u)-F^-_{m,n}(u)]\mathrm{e}^{-q\frac{G^3}{2}u}du+(-i)\mathrm{e}^{q\frac{G^3}{3}}\int_{\Gamma_3}f^q_{m,n}(\tau)d\tau\right] \tag{105}$$

where

$$d_{m,n}=i2^{m+n}\binom{-1/2}{n}\binom{-1/2}{m} \tag{106}$$

$$F^{\pm}_{m,n}(u)=\frac{1}{(\tau^{\pm}(u)-i)^{2m+1}(\tau^{\pm}(u)+i)^{2n+1}} \tag{107}$$

$$f^q_{m,n}(\tau)=\frac{\mathrm{e}^{q\frac{G^3}{2}h(\tau)}}{(\tau-i)^{2m}(\tau+i)^{2n}}, \tag{108}$$

where $h(\tau)$ is given in (96). Now several helpful lemmas follow.

Lemma 13. Let $m,n\in\mathbb{Z}$, $m,n\geq 0$ and $d_{m,n}$ be defined by equation (106). Then

$$|d_{m,n}|\leq\mathrm{e}^{-1/2}2^{m+n}\qquad\text{if } m+n>0$$

Proof. Let $s \in \mathbb{N}$, then

$$\left|\binom{-1/2}{s}\right| = \left|\frac{(-1)^s}{s!}\Big(\frac{1}{2}\Big)\Big(\frac{1}{2}+1\Big)\cdots\Big(\frac{1}{2}+s-1\Big)\right| = \frac{1}{2^s}\Big[1\cdot\frac{3}{2}\cdots\frac{2s-1}{s}\Big]\Big]$$
$$\leq \frac{1}{2^s}\Big(2-\frac{1}{s}\Big)^s = \Big(1-\frac{1}{2s}\Big)^s \leq \lim_{s\to\infty}\Big(1-\frac{1}{2s}\Big)^s = \mathrm{e}^{-1/2}$$

□

Next lemma gives information about the functions $F^{\pm}_{m,n}(u)$.

Lemma 14. The function $F^{\pm}_{m,n}(u)$ defined in (107) has the expansion

$$F^{\pm}_{m,n}(u) = (\pm\sqrt{u})^{-2m-1}\sum_{j=0}^{\infty} d_j^{m,n}(\pm\sqrt{u})^j \tag{109}$$

where the coefficients $d_j^{m,n}$ satisfy

$$d_0^{m,n} = 1/(2i)^{2n+1}, \quad |d_j^{m,n}| \leq \left(\frac{4}{3}\right)^m \left(\frac{3}{2}\right)^{\frac{j+3}{2}}. \tag{110}$$

Consequently, the series (109) is convergent for $|\sqrt{u}| < \sqrt{2/3}$.

Proof. Let us introduce the function

$$T^{\pm}_{m,n}(x) := (\pm)x^{2m+1}F^{\pm}_{m,n}(x^2) = \sum_{j=0}^{\infty} d_j^{m,n}(\pm x)^j,$$

which is well defined since $u = x^2$ is a good change of variables in $\mathbb{R}^+$ and has the two inverses $x = \pm\sqrt{u}$. To bound the coefficients $d_j^{m,n}$ we use Cauchy formula:

$$(\pm 1)^j d_j^{m,n} = \frac{1}{2\pi i}\int_{|x|=\varepsilon}\frac{T^{\pm}_{m,n}(x)}{x^{j+1}}dx = \frac{-1}{2\pi i}\int_{|x|=\varepsilon}\frac{F^{\pm}_{m,n}(x^2)}{x^{j-2m}}dx.$$

Applying the change of variables

$$x = \pm\sqrt{(\tau-i)^2 - \frac{i}{3}(\tau-i)^3} = \pm(\tau-i)\sqrt{(1-\frac{i}{3}(\tau-i))} = \pm\frac{\tau-i}{\sqrt{3}}(\sqrt{2-i\tau}), \tag{111}$$

we obtain

$$\begin{aligned}(\pm 1)^j d_j^{m,n} &= \mp\frac{1}{2\pi i}\int_{|\tau-i|=\rho}\frac{(\tau-i)^{2m-j}}{3^{\frac{2m-j}{2}}}(2-i\tau)^{\frac{2m-j}{2}}\frac{1}{(\tau-i)^{2m+1}(\tau+i)^{2n+1}}\frac{3(1-i\tau)}{2\sqrt{3}(2-i\tau)^{\frac{1}{2}}}d\tau \\ &= \mp\frac{1}{2}\frac{i^{\frac{j+1}{2}-m}}{2\pi 3^{m-\frac{j+1}{2}}}\int_{|\tau-i|=\rho}\frac{d\tau}{(\tau-i)^{j+1}(\tau+i)^{2n}(\tau+2i)^{\frac{j+1-2m}{2}}}.\end{aligned}$$

Now, taking $\rho = 1$ and using that $|\tau+i| \geq 1$ and that $2 \leq |\tau+2i| \leq 4$ we have

$$|d_j^{m,n}| \leq \left(\frac{4}{3}\right)^m \left(\frac{3}{2}\right)^{\frac{j+3}{2}},$$

which is the desired bound. From this bound it is clear that the series defining $T^{\pm}_{m,n}(x)$ is convergent for $|x| < \sqrt{2/3}$ and therefore the one for $F^{\pm}_{m,n}(u)$ is convergent for $\sqrt{u} < \sqrt{2/3}$. □

From equation (109) we have

$$F^{\pm}_{m,n}(u) = (\pm\sqrt{u})^{-2m-1}\sum_{j=0}^{2m} d_j^{m,n}(\pm\sqrt{u})^j + g^{\pm}_{m,n}(\pm\sqrt{u}), \tag{112}$$

where the regular part of the function $F^{\pm}_{m,n}(u)$ is given by

$$g^{\pm}_{m,n}(\pm\sqrt{u}) = (\pm\sqrt{u})^{-2m-1}\sum_{j=2m+1}^{\infty} d_j^{m,n}(\pm\sqrt{u})^j \tag{113}$$

and $d_j^{m,n}$ are defined by equation (109) and satisfy bounds (110).

Lemma 15. Let $g^{\pm}_{m,n}(\pm\sqrt{u})$ as in equation (113), $0<\beta<1$ and $0<\sqrt{u}<\beta\sqrt{2/3}$. Then

$$\left|g^{\pm}_{m,n}(\pm\sqrt{u})\right| < \frac{9}{1-\beta}2^{m-2}.$$

Proof. It is clear from equation (113) that

$$g^{\pm}_{m,n}(\pm\sqrt{u}) = \sum_{s=0}^{\infty} d^{m,n}_{s+2m+1}(\pm\sqrt{u})^s.$$

Since by hypothesis $0<\sqrt{u}<\beta\sqrt{2/3}$ with $\beta<1$, we can apply lemma 14 to get

$$\begin{aligned}|g^{\pm}_{m,n}(\pm\sqrt{u})| &\le \left(\frac{4}{3}\right)^m\left(\frac{3}{2}\right)^{\frac{2m+4}{2}}\sum_{s=0}^{\infty}\left(\frac{3}{2}\right)^{\frac{s}{2}}(\sqrt{u})^s\\ &\le \left(\frac{4}{3}\right)^m\left(\frac{3}{2}\right)^{\frac{2m+4}{2}}\sum_{s=0}^{\infty}\left(\frac{3}{2}\right)^{\frac{s}{2}}\left(\beta\sqrt{2/3}\right)^s = \frac{9}{1-\beta}2^{m-2}\end{aligned}$$

which proves the lemma. □

Next proposition gives a bound for $N(q,m,n)$.

Proposition 16. Let $N(q,m,n)$ as defined in (105) for $q>0$, $m,n\ge 0$, $m+n>0$, $G>1$. Then

$$|N(q,m,n)| \le 2^{n+m+3}\,\mathrm{e}^q\,G^{m-2n-1/2}\,\mathrm{e}^{-qG^3/3}.$$

Proof. We will bound the integrals of $N(q,m,n)$ choosing

$$\varepsilon = G^{-3/2},\quad G>1.$$

We write down then, using (103) and that $k_\varepsilon > 1$,

$$\begin{aligned}&\left|\int_{u(C_\varepsilon)}^{\infty} F^{\pm}_{m,n}(u)\mathrm{e}^{-q\frac{G^3}{2}u}\,du\right| \le \int_{G^{-3}k_\epsilon}^{\infty}|F^{\pm}_{m,n}(u)|\mathrm{e}^{-q\frac{G^3}{2}u}\,du \le \int_{G^{-3}}^{\infty}|F^{\pm}_{m,n}(u)|\mathrm{e}^{-q\frac{G^3}{2}u}\,du\\ &\le |F^{\pm}_{m,n}(u(C_\varepsilon))|\int_{G^{-3}}^{\infty}\mathrm{e}^{-q\frac{G^3}{2}u}\,du \le \frac{G^{3m+\frac{3}{2}}}{(2-(G^{-\frac{3}{2}}))^{2n+1}}\frac{2}{qG^3}\left[\mathrm{e}^{-\frac{q}{2}}\right] \le 2G^{3m-\frac{3}{2}}.\end{aligned} \tag{114}$$

