

THE METHOD OF SPREADING CUMULATIVE TWIST AND ITS APPLICATION TO THE RESTRICTED CIRCULAR PLANAR THREE BODY PROBLEM

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ABSTRACT. The purpose of this paper is twofold. First we show that the dynamics of a Sun-Jupiter-Comet system and under some simplifying assumptions has a semi-infinite region of instability. This is done by reducing the dynamics to the study of a certain exact area preserving (EAP) map and showing applicability of Aubry–Mather theory. Second, we give a sufficient “cumulative twist” condition to verify that an EAP map is an EAP twist (EAPT) map in a certain coordinate system. We construct such a coordinate system by “spreading the cumulative twist” which arises from the long term dynamics of system.

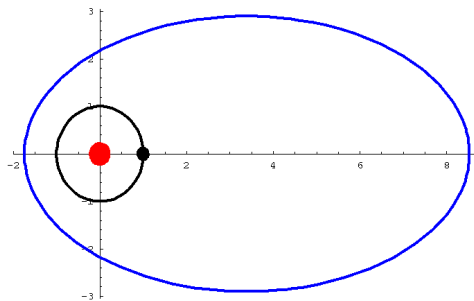
These results along with the prequel paper [GK1] show that the aforementioned EAPT map admits an application of Aubry–Mather theory and has no invariant curves in a certain semi-infinite region. This, in particular, leads to existence of orbits of the Comet with initial eccentricity 0.66 and “visible in the Solar system” (see figure1) which get ejected to infinity. Alternatively certain orbits of the Comet can come from infinity and be captured so that they approach orbits of eccentricity 0.66 in the future. More generally, Aubry–Mather theory implies that in the instability region above eccentricity 0.66 there are all possible Chazy instabilities (see Section 1.1 for details).

1. INTRODUCTION

Poincaré return maps of Hamiltonian systems of 1.5 or 2 degrees of freedom are exact area preserving (EAP) maps of a cylinder $\mathbb{A} = \mathbb{T} \times \mathbb{R}$. For exact area preserving *twist* (EAPT) maps there is Aubry–Mather theory which describes much of the dynamics. The EAP property is invariant under canonical coordinate changes. Twist however is not. For a concrete

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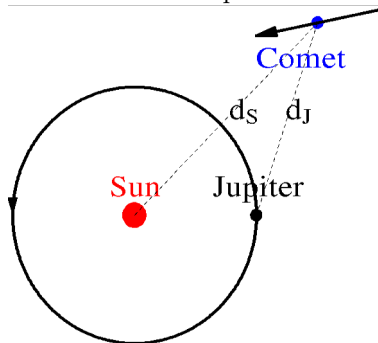
FIGURE 1. Ellipse of Eccentricity $e = 0.66$



Hamiltonian system it is often a problem to determine a coordinate system with the twist property. In this paper simple sufficient conditions are given to guarantee that an EAP map is actually an EAPT map. We use the method of *spreading cumulative twist* to prove this result. The main application is construction of instabilities for the restricted circular planar three body problem (RCP3BP). See Theorem 1.2 and Corollary 1.3.

Let us now consider the restricted circular planar three body problem (RCP3BP) with two massive primaries, which we call the Sun and Jupiter, that perform uniform circular motion about their center of mass (see fig. 2). The system is normalized to mass one so the Sun has mass $1 - \mu$ and Jupiter has mass μ . We further normalize so that Jupiter rotates with period 2π , and the distance from the Sun to Jupiter is constant and also normalized to one. Our goal is to understand the behavior of the massless comet whose position in polar coordinates is denoted (r, ψ) . It is convenient to consider the system in a rotating frame of reference which rotates with unit speed in the same direction as Jupiter. In this system, the Sun and Jupiter are fixed points on the x -axis corresponding to $\psi = 0$. We let $(r, \varphi) = (r, \psi - t)$ denote the motion of the comet in the rotating frame of reference.

FIGURE 2. The Sun-Jupiter-Comet system



The RCP3BP has a conserved quantity known as the *Jacobi constant*.

$$J(r, \varphi, \dot{r}, \dot{\varphi}) = \frac{r^2}{2} + \frac{\mu}{d_J} + \frac{1-\mu}{d_S} - \frac{\dot{r}^2 + r^2\dot{\varphi}^2}{2} =: U(r, \varphi) - \frac{\dot{r}^2 + r^2\dot{\varphi}^2}{2},$$

where d_J and d_S are distances to Jupiter and the Sun respectively.

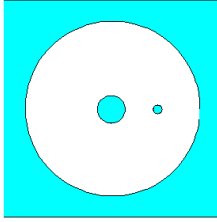
$$(1) \quad \begin{aligned} d_J(r, \varphi) &:= (r^2 - 2(1-\mu)r \cos(\varphi) + (1-\mu)^2)^{\frac{1}{2}} \\ d_S(r, \varphi) &:= (r^2 + 2\mu r \cos(\varphi) + \mu^2)^{\frac{1}{2}} \end{aligned}$$

Denote by

$$\mathcal{H}(J_0) := \{(r, \varphi) : U \geq J_0\}$$

a set of points in the plane of motion (configuration space). Points in this set are called *the Hill regions* associated to the Jacobi constant J_0 . These regions are the set of the admissible

FIGURE 3. Disjoint Hill Regions



locations of the comet in the (r, φ) plane (e.g. shaded regions in Figure 3).

Fixing the Jacobi constant restricts dynamics to an invariant energy surface, denoted

$$\mathcal{S}(J_0) := \{(r, \varphi, \dot{r}, \dot{\varphi}) : J(r, \varphi, \dot{r}, \dot{\varphi}) = J_0\}$$

Most of these surfaces are smooth 3-dimensional manifolds. Denote by $RCP3BP(\mu, J_0)$ the RCP3BP with Sun-Jupiter mass ratio μ and dynamics restricted to the surface $\mathcal{S}(J_0)$.

It turns out that for $\mu \leq 10^{-3}$ and $J_0 \geq 1.52$ the set $\mathcal{H}(J_0)$ consists of three disjoint connected components (see fig. 3): a region around the Sun called the *inner Hill region*, a region around Jupiter called the *lunar Hill region*, and non-compact region called the *outer Hill region*. The boundary of these regions can be found by considering the “zero velocity” curves which are on the boundary of the Hill regions [AKN]. In this paper we consider only orbits contained in the outer Hill region, denoted by $\mathcal{H}^{out}(J_0)$. For convenience, denote $\mathcal{S}^{out}(J_0) = \mathcal{H}^{out}(J_0) \cap \mathcal{S}(J_0)$ and when dynamics in $\mathcal{S}^{out}(J_0)$ is considered, we refer exclusively to the case when the outer Hill region is disjoint from the other two.

Lemma 1.1. *If an orbit $(r, \psi)(t)$ in $\mathcal{H}^{out}(J_0)$ makes more than one complete rotation about the origin, e.g. $|\psi(T) - \psi(0)| > 4\pi$ for some $T > 0$, then $\frac{J_0^2}{2} - 8\mu \leq r(t)$ for all $0 \leq t \leq T$.*

This lemma is proven in [GK1]. If an orbit makes less than one rotation, then one can show that it escapes to infinity and we are not interested in these orbits.

As the position of Jupiter is at radius $1 - \mu$, then this lemma implies that for $\mu \leq 10^{-3}$ and $J_0 \geq 1.52$ that if the comet is in an elliptic or parabolic orbit in the outer Hill region, then it remains bounded away from collisions with the Sun and Jupiter by a distance at least 14.5% of the Sun-Jupiter distance.

For small μ and away from collisions, the RCP3BP is nearly integrable and can be approximated with the Sun-Comet two body problem (2BP(SC)) corresponding to $\mu = 0$. Elliptic motions of a 2BP have two special points where the radial velocity \dot{r} of the comet is zero. The *perihelion* is the closest point to the Sun¹, denoted r^{perih} , and the *apohelion* is the farthest point from the Sun, denoted r^{apoh} . Define the *osculating (or instantaneous) eccentricity* $e(t)$

¹To be pedantic, the perihelion is technically defined to be a point in the orbit when $r \leq J_0^2$ and $\dot{r} = 0$. It is **not** necessarily the closest point to the Sun. Rather it is when the comet is at the closest point to the center of mass of the system. The Sun is within μ of the center of mass. It turns out that in our Solar

for the RCP3BP to be the eccentricity of the comet in the unperturbed 2BP(SC) system with initial conditions taken to be those of comet in the RCP3BP at time t .

Theorem 1.2. *Consider the RCP3BP(μ, J_0) with dynamics in $\mathcal{S}^{\text{out}}(J_0)$. There exists a function $e^* = e^*(\mu, J_0)$ and there exist trajectories of a comet with initial eccentricity $e_0 \geq e^*(\mu, J_0)$ that increase in eccentricity to beyond one in a manner so that the comet escapes the solar system to infinity. For example $e^*(10^{-3}, 1.8) \leq 0.66$.*

An informal way to rephrase this Theorem is as follows: We show that the RCP3BP(μ, J_0) admits application of Aubry–Mather theory and has no invariant 2-dimensional tori of eccentricity above $e^*(\mu, J_0)$. Thus, there is a semi-infinite region of instability. There is a nice set of conclusions from this theorem.

1.1. Chazy Motions. In 1922, Chazy gave a complete classification of the final motions of the spatial 3BP, i.e. a description of all possible states that a three body problem can approach as time goes to infinity. It turns out that there are seven types of final motions (see section 2.4 [AKN]). For the RCP3BP there are only four possible types of motion that the comet can exhibit. The four types of final motions can be defined as follows.

- H^\pm (*hyperbolic*): $r \rightarrow \infty, \dot{r} \rightarrow c > 0$ as $t \rightarrow \pm\infty$
- P^\pm (*parabolic*): $r \rightarrow \infty, \dot{r} \rightarrow 0$ as $t \rightarrow \pm\infty$
- B^\pm (*bounded*): $\limsup_{t \rightarrow \pm\infty} r = R_B < \infty$
- OS^\pm (*bounded*): $\limsup_{t \rightarrow \pm\infty} r = \infty, \liminf_{t \rightarrow \pm\infty} r = R_{OS} < \infty$

We say that the RCP3BP has a *full set of Chazy motions* if $H^- \cap H^+, H^\pm \cap P^\mp, H^\pm \cap B^\mp, H^\pm \cap OS^\mp, P^+ \cap P^-, P^\pm \cap B^\mp, P^\pm \cap OS^\mp, B^+ \cap B^-, B^\pm \cap OS^\mp, OS^+ \cap OS^-$ are all nonempty intersections, i.e. if any possible past and future of the comet’s motion can be realized by a trajectory in the RCP3BP. A corollary to our main theorem states that this is possible for comets in the outer Hill region on an energy surface with $J_0 \geq 1.52$. Additionally it possible to spend arbitrarily long amounts of time in between approach of the final motions.

Corollary 1.3. *There is a full set of Chazy instabilities in the region $e \geq e^*(\mu, J_0)$*

This paper is the second in the sequence of three papers on instabilities for the restricted circular planar three body problem. In the prequel [GK1], we proved Theorem 1.2 under the artificial constraint that $e_0 \leq e_{\text{max}}(\mu, J_0)$. For example $e_{\text{max}}(10^{-3}, 1.8) = 0.96$. In the present work this constraint is removed.

The primary tools for this result are Aubry–Mather theory and rigorous numerical integration. It is not trivial to apply Aubry–Mather theory to the restricted circular planar three body problem since the typical usage requires regions of instability to be invariant domains and this does not hold in our case as there are orbits escaping to infinity. We stress that trajectories are **not** constructed by means of numerical integration. After a mathematical framework is developed, we derive a list of inequalities. To have an explicit value of e^* , we use a computer to verify the range of validity of the inequalities, which are of two types: *analytic* and *dynamic*. Analytic inequalities **do not** make use to integration of the equations of motion. Dynamical inequalities **do involve integration**, but only over short periods of

System, the radius of the Sun is approximately 0.00089 the Sun–Jupiter distance, so we allow this slight abuse in terminology for small μ [NASA].

time. We use software which can handle both types of inequalities in a mathematically rigorous way (see appendix 14).

Relying on Mather's variational method ([Ma1],[X]) we show that there is a full set of Chazy instabilities [AKN]. We would like to point out that existence of ejection orbits and Chazy instabilities for RCP3BP was established by Llibre and Simo [LS] and later by [X2]. We estimate their $e^*(0.001, 1.8) \approx 0.995$, however their motions belong to a horseshoe, while ours have a fairly different nature: our orbits are *local action minimizers and shadow closely a collection of Aubry-Mather sets*. The idea of constructing Chazy instabilities originated in the famous paper by Sitnikov [Si] (see also [Mo2] for conceptual and transparent exposition of Sitnikov's work). Alekseyev constructed oscillatory motions for the full spatial three body problem [Al].

2. PLAN OF THE PROOF

Recall that motions of the comet in rotating polar coordinates (r, φ) can be viewed as the solutions to Hamilton's equations with a Hamiltonian of the form

$$(2) \quad H_{Polar} = H_{2BP(SC)} + \Delta H(r, \varphi) := \frac{P_r^2}{2} + \frac{P_\varphi^2}{2r^2} - P_\varphi - \frac{1}{r} + \Delta H(r, \varphi; \mu),$$

where P_r and P_φ are the momenta variables conjugate to r and φ respectively (see e.g. [AKN]) and ΔH is the μ -small perturbation of the associated Sun-Comet two body problem (2BP(SC)). This system arises by initially considering the planar 3BP where the comet has mass m , and letting $m \rightarrow 0$. With the notations in (1), ΔH can be written

$$\Delta H(r, \varphi; \mu) := \frac{1}{r} - \frac{\mu}{d_J} - \frac{1-\mu}{d_S} = \frac{\mu(\mu-1)(1+3\cos(2\varphi))}{4r^3} + O\left(\frac{\mu}{r^4}\right)$$

The proof starts with expressing equations of motion of RCP3BP in so called *Delaunay variables* (formally defined in section 3). These are action-angle variables of the $2BP(SC)$ or, equivalently, of $RCP3BP$ with $\mu = 0$, and have two angular variables ℓ and g in \mathbb{T} , and two action variables $0 \leq G \leq L$. There is a canonical transformation

$$\mathcal{D} : (\ell, L, g, G) \rightarrow (r, \varphi, P_r, P_\varphi)$$

which converts Delaunay coordinates into symplectic polar coordinates. The image consists of all bounded motions of the $2BP(SC)$. The map \mathcal{D} is described in section 3.

It turns out that there is a good 2-dimensional Poincaré section $\Gamma \subset \mathcal{S}^{out}(J_0) = \{H = -J_0\}$ of the dynamics of $RCP3BP(\mu, J_0)$ in the outer Hill region. In other words, a Poincaré map $\mathcal{F}_\mu : U \rightarrow \Gamma$ is well-defined on an open set $U \subset \Gamma$ homeomorphic to an annulus (see section 4, formula (5)). For $\mu = 0$ there are natural coordinates on $\Gamma \simeq \mathbb{T} \times \mathbb{R}_+ \ni (\ell, L)$ with $\ell \in \mathbb{T}$ and $L \geq 0$. It turns out that for $\mu = 0$ and $J_0 > 1.52$ the quantities

$$L = L(e, J_0) \quad \text{and} \quad e = e(L, J_0)$$

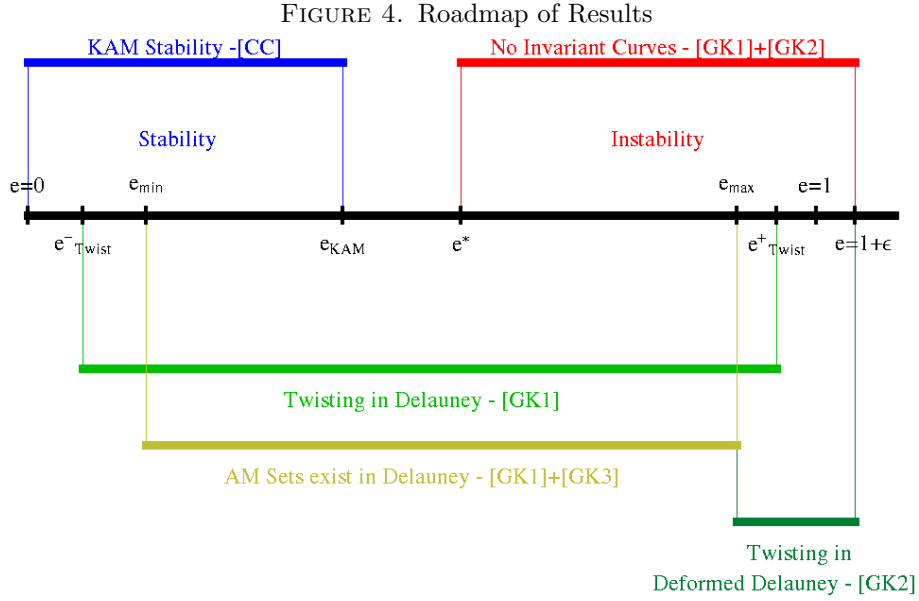
are monotone implicit functions of each other. On the energy surface $\mathcal{S}^{out}(J_0)$ they satisfy the implicit relation $J_0 = 1/(2L^2) + L\sqrt{1-e^2}$. Moreover, $L \rightarrow \infty$ as $e \rightarrow 1$ and vice versa

on $\mathcal{S}^{out}(J_0)$. Below we shall use either e or L -parametrization of the vertical (i.e. action) coordinate. When $\mu = 0$, the Poincaré map \mathcal{F}_μ has the form

$$\mathcal{F}_0 : (\ell, L) \rightarrow (\ell + 2\pi L^{-3}, L).$$

This map is clearly a twist map. (See [MF], [Ban], [Mo1] for discussion of twist maps). For small μ , the corresponding Poincaré map \mathcal{F}_μ is a small perturbation of \mathcal{F}_0 *only for e separated away from 1*.

In order to prove all the results stated above, it is sufficient to perform a detailed analysis of \mathcal{F}_μ . Analysis of \mathcal{F}_μ naturally divides into the following stages. ([GK2] refers to the present work in fig 4.)



Stage 1. Determine a *twist region*, denoted $Tw^{Del} = \{e_{\text{twist}}^- \leq e \leq e_{\text{twist}}^+\}$, where \mathcal{F}_μ is a twist map.

This is done by derivation of sufficient condition to check an infinitesimal twist condition holds locally uniformly. This condition says that a function of certain first and second partial derivatives of H has to be strictly negative. See section 4 for details. The values e_{twist}^- and e_{twist}^+ are computed by numerical extremization of value of this function. It is important to notice that Tw^{Del} is **not invariant**, but is however compact. Even though \mathcal{F}_μ twists in Tw^{Del} , a priori there may be no invariant sets in Tw^{Del} at all.

Stage 2. Show that for each $n \geq N_0$ and each rotation number $\omega \in [\frac{1}{n+1}, \frac{1}{n}] \subset \mathbb{R}$ the corresponding Aubry-Mather set Σ_ω of \mathcal{F}_μ has small vertical L -deviations on the cylinder,

i.e. $\Sigma_\omega \subset \{(\ell, L) : L_n^- < L < L_n^+\}$.

This stage is done in [GK3] using the ordering condition from Aubry-Mather theory. This implies that in a region slightly smaller than Tw^{Del} there are Aubry-Mather sets.

Stage 3. Rule out invariant curves to show the existence of a region of instability $\{e^* \leq e \leq e_{max}\} \subset Tw^{Del}$. This step is done in [GK1] through the method of comparison of actions.

Combining steps 1, 2 and 3 with Mather's variational techniques, we obtain a proof of Theorem 1.2 *only in the region* Tw^{Del} . It might not be surprising that the twist region Tw^{Del} is compact. Action-angle (Delaunay) variables are designed to describe the compact part of the dynamics and as motions approach unbounded (parabolic) motions, usage of these coordinates becomes less and less reliable. For example, they are not defined for Aubry-Mather sets $\tilde{\Sigma}_\omega$ with very small rotation numbers ω (see [GK1]). Thus, in order to prove existence of ejection/capture orbits we need to prove the existence of a semi-infinite region of instability in the L direction for the map \mathcal{F}_μ . This leads to analysis of the non-compact part "above" Tw^{Del} , denoted Tw^∞ .

Stage 4. Construction of symplectic deformation of Delaunay variables so that \mathcal{F}_μ is a twist map for nearly parabolic motions.²

This is done through analysis of the dynamics of the RCP3BP in symplectic polar coordinates where the coordinate system is well-defined for all non-collision motions. It turns out that the arguments of Stage 3 apply to Aubry-Mather sets in Tw^∞ (near the "top" boundary of Tw^{Del}) and exclude the possibility of invariant curves of any small rotation number. This shows that a region of instability which contains $\{e \geq e^*\}$ is semi-infinite (in L).

Delaunay coordinates are action-angle coordinates for the 2BP. The reason Stage 4 is needed is that these action-angle coordinates are only well defined for a compact region of the phase space. As $e \rightarrow 1$ motions tend to unbounded ones and Delaunay coordinates degenerate.

Construction of a modified coordinate system to correct these degeneracies is the primary focus of this paper.

The construction is of a fairly general nature and can be applied to other Hamiltonian systems. Throughout we make specific comments on the applicability of the construction to RCP3BP. Stage 4 is broken up as follows.

Stage 4.1: Algebraic deformations of action-angle variables

²For an EAP map to be twisting in some symplectic coordinate system is far from granted. Pick an EAP with two elliptic islands which twist in different directions; such a map has no such symplectic coordinates.