It remains only the last integral of (105) where the integrand is given in (108) and the domain Γ_3 in (99). The path Γ_3 can be parameterized by

$$\tau = i + G^{-\frac{3}{2}}\mathrm{e}^{\mathrm{i}\theta}\quad \text{with } \theta\in[\theta_1,\theta_2] = [\pi-\theta_\varepsilon,\theta_\varepsilon], \tag{115}$$

with θ_ε given in (102). If we define

$$\tilde{h}(\theta) = h(\tau(\theta)) = i\left(\frac{\tau(\theta)^3}{3}+\tau(\theta)\right),$$

a straightforward computation using (101) shows that

$$\tilde{h}(\theta) = -\frac{2}{3} - G^{-3}\Big(\mathrm{e}^{2i\theta} + \frac{1}{3i}G^{-\frac{3}{2}}\mathrm{e}^{3i\theta}\Big)$$

and then, as $G > 1$:

$$\big|\mathrm{e}^{q\frac{G^3}{2}\tilde{h}(\theta)}\big| = \mathrm{e}^{-\frac{q}{3}G^3}\mathrm{e}^{-\frac{q}{2}(\cos 2\theta + \frac{1}{3}G^{-\frac{3}{2}}\sin 3\theta)} \leq \mathrm{e}^{-\frac{q}{3}G^3}\mathrm{e}^{\frac{q}{2}(1+\frac{1}{3}G^{-\frac{3}{2}})} \leq \mathrm{e}^{-\frac{q}{3}G^3}\mathrm{e}^{q}. \tag{116}$$

Note that, by (115), over Γ_3 we have that $|\tau - i| = G^{-\frac{3}{2}} < 1$ and therefore $|\tau + i| > 1$, and we can bound the last integral of (105) using (116):

$$\left|\int_{\Gamma_3} \frac{\mathrm{e}^{q\frac{G^3}{2}h(\tau)}}{(\tau - i)^{2m}(\tau + i)^{2n}}\, d\tau\right| = \left|\int_{\theta_1}^{\theta_2} \frac{\mathrm{e}^{q\frac{G^3}{2}\tilde{h}(\theta)}}{(\tau(\theta) - i)^{2m}(\tau(\theta) + i)^{2n}} i\, G^{-\frac{3}{2}}\mathrm{e}^{i\theta}\, d\theta\right|$$

$$\leq \int_{\theta_1}^{\theta_2} \frac{|\mathrm{e}^{q\frac{G^3}{2}\tilde{h}(\theta)}|}{|G^{-3/2}|^{2m}} G^{-\frac{3}{2}}\, d\theta \leq \int_{\theta_1}^{\theta_2} \frac{\mathrm{e}^{-\frac{q}{3}G^3}\mathrm{e}^{q}}{G^{-3m}} G^{-\frac{3}{2}}\, d\theta \leq \pi G^{3m-3/2}\mathrm{e}^{-\frac{q}{3}G^3}\mathrm{e}^{q}. \tag{117}$$

From lemma 13 and the bounds (114) and (117), we can bound $N(q,m,n)$ by equation (105) as follows

$$|N(q,m,n)| \leq \mathrm{e}^{-1/2}2^{m+n}\mathrm{e}^{-q\frac{G^3}{3}}G^{m-2n-\frac{1}{2}}\Big(4 + \pi\mathrm{e}^{q}\Big) \leq 2^{m+n+3}\mathrm{e}^{q}\mathrm{e}^{-q\frac{G^3}{3}}G^{m-2n-\frac{1}{2}}.$$

□

Next proposition provides an asymptotic expression for $N(q,m,n)$.

Proposition 17. Let $n + m > 0$ and the constants $d_j^{m,n}$ be defined by equation (109) and $d_{n,m}$ by equation (106). For $q > 0$, $m, n \geq 0$ and $G > 1$ we have

$$N(q,m,n) = \frac{d_{m,n}\mathrm{e}^{-q\frac{G^3}{3}}}{G^{2m+2n-1}}\left[\sum_{s=0}^{m}(-1)^s\sqrt{\pi}\frac{2^{\frac{3}{2}}q^{s-\frac{1}{2}}}{(2s-1)!!}d_{2m-2s}^{m,n}G^{3s-\frac{3}{2}} + T_{m,n}^q + R_{m,n}^q\right]$$

where

$$|T_{m,n}^q| \leq 45\, 2^{2m+2}\cdot G^{-3} \qquad |R_{m,n}^q| \leq 18\, q^{m-1}G^{3m-3}.$$

When $s = 0$ the factor $1/(2s-1)!!$ in the formula above should be replaced by 1.

Proof. We proceed as in the proof of proposition 16 changing the path of integration to the path Γ defined in (98) leading to the integral (105). The important fact is that the integral (105) does not depend on ε. So, we will compute only the ε-independent terms of that integral. We will follow a series of lemmas leading to the proof of the statement.

Lemma 18. Let $0 < \varepsilon < 1$ and $u(C_\varepsilon)$ be as in equation (103), $F_{m,n}^{\pm}$ defined by (107). For any $\varepsilon > 0$ small enough we have, if $G > 1$:

$$\int_{u(C_\varepsilon)}^{\infty} F_{m,n}^{\pm}(u)\mathrm{e}^{-q\frac{G^3}{2}u}du = \sum_{j=0}^{2m}\int_{u(C_\varepsilon)}^{\infty}\mathrm{e}^{-q\frac{G^3}{2}u}d_j^{m,n}(\pm\sqrt{u})^{-2m-1+j}du + \widehat{E}$$

where the constants $d_j^{m,n}$ are defined by equation (109) and $\widehat{E}$ satisfies

$$|\widehat{E}| \leq 45\, 2^{2m+2}\, G^{-3}.$$

Proof. Let us take $\sqrt{u_*} = \beta\sqrt{2/3}$ with $\beta = -1 + \frac{\sqrt{11}}{4}\sqrt{3 + \sqrt{11}/2} \simeq 0.79$. A simple calculation using (101) shows that $|\tau^{\pm}(u^*) - i| = \frac{1}{2}$. By definition, for $\varepsilon > 0$ small enough we have that $0 < u(C_\varepsilon) < u_* < \sqrt{u_*} < \sqrt{2/3}$, so

$$\int_{u(C_\varepsilon)}^{\infty} F_{m,n}^{\pm}(u)\mathrm{e}^{-q\frac{G^3}{2}u}du = \int_{u(C_\varepsilon)}^{u_*} F_{m,n}^{\pm}(u)\mathrm{e}^{-q\frac{G^3}{2}u}du + \widehat{E}_1$$

with

$$\widehat{E}_1=\int_{u_*}^{\infty}F^{\pm}_{m,n}(u)\mathrm{e}^{-q\frac{G^3}{2}u}du,$$

which can be bounded as

$$\begin{aligned}|\widehat{E}_1| &= \left|\int_{u_*}^{\infty}F^{\pm}_{m,n}(u)\mathrm{e}^{-q\frac{G^3}{2}u}\,du\right|\leq\int_{u_*}^{\infty}\frac{\mathrm{e}^{-q\frac{G^3}{2}u}}{|(\tau^{\pm}(u)-i)^{2m+1}(\tau^{\pm}(u)+i)^{2n+1}|}\,du\\
&\leq \frac{2\mathrm{e}^{-q\frac{G^3}{2}u_*}}{qG^3}\frac{1}{|\tau^{\pm}(u_*)-i|^{2m+1}}\frac{1}{|\tau^{\pm}(u_*)+i|^{2n+1}}\\
&\leq 2^{2m+2}G^{-3}\mathrm{e}^{-q\frac{G^3}{2}u_*}\leq 2^{2m+2}G^{-3}.\end{aligned}\tag{118}$$

By lemma 14 and equation (112) we have

$$\int_{u(C_\varepsilon)}^{u_*}F^{\pm}_{m,n}(u)\mathrm{e}^{-q\frac{G^3}{2}u}du=\sum_{j=0}^{2m}\int_{u(C_\varepsilon)}^{u_*}d_j^{m,n}\mathrm{e}^{-q\frac{G^3}{2}u}(\pm\sqrt{u})^{-2m-1+j}du+\widehat{E}_2$$

where

$$\widehat{E}_2=\int_{u(C_\varepsilon)}^{u_*}g^{\pm}_{m,n}(\pm\sqrt{u})\mathrm{e}^{-q\frac{G^3}{2}u}du.$$

Using that $\sqrt{u_*}=\beta\sqrt{2/3}$, by lemma 15 we have that, for any $\varepsilon>0$ small enough,

$$\begin{aligned}|\widehat{E}_2| &\leq\int_{u(C_\varepsilon)}^{u_*}|g^{\pm}_{m,n}(\pm\sqrt{u})|\mathrm{e}^{-q\frac{G^3}{2}u}du\leq 9\,\frac{2^{m-2}}{1-\beta}\int_0^{\infty}\mathrm{e}^{-q\frac{G^3}{2}u}du\\
&\leq 9\,\frac{2^{m-1}}{q(1-\beta)}G^{-3}\leq 9\,\frac{2^{m-1}}{1-\beta}G^{-3}.\end{aligned}$$

Finally,

$$\int_{u(C_\varepsilon)}^{u_*}d_j^{m,n}\mathrm{e}^{-q\frac{G^3}{2}u}(\pm\sqrt{u})^{-2m-1+j}du=\int_{u(C_\varepsilon)}^{\infty}d_j^{m,n}\mathrm{e}^{-q\frac{G^3}{2}u}(\pm\sqrt{u})^{-2m-1+j}du+\widehat{E}_3(j)\tag{119}$$

where

$$\widehat{E}_3(j)=-\int_{u_*}^{\infty}d_j^{m,n}\mathrm{e}^{-q\frac{G^3}{2}u}(\pm\sqrt{u})^{-2m-1+j}du.$$

Let us bound $\widehat{E}_3(j)$ using the inequalities of Lemma 14:

$$\begin{aligned}|\widehat{E}_3(j)| &\leq |d_j^{m,n}|(\sqrt{u_*})^{-2m-1+j}\int_{u_*}^{\infty}\mathrm{e}^{-q\frac{G^3}{2}u}du\leq|d_j^{m,n}|(\sqrt{u_*})^{-2m-1+j}2\mathrm{e}^{-q\frac{G^3}{2}u_*}\frac{G^{-3}}{q}\\
&\leq 2|d_j^{m,n}|\left(\beta\sqrt{2/3}\right)^{-2m-1+j}G^{-3}\leq 2\left(\frac{4}{3}\right)^m\left(\frac{3}{2}\right)^{\frac{j+3}{2}}\left(\beta\sqrt{2/3}\right)^{-2m-1+j}G^{-3}\\
&= 9\,2^{m-1}\beta^{-2m-1+j}G^{-3},\end{aligned}$$

then, calling $\widehat{E}_3=\sum_{j=1}^{2m}\widehat{E}_3(j)$, we have

$$|\widehat{E}_3|\leq 9\,2^{m-1}G^{-3}\sum_{j=0}^{2m}\beta^{-2m-1+j}\leq 9\,2^{m-1}G^{-3}\frac{\beta^{-2m-1}}{1-\beta}.$$

Now the lemma is proven using that $1/\beta<\sqrt{2}$ and

$$|\widehat{E}|=|\widehat{E}_1+\widehat{E}_2+\widehat{E}_3|$$

□

The next lemma is a straightforward application of the last one.