Given a set of action-angle variables, there may be degeneracies which spoil the reduction of dynamics to an EAPT. We exhibit a method to produce a set of action-angle variables where a Poincaré return map is well defined. The method amounts to an algebraic deformation of existing action-angle variables. For the RCP3BP, the deformed action-angle variables are similar to Delaunay without the singularities arising for $e \approx 1$ and they allow for a representation of nearly parabolic motions. While the algebraically deformed variables allow a return map to be defined, a priori it is not true that the map is twisting. For RCP3BP, the twisting fails in the algebraically deformed Delaunay variables for exactly the same reason as for Delaunay variables.

Stage 4.2: Sufficient conditions for twist

Sufficient conditions for twisting are stated. The key condition may be described as the property that long term, i.e. after several iterates, the map \mathcal{F}_μ is twisting. We call this *cumulative twist*. In polar coordinates for RCP3BP, the condition says transition times from the apohelion to the perihelion increase as the apohelion distance increases. The plan is to use the long term information about cumulative twist to create a coordinate system which has an EAPT map.

Stage 4.3: Construction of a dynamically defined direction field

We build a new direction field dynamically. The main idea is to ‘spread the cumulative twist’ along trajectories which comes from the sufficient condition in stage 4.2. This procedure works in a fairly general setting and is applicable to other Hamiltonian systems.

Stage 4.4: Construction of dynamically deformed coordinates

The vector field in stage 4.3 is used to construct a dynamic deformation of action-angle variables. The new coordinates are used to define a return map which is an EAPT. Once the property of twist is ensured, we are in a position to apply Aubry-Mather theory.

2.1. Organization of Paper.

- In section 3, Delaunay variables and their singularities are introduced as well as a method of algebraically deforming them.
- In section 4 we discuss twisting in Delaunay variables.
- In section 5, the method of spreading of cumulative twist is introduced for general systems. It is applied to the RCP3BP in section 12.
- In section 5.1 the so called *dynamically deformed variables* are constructed. Using these coordinates we establish that an EAP map for the RCP3BP is twisting.
- In section 6 details how Aubry-Mather theory may be applied to the RCP3BP to produce Chazy instabilities.

3. AN ALGEBRAIC DEFORMATION OF ACTION-ANGLE VARIABLES

In this section, we introduce a method to produce an algebraic deformation of action-angle variables. First we introduce the class of systems considered and offer a formulation of

action-angle variables for this class. Consider an integrable C^r Hamiltonian ($r \geq 3$)

$$H_0(q_1, q_2, p_1, p_2) = \frac{p_1^2}{2} + V_{\text{eff}}(q_1, p_2)$$

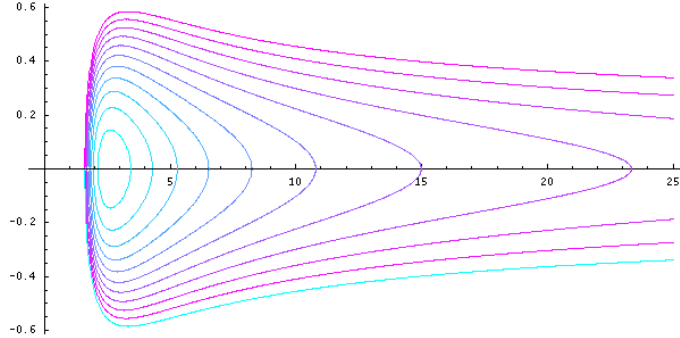
with 2 degrees of freedom, where V_{eff} is a C^r ($r \geq 3$) function with a unique minimum in the variable q_1 for a fixed p_2 , and suppose the minimum is strictly negative. (If it is not strictly negative, it suffices to add a constant to V_{eff} to ensure this property). The function V_{eff} is typically known as the effective potential.

Suppose $q_1^* = q_1^*(p_2)$ is the unique point minimum point of V_{eff} , i.e. where $\frac{\partial V_{\text{eff}}}{\partial q_1}(q_1^*(p_2), p_2) = 0$. Let $E_{\min} = V_{\text{eff}}(q_1^*(p_2), p_2)$ be the minimum value for V_{eff} . Fixing an energy surface $H_0 = E$ implicitly defines one of the variables, say p_2 . Let $S_E = \{(q_1, p_1) : H_0(q_1, q_2, p_1, p_2) = E\}$ be the level sets of the Hamiltonian (see fig. 5). For $E = E_{\min}$ the curve S_E is degenerate and consists of a single point. For sufficiently small $E \geq E_{\min}$, the set S_E is compact and there exists points $q_1^-(E) \leq q_1^*(p_2(E)) \leq q_1^+(E)$ where S_E intersects the axis $p_1 = 0$. Such points exist since H_0 is convex in p_1 . Denote the area under the curve S_E by

$$A(E) = \int_{q_1^-(E)}^{q_1^+(E)} \sqrt{2(E - V_{\text{eff}}(q, p_2))} dq$$

Let $L(E) = \frac{A(E)}{2\pi}$. This formula can be inverted to solve for E . Define the inversion h so

FIGURE 5. Level sets for the 2BP(SC) system in (r, P_r) variables



that $h(L; p_2) = E$. By Arnold-Liouville, this produces a generating function

$$S(q_1, q_2, L, G) = q_2 G + \int_{q_1^-(L, G)}^{q_1} \sqrt{2(h(L, G) - V_{\text{eff}}(q, G))} dq$$

and the generating function defines a symplectic change of coordinates from (q_1, q_2, p_1, p_2) to action-angle coordinates (L, ℓ, G, g) . Call the change of coordinates Ψ . Notice that Ψ is only well defined when the area $A(E)$ is finite. If the effective potential is such that $A(E) = \infty$ for some $E = E_\infty$, then action-angle variables are not well defined; namely $L \rightarrow \infty$ in some spots near $E = E_\infty$. Suppose there exists such an $E_\infty < \infty$. Then action-angle variables are only defined in the region $\Omega_0 = \bigcup_{E \in [E_{\min}, E_\infty]} S_E$.

Now if $H = H_0 + \Delta H$, where ΔH is a small perturbation of the integrable Hamiltonian H_0 , then the generating function still allows us to convert to action-angle variables in the region Ω_0 . However, this region is **not invariant** under the Hamiltonian flow and it is possible for solutions to enter into a region where action-angle coordinates are not well defined. Later in this section, we develop a method to deform the action-angle variables so that they are well defined for a region larger than Ω_0 .

3.1. Delaunay Variables. As a model for this type of system, we introduce the action-angle variables for the 2BP which are classically known as Delaunay variables. These variables were originally used to describe bounded motions (i.e. $e \in [0, 1)$) of the 2BP and hence when applied to RCP3BP, Delaunay variables have singularities for motions near $e = 1$, the so called *nearly parabolic motions*. However this is precisely the region we are interested in proving that there is diffusion in. We remark that in the rotating frame of coordinates, the Hamiltonian has an additional term from the gyroscopic force. This causes the degeneracies at $e = 1$ to appear when $H_{2BP(SC)} + P_\varphi = 0$, i.e when $P_\varphi = -H_{2BP(SC)} = J_0$. We typically think of fixing an energy surface, then consider degeneracies near $P_\varphi = J_0$. A derivation of Delaunay variables can be found in [GPS], [SS] (also see [AKN] and [CC] for some nice exposition). In short, they arise by considering the generating function

$$S(r, \varphi, L, G) = \varphi G + \int_{r^{perih}(L, G)}^r \left(\sqrt{\frac{-1}{L^2} - \frac{G^2}{r^2} + \frac{2}{r}} \right) dr$$

This gives the canonical transformation $\mathcal{D}(\ell, g, L, G) = (r, \varphi, P_r, P_\varphi)$ from Delaunay variables to symplectic polar variables where $r^{perih} = L^2(1 - \sqrt{1 - \frac{G^2}{L^2}})$ is the perihelion of the 2BP(SC) expressed in terms of L, G . The image of \mathcal{D} is only defined for bounded motions of the 2BP(SC) with $(\ell, g) \in \mathbb{T}^2$ and $0 \leq G \leq L$.

For the 2BP, L^2 is the semi-major axis of the ellipse of the orbit, so by Kepler's Third Law, the period $T = 2\pi L^3$. Upon examination of the generating function observe $G = P_\varphi$ is *angular momentum*, or alternatively LG is the semi-minor axis of the ellipse of the orbit. The variable $\ell \in \mathbb{T}$ is the *mean anomaly* which is $\ell = \pi \bmod 2\pi$ at the apohelion, $\ell = 0 \bmod 2\pi$ at the perihelion, and in general $(\ell - \ell_0) = \frac{2\pi}{T}t$. The quantity $(g + t)$ can be interpreted as the perihelion angle (in non-rotating coordinates g itself plays this role). The radius r can be expressed in Delaunay coordinates by $r = L^2(1 - e \cos(u))$ where the eccentricity $e = \sqrt{1 - \frac{G^2}{L^2}}$, and u , called *the mean anomaly*, is given implicitly by the *Kepler equation*

$$u - e \sin(u) = \ell.$$

More description of Delaunay variables can be found in [AKN] or [CC]. Applying the canonical transformation \mathcal{D} to the Hamiltonian for the 2BP(SC) in rotating polar coordinates gives

$$H_{2BP(SC)} \circ \mathcal{D}^{-1} = -\frac{1}{2L^2} - G$$

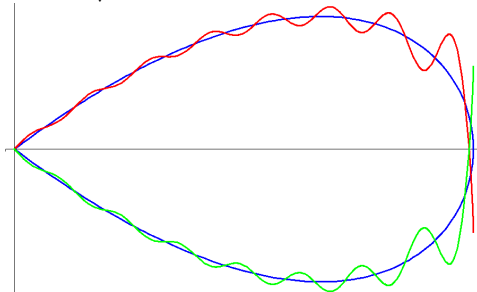
Note that S satisfies $\det\left(\frac{\partial^2 S}{\partial(r, \phi) \partial(L, G)}\right) = \frac{L^3}{P_r} \neq 0$. Hence in general there exists a canonical transformation from polar to Delaunay. It is provided by the above generating function and

is well defined inside of the homoclinic loop (see Figure 6). Hence where it is well defined, one gets Delaunay variables for RCP3BP using the generating function S . This yields

$$H_{Del} = H_{Polar} \circ \mathcal{D}^{-1} = -\frac{1}{2L^2} - G + \Delta H(L, G, \ell, g),$$

where the perturbation term is converted to Delaunay. As the “where it makes sense” indicates, Delaunay variables are not defined for the RCP3BP for nearly parabolic motions. More specifically Delaunay variables are not defined very close to separatrices corresponding to nearly parabolic motions in RCP3BP. It is possible that the perturbation can push a highly elliptic orbit into a hyperbolic orbit with $e > 1$ (in fact we desire this in Theorem 1.2). In Delaunay variables, this corresponds to $L \rightarrow \infty$ and occurs near places where the separatrices leave the homoclinic loop of the 2BP(SC) in which Delaunay variables are defined (see Figure 6).

FIGURE 6. Deviation of RCP3BP Separatrices (colored) from 2BP(SC) Homoclinic loop (blue) in $(\frac{1}{\sqrt{r}}, P_r)$ variables



3.2. An Algebraic Deformation. We overcome the technical issue that action-angle variables are not well defined near separatrices. Geometrically the approach corresponds to encapsulating the separatrices in a larger domain (see Figure 7). Mathematically, the trick reduces to applying the Arnold-Liouville theorem with a different integrable Hamiltonian on a nearby energy surface to produce a different domain where action-angle-variables are well defined. The energy should be chosen so that the size of the new domain is large enough to capture behaviors of trajectories which are close to the separatrices. We illustrate the trick with RCP3BP and postpone the technical estimates until the appendices.

Consider action-angle variables for the Hamiltonian

$$(3) \quad H_\nu(r, P_r, P_\varphi) := \frac{P_r^2}{2} + \frac{(P_\varphi - \nu)^2}{2r^2} - \frac{1}{r}$$

Action-angle variables are defined inside of bounded level sets corresponding to $H_\nu < 0$. We shall prove that these sets contains a large enough part of nearly parabolic motions for RCP3BP(μ, J_0).

The parabolic separatrices for the RCP3BP, denoted by \mathcal{P}_μ^+ (resp. \mathcal{P}_μ^-), are defined to be the set of points (r, φ, P_r) such that $(r, \varphi, P_r)(0) = (r_0, \varphi_0, 0)$, $P_r(t) > 0$ for all $t > 0$ (resp.

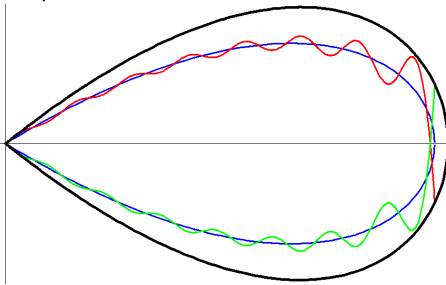
$P_r(t) < 0$ for all $t < 0$), and $\lim_{t \rightarrow \infty} P_r(t) = 0$ (resp. $\lim_{t \rightarrow -\infty} P_r(t) = 0$). In the case of 2BP(SC), then separatrices on the energy surface $\mathcal{S}^{out}(J_0)$ are given for all $\varphi_0 \in \mathbb{T}$ by

$$(4) \quad (r, P_r)^\pm = \left(r, \pm \sqrt{\frac{2}{r} - \frac{J_0^2}{r^2}} \right)$$

A theorem of McGehee [McG] guarantees these objects exist and are one dimensional C^∞ smooth manifolds for all μ . In the case of μ small, we can say more.

Theorem 3.1. *There exists $\nu = \nu(\mu, J_0)$ such that the forward and backward separatrices $(r, P_r)^\pm(\varphi_0)$ of RCP3BP(μ, J_0) are contained inside the homoclinic loop given by $H_\nu(r, P_r, J_0) = 0$ for all $\varphi_0 \in \mathbb{T}^1$. In particular, when $J_0 = 1.8$ and $\mu = 0.001$, then $\nu \leq 2.8\mu$.*

FIGURE 7. Containment of RCP3BP separatrices inside the larger homoclinic loop of H_ν in $(\frac{1}{\sqrt{r}}, P_r)$ variables



Proof of this theorem is contains in appendix 9. Theorem 3.1 encloses the nearly parabolic solutions of the RCP3BP on the energy surface $\mathcal{S}^{out}(J_0)$ inside of the homoclinic loop defined by $H_\nu(r, P_r, J_0) = 0$. We can make a canonical change of variables to action-angle coordinates inside of this loop via the Arnold-Liouville theorem. Call the coordinates given by the map $\mathcal{D}_\nu(L_\nu, G_\nu, \ell_\nu, g_\nu) = (r, \varphi, P_r, P_\varphi)$ algebraically deformed Delaunay variables (ADDV). Note that $\mathcal{D}_\nu = \mathcal{D} \circ (P_\varphi \mapsto P_\varphi - \nu)$ so \mathcal{D}_ν is clearly a canonical change of coordinates. Specifically $dr \wedge dP_r + d\varphi \wedge dP_\varphi = dL_\nu \wedge d\ell_\nu + dG_\nu \wedge dg_\nu$.

Remark: It is important to note that we are not using any dynamics from the formal Hamiltonian H_ν . It is only used to define the domain of definition \mathcal{D}_ν .

The new homoclinic loop defined by $H_\nu(r, P_r, J_0) = 0$ inside of which ADDV are well defined contains the homoclinic loop $H_{2BP(SC)}(r, P_r, J_0) = -J_0$ (see formula ?? for defintion of $H_{2BP(SC)}$) in which the original Delaunay variables were defined (see fig. 7). Because the new loop is larger, some solutions whose positions in polar coordinates that could not be expressed in Delaunay now have representations in ADDV. Note that ADDV also becomes undefined, however, it does so away from the separatrices of H_{Polar} on the energy surface $\mathcal{S}^{out}(J_0)$.

3.2.1. *The relation between Delaunay and ADDV.* One might wonder how Algebraically Deformed Delaunay Variables (ADDVs) are related to the usual Delaunay variables. By definition, the ADDVs $(L_\nu, G_\nu, \ell_\nu, g_\nu)$ are the action-angle variables for the Hamiltonian H_ν which are ‘close’ to those of the 2BP Hamiltonian. By construction one can see that immediately that $G_\nu = G - \nu$. From examination of the Hamiltonians, it can also be shown that $\frac{1}{L_\nu^2} = \frac{1}{L^2} + 2\nu$. Geometrically, for a fixed energy, the lobe in ADDV has more area than the lobe in Delaunay and the variables L, L_ν measure the area of such lobes. This has the effect that L_ν is finite in regions where L has become infinity. The relations to convert from symplectic polar to Delaunay can be derived by noting $\mathcal{D}^{-1} \circ \mathcal{D}_\nu = (P_\varphi \mapsto P_\varphi - \nu)$. For example, $r = L_\nu^2(1 - e_\nu \cos(u_\nu))$, where $u_\nu - e_\nu \sin(u_\nu) = \ell_\nu$ and $e_\nu = \sqrt{1 - \frac{G_\nu^2}{L_\nu^2}}$.

4. TWIST IN DELAUNAY VARIABLES AND ADDV

Having constructed a suitable action-angle coordinate system for RCP3BP, we now focus on the issue of reducing the dynamics to that of an exact area preserving twist map (EAPT) (see [MF], [Ban], [Mo1] for exposition on EAPTs). We illustrate the difficulties using Delaunay variables, then comment on algebraically deformed Delaunay variables.

Consider the Poincaré section $P = \{g = 0 \pmod{2\pi}\} \subset \mathcal{S}(J_0)$. In [GK1], Lemma 10.1 we show that $-1.025 \leq \dot{g} \leq -0.9975$ for $J_0 = 1.8$ and $\mu \leq 10^{-3}$. Hence, P is a well defined section. Consider the Poincaré return map $\mathcal{F} : \mathcal{S}^{out}(J_0) \cap P \mapsto \mathcal{S}^{out}(J_0) \cap P$ defined by

$$(5) \quad \mathcal{F} = \mathcal{F}_{\mu, J_0} : (\ell_0, L_0) \mapsto (\ell_1, L_1) = (\ell(t_P, \ell_0, L_0), L(t_P, \ell_0, L_0))$$

where $t_P > 0$ is the first return time to P . The map \mathcal{F} is called *twisting* or *satisfies the twist property* in a domain $\Omega \subset \mathbb{A}$ if for all $(\ell_0, L_0) \in \Omega$,

$$\text{sign} \left(\frac{\partial \ell_1}{\partial L_0} \right) = \text{const} \neq 0$$

The twist property can be interpreted geometrically to say that the image of a vertical line (i.e. in the L direction) under \mathcal{F} is twisted to left (or right).

In the case $\mu = 0$, we have $\frac{\partial \ell_1}{\partial L_0} = -\frac{3}{L_0^4} \cdot 2\pi < 0$. In terms of the motions of the comet, this says that higher eccentricity comets revolve around the sun more slowly than their low eccentricity counterparts. This is roughly Kepler’s third law.

For small $\mu > 0$, Jupiter’s effects on the comet are negligible far away from the Sun and we still expect increasing eccentricity to slow down the motion of the comet. The twist property in Delaunay variables for the RCP3BP is then

$$\frac{\partial \ell_1}{\partial L_0} = -\frac{3}{L_0^4} \cdot 2\pi + O(\mu) < 0$$

It is possible for large L_0 that the $O(\mu)$ perturbation terms overwhelm the $-\frac{3}{L_0^4}$ term from the 2BP(SC) and change the sign. This is *why* twisting can fail.

In [GK1], section 6.1, an explicit twisting condition is developed that allows a computer to look for sign changes in the twist term. This condition boiled down to checking the sign of a complicated expression (the so called *twist term* in [GK1]) over a domain. A computer is used to find sign changes of the twist term, and produce the following lemma.

Lemma 4.1. *In Delaunay variables for RCP3BP(0.001, 1.8), the map \mathcal{F}_μ is twisting for $e_{twist}^-(0.001, 1.8) \leq 0.07 \leq e \leq 0.994 \leq e_{twist}^+(0.001, 1.8)$ and is not twisting for some values corresponding to $e > 0.994$.*

Denote by $\mathcal{F}^\nu = \mathcal{F}_\mu^\nu(J_0)$ an EAP map arising from RCP3BP(μ, J_0) in ADDV coordinates on the section $\{g_\nu = 0 \pmod{2\pi}\}$. It is possible to derive an expression to check for twisting in ADDV. This is done in exactly the same manner as in [GK1]. A similar lemma holds.

Lemma 4.2. *In ADDV for RCP3BP(0.001, 1.8) let $\nu = 2.8\mu$. Then the map \mathcal{F}_μ^ν is twisting for $0 \leq e \leq 0.984$ and is not twisting for some values corresponding to $e > 0.984$.*

While it is true that \mathcal{F}^ν is *defined* for nearly parabolic motions, without the property of twist, it is useless for the purposes of applying Aubry-Mather theory. The next several sections are devoted to developing a coordinate system in which the twist property holds for \mathcal{F}^ν .

5. THE METHOD OF SPREADING CUMULATIVE TWIST

In this section, we study a time-periodic Hamiltonian $H(\ell, L, t)$ and its convexity with respect to L . Presence of such convexity implies that the natural time 2π -map Poincaré map $F : (\ell, L) \rightarrow (\ell', L')$ is twisting. Twisting enable us to apply Aubry-Mather theory.

We state sufficient conditions on H for existence of a time-periodic canonical change of coordinates Ψ such that $H \circ \Psi(\ell, L, t)$ is convex in L . For the RCP3BP, these conditions essentially reduce to $\frac{\partial T}{\partial r_{apoh}} > 0$, i.e. increasing the apohelion radius increases the period of revolution. For the duration of this section we work in a general framework as we believe there are *further applications of the result of this section*. Comments considering applicability to RCP3BP are scattered throughout and the actual application to the RCP3BP is carried out in Appendix 12 after a certain set of estimates are established in earlier appendices.