Lemma 19. Let $0 < \varepsilon < 1$ and $u(C_\varepsilon)$ be as in equation (103), $F^{\pm}_{m,n}$ defined by (107) for any $\varepsilon > 0$ small enough we have, if $G > 1$:

$$\int_{u(C_\varepsilon)}^{\infty} \left[F^{+}_{m,n}(u) - F^{-}_{m,n}(u)\right] \mathrm{e}^{-q\frac{G^3}{2}u} du = 2\sum_{s=0}^{m} \int_{u(C_\varepsilon)}^{\infty} \mathrm{e}^{-q\frac{G^3}{2}u} d^{m,n}_{2m-2s} (\sqrt{u})^{-2s-1} du + 2\widehat{E}$$

where $\widehat{E}$ is the same as in lemma 18.

Proof. By lemma 18 we have

$$\int_{u(C_\varepsilon)}^{\infty} \left[F^{+}_{m,n}(u) - F^{-}_{m,n}(u)\right] \mathrm{e}^{-q\frac{G^3}{2}u} du = \sum_{j=0}^{2m} \int_{u(C_\varepsilon)}^{\infty} \mathrm{e}^{-q\frac{G^3}{2}u} d^{m,n}_{j} [1-(-1)^{-2m-1+j}](\sqrt{u})^{-2m-1+j} du + 2\widehat{E}$$

then the non trivial terms in the sum are given when $-2m - 1 + j = -2s - 1$ with $s = 0, \ldots, m$. This observation proves the lemma. ☐

Lemma 20. Let $0 < \varepsilon < 1$ and $u(C_\varepsilon)$ be as in equation (105). Then the ε-independent term of

$$\int_{u(C_\varepsilon)}^{\infty} \mathrm{e}^{-q\frac{G^3}{2}u} d^{m,n}_{2m-2s} (\sqrt{u})^{-2s-1} du$$

is

$$(-1)^s 2^{s+\frac{3}{2}} (2s+1) \frac{(s+1)!}{(2s+2)!} d^{m,n}_{2m-2s} q^{s-\frac{1}{2}} G^{3s-\frac{3}{2}}\, \Gamma(1/2)$$

Proof. By equation (103) we know that $u(C_\varepsilon) = O(\varepsilon^2)$ and then the following definitions make sense, calling $\delta = \frac{G^3}{2}$:

$$I_{p,s}(\varepsilon) = \int_{u(C_\varepsilon)}^{\infty} \mathrm{e}^{-q\delta u}\, u^{p-(2s+1)/2}\, du$$
$$f_{p,s}(\varepsilon) = u(C_\varepsilon)^{p-(2s+1)/2}\, \mathrm{e}^{-q\delta u(C_\varepsilon)}.$$

Using this notation and integrating by parts we have

$$\begin{aligned} I_{p-1,s}(\varepsilon) &= \frac{q\delta}{p-s-1/2} \int_{u(C_\varepsilon)}^{\infty} \mathrm{e}^{-q\delta u} u^{p-(2s+1)/2}\, du - \frac{u(C_\varepsilon)^{p-(2s+1)/2}\, \mathrm{e}^{-q\delta u(C_\varepsilon)}}{p-s-1/2} \\ &= \frac{1}{p-s-1/2} \; (q\delta I_{p,s}(\varepsilon) - f_{p,s}(\varepsilon)) \end{aligned} \tag{120}$$

and also

$$\int_{u(C_\varepsilon)}^{\infty} \mathrm{e}^{-q\frac{G^3}{2}u} d^{m,n}_{2m-2s} (\sqrt{u})^{-2s-1} du = d^{m,n}_{2m-2s} I_{0,s}(\varepsilon). \tag{121}$$

Now, in the case where $s > 0$, using equation (120) s times we get

$$I_{0,s}(\varepsilon) = \frac{(q\delta)^s}{(-s-\frac{1}{2}+1)(-s-\frac{1}{2}+2)\cdots(-\frac{1}{2})} I_{s,s}(\varepsilon) - \sum_{p=1}^{s} \frac{(q\delta)^{p-1} f_{p,s}(\varepsilon)}{(-s-\frac{1}{2}+1)\cdots(-s-\frac{1}{2}+p)}.$$

The ε-independent term of $I_{0,s}(\varepsilon)$ is given by

$$\begin{aligned} \frac{(q\delta)^s}{(-s-\frac{1}{2}+1)(-s-\frac{1}{2}+2)\cdots(-\frac{1}{2})} \lim_{\varepsilon\to 0} I_{s,s}(\varepsilon) &= \frac{(q\delta)^s}{(-s-\frac{1}{2}+1)(-s-\frac{1}{2}+2)\cdots(-\frac{1}{2})} \frac{1}{\sqrt{q\delta}}\; \Gamma(1/2) \\ &= \frac{(\sqrt{q\delta})^{2s-1}}{(-s-\frac{1}{2}+1)(-s-\frac{1}{2}+2)\cdots(-\frac{1}{2})} \Gamma(1/2). \end{aligned}$$

Then the ε-independent term of the integral in equation (121) is

$$\frac{d^{m,n}_{2m-2s}(\sqrt{q\delta})^{2s-1}}{(-s-\frac{1}{2}+1)(-s-\frac{1}{2}+2)\cdots(-\frac{1}{2})}\,\Gamma(1/2)$$

when $s>0$.

In the same way, we have that the ε-independent term of

$$I_{0,0}(\varepsilon)=\int_{u(C_\varepsilon)}^{\infty}\mathrm{e}^{-q\frac{G^3}{2}u}d^{m,n}_{2m}(\sqrt{u})^{-1}du$$

is $d^{m,n}_{2m}(\sqrt{q\delta})^{-1}\,\Gamma(1/2)$. Therefore the lemma is proved if we notice that

$$\begin{aligned}&\big(-s-\frac{1}{2}+1\big)\big(-s-\frac{1}{2}+2\big)\cdots\big(-\frac{1}{2}\big)=\frac{(-1)^s}{2^s}(2s-1)(2s-3)\cdots(1)\\&=\frac{(-1)^s}{2^s}\frac{(2s+1)!!}{2s+1}=\frac{(-1)^s}{2^{2s+1}(2s+1)}\frac{(2s+2)!}{(s+1)!}\end{aligned}$$

where we have used that

$$(2s+1)!!=\frac{(2s+2)!}{2^s(s+1)!}\,.$$

This expression allow us to write the cases $s>0$ and $s=0$ in one equation which completes the proof. ☐

Next Lemma is a straightforward application of lemmas 19 and 20.

Lemma 21. Let $u(C_\varepsilon)$ given in equation (103) and $F^{\pm}_{m,n}$ defined by (107), then the ε-independent terms of

$$\int_{u(C_\varepsilon)}^{\infty}\big[F^{+}_{m,n}(u)-F^{-}_{m,n}(u)\big]\mathrm{e}^{-q\frac{G^3}{2}u}du$$

are given by

$$\sum_{s=0}^{m}(-1)^s2^{s+\frac{5}{2}}(2s+1)\frac{(s+1)!}{(2s+2)!}d^{m,n}_{2m-2s}q^{s-\frac{1}{2}}G^{3s-\frac{3}{2}}\,\Gamma(1/2)+2\widehat{E}$$

where $\widehat{E}$ is the same as in lemma 18.

Lemma 22. Let $f^{q}_{m,n}$ be defined in equation (108), then

$$\mathrm{Res}(f^{q}_{m,n}(\tau),i)=2i\,\mathrm{e}^{-qG^3/3}\sum_{l=0}^{m-1}\frac{1}{l!}\left(\frac{-qG^3}{2}\right)^{l}d^{m,n}_{2m-1-2l}\tag{122}$$

Proof. We use the definition of $f^{q}_{m,n}$ given in (108), with $h(\tau)$ given in (96). Now, using (101),

$$h(\tau)=-2/3-(\tau-i)^2+i(\tau-i)^3/3$$

and we have, taking any $\delta>0$ small enough,

$$\mathrm{Res}(f^{q}_{m,n}(\tau),i)=\frac{1}{2\pi i}\int_{|\tau-i|=\delta}f^{q}_{m,n}(\tau)d\tau=\frac{1}{2\pi i}\int_{|\tau-i|=\delta}\frac{\mathrm{e}^{q\frac{G^3}{2}h(\tau)}}{(\tau-i)^{2m}(\tau+i)^{2n}}\,d\tau.$$

We use again one of the changes (111), for instance

$$x=\sqrt{h(i)-h(\tau)}=\frac{\tau-i}{\sqrt{3}}(\sqrt{2-i\tau}),$$

obtaining

$$\operatorname{Res}\left(f^q_{m,n}(\tau), i\right) = \frac{\mathrm{e}^{-qG^3/3}}{\pi}\int_{|x|=\bar\delta}\frac{\mathrm{e}^{-qG^3x^2/2}}{(\tau_+(x)-i)^{2m+1}(\tau_+(x)+i)^{2n+1}}\,x\,d\tau$$
$$= 2i\,\mathrm{e}^{-qG^3/3}\operatorname{Res}\left(xF^{n,m}_+(x^2)\mathrm{e}^{-q\frac{G^3}{2}x^2}, 0\right).$$

Now we can use the Taylor expansion of the function $F^{n,m}_+(x^2) = \sum_{j\geq 0} d_j^{m,n}x^{j-2m-1}$ and the expansion of $\mathrm{e}^{-qG^3x^2/2} = \sum_{l\geq 0}\frac{1}{l!}\left(-qG^3x^2/2\right)^l$ to obtain the desired formula.