Notice that existence of twisting coordinate system is *not granted*³. One can also work out the construction below for exact area-preserving maps, i.e. for the discrete time.

Consider a C^r Hamiltonian $H(\ell, L, s)$ periodic in ℓ and s with period 2π and well defined in a region $\mathcal{U} \subset \mathbb{A} \times \mathbb{T} = \mathbb{T} \times \mathbb{R} \times \mathbb{T}$ with $r \geq 3$. Let $\Phi_t(\ell, L, s)$ be the flow of H . In particular, the time-component of $\pi_3 \circ \Phi_s$ satisfies $\pi_3 \circ \Phi_t(\ell, L, s) = s + t$. This implies that the equations of variation of $H(\ell, L, s)$ preserve 2-dimensional subspaces tangent to the cylinder component. Namely,

$$d\Phi_t : T_{(\ell, L, s)}\mathbb{A} \rightarrow T_{\Phi_t(\ell, L, s)}\mathbb{A}.$$

Thus, the tangent space to the cylinder \mathbb{A} at (ℓ, L, s) is mapped into the tangent space to \mathbb{A} at $\Phi_t(\ell, L, s)$. Denote the restriction of $d\Phi_t$ to $T_{(\ell, L, s)}\mathbb{A}$ by $d\Phi_t^*$. It can also be defined using the

³See footnote on page 6 for an easy example.

following commutative diagram. Let $\pi : T_{(\ell, L, t)}(\mathbb{A} \times \mathbb{T}) \rightarrow T_{(\ell, L, t)}\mathbb{A}$ be the natural projection. Then

$$\begin{array}{ccc} T_{(\cdot, \cdot, t)}(\mathbb{A} \times \mathbb{T}) & \xrightarrow{d\Phi_t} & T_{(\cdot, \cdot, t+s)}(\mathbb{A} \times \mathbb{T}) \\ \downarrow \pi & & \downarrow \pi \\ T_{(\cdot, \cdot, s)}\mathbb{A} & \xrightarrow{d\Phi_t^*} & T_{(\cdot, \cdot, t+s)}\mathbb{A}. \end{array}$$

Here we study *evolution of the tangent space to the cylinder* $T_{(\cdot, \cdot, \cdot)}\mathbb{A}$ naturally embedded into the ambient tangent space $T_{(\cdot, \cdot, t)}(\mathbb{A} \times \mathbb{T})$.

A time-periodic Hamiltonian $H(\ell, L, t)$ can arise from an autonomous two degree of freedom Hamiltonian \overline{H} . When one restricts \overline{H} to an energy surface and reduces order (see e.g [A] sect. 45), it leads to a time-periodic Hamiltonian. For example this can be done to Hamiltonians from section 3.

Fix a section Σ , e.g. $\{\ell = 0 \pmod{2\pi}\}$, and define the return times of $(0, L, t) \in \Sigma$ to be

$$\begin{aligned} T_+(L, t) &= \min\{t^* > t \text{ such that } \ell_{t^*} = 0\} \\ T_-(L, t) &= \max\{t^* < t \text{ such that } \ell_{t^*} = 0\} \end{aligned}$$

Definition 5.1. Let \mathcal{W} be the set of $(\ell, L, t) \in \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T}$ such that

- (1) the return time to the section Σ is finite: $T_{\pm}(L, t) < \infty$ for any $(0, L, t) \in \Sigma$.
- (2) every point inside $(\ell, L, t) \in \mathcal{W}$ arises by flowing from a point in the section Σ : $\exists t_{\pm}$ such that $\ell_{t_{\pm}} = 0 \pmod{2\pi}$ for $t_- \leq t \leq t_+$
- (3) the angle of twisting is uniformly bounded away by $\frac{\pi}{2}$ for all time between $T_-(L, t)$ and $T_+(L, t)$. More exactly, there exists $\kappa > 0$ such that for any $(0, L, s) \in \mathcal{W}$,

$$\begin{aligned} d\Phi_t^*(0, L, s)(0, 1) \cdot (0, 1) &\geq \kappa \|d\Phi_t^*(0, L, s)(0, 1)\| && \text{for any } 0 \leq t \leq T_+(L, s) \\ d\Phi_{-t}^*(0, L, s)(0, 1) \cdot (0, 1) &\geq \kappa \|d\Phi_{-t}^*(0, L, s)(0, 1)\| && \text{for any } 0 \leq t \leq T_-(L, s). \end{aligned}$$

- (4) moving in the action L direction on the section $\Sigma \cap \mathcal{W}$ decreases the return time T : $\frac{\partial T_{\pm}}{\partial L}(L_{t_{\pm}}, t_{\pm})|_{L=L_{t_{\pm}}} < 0$.

Notice that the region \mathcal{W} can be *non-invariant* and *non-compact*⁴. The first 2 conditions are nothing more than an abstract definition of a non-invariant region to be investigated. For the RCP3BP it corresponds to “inside parabolic” initial conditions (see Appendix 12). The third condition prohibits the dynamics from twisting a vertical vector by more than $\frac{\pi}{2}$ ⁵. The fourth condition is the “sufficient condition” needed to ensure twist. It says that there is *cumulative twist*.

Theorem 5.2. Suppose \mathcal{W} , defined above, is non-empty for the flow of a C^r ($r \geq 3$) Hamiltonian H . Then there is a C^{r-1} smooth periodic family $\{\Psi_s\}_{s \in \mathbb{T}}$ of canonical coordinate changes $\Psi : (\ell, L; s) \rightarrow (\ell_s^{dyn}, L_s^{dyn}; s)$ such that $\forall s \in \mathbb{T}$ the composition $H \circ \Psi^{-1}(\ell_s^{dyn}, L_s^{dyn}; s)$ is convex with respect to L_s^{dyn} in $\Psi(\mathcal{W})$; specifically $\partial_{L_s^{dyn}}^2 (H \circ \Psi^{-1})|_{\Psi(\mathcal{W})} > 0$.

⁴We remark that in the case \mathcal{W} is known to be compact, then the first two conditions can be replaced with that condition that angular ℓ component moves with positive velocity: $\dot{\ell} = \partial_L H > 0$.

⁵We believe that this condition can be removed, but did not pursue this seriously

On one side having a possibly non-invariant region \mathcal{W} gives more flexibility for applications of Aubry–Mather theory, on the other side this causes more concerns in proving existence of Aubry–Mather sets. Usually Aubry–Mather theory is done inside of an invariant region. We handle the issue of non-invariance for the RCP3BP in section 6.

Up to non-invariance conditions (1–2) and no overtwisting (3), Theorem 5.2 says that

If there is a cumulative twist, then after a canonical coordinate change there is a twist.

Consider a C^{r-2} ($r \geq 3$) smooth direction field in $T\mathbb{A} \times \mathbb{T}$, i.e. a family of directions $v_{(\ell,L,s)} \in T_{(\ell,L,s)}\mathbb{A}$ with $(\ell, L, s) \in \mathcal{W}$ which is C^{r-2} smooth in (ℓ, L, s) . Since $d\Phi_t^*$ preserves the tangent spaces to the cylinder $T_{\mathcal{W}}\mathbb{A}$, both $d\Phi_t^*(\ell, L, s)v_{(\ell,L,s)}$ and $v_{\Phi_t(\ell,L,s)}$ belong to the 2-dimensional space with induced orientation. Therefore, the sign of the wedge product is well defined. Define the function

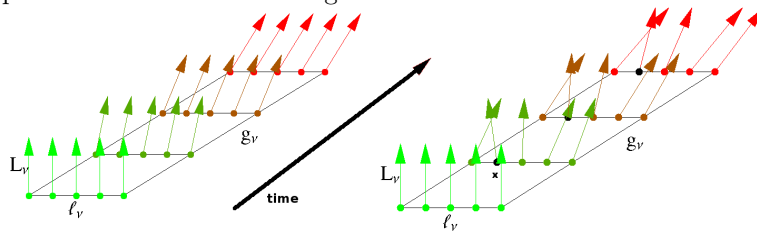
$$\delta(\ell, L, s) := \lim_{t \rightarrow 0} \frac{d\Phi_t^*(\ell, L, s)v_{(\ell,L,s)} \wedge v_{\Phi_t(\ell,L,s)}}{t}.$$

Definition 5.3. *A vector field is called twisted (or a twisting direction field) if the sign of $\delta(\ell, L, s)$ is constant and nonzero $\forall (\ell, L, s) \in \mathcal{W}$.*

Geometrically twist means a vector makes an angle with its image under the flow of the equations of variation. $\delta(\ell, L, s)$ measures the rate of twisting at the point (ℓ, L, s) . Notice that for convex Hamiltonians, the vertical direction field $v_{(\ell,L,s)} \equiv (0, 1)$ is twisted. For example, for the 2BP(SC) we have $\delta(\ell, L, s) = -\frac{3}{L^4}$. The failure of twist for the RCP3BP geometrically indicates that there are spots where the vertical vector is pointing in the incorrect direction. See figure 8.

The proof of Theorem 5.2 consists of two steps. The first step is to **construct** Γ , a twisting direction field. This is done by via “spreading twist” to produce a direction field along a single orbit. This construction extends smoothly to the whole set \mathcal{W} . The second step is to use the direction field Γ to construct a smooth canonical coordinate change. This is accomplished by straightening the direction field Γ .

FIGURE 8. A twisted direction field (left) and the failure of twist (right) at the point x and its forward images



Lemma 5.4. *There exists a C^{r-2} smooth twisting direction field Γ on $T_{\mathcal{W}}\mathbb{A}$.*

Proof: First, let us show how to spread twist along a single trajectory and construct a twisting direction field. Let us first start by introducing some notation.

Consider a trajectory with initial conditions $(0, L_0, t_0) \in \Sigma \cap \mathcal{W}$. Let $p_t := \Phi_t(0, L_0, t_0)$ be a parametrization of points along the trajectory. Such a parameterization exists by conditions (1) and (2). By condition (1) we have $T = T_+(L_0, t_0) < +\infty$. Define the tangent vectors $v_0^* := (0, 1) \in T_{(0, L_0, t_0)}\mathbb{A}$ and $v_T^* := (0, 1) \in T_{(\ell_T, L_T, t_0+T)}\mathbb{A}$. Introduce also a C^∞ function $\lambda : [0, 1] \rightarrow \mathbb{R}_+$ such that $\lambda|_{[0, 1/3]} \equiv 1/T_+(L_0, t_0)$ and $\lambda|_{[2/3, 1]} \equiv 1/T_-(L_0, t_0)$.

Let us construct a C^{r-2} smooth family $\{v_t\}_{0 \leq t \leq T}$ where v_t 's are non-vanishing tangent vectors at p_t 's. Suppose $w : [0, 1] \rightarrow [0, 1]$ is a C^{r-2} smooth monotone strictly increasing function with $w(0) = 0, w(1) = 1, w^{(k)}(0) = w^{(k)}(1) = 0$ for any $1 \leq k \leq r-2$. Define for $0 \leq \alpha \leq 1$

$$(6) \quad v_{\alpha T} := \lambda(\alpha) \left((1 - w(\alpha)) d\Phi_{\alpha T}^*(v_0^*) + w(\alpha) d\Phi_{-(1-\alpha)T}^*(v_T^*) \right).$$

Notice the vector field $\{v_t\}_{0 \leq t \leq T}$ has non-vanishing vertical component due to no over twisting condition 3.

To see smoothness of the direction field along an orbit away from Σ , note that $d\Phi^*$ and $T = T(L_0, t_0)$ are C^{r-2} -smooth by smooth dependence in initial conditions. Since w is C^{r-2} smooth, then dependence on α is C^{r-2} -smooth and hence $\{v_t\}_{t \in [0, T]}$ is a C^{r-2} -smooth family along the trajectory away from Σ . At the section Σ , note that v_0 and v_T are parallel to $(0, 1)$ so the *direction* field is continuous.

To prove C^{r-2} smoothness on Σ recall that $\lambda'(\alpha) \equiv 0$ for $\alpha \in [0, 1/3] \cup [2/3, 1]$ and for these α 's consider

$$\begin{aligned} \frac{\partial}{\partial \alpha}(v_{\alpha T}) &= \lambda(\alpha) \left(d\Phi_{\alpha T}^*(v_0) \cdot (-w'(\alpha)) + (1 - \alpha) \frac{\partial}{\partial \alpha} \left(d\Phi_{\alpha T}^*(v_0) \right) \cdot T \right. \\ &\quad \left. + d\Phi_{-(1-\alpha)T}^*(v_T) \cdot w'(\alpha) + w(\alpha) \frac{\partial}{\partial \alpha} \left(d\Phi_{-(1-\alpha)T}^*(v_T) \right) \cdot T \right). \end{aligned}$$

and hence

$$\left(\frac{\partial}{\partial \alpha}(v_{\alpha T}) \right) \Big|_{\alpha=0} = \frac{\partial}{\partial t} \left(d\Phi_t^*(v_0) \right) \Big|_{t=0} \quad \text{and} \quad \left(\frac{\partial}{\partial \alpha}(v_{\alpha T}) \right) \Big|_{\alpha=1} = \frac{\partial}{\partial t} \left(d\Phi_t^*(v_T) \right) \Big|_{t=0}.$$

To have C^1 smoothness on Σ we need to have $\frac{\partial}{\partial t} v_t|_\Sigma$ match for $t > 0$ and $t < 0$ at every point in $\Sigma \cap \mathcal{W}$. The above procedure views every point on $\Sigma \cap \mathcal{W}$ as either $(0, L_0, t_0)$ or as the image $\Phi_{T'}(0, L'_0, t'_0)$ of a different point $(0, L'_0, t'_0) \in \Sigma \cap \mathcal{W}$, where $T' = T_+(0, L'_0, t'_0)$. Thus, C^1 smoothness of $\{v_t\}_{t \in [0, T]}$ follows from the above formula. Similarly one can prove smoothness of higher order.

It turns out that on the section Σ the vector field $\{v_t\}_{t \in [0, T]}$ is not twisting. To rectify it we introduce its modification. Fix $\epsilon \ll T$, and suppose $u : \mathbb{R} \rightarrow \mathbb{R}$ is a C^{r-2} smooth function such that $u(\pm\epsilon) = u(0) = 0, u'(0) > 0, u^{(k)}(\pm\epsilon) = 0$ for any $1 \leq k \leq r-2, u$ is nonzero everywhere else inside of $(-\epsilon, \epsilon)$, and u is identically zero outside of $(-\epsilon, \epsilon)$. Consider the

following perturbation of $\{v_t\}_{t \in [0, T]}$:

$$v'_t = (1 - u(t))v_t + u(t)v_t^\perp$$

where v_t^\perp denotes a unit vector orthogonal to v_t so that $v_t \wedge v_t^\perp > 0$. Clearly the new direction field $\{v'_t\}$ is a smooth perturbation of $\{v_t\}$ since u is smooth and v_t^\perp is smooth since v_t is smooth.

To prove twist, consider

$$\begin{aligned} & d\Phi_t^*(\ell, L, s)v'_{(\ell, L, s)} \wedge v'_{\Phi_t(\ell, L, s)} \\ &= d\Phi_t^*(\ell, L, s) \left((1 - u(s))v_{(\ell, L, s)} + u(s)v_{(\ell, L, s)}^\perp \right) \wedge \left((1 - u(t+s))v_{\Phi_t(\ell, L, s)} + u(t+s)v_{\Phi_t(\ell, L, s)}^\perp \right) \\ &= (1 - u(s))(1 - u(t+s))d\Phi_t^*(\ell, L, s)v_{(\ell, L, s)} \wedge v_{\Phi_t(\ell, L, s)} \\ &+ u(s)(1 - u(t+s))d\Phi_t^*(\ell, L, s)v_{(\ell, L, s)}^\perp \wedge v_{\Phi_t(\ell, L, s)} \\ &+ (1 - u(s))u(t+s)d\Phi_t^*(\ell, L, s)v_{(\ell, L, s)} \wedge v_{\Phi_t(\ell, L, s)}^\perp \\ &+ u(s)u(t+s)d\Phi_t^*(\ell, L, s)v_{(\ell, L, s)}^\perp \wedge v_{\Phi_t(\ell, L, s)}^\perp, \end{aligned}$$

then divide the above quantities by t and take the limit as $t \rightarrow 0$ to compute the function δ for $\{v'_t\}$. Denote this quantity by $\delta_{v'}$ and the twist term for $\{v_t\}_{t \in [0, T]}$ as δ_v . Notice that $d\Phi_t^*(\ell, L, s)v_{(\ell, L, s)}^\perp \wedge v_{\Phi_t(\ell, L, s)}^\perp$ and $d\Phi_t^*(\ell, L, s)v_{(\ell, L, s)} \wedge v_{\Phi_t(\ell, L, s)}$ are the same. This produces

$$\begin{aligned} \delta_{v'}(\ell, L, s) &= (1 - u(s))^2 \delta_v(\ell, L, s) + u(s)^2 \delta_v(\ell, L, s) \\ &+ \left(\lim_{t \rightarrow 0} \frac{u(s)(1 - u(t+s)) - (1 - u(s))u(t+s)}{t} \right) v_{(\ell, L, s)}^\perp \wedge v_{(\ell, L, s)} \\ &= \left((1 - u(s))^2 + u(s)^2 \right) \delta_v(\ell, L, s) + u'(s). \end{aligned}$$

Notice that condition (4) implies that $d\Phi_T^*v_0 \wedge v_T > 0$. Moreover, for any $0 < \alpha < 1$ we have $d\Phi_{\alpha T}^*v_0 \wedge d\Phi_{-(1-\alpha)T}^*v_T > 0$ uniformly in α , because $d\Phi_{-(1-\alpha)T}^*$ is a orientation preserving and non-degenerate. Replace v by $\lambda(\alpha)d\Phi_{\alpha T}^*v_0$, v^\perp by $\lambda(\alpha)d\Phi_{-(1-\alpha)T}^*v_T$, t by αT , and u by w in the calculation above. Then

$$\delta_v(\Phi_{\alpha T}(0, L, s)) = w'(\alpha) \lambda^2(\alpha) d\Phi_{\alpha T}^*v_0 \wedge d\Phi_{-(1-\alpha)T}^*v_T.$$

Note that on the section Σ we have $u'(0) > 0$ by choice of u , so $\{v'_t\}$ is twisting on Σ . Furthermore, since the vector field $\{v_t\}$ is twisting away from Σ , i.e. $\delta_v(\ell, L, s) > 0$ off of Σ , then we may choose u on the compact interval $[-\epsilon, \epsilon]$ so that $\delta_{v'}(\ell, L, s)$ is strictly positive.

Hence $\{v'_t\}_{t \in [0, T]}$ is a twisting vector field along the trajectory p_t . Note that v_t , and as a result v'_t , are not continuous across Σ . While the length of v_t experiences jump, the *direction does not*. Indeed, size λ is identically constant on each side of Σ separately. The magnitude of λ is selected to match time derivatives on both sides so the vector field defines a smooth *direction field* across Σ . Note the construction of the vector field for a single trajectory p_t extends smoothly to all of \mathcal{W} ; we use this to induce the direction field that is Γ . \square

5.1. Dynamically Deformed Variables. In this section, we create a new canonical coordinate system which respects the property of twist. We denote the new variables (L_{dyn}, ℓ_{dyn}) and call them *dynamically deformed variables*. One way to think of dynamically deformed variables is to think that at each point in the flow, the coordinate system is dynamically changed to one in which the twist property holds infinitesimally. The new action direction arises from a straightening of the direction field Γ . Stated as a lemma:

Lemma 5.5. *Suppose Γ is the smooth direction field as constructed in Lemma 5.4. Then there exists a C^{r-2} -smooth time-periodic family of symplectic maps $\Psi_s : (\ell, L, s) \rightarrow (\ell_{dyn}, L_{dyn}, s)$ which straightens the direction field Γ .*

Proof: Note that the direction field Γ is defined only on \mathcal{W} . Moreover, angle with the horizontal component is strictly positive (see formula (6) and comments below it). We extend Γ smoothly to whole cylinder in the following manner. Let U_ϵ be an ϵ neighborhood, $\epsilon > 0$ of the set \mathcal{W} . On $U_\epsilon \cap \partial\mathcal{W}$, leave the direction field Γ as defined on $\partial\mathcal{W}$. On $\mathbb{A} - U_\epsilon^{int}$, define the direction field Γ to be the vertical direction field which is identically $(0, 1)$ everywhere. On $U_\epsilon - \mathcal{W}$ define the vector field Γ to be a smooth interpolation between vectors on the boundaries, i.e. smoothly interpolate between $\Gamma|_{\partial\mathcal{W}}$ and $(0, 1)$.

Suppose $\mathcal{C}_0 = \{(L^*, \ell) | \ell \in \mathbb{T}\}$ is a horizontal circle which intersects the section Σ at height L^* . Consider the images of \mathcal{C}_0 after integration along Γ for time t , where we use unit velocity along Γ . Denote them \mathcal{C}_t . Clearly the \mathcal{C}_t are diffeomorphic to circles. Additionally, since $\Gamma|_{\mathcal{W}} = (0, 1)$, then for all L_0 there exists t_0 such that $\mathcal{C}_{t_0} \cap \mathcal{W} = L_0$. Hence one can introduce the parameterization $y(L_0) = \mathcal{C}_{t_0}$. Furthermore condition (3) along with the exact form of construction (6) implies that $\{y(L_0)\}_{L_0 \geq L^*}$ is a foliation of the annulus $[L^*, \infty) \times \mathbb{T}$ and that curves $y(L_0)$ always intersect the direction field Γ transversally. Define

$$L_{dyn}(L_0) := \text{the area between } y(L_0) \text{ and } \mathcal{C}_0.$$

Since there is already a symplectic form $d\ell \wedge dL$ on the cylinder, formally we can define the dual angular variable ℓ_{dyn} so that

$$d\ell \wedge dL = d\ell_{dyn} \wedge dL_{dyn}.$$

This is well defined since vectors in Γ always make a nonzero angle with the curves $y(L_0)$.