□

From this lemma one and the bounds for $d_j^{m,n}$ given in (110), one has

$$\left|\operatorname{Res}\left(f^q_{m,n}(\tau), i\right)\right| \leq 3\,2^m\mathrm{e}^{-qG^3/3}\sum_{l=0}^{m-1}\frac{1}{l!}\left(\frac{qG^3}{3}\right)^l$$
$$\leq 3\,2^{m+1}\mathrm{e}^{-qG^3/3}\left(\frac{qG^3}{3}\right)^{m-1} = \frac{2^{m+1}q^{m-1}G^{3m-3}}{3^{m-2}}\mathrm{e}^{-qG^3/3}. \tag{123}$$

Now we can prove proposition 17. $N(q,m,n)$ is given in (105), and since it does not depend on ε we can apply lemmas 21 and 22 and bound (123) to obtain

$$N(q,m,n) = \frac{d_{m,n}\mathrm{e}^{-q\frac{G^3}{3}}}{G^{2m+2n-1}}\left[\sum_{s=0}^m(-1)^s2^{s+\frac{5}{2}}(2s+1)\frac{(s+1)!}{(2s+2)!}d^{m,n}_{2m-2s}q^{s-\frac{1}{2}}G^{3s-\frac{3}{2}}\,\Gamma(1/2)+T^q_{m,n}+R^q_{m,n}\right]$$

where

$$R^q_{m,n} = (-i)\mathrm{e}^{q\frac{G^3}{3}}\int_{\Gamma_3} f^q_{m,n}(\tau)d\tau$$

and by lemma 21

$$|T^q_{m,n}| = 2\widehat{E} \leq 45\,2^{2m+2}\cdot G^{-3}.$$

By lemma 22

$$|R^q_{m,n}| \leq \frac{2^{m+1}q^{m-1}}{3^{m-2}}G^{3m-3} < 18\,q^{m-1}G^{3m-3}.$$

Using that $2^{s+1}(s+1)!(2s+1)!! = (2s+2)!$ to show that

$$\frac{(2s+1)(s+1)!}{(2s+2)!} = \frac{1}{2^{s+1}(2s-1)!!}.$$

completes the proof of the proposition 17. Due to the fact that the right hand side of this last expression is not defined when $s=0$ but the left hand side is and is equal to one, we need to point out that when $s=0$, the term $1/(2s-1)!!$ in the final formula should be replaced by 1. □

The proof of Theorem 7 will be done constructively through the following series of lemmas and propositions.

Let us first compute some coefficients $c_q^{n,m}$, more precisely $c_1^{3,1}$, $c_1^{2,2}$, $c_0^{2,0}$ and $c_0^{3,1}$

Lemma 23. Let $c_q^{n,m}$ be defined by (81). Then

$$c_1^{3,1} = 1+Q_1, \qquad c_1^{2,2} = -3e+Q_2, \qquad c_0^{2,0} = 1+Q_3, \qquad c_0^{3,1} = -\frac{5}{2}\,e+Q_4,$$

with

$$|Q_i| \leq 98e^2, \quad i = 1,2,3,4.$$

Proof. From its definition given in (81) and using the change of variable $t = E - e\sin E$ we have

$$c_1^{3,1} = \frac{1}{2\pi}\int_0^{2\pi}\left(r\mathrm{e}^{if(E)}\right)r^3\mathrm{e}^{-it}dE, \qquad c_1^{2,2} = \frac{1}{2\pi}\int_0^{2\pi}\left(r\mathrm{e}^{if(E)}\right)^2 r\mathrm{e}^{-it}dE,$$
$$c_0^{2,0} = \frac{1}{2\pi}\int_0^{2\pi} r^3 dE, \qquad c_0^{3,1} = \frac{1}{2\pi}\int_0^{2\pi}\left(r\mathrm{e}^{if(E)}\right)r^3 dE.$$

From equations (95) we have

$$c_1^{3,1} = \frac{1}{2\pi}\int_0^{2\pi}[a^2\mathrm{e}^{iE} - e + \frac{e^2}{4a^2}\mathrm{e}^{-iE}](1-e\cos E)^3\mathrm{e}^{-iE}\mathrm{e}^{ie\sin E}dE \tag{124}$$
$$c_1^{2,2} = \frac{1}{2\pi}\int_0^{2\pi}[a^2\mathrm{e}^{iE} - e + \frac{e^2}{4a^2}\mathrm{e}^{-iE}]^2(1-e\cos E)\mathrm{e}^{-iE}\mathrm{e}^{ie\sin E}dE \tag{125}$$
$$c_0^{2,0} = \frac{1}{2\pi}\int_0^{2\pi}(1-e\cos E)^3 dE \tag{126}$$
$$c_0^{3,1} = \frac{1}{2\pi}\int_0^{2\pi}[a^2\mathrm{e}^{iE} - e + \frac{e^2}{4a^2}\mathrm{e}^{-iE}](1-e\cos E)^3 dE\,. \tag{127}$$

In what follows we will use (see (95b)) that

$$0 \le e \le 1, \qquad \frac{1}{2} \le a^2 \le 1, \qquad a^2 + \frac{e^2}{4a^2} = 1. \tag{128}$$

To bound $c_1^{3,1}$ we use equation (124). It is easy to see that

$$a^2\mathrm{e}^{iE} - e + \frac{e^2}{4a^2}\mathrm{e}^{-iE} = \mathrm{e}^{iE} - e + \bar{E}_1, \tag{129a}$$
$$(1-e\cos E)^3 = 1 - 3e\cos E + \bar{E}_2, \quad \mathrm{e}^{ie\sin E} = 1 + ie\sin E + \bar{E}_3, \tag{129b}$$

where

$$\bar{E}_1 = (a^2-1)\mathrm{e}^{iE} + \frac{e^2}{4a^2}\mathrm{e}^{-iE}, \qquad \bar{E}_2 = 3e^2\cos^2 E - e^3\cos^3 E, \qquad \bar{E}_3 = \frac{1}{2}\sum_{j=0}^{\infty} 2\frac{(ie\sin E)^{j+2}}{(j+2)!},$$

satisfy

$$\left|\bar{E}_1\right| \le \frac{e^2}{2} + \frac{e^2}{2} = e^2, \qquad \left|\bar{E}_2\right| \le 4e^2, \qquad \left|\bar{E}_3\right| \le \frac{e^2}{2}\mathrm{e}^e \le e^2\frac{\mathrm{e}}{2} \le 2e^2.$$

Using equations (129), we have from equation (124) that $c_1^{3,1}$ is the Fourier coefficient of order 1 of the function

$$(\mathrm{e}^{iE} - e + \bar{E}_1)(1 - 3e\cos E + \bar{E}_2)(1 + ie\sin E + \bar{E}_3) = \\ \mathrm{e}^{iE} - e - 3e\cos E\mathrm{e}^{iE} + ie\sin E\mathrm{e}^{iE} + \tilde{Q}_1(E)$$

where

$$\begin{aligned}\tilde{Q}_1(E) =& \bar{E}_1 - 3e^2\cos E - 3e\bar{E}_1\cos E + \bar{E}_2(\mathrm{e}^{iE} - e + \bar{E}_1) - ie^2\sin E - 3ie^2\cos E\sin E\mathrm{e}^{iE} \\ &- 3ie^3\cos E\sin E - 3ie^2\sin E\cos E\bar{E}_2 + ie\sin E\bar{E}_2(\mathrm{e}^{iE} - e + \bar{E}_1) \\ &+ \bar{E}_3(\mathrm{e}^{iE} - e + \bar{E}_1 - 3e\cos E\mathrm{e}^{iE} - 3e^2\cos E - 3e\bar{E}_1\cos E + \bar{E}_2(\mathrm{e}^{iE} - e + \bar{E}_1)),\end{aligned}$$

which implies that, up to order one in e, the Fourier coefficient $c_1^{3,1}$ is exactly 1. From the bounds for $\bar{E}_1$, $\bar{E}_2$ and $\bar{E}_3$ we find $|\tilde{Q}_1(E)| \le 98e^2$, which implies the lemma for $c_1^{3,1}$.