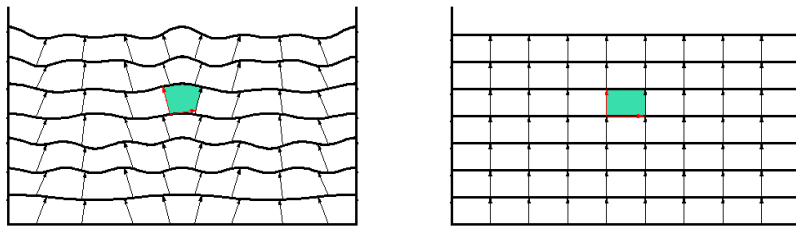


FIGURE 9. Γ (left) and its straightening (right) under canonical change of coordinates

To see this geometrically, consider curves $y(L_0)$ and $y(L_0 + \epsilon)$ with ϵ sufficiently small. At the point $(\ell, L, s) \in y(L_0)$, the direction field Γ defines a unique direction $v = v_{(\ell, L, s)}$

pointing from the circle $y(L_0)$ to the circle $y(L_0 + \epsilon)$. Suppose we take a vector w of length ϵ tangent to $y(L_0)$ at the point (ℓ, L, s) . Then since Γ makes a nonzero angle with $y(L_0)$ we have $v \wedge w = O(\epsilon^2)$ is a nonzero area element.

At the point (ℓ, L, s) , we think of straightening the direction field Γ so that the vector v is pointing vertically, i.e. think of v as the a new action direction, denoted L_{dyn} , and think of the ℓ_{dyn} direction as being specified by straightening the vector w to point horizontally. We may scale length of the vector in the ℓ_{dyn} direction to preserve the area $v \wedge w$ up to order $O(\epsilon^2)$. As $\epsilon \rightarrow 0$, this produces a smooth area form which induces a smooth canonical change of coordinates $\Psi_s : (\ell, L, s) \rightarrow (\ell_{dyn}, L_{dyn}, s)$. \square

By construction, the dynamically defined coordinate system (ℓ_{dyn}, L_{dyn}, s) has the property of twist along trajectories of H . Hence convexity of H in dynamically deformed variables follows.

6. APPLICATION OF AUBRY-MATHER THEORY TO RCP3BP

Definition: A compact invariant region C bounded by the rotationally invariant curves C_- and C_+ is called a *Birkhoff Region of Instability* (BRI).

In such BRIs, Birkhoff showed the existence of orbits passing arbitrarily close to C_- and then to C_+ and vice versa [MF], §17. We need a similar, but stronger result given by Mather [Ma1].

Theorem 6.1. (*Mather Connecting Theorem*) *Suppose $\omega_1 < \alpha_1, \alpha_2 < \omega_2$ and suppose there are no rotationally invariant curves with rotation number $\omega \in (\omega_1, \omega_2)$ in a BRI. Then there is a trajectory in the phase space whose α -limit set lies in the Aubry-Mather set Σ_{α_1} and whose ω -limit sets lies in Σ_{α_2} . Moreover, for a sequence of rotation numbers $\{\alpha_i\}_{i \in \mathbb{Z}}$, $\alpha_i \in (\omega_1, \omega_2)$ and a sequence of positive numbers $\{\epsilon_i\}$, there exists an orbit in the phase space $\{p_j\}$ and an increasing bi-infinite sequence of integers $j(i)$ such that the $\text{dist}(\Sigma_{\alpha_i}, p_{j(i)}) < \epsilon_i$ for all $i \in \mathbb{Z}$.*

Recall that by construction both algebraically deformed Delaunay variables (ADDV) are defined for nearly parabolic motions, and we use the results of appendix 12 to produce dynamically deformed Delaunay variables on a set \mathcal{W} contains a large subset of nearly parabolic motions. Using these variables, the dynamics may be reduced to that of an EAPT map \mathcal{F}_{dyn} defined in a semi-infinite domain, denoted Tw^∞ . It follows from the results in [GK1] that while Tw^∞ is **not** invariant, it **is** free of invariant curves. It turns out that the hypothesis of a BRI in Mather Connecting Theorem can be relaxed slightly without affecting the conclusion. We shall do so using the EAPT \mathcal{F}_{dyn} and the domain Tw^∞ . The key is to specify the location of local and global minimizers. This is the aim of the next several lemmas.

Lemma 6.2. *For any $c > 0$ there is a c -neighborhood $N(\omega, c)$ of the Aubry Mather set Σ_ω for $\omega > 0$ which has well defined action-angle (ADDV) coordinates.*

Proof: After passing through a perihelion, a solution either has an apohelion in its future, or it does not. If it does not have an apohelion in the future, the comet must exit parabolically or hyperbolically. Solutions which exit the Solar System hyperbolically do not have well defined rotation numbers since they must eventually leave the neighborhood of the separatrices

where algebraically deformed Delaunay variables (ADDV) are defined. Parabolic solutions have rotation number $\omega = 0$ since they correspond to separatrices which are converging to the fixed point at infinity in $(\frac{1}{\sqrt{r}}, P_r)$ variables (see fig. 6).

The separatrices are contained in the domain of definition of ADDV by construction (see fig 14). Furthermore, all other trajectories in the domain of definition contain an apohelion in their futures and trajectories remain inside the domain of definition until that apohelion is reached. Hence these solutions have well defined algebraically deformed variables.

Solutions with apohelions stay some bounded distance way from the separatrices and the boundary of the domain of definition. When expressed in ADDV, then these points in the domain of definition have a finite L_ν value (recall $L_\nu \rightarrow \infty$ at the boundary of the domain of definition). See figures 14 and 7. But then these solutions are in neighborhoods of Aubry-Mather sets which have rotation numbers $\omega > 0$. Since L_ν is bounded, then we can take as large of a neighborhood in the L_ν direction as we please around the Aubry-Mather set Σ_ω for $\omega > 0$. \square

The following lemma follows from results in [GK3].

Lemma 6.3. *Let $\omega_{max} = \sup\{\omega : \Sigma_\omega \subset Tw^{Del}\}$ be the maximal rotation number for Aubry-Mather sets which are contained in the twist region $Tw^{Del} \cup Tw^\infty$. There is a continuous function $c(\omega) > 0$ such that for all $\omega \in (0, \omega_{max})$, there is a $c(\omega)$ -neighborhood $N(\omega, c)$ of Σ_ω contained in $Tw^{Del} \cup Tw^\infty$.*

Consider the collection of neighborhoods for $\omega, \omega' \in (0, \omega_{max})$:

$$N(\omega, \omega', c) := \bigcup_{\rho \in (\omega, \omega')} N(\rho, c).$$

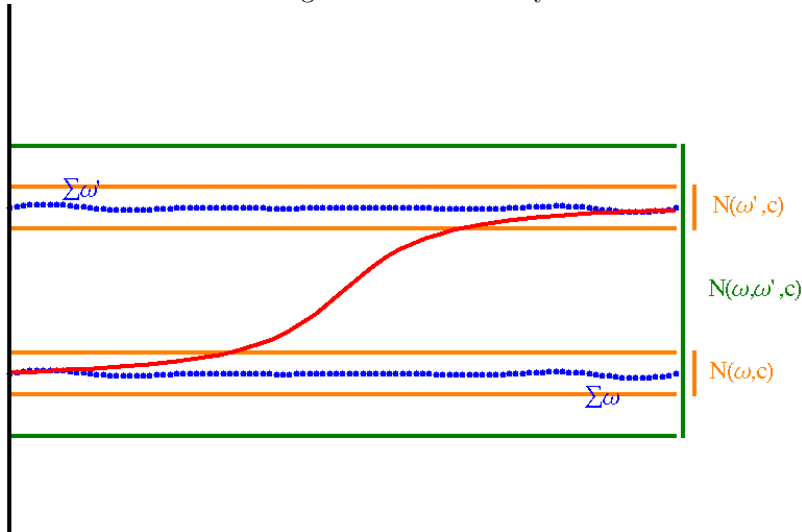
Lemma 6.4. *There exists $c > 0$ such that an orbit of \mathcal{F}_{dyn} connecting Σ_ω and $\Sigma_{\omega'}$ belongs $N(\omega, \omega', c)$.*

The lemma follows from careful study of Mather's original proofs. [X] contains a simpler approach to these results. Simply follow Xia's exposition and note that almost connecting orbits stay in a $2c$ neighborhood of Σ_ω and $\Sigma_{\omega'}$. Such a c is guaranteed to exist by Lemma 6.2 and the Mather Connecting Theorem may be applied in the non-invariant region $N(\omega, \omega', c)$. We are in position to prove the main theorem.

Proof of Theorem 1.2: It is known that every rotation number $\omega \geq 0$ has a non-empty Aubry-Mather set Σ_ω associated to it. Furthermore it is known that Aubry-Mather sets are ordered by rotation number ω with Σ_ω and $\Sigma_{\omega'}$ close in the sense of Hausdorff distance for ω and ω' close [Ban],[MF]. Smaller rotation numbers correspond to slower rotation around the base \mathbb{T} in the ℓ direction, but this is to say that smaller rotation numbers correspond to higher eccentricities.

To get a diffusing orbit in polar coordinates, we use the Mather Connecting Theorem to provide the existence of an orbit of \mathcal{F}_{dyn} with specified properties. Specifically, let us choose the rotation numbers $\{\omega_i\}_{i < 0}$ corresponding to eccentricities near $e = e^*(\mu, J_0) \ll 1$

FIGURE 10. Neighborhoods of Aubry-Mather Sets



and pick the $\{\epsilon_i\}_{i < 0}$ to come arbitrarily close to the desired rotation numbers. This is to say that in negative time we limit on to a bounded Aubry-Mather set of positive rotation number.

To diffuse upwards, we may pick a finite sequence of ω_i 's sufficiently close and by Lemma 6.4, connecting orbits remain in a neighborhood $N(\omega, \omega', c)$ of the Aubry-Mather sets for some c , where c exists by Lemma 6.4.

We claim there exists a ω^* such that solutions in a neighborhood of Σ_{ω^*} escape to infinity after a final passage by the Sun-Jupiter system. Provided such an ω^* exists, select the ω_i 's in Mather's Connecting theorem to approach this ω^* in order to escape. We remark that Lemma 6.4 holds for almost connecting orbits between $\omega = \omega^*$ and $\omega = 0$. Simply pick neighborhoods for $\omega, \omega' \rightarrow 0$ with $c \rightarrow 0$. It remains to show such a ω^* exists.