From equation (129), it is easy to see that

$$\left[a^2\mathrm{e}^{iE} - e + \frac{e^2}{4a^2}\mathrm{e}^{-iE}\right]^2 = \left[\mathrm{e}^{iE} - e + \bar{E}_1\right]^2 = \mathrm{e}^{2iE} - 2e\mathrm{e}^{iE} + \bar{E}_4$$

where

$$\bar{E}_4 = e^2 + 2\bar{E}_1(\mathrm{e}^{iE} - e) + \bar{E}_1^2$$

can be bounded, in regard of equations (128) and the bound for $\tilde{E}_1$, as

$$|\bar{E}_4| \leq e^2 + 2e^2(1+e) + e^4 \leq 6e^2.$$

Using equation (129), we see from equation (125) that $c_1^{2,2}$ is the Fourier coefficient of order 1 of the function

$$(\mathrm{e}^{2iE} - 2e\mathrm{e}^{iE} + \bar{E}_4)(1 - e\cos E)(1 + ie\sin E + \bar{E}_3) = \\ \mathrm{e}^{2iE} - e\cos E\mathrm{e}^{2iE} - 2e\mathrm{e}^{iE} + ie\sin E\mathrm{e}^{2iE} + \tilde{Q}_2(E)$$

where

$$\begin{aligned}\tilde{Q}_2(E) &= 2e^2\cos E\mathrm{e}^{iE} + \bar{E}_4 - e\bar{E}_4\cos E \\ &= ie\sin E(-e\cos E\mathrm{e}^{2iE} - 2e\mathrm{e}^{iE} + 2e^2\cos E\mathrm{e}^{iE} + \bar{E}_4 - e\bar{E}_4\cos E) \\ &= \bar{E}_3(\mathrm{e}^{2iE} - e\cos E\mathrm{e}^{2iE} - 2e\mathrm{e}^{iE} + 2e^2\cos E\mathrm{e}^{iE} + \bar{E}_4 - e\bar{E}_4\cos E).\end{aligned}$$

From the above expressions we conclude that, up to order one in e, the Fourier coefficient $c_1^{2,2}$ is exactly $-3e$, and from the bounds for $\bar{E}_4$ and $\bar{E}_3$ we find that $|\tilde{Q}_2(E)| \leq 50e^2$ which implies the lemma for $c_1^{2,2}$.

We compute $c_0^{2,0}$ using equation (126), as well as equation (129b) to get

$$c_0^{2,0} = \frac{1}{2\pi}\int_0^{2\pi}(1 - 3e\cos E + \bar{E}_2)dE = 1 + Q_3$$

with

$$Q_3 = \frac{1}{2\pi}\int_0^{2\pi}\bar{E}_2 dE$$

and we have immediately, using the bound for $\bar{E}_2$, that $|Q_3| \leq 4e^2$, the desired result for $c_0^{2,0}$.

Finally, we compute $c_0^{3,1}$ using equation (127), as well as equations (129b)

$$c_0^{3,1} = \frac{1}{2\pi}\int_0^{2\pi}(\mathrm{e}^{iE} - e + \bar{E}_1)(1 - 3e\cos E + \bar{E}_2)dE.$$

Now, we want to find, up to order e the Fourier coefficient of order zero of the function

$$(\mathrm{e}^{iE} - e + \bar{E}_1)(1 - 3e\cos E + \bar{E}_2) = \mathrm{e}^{iE} - 3e\mathrm{e}^{iE}\cos E - e + \bar{E}_5,$$

where

$$\bar{E}_5 = \bar{E}_2\mathrm{e}^{iE} + 3e^2\cos E - e\bar{E}_2 + \bar{E}_1 - 3e\bar{E}_1\cos E + \bar{E}_2\bar{E}_1,$$

from where we find

$$c_0^{3,1} = -\frac{5}{2}e + Q_4$$

with

$$Q_4 = \frac{1}{2\pi}\int_0^{2\pi}\bar{E}_5 dE,$$

and using the bounds for $\bar{E}_2$ and $\bar{E}_1$, we find $|Q_4| \leq 19e^2$. □

Lemma 24. Let $G \leq 32$. If $q, k \in \mathbb{N}$, $q \geq 1$, $k \geq 2$, then the Fourier coefficients of the Melnikov potential (52) verify the following bounds:

$$
\begin{aligned}
|L_{q,\ 0}| &\leq 2^9 \left(2\mathrm{e}^2\right)^q e^q\, G^{-3/2}\, \mathrm{e}^{-qG^3/3} \\
|L_{q,\ 1}| &\leq 2^7 \mathrm{e}^q (1+e)^4\, G^{-7/2} \mathrm{e}^{-qG^3/3} \\
|L_{q,-1}| &\leq 2^9 \left(2\mathrm{e}^2\right)^q\ e^{|1-q|}\, G^{-1/2} \mathrm{e}^{-qG^3/3} \\
|L_{q,\ k}| &\leq 2^5 2^k \mathrm{e}^q (1+e)^k\, G^{-2k-1/2} \mathrm{e}^{-qG^3/3} \\
|L_{q,-k}| &\leq 2^5 2^{2k} \left(2\mathrm{e}^2\right)^q e^{|k-q|} G^{k-1/2} \mathrm{e}^{-qG^3/3}
\end{aligned}
$$

Proof. From equations (84) by propositions 12 and 16 we have

$$
\begin{aligned}
|L_{q,\ 0}| &\leq 2^4 \mathrm{e}^q \mathrm{e}^{-qG^3/3} (2e\mathrm{e}^{\sqrt{1-e^2}})^q G^{-1/2} \sum_{l\geq 1} (2^4 G^{-1})^l \\
|L_{q,\ 1}| &\leq 2^2 \mathrm{e}^q \mathrm{e}^{-qG^3/3} G^{-3/2} \sum_{l\geq 2} ((1+e)^2 2^2 G^{-1})^l \\
|L_{q,-1}| &\leq \mathrm{e}^q \mathrm{e}^{-qG^3/3} \mathrm{e}^{q\sqrt{1-e^2}} 2^q e^{|1-q|} G^{3/2} \sum_{l\geq 2} (2^4 G^{-1})^l \\
|L_{q,\ k}| &\leq 2^4 2^{-k} \mathrm{e}^q \mathrm{e}^{-qG^3/3} (1+e)^{-k} G^{-k-1/2} \sum_{l\geq k} ((1+e)^2 2^2 G^{-1})^l \\
|L_{q,-k}| &\leq 2^4 2^{-2k} \mathrm{e}^q \mathrm{e}^{-qG^3/3} \mathrm{e}^{q\sqrt{1-e^2}} 2^q e^{|k-q|} G^{2k-1/2} \sum_{l\geq k} (2^4 G^{-1})^l
\end{aligned}
$$

since by hypothesis $2^4/G \leq 1/2$. Since all these series converge we have proven the lemma using that $0 \leq e \leq 1$. □

Lemma 25. If $q \in \mathbb{N}$, $q \geq 2$. Assume $G \geq 32$, $eG \leq 1/8$, Then for $q \geq 2$

$$
|L_q| \leq \sum_{k\in\mathbb{Z}} |L_{q,k}| \leq 2^{13} \mathrm{e}^{-qG^3/3} (\mathrm{e}^2 2^3 G)^q G^{-1/2} \tag{131}
$$

Proof. From lemma (24) we have:

$$
\begin{aligned}
\sum_{k\in\mathbb{Z}} |L_{q,k}| &\leq |L_{q,0}| + |L_{q,1}| + |L_{q,-1}| + \sum_{k\geq 2} \big(|L_{q,k}| + |L_{q,-k}|\big) \\
&\leq \mathrm{e}^{-qG^3/3} \Big[2^9 2^q e^q \mathrm{e}^{2q} G^{-3/2} + 2^7 (1+e)^4 \mathrm{e}^q G^{-7/2} + 2^9 2^q e^{q-1} \mathrm{e}^{2q} G^{-1/2} \\
&\quad + 2^5 \sum_{k\geq 2} \Big(2^k (1+e)^k \mathrm{e}^q G^{-2k-1/2} + 2^{2k+q} \mathrm{e}^{2q} e^{|k-q|} G^{k-1/2} \Big) \Big] \\
&\leq \mathrm{e}^{-qG^3/3} \Big[2^{10} 2^q e^{q-1} \mathrm{e}^{2q} G^{-1/2} + 2^7 (1+e)^4 \mathrm{e}^q G^{-7/2} + 2^5 \mathrm{e}^q G^{-1/2} \sum_{k=2}^{\infty} (2(1+e)G^{-2})^k \\
&\quad + 2^5 \mathrm{e}^{2q} G^{-1/2} 2^q e^q \sum_{k=2}^{q-1} (4Ge^{-1})^k + 2^5 \mathrm{e}^{2q} G^{-1/2} e^{-q} 2^q \sum_{k=q}^{\infty} (4eG)^k \Big].
\end{aligned}
$$

Using that $eG \leq 1/8$

$$\sum_{k\in\mathbb{Z}} |L_{q,k}| \leq \mathrm{e}^{-qG^3/3}\Bigg[2^{10}2^q e^{q-1}\mathrm{e}^{2q}G^{-1/2} + 2^7 G^{-7/2} + 2^{10}G^{-9/2}$$

$$+ 2^4 2^{3q}\mathrm{e}^{2q}G^{q-3/2}e + 2^6\mathrm{e}^{2q}G^{q-1/2}2^{3q}\Bigg]$$

$$\leq 2^{10}\mathrm{e}^{-qG^3/3}\mathrm{e}^{2q}2^{3q}G^{q-1/2}\Bigg[2^{-2q}e^{q-1}G^{-q} + 2^{-3q}\mathrm{e}^{-2q}G^{-3-q}$$

$$+ 2^{-3q}\mathrm{e}^{-2q}G^{-4-q} + G^{-1}e + 1\Bigg] \leq 2^{13}\mathrm{e}^{-qG^3/3}(\mathrm{e}^2 2^3 G)^q G^{-1/2}$$

which concludes the proof. □

The Melnikov potential $\mathcal{L}$ (51) has a Fourier Cosine series (52) which can be split with respect to the variable s as $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \cdots$, like in (54-55), as well as a complex Fourier series (83) $\mathcal{L} = \sum_{q\in\mathbb{Z}} L_q \mathrm{e}^{iqs}$. Both series are related through $\mathcal{L}_0 = L_0$ and $\mathcal{L}_q = 2\Re\left\{\mathrm{e}^{iqs}L_q\right\}$ for $q \geq 1$. In the next lemma we see rather easily that the terms $\mathcal{L}_{\geq 2} = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \cdots$ of second order with respect to s satisfy a very exponentially small bound in G.

Lemma 26. For $G \geq 32$, $eG \leq 1/8$, $\mathcal{L}_{\geq 2}(\alpha, G, s; e) = 2\sum_{q\geq 2}\sum_{k\in\mathbb{Z}} L_{q,k}\cos(qs + k\alpha)$ is bounded as

$$|\mathcal{L}_{\geq 2}(\alpha, G, s; e)| \leq 2^{28}G^{3/2}\mathrm{e}^{-2G^3/3}.$$

Proof. By lemma 25

$$|\mathcal{L}_{\geq 2}| \leq 2^{13}G^{-1/2}\sum_{q\geq 2}\left[\mathrm{e}^{-G^3/3}\mathrm{e}^2 2^3 G\right]^q \leq 2^{20}\mathrm{e}^4 G^{3/2}\mathrm{e}^{-2G^3/3}$$

where the last bound holds if

$$\mathrm{e}^{-2G^3/3}\mathrm{e}^2 2^3 G \leq 1/2 \tag{132}$$

which is true for every $G \geq 32$. Now, using that $\mathrm{e} < 4$ we get the result. □

The next step provides an asymptotic formula for $\mathcal{L}_1 = 2\Re\left\{\mathrm{e}^{is}L_1\right\}$.

Lemma 27. For $G \geq 32$ and $eG \leq 1/8$ we have the following formula for L_1 (83)

$$\Re\left\{\mathrm{e}^{is}L_1\right\} = \Re\left\{\mathrm{e}^{is}\left(\left(c_1^{3,1}N(1,2,1) + E_3\right)\mathrm{e}^{-i\alpha} + \left(c_1^{2,2}N(1,2,0) + E_{1-,2}\right)\mathrm{e}^{-2i\alpha} + E_1\right)\right\} \tag{133}$$

where

$$|E_1(\alpha, G; e)| \leq 2^{18}\mathrm{e}^{-G^3/3}\left[eG^{-3/2} + G^{-7/2} + e^2G^{5/2}\right]$$

$$|E_3(\alpha, G; e)| \leq 2^{20}\mathrm{e}^{-G^3/3}G^{-3/2}$$

$$|E_4(\alpha, G; e)| \leq 2^{18}\mathrm{e}^{-G^3/3}e\,G^{1/2}.$$

Proof. From equation (83), we have that

$$L_1 = L_{1,0} + \sum_{k\geq 1}\left(L_{1,k}\mathrm{e}^{ik\alpha} + L_{1,-k}\mathrm{e}^{-ik\alpha}\right)$$

$$= L_{1,-1}\mathrm{e}^{-i\alpha} + L_{1,-2}\mathrm{e}^{-2i\alpha} + \sum_{k\geq 0} L_{1,k}\mathrm{e}^{ik\alpha} + \sum_{k\geq 3} L_{1,-k}\mathrm{e}^{-ik\alpha}$$

Now, setting

$$E_1 = \sum_{k\geq 0} L_{1,k}\mathrm{e}^{ik\alpha} + \sum_{k\geq 3} L_{1,-k}\mathrm{e}^{-ik\alpha} \tag{134}$$

we can write

$$\Re\Big\{L_1 e^{is}\Big\}= \Re\Big\{\big(L_{1,-1}e^{-i\alpha} + L_{1,-2}e^{-2i\alpha} + E_1\big)e^{is}\Big\}. \tag{135}$$

By definitions (84) we have

$$L_{1,-1} = c_1^{3,1} N(1,2,1) + \sum_{l\geq 3} c_1^{2l-1,1} N(1,l,l-1) \tag{136a}$$

$$L_{1,-2} = c_1^{2,2} N(1,2,0) + \sum_{l\geq 3} c_1^{2l-2,2} N(1,l,l-2) \tag{136b}$$

If we set

$$E_3 = \sum_{l\geq 3} c_1^{2l-1,1} N(1,l,l-1) \tag{137a}$$

$$E_4 = \sum_{l\geq 3} c_1^{2l-2,2} N(1,l,l-2) \tag{137b}$$

we obtain just (133) from equations (136) and (135). Once we have obtained the formula (133), it only remains to bound properly the errors E_1, E_3 and E_4. From equation (134), by the triangle inequality and lemma 24 we have

$$\begin{aligned}
|E_1| &\leq |L_{1,0}| + |L_{1,1}| + \sum_{k\geq 2} |L_{1,k}| + \sum_{k\geq 3} |L_{1,-k}| \\
&\leq e^{-G^3/3}\Bigg[2^{10} e e^2 G^{-3/2} + 2^7(1+e)^4 G^{-7/2} + 2^5 e \sum_{k\geq 2} 2^k (1+e)^k G^{-2k-1/2} \\
&+ 2^6 e^2 \sum_{k\geq 3} 2^{2k} e^{k-1} G^{k-1/2}\Bigg] \\
&\leq e^{-G^3/3}\Bigg[2^{10} e e^2 G^{-3/2} + 2^7 e(1+e)^4 G^{-7/2} + 2^8 e(1+e)^2 G^{-9/2} + 2^{13} e^2 e^2 G^{5/2}\Bigg] \\
&\leq 2^{17} e^{-G^3/3}\Bigg[e G^{-3/2} + G^{-7/2} + G^{-9/2} + e^2 G^{5/2}\Bigg] \\
&\leq 2^{18} e^{-G^3/3}\Bigg[e G^{-3/2} + G^{-7/2} + e^2 G^{5/2}\Bigg].
\end{aligned} \tag{138}$$

Now we proceed with E_3 and E_4. By propositions 12 and 16, from equations (137)

$$|E_3| \leq \sum_{l\geq 3} |c_1^{2l-1,1} N(1,l,l-1)| \leq 2^3 e^{\sqrt{1-e^2}}\, e\, e^{-G^3/3} G^{3/2} \sum_{l\geq 3} (2^4 G^{-1})^l \leq 2^{16} e^2 e^{-G^3/3} G^{-3/2},$$

$$|E_4| \leq \sum_{l\geq 3} |c_1^{2l-2,2} N(1,l,l-2)| \leq 2e e^{\sqrt{1-e^2}}\, e\, e^{-G^3/3} G^{7/2} \sum_{l\geq 3} (2^4 G^{-1})^l \leq 2^{14} e^2 e^{-G^3/3} e G^{1/2}.$$

The two estimates above, together with estimate (138) provide the desired bounds for the errors of equation (133). □

Putting together Lemmas 26 and 27 we already have

$$\mathcal{L} = L_0 + 2\Re\big\{[(c_1^{3,1} N(1,2,1) + E_3)e^{-i\alpha} + (c_1^{2,2} N(1,2,0) + E_4)e^{-2i\alpha} + E_1]e^{is}\big\} + \mathcal{L}_{\geq 2} \tag{139}$$

with

$$\begin{aligned}
|E_1(\alpha, G; e)| &\leq 2^{18}\left(G^{-7/2} + e^2 G^{5/2} + e G^{-3/2}\right) e^{-G^3/3} \\
|E_3(\alpha, G; e)| &\leq 2^{20} G^{-3/2}\, e^{-G^3/3} \\
|E_4(\alpha, G; e)| &\leq 2^{18} e\, G^{1/2}\, e^{-G^3/3} \\
|\mathcal{L}_{\geq 2}(\alpha, G, s; e)| &\leq 2^{28} G^{3/2}\, e^{-G^3 \frac{4}{9}}.
\end{aligned} \tag{140}$$

We now compute $N(1,2,1)$ and $N(1,2,0)$.

Lemma 28. Let $N(q,m,n)$ be defined by equations (85).Then

$$N(1,2,1) = \frac{1}{4}\sqrt{\frac{\pi}{2}}G^{-1/2}\mathrm{e}^{-G^3/3} + {}^1E_{TT}$$
$$N(1,2,0) = \sqrt{\frac{\pi}{2}}G^{3/2}\mathrm{e}^{-G^3/3} + {}^2E_{TT}$$

where

$$|{}^1E_{TT}| \le 2^6\,9\,G^{-2}\mathrm{e}^{-G^3/3}, \qquad |{}^2E_{TT}| \le 2^5\,9\,\mathrm{e}^{-G^3/3}.$$

Proof. From proposition 17 we have

$$N(1,2,1) = \frac{d_{2,1}}{G^5}\mathrm{e}^{-G^3/3}\Big[d_4^{2,1}\sqrt{\pi}\Big(\frac{2}{G}\Big)^{3/2} - 2^2 d_2^{2,1}\sqrt{\pi}\sqrt{\frac{G^3}{2}} + \frac{2^3}{3}d_0^{2,1}\sqrt{\pi}(\sqrt{\frac{G^3}{2}})^3 + T^1_{2,1} + R^1_{2,1}\Big] \tag{141}$$

where

$$|T^1_{2,1}| \le 45\,2^6 G^{-3} \qquad |R^1_{2,1}| \le 18\,G^3$$

and

$$N(1,2,0) = \frac{d_{2,0}}{G^3}\mathrm{e}^{-G^3/3}\Big[2d_4^{2,0}\sqrt{\pi}(\sqrt{\frac{G^3}{2}})^{-1} - 2^2 d_2^{2,0}\sqrt{\pi}\sqrt{\frac{G^3}{2}} + \frac{2^3}{3}d_0^{2,0}\sqrt{\pi}(\sqrt{\frac{G^3}{2}})^3 + T^1_{2,0} + R^1_{2,0}\Big] \tag{142}$$

where

$$|T^1_{2,0}| \le 45\,2^6 G^{-3} \qquad |R^1_{2,0}| \le 18\,G^3.$$

Taking the dominant terms in (141) and (142) we get:

$$N(1,2,1) = d_{2,1}d_0^{2,1}\frac{2\sqrt{2}}{3}\sqrt{\pi}G^{-1/2}\mathrm{e}^{-G^3/3} + {}^1E + {}^1E_{TR} \tag{143}$$

where

$${}^1E = 2^{\frac{3}{2}}d_{2,1}\sqrt{\pi}\big(d_4^{2,1}G^{-\frac{13}{2}} - d_2^{2,1}G^{-\frac{7}{2}}\big)\mathrm{e}^{-G^3/3} \qquad {}^1E_{TR} = (T^1_{2,1} + R^1_{2,1})d_{2,1}G^{-5}\mathrm{e}^{-G^3/3}$$

and

$$N(1,2,0) = d_{2,0}d_0^{2,0}\frac{2\sqrt{2}}{3}\sqrt{\pi}G^{3/2}\mathrm{e}^{-G^3/3} + {}^2E + {}^2E_{TR} \tag{144}$$

where

$${}^2E = 2^{\frac{3}{2}}d_{2,0}\sqrt{\pi}\big(d_4^{2,0}G^{-\frac{9}{2}} - d_2^{2,0}G^{-\frac{3}{2}}\big)\mathrm{e}^{-G^3/3} \qquad {}^2E_{TR} = (T^1_{2,0} + R^1_{2,0})d_{2,0}G^{-3}\mathrm{e}^{-G^3/3}.$$

Using the bounds given in Lemma 14 for $d_j^{m,n}$ and the bounds given in Lemma 13 for $d_{m,n}$:

$$\begin{aligned} |{}^1E| &\le 2^{\frac{3}{2}}|d_{2,1}|\sqrt{\pi}(|d_4^{1,2}| + |d_2^{2,1}|)G^{-\frac{7}{2}}\mathrm{e}^{-G^3/3} \le 2^7\,9\,G^{-\frac{7}{2}}\mathrm{e}^{-G^3/3} \\ |{}^1E_{TR}| &\le |d_{2,1}|\,36\,G^{-2}\mathrm{e}^{-G^3/3} \le 2^5 9\,G^{-2}\mathrm{e}^{-G^3/3} \end{aligned}$$

and also:

$$\begin{aligned} |{}^2E| &\le 2^{\frac{3}{2}}|d_{2,0}|\sqrt{\pi}(|d_4^{2,0}| + |d_2^{2,0}|)G^{-\frac{3}{2}}\mathrm{e}^{-G^3/3} \le 2^6 9\,G^{-\frac{3}{2}}\mathrm{e}^{-G^3/3} \\ |{}^2E_{TR}| &\le |d_{2,0}|\,36\mathrm{e}^{-G^3/3} \le 2^4 9\,\mathrm{e}^{-G^3/3}. \end{aligned}$$

Using Lemma 14, $d_0^{m,n} = 1/(2i)^{2n+1}$ and by definition (106) for $d_{m,n}$ we have that

$$d_{2,1}d_0^{2,1} = -i2^3\binom{-1/2}{2}\binom{-1/2}{1}\Big(\frac{i}{2^3}\Big) = -\frac{3}{2^4}$$
$$d_{2,0}d_0^{2,0} = i2^2\binom{-1/2}{2}\Big(-\frac{i}{2}\Big) = \frac{3}{2^2}.$$

We can then write equation (143) as

$$N(1,2,1) = \frac{1}{4}\sqrt{\frac{\pi}{2}}G^{-1/2}\mathrm{e}^{-G^3/3} + {}^1E_{TT} \tag{145}$$

where

$${}^1E_{TT} = {}^1E + {}^1E_{TR},$$

satisfies

$$|{}^1E_{TT}| \leq 2^7\, 9\, G^{-\frac{7}{2}}\mathrm{e}^{-G^3/3} + 2^5 9\, G^{-2}\mathrm{e}^{-G^3/3} \leq 2^6 9\, G^{-2}\mathrm{e}^{-G^3/3}.$$

In an analogous way, equation (144) can be written as

$$N(1,2,0) = \sqrt{\frac{\pi}{2}}G^{3/2}\mathrm{e}^{-G^3/3} + {}^2E_{TT} \tag{146}$$

where

$${}^2E_{TT} = {}^2E + {}^2E_{TR}$$

satisfies

$$|{}^2E_{TT}| \leq 2^6 9\, G^{-\frac{3}{2}}\mathrm{e}^{-G^3/3} + 2^4 9\,\mathrm{e}^{-G^3/3} \leq 2^5 9\,\mathrm{e}^{-G^3/3}$$

and this proves the lemma. □

Using the approximations given in Lemma 28 we have from lemmas 26 and 27

Lemma 29. For $G \geq 32$ and $eG \leq 1/8$, the Melnikov potential $\mathcal{L}$ (83) is given by

$$\mathcal{L} = L_0 + \cos(s-\alpha)\left(c_1^{3,1}\sqrt{\frac{\pi}{8}}G^{-1/2}\mathrm{e}^{-G^3/3} + E_3 + E_5\right)$$
$$+ \cos(s-2\alpha)\left(c_1^{2,2}\sqrt{2\pi}G^{3/2}\mathrm{e}^{-G^3/3} + E_4 + E_6\right) + 2\Re\{E_1\mathrm{e}^{is}\} + \mathcal{L}_{\geq 2}$$

where $\mathcal{L}_{\geq 2}$ and E_k with $k = 1, 3, 4$ are given in equations (140) and

$$|E_5| \leq 2^{13}\, 9\, G^{-2}\mathrm{e}^{-G^3/3}, \qquad |E_6| \leq 2^{11}\, 9\, eK\mathrm{e}^{-G^3/3}.$$

Proof. By lemma 28 we have that $N(1,2,1)$ and $N(1,2,0)$ are real and then coincide with their real part. Equation (139) gives the correct estimation of $\mathcal{L}$. To complete the proof is enough to take

$$E_5 = c_1^{3,1} \cdot {}^1E_{TT} \qquad \text{and} \qquad E_6 = c_1^{2,2} \cdot {}^2E_{TT}$$

where ${}^1E_{TT}$ and ${}^2E_{TT}$ are given in lemma 28. Therefore by proposition 12 we find directly the bounds of E_5 and E_6.

□

Lemma 30. For $G \geq 32$ and $eG \leq 1/8$, the Melnikov potential $\mathcal{L}$ (83) is given by

$$\mathcal{L} = L_0 + \cos(s-\alpha)\left(\sqrt{\frac{\pi}{8}}G^{-1/2}\mathrm{e}^{-G^3/3} + E_3 + E_5 + E_7\right)$$
$$- \cos(s-2\alpha)\left(3\sqrt{2\pi}eG^{3/2}\mathrm{e}^{-G^3/3} + E_4 + E_6 + E_8\right) + 2\Re\{E_1\mathrm{e}^{is}\} + \mathcal{L}_{\geq 2}$$

where $\mathcal{L}_{\geq 2}$ and E_k with $k = 1, 3, \ldots 6$ are given in equations (140) and

$$|E_7| \leq 98e^2G^{-1/2}\mathrm{e}^{-G^3/3} \qquad |E_8| \leq 982^2e^2G^{3/2}\mathrm{e}^{-G^3/3}$$

Proof. From lemma 23 we have

$$c_1^{3,1}\sqrt{\frac{\pi}{8}}G^{-1/2}\mathrm{e}^{-G^3/3} = \sqrt{\frac{\pi}{8}}G^{-1/2}\mathrm{e}^{-G^3/3} + E_7$$
$$c_1^{2,2}\sqrt{2\pi}G^{3/2}\mathrm{e}^{-G^3/3} = -3\sqrt{2\pi}eG^{3/2}\mathrm{e}^{-G^3/3} + E_8$$

with

$$E_7 = Q_1\sqrt{\frac{\pi}{8}}G^{-1/2}\mathrm{e}^{-G^3/3}$$
$$E_8 = Q_2\sqrt{2\pi}G^{3/2}\mathrm{e}^{-G^3/3}$$

Therefore by lemma (29) and the bounds of Q_1 and Q_2 given in lemma 23 we conclude the proof. □

It only remains to estimate the Fourier coefficient $L_0 = \mathcal{L}_0$ defined in (55) or (83).

Lemma 31. Let $N(q, m, n)$ be defined by equations (85). Then for $m, n \in \mathbb{N}$, $m + n > 0$,

$$|N(0, m, n)| \leq 2^{m+n+2}G^{-2m-2n+1}.$$

Proof. Since $\tau \in \mathbb{R}$ in the integral (85), it is easy to see that

$$\frac{1}{|\tau + i|}, \frac{1}{|\tau - i|} \leq 1$$

and then

$$\frac{1}{|\tau + i|^{2n}}\frac{1}{|\tau - i|^{2m}} \leq \frac{1}{1 + \tau^2}\ .$$

For $n, m > 0$, using equation (85) and lemma 13 to bound $d_{m,n}$, the lemma follows:

$$|N(0, m, n)| \leq 2^{m+n}G^{-2m-2n+1}\mathrm{e}^{-1/2}\int_{-\infty}^{\infty}\frac{d\tau}{1 + \tau^2}$$
$$= 2^{m+n}G^{-2m-2n+1}\mathrm{e}^{-1/2}\pi \leq 2^{m+n+2}G^{-2m-2n+1}.$$

□

Lemma 32. Let $k \in \mathbb{N}$ and $L_{0,k}$ defined by equation (55). Then

$$L_{0,k} = \sum_{l \geq k+1} c_0^{2l-k,-k}N(0, l - k, l)$$

Proof. From equations (84), we have just to prove that for $k \geq 2$

$$N(0, 0, k) = N(0, k, 0) = 0.$$

By equations (85) this reduces to show that

$$\int_{-\infty}^{\infty}\frac{d\tau}{(\tau \pm i)^{2k}} = 0$$

where the positive sign in the denominator correspond to $I(0, 0, k)$ and the negative to $I(0, k, 0)$. Since the variable $\tau \in \mathbb{R}$ this integral is trivial

$$\int_{-\infty}^{\infty}\frac{d\tau}{(\tau \pm i)^{2k}} = -\frac{1}{2k - 1}\frac{1}{(\tau \pm i)^{2k-1}}\Bigg|_{-\infty}^{\infty} = 0$$

this proves the lemma. □

Lemma 33. Let $L_{0,k}$ be defined by equations (84) for $k \geq 0$. If $G \geq 32$,

$$|L_{0,k}| \leq 2^{2k+8} e^k G^{-2k-3}.$$

Proof. From lemma 32 we have

$$|L_{0,k}| \leq \sum_{l \geq k+1} |c_0^{2l-k,-k}||N(0, l-k, l)|,$$

and by propositions 12 and 16,

$$|L_{0,\pm k}| \leq 2^{-2k+3} e^k G^{2k+1} \sum_{l \geq k+1} (2^4 G^{-4})^l \leq e^k 2^{2k+8} G^{-2k-3}.$$

□

Lemma 34. Let $L_0 = \mathcal{L}_0$ be defined by equations (55) or (83). Then for $G \geq 32$

$$\begin{aligned} L_0 &= L_{0,0} + (c_0^{3,1} \frac{3}{4} \pi G^{-5} + F_2) \cos(\alpha) + F_3 \\ L_{0,0} &= c_0^{2,0} \frac{\pi}{2} G^{-3} + F_1 \end{aligned}$$

where

$$|F_1| \leq 2^{12} G^{-7}, \qquad |F_2| \leq 2^{15} e G^{-9}, \qquad |F_3| \leq 2^{14} e^2 G^{-7}.$$

Proof. From proposition 11 we know that

$$L_0 = L_{0,0} + 2 \sum_{k \geq 1} L_{0,k} \cos k\alpha,$$

and from lemma 32 we have that

$$L_{0,0} = c_0^{2,0} N(0,1,1) + \sum_{l \geq 2} c_0^{2l,0} N(0,l,l) \tag{147a}$$

$$L_{0,1} = c_0^{3,-1} N(0,1,2) + \sum_{l \geq 3} c_0^{2l-1,-1} N(0,l-1,l) \tag{147b}$$

$$L_{0,k} = \sum_{l \geq k+1} c_0^{2l-k,-k} N(0,l-k,l) \qquad \text{for } k \geq 2. \tag{147c}$$

Introducing

$$F_1 = \sum_{l \geq 2} c_0^{2l,0} N(0,l,l), \quad F_2 = 2 \sum_{l \geq 3} c_0^{2l-1,-1} N(0,l-1,l), \quad F_3 = 2 \sum_{k \geq 2} \cos k\alpha L_{0,k},$$

by lemmas 31, 33 and proposition 12, using the hypothesis on G we have

$$\begin{aligned} |F_1| &\leq 2^3 G \sum_{l \geq 2} (2^4 G^{-4})^l \leq 2^{12} G^{-7} \\ |F_2| &\leq 2^2 e G^3 \sum_{l \geq 3} (2^4 G^{-4})^l \leq 2^{15} e G^{-9} \\ |F_3| &\leq 2 \sum_{k \geq 2} |L_{0,k}| \leq 2^{14} e^2 G^{-7}. \end{aligned}$$

Now, from definition (85) we have that

$$\begin{aligned} N(0,1,1) = &\ \frac{2^2}{G^3} \binom{-1/2}{1} \binom{-1/2}{1} \int_{-\infty}^{\infty} \frac{d\tau}{(\tau^2+1)^2} = 2^2 \Big(-\frac{1}{2}\Big)\Big(-\frac{1}{2}\Big) G^{-3} = \frac{\pi}{2} G^{-3}, \\ N(0,1,2) = &\ \frac{2^3}{G^5} \binom{-1/2}{1} \binom{-1/2}{2} \int_{-\infty}^{\infty} \frac{d\tau}{(\tau - i)(\tau + i)^2} = 2^3 \Big(-\frac{1}{2}\Big)\Big(\frac{3}{2^3}\Big)\Big(-\frac{\pi}{4}\Big) G^{-5} = \frac{3}{8} \pi G^{-5}. \end{aligned}$$

From these equations, substituting equations (147) in the definition of L_0 and the bounds given in equations (148) we have proven this lemma. □

A refinement of this lemma is

Lemma 35. Let $L_0 = \mathcal{L}_0$ be defined by equations (55) or (83). Then if $G \geq 2^{3/2}$

$$L_0 = L_{0,0} + (-\frac{15}{8}\pi e G^{-5} + F_2 + F_5)\cos(\alpha) + F_3$$
$$L_{0,0} = \frac{\pi}{2}G^{-3} + F_1 + F_4$$

where F_1, F_2 and F_3 are given in lemma 34 and

$$|F_4| \leq 2\,98\,G^{-3}e^2, \qquad |F_5| \leq 2^2\,98\,G^{-5}e^2.$$

Proof. In lemma 23 we have computed the constants $c_0^{2,0}$ and $c_0^{3,1}$, then by setting

$$F_4 = \frac{\pi}{2}Q_3 G^{-3}, \qquad F_5 = \frac{3}{4}\pi Q_4 G^{-5}\cos\alpha,$$

and using the bounds for Q_3 and Q_4 we find the desired bound for F_4 and F_5. □

With this lemma we can rewrite lemma 30 exactly as Theorem 7, and so we have proved it.

Acknowledgments

The authors are indebted to Marcel Guàrdia, Pau Martín, Regina Martínez and Carles Simó for helpful discussions.

References

[Arn64] V. I. Arnold. Instability of dynamical systems with several degrees of freedom. *Soviet Mathematics*, 5(5):581–585, 1964.

[Bol06] S. Bolotin. Symbolic dynamics of almost collision orbits and skew products of symplectic maps. *Nonlinearity*, 19(9):2041–2063, 2006.

[CZ11] M. J. Capiński and P. Zgliczyński. Transition tori in the planar restricted elliptic three-body problem. *Nonlinearity*, 24(5):1395–1432, 2011.

[DG00] A. Delshams and P. Gutiérrez. Splitting potential and the Poincaré-Melnikov method for whiskered tori in Hamiltonian systems. *Journal of Nonlinear Science*, 10:433–476, 2000.

[DGR13] A. Delshams, M. Gidea and P. Roldán. Transition map and shadowing lemma for normally hyperbolic invariant manifolds. *Discrete Contin. Dyn. Syst.*, 33(3):1089–1112, 2013.

[DLS06] A. Delshams, R. de la Llave and T. M. Seara. A geometric mechanism for diffusion in Hamiltonian systems overcoming the large gap problem: heuristics and rigorous verification on a model. *Mem. Amer. Math. Soc.*, 179(844):viii+141, 2006.

[DLS08] —. Geometric properties of the scattering map of a normally hyperbolic invariant manifold. *Adv. Math.*, 217(3):1096–1153, 2008.

[Erd56] A. Erdélyi. *Asymptotic expansions.* Dover Publications, Inc., New York, 1956.

[FGKR14] J. Fejoz, M. Guardia, V. Kaloshin and P. Roldán. Kirkwood gaps and diffusion along mean motion resonances in the restricted planar three-body problem. *To Appear in J. Eur. Math. Soc.*, 2014.

[GL06] M. Gidea and R. de la Llave. Topological methods in the instability problem of Hamiltonian systems. *Discrete Contin. Dyn. Syst.*, 14(2):295–328, 2006.

[GLS14] M. Gidea, R. d. l. Llave and T. M. Seara. A general mechanism of diffusion in hamiltonian systems: Qualitative results. *preprint, arXiv:1405.0866*, 2014.

[GMS12] M. Guardia, P. Martíin and T. M. Seara. Oscillatory motions for the restricted planar circular three-body problem. *preprint, arXiv:1207.6531*, 2012.

[GMS15] —. A $\mathcal{C}^0$ lambda lemma and oscillatory orbits for the restricted elliptic planar three body problem. *preprint*, 2015.

[LMS85] J. Llibre, R. Martínez and C. Simó. Tranversality of the invariant manifolds associated to the Lyapunov family of periodic orbits near L_2 in the restricted three-body problem. *J. Differential Equations*, 58(1):104–156, 1985.

[LS80] J. Llibre and C. Simó. Oscillatory solutions in the planar restricted three-body problem. *Math. Ann.*, 248(2):153–184, 1980.

[McG73] R. McGehee. A stable manifold theorem for degenerate fixed points with applications to celestial mechanics. *J. Differential Equations*, 14:70–88, 1973.

[Moe02] R. Moeckel. Generic drift on Cantor sets of annuli. In *Celestial mechanics (Evanston, IL, 1999)*, volume 292 of *Contemp. Math.*, pages 163–171. Amer. Math. Soc., Providence, RI, 2002.

[Moe07] —. Symbolic dynamics in the planar three-body problem. *Regul. Chaotic Dyn.*, 12(5):449–475, 2007.

[MP94] R. Martínez and C. Pinyol. Parabolic orbits in the elliptic restricted three body problem. *J. Differential Equations*, 111(2):299–339, 1994.

[MS14] R. Martínez and C. Simó. Invariant manifolds at infinity of the RTBP and the boundaries of bounded motion. *Regul. Chaotic Dyn.*, 19(6):745–765, 2014.

[Rob84] C. Robinson. Homoclinic orbits and oscillation for the planar three-body problem. *J. Differential Equations*, 52(3):356–377, 1984.

[Rob88] —. Horseshoes for autonomous Hamiltonian systems using the Mel′nikov integral. *Ergodic Theory Dynam. Systems*, 8*(Charles Conley Memorial Issue):395–409, 1988.

[Win41] A. Wintner. *The Analytical Foundations of Celestial Mechanics*. Princeton Mathematical Series, v. 5. Princeton University Press, Princeton, N. J., 1941.

[Xia92] Z. Xia. Melnikov method and transversal homoclinic points in the restricted three-body problem. *J. Differential Equations*, 96(1):170–184, 1992.

[Xia93] —. Arnol′d diffusion in the elliptic restricted three-body problem. *J. Dynam. Differential Equations*, 5(2):219–240, 1993.