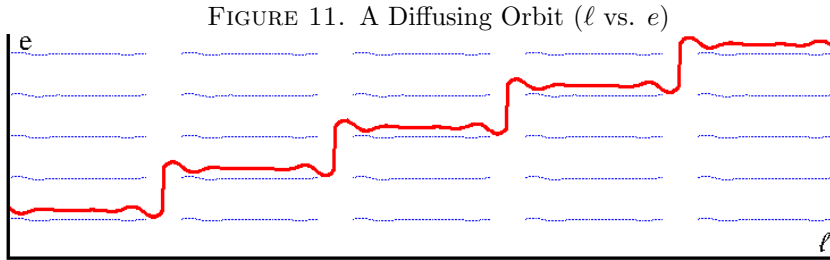
Consider solutions that have $e(t) < 1$ for $t \leq 0$, but eventually leave the homoclinic loop where action-angle variables are well defined. Such solutions have well defined action-angle variables as long as they remain inside the homoclinic loop in figure 7. Escape is possible if $e(t) \geq 1$ for all sufficiently large t . In order to produce such solutions, note that if a comet is exiting the solar system, then at some point, it must make a last passage by the Sun-Jupiter system. During this passage, Jupiter perturbs the eccentricity by some quantity Δe . If the comet has initial eccentricity e^{exit} sufficiently close to one before a close passage, and after this passage $e = e^{exit} + \Delta e > 1$ is large enough, then the comet exits the Solar system hyperbolically.

To verify this magnitude of jump is possible, we phrase the jump in eccentricity in terms of angular momentum to apply the results in appendix 8. It turns out that to exit after a passage

by the Sun on $\mathcal{S}^{out}(1.8)$ then $P_\varphi > 1.8 + 0.04\mu$ when $r = 5$. When passing by the Sun-Jupiter system, the jump in angular momentum can be as large as μ (this follows from the rigorous numerics in [GK1]). It follows that Δe can be as large as 0.07μ for nearly parabolic motions. Hence for $e^{exit} = 1 - 0.07\mu$ escape is possible after a single passage by the Sun-Jupiter system.

Since $e^{exit} < 1$, then a solutions with e^{exit} have a finite L_ν value, denoted L_ν^{exit} , since $L_\nu = L_\nu(J_0, e)$. Since $L_\nu^{exit} < \infty$ there is an Aubry-Mather set Σ_{ω^*} with rotation number $\omega^* > 0$ and a neighborhood $N(\omega^*, c) \subset [L_\nu^{exit}, \infty]$. \square

One could visualize the Aubry-Mather sets as the remainders of tori after a perturbation has been filled them with infinitely many small holes. To envision a diffusing orbit, first imagine unrolling the cylinder on the real plane. A diffusing orbit will be one which “climbs a set of stairs”, i.e. increases in the holes of the Aubry-Mather set, and then follows the remnants of a torus of higher rotation number for a while. The largest increase (“taking a step”) occurs primarily at times when the comet is at the perihelion.



We remark that while elements of the proof explicitly used constants derived from $J_0 = 1.8$ and $\mu = 10^{-3}$, the software is robust enough to handle other constants.

Proof of Corollary 1.3: Realization of Chazy Motions is an easily consequence of the application of Mather’s variational methods. It suffices to describe how to get the desired behavior in one direction, since achieving different behaviors in forward and backward time is obtained by concatenation of two one-sided sequences of rotation numbers.

In order to achieve bounded motions, pick diffusing orbits to limit to an Aubry-Mather set Σ_ω with $\omega > \omega^* > 0$ where ω^* is defined in the proof of Theorem 1.2. More specifically pick sequences $\omega_i \rightarrow \omega$ and $\epsilon_i \rightarrow 0$ in Mather’s Connecting Theorem. For example, ω could correspond to the rotation number for some rotationally invariant curve of low eccentricity, say a KAM curve.

In order to get unbounded hyperbolic motions, use the method described in the proof of Theorem 1.2. Namely, approach the rotation number ω^* before a perihelion, and after a perihelion, a neighborhood of these solutions is carried to infinity hyperbolically.

Since $\mathcal{U} = N(\omega^*, c(\omega^*)) \cap \{P_r = 0\}$ contains solutions with $|\dot{r}| \rightarrow c > 0$ as $t \rightarrow \infty$, then in some larger neighborhood $\hat{\mathcal{U}}$ there are points which escapes with $|\dot{r}| \rightarrow 0$, i.e. parabolically, by smooth dependence on initial conditions. In the unperturbed case \mathcal{F}_0 has a fixed point at infinity and the homoclinic loop at infinity corresponding to parabolic motion. In the perturbed case $\mu > 0$, the separatrices no longer overlap, and some of points on the separatrices have eccentricities less than one. While the separatrices themselves have rotation number $\omega = 0$, they lay in the neighborhood of Σ_{ω^*} . To limit to parabolic motions, it is possible to pick a finite sequence of $\omega_i \rightarrow \omega^*$ and ϵ_i sufficiently small solutions to constrain solutions into a neighborhood where escape is possible. Then select as the tail of the sequence rotation numbers $\omega_i \rightarrow 0$ and $\epsilon_i \rightarrow 0$. The diffusing trajectory has a parabolic tail.

To get oscillatory orbits, pick a sequence of rotation numbers ω_i with a subsequence $\omega_{i_k} \rightarrow 0$ and every other term in the sequence bounded away from rotation number $\omega = 0$. Pick the ϵ_i so that the neighborhood of ω_{i_k} is so small that the neighborhood never intersects the separatrices when leaving the sun. Since the time between an apohelion and perihelion is finite, this is possible. Physically this constrains the comet to always have an apohelion and turn around to make another pass by the Sun-Jupiter system. \square

The remainder of the paper is dedicated to careful analysis of the RCP3BP to show the applicability of Theorem 5.2. Specifically our goal is to show applicability in the case of $\mu = 10^{-3}$ and $J_0 = 1.8$.

7. APPENDIX: ESTIMATES ON PERTURBATION TERMS

It turns out that the perturbation term can be written

$$\Delta H(r, \varphi; \mu) = \sum_{i=1}^{\infty} (-1)^i \frac{\mu(1-\mu)(\mu^i - (\mu-1)^i)}{r^{i+2}} P_{i+1}(\cos(\varphi))$$

where P_i is the i^{th} Legendre polynomial. The bounds on the Legendre polynomials can be used to produce upper bounds on the perturbation terms which are independent of φ . See [GK1]. It turns out that

$$\begin{aligned} \max_{\varphi} |\Delta H(r, \varphi; \mu)| &\leq (|\Delta H|)^+(r) := \frac{\mu(1-\mu)}{r(r-1+\mu)(r+\mu)} \\ \max_{\varphi} |\partial_{\varphi} \Delta H(r, \varphi; \mu)| &\leq (|\frac{\partial \Delta H}{\partial \varphi}|)^+(r) := \frac{\mu(1-\mu)r(1+3r(r-1)+\mu(6r-3)+3\mu^2)}{(r-1+\mu)^3(r+\mu)^3} \\ \max_{\varphi} |\partial_r \Delta H(r, \varphi; \mu)| &\leq (|\frac{\partial \Delta H}{\partial r}|)^+(r) := -\frac{1}{r^2} + \frac{\mu}{(r-1+\mu)^2} + \frac{1-\mu}{(r+\mu)^2} \\ \max_{\varphi} |\partial_{r\varphi}^2 \Delta H(r, \varphi; \mu)| &\leq (|\frac{\partial^2 \Delta H}{\partial r \partial \varphi}|)^+ := \mu(1-\mu) \left(\frac{3(1-\mu)}{(r-1+\mu)^4} + \frac{2}{(r-1+\mu)^3} + \frac{3\mu}{(\mu+r)^4} - \frac{2}{(\mu+r)^3} \right) \\ \max_{\varphi} |\partial_{\varphi\varphi}^2 \Delta H(r, \varphi; \mu)| &\leq (|\frac{\partial^2 \Delta H}{\partial \varphi^2}|)^+ := 3\mu(1-\mu) \left(\frac{(1-\mu)^3}{(r-1+\mu)^5} + \frac{\mu^3}{(\mu+r)^5} \right) \\ \max_{\varphi} |\partial_{rr}^2 \Delta H(r, \varphi; \mu)| &\leq (|\frac{\partial^2 \Delta H}{\partial r^2}|)^+ := -\frac{2}{r^3} + \frac{2\mu}{(r-1+\mu)^3} + \frac{2(1-\mu)}{(\mu+r)^3} \end{aligned}$$

Remark: All of these estimates are independent of the Jacobi constant and are $O(\frac{\mu}{r^3})$ or better. For $r \geq 1.5$ and $\mu = 0.001$, we can be even more explicit.

$$\begin{aligned} \max_{\varphi} |\Delta H(r, \varphi; \mu)| &\leq \frac{3\mu}{r^3} & \max_{\varphi} |\partial_{\varphi} \Delta H(r, \varphi; \mu)| &\leq \frac{39\mu}{r^3} \\ \max_{\varphi} |\partial_r \Delta H(r, \varphi; \mu)| &\leq \frac{15\mu}{r^4} & \max_{\varphi} |\partial_{r\varphi}^2 \Delta H(r, \varphi; \mu)| &\leq \frac{319\mu}{r^4} \\ \max_{\varphi} |\partial_{\varphi\varphi}^2 \Delta H(r, \varphi; \mu)| &\leq \frac{719\mu}{r^5} & \max_{\varphi} |\partial_{rr}^2 \Delta H(r, \varphi; \mu)| &\leq \frac{108\mu}{r^5} \end{aligned}$$

7.1. Estimates on terms involving Delaunay. In this subsection, we collect estimates involving perturbation terms and Delaunay variables. Estimates throughout this section use $\mu = 10^{-3}$ and the above estimates on perturbation terms in polar. Additionally we require $J_0 = 1.8$ and $G \in [1.67, 1.81]$ which all solutions of interest satisfy. For these parameters, one can show that eccentricity $e \in [0.52, 1.04]$. In [GK1] we established that for these parameters that $r \geq 1.6$.

We remark that some estimates are carried in Delaunay and others in algebraically deformed Delaunay (ADDV). If an estimate is done using Delaunay, it is not hard to convert it to ADDV by simply attaching subscript ν 's to all Delaunay variables. For the definition of ADDV and since $\nu = 2.8\mu$ is small, then it is not hard to show that $G_{\nu} \in [1.67 - \nu, 1.81 - \nu]$ and $e_{\nu} \in [0.5, 1.03]$. Furthermore if $e \leq 1$, then $e_{\nu} \leq 0.998 < 1$.

Lemma 7.1. *If $J_0 = 1.8$, $\mu = 10^{-3}$, and $G \in [1.67, 1.81]$, then $|\frac{\partial \Delta H}{\partial L_{\nu}}| \leq 0.105512$. Furthermore $|\frac{\partial \Delta H}{\partial L_{\nu}}| \leq \frac{433\mu}{r^3}$ for $r \geq 1.6$.*

Proof:

$$\frac{\partial \Delta H}{\partial L_{\nu}} = \left(\frac{\partial \Delta H}{\partial r} \right) \left(\frac{\partial r}{\partial L_{\nu}} \right) + \left(\frac{\partial \Delta H}{\partial \varphi} \right) \left(\frac{\partial \varphi}{\partial L_{\nu}} \right)$$

We can use the quantities in the above subsection to produce upper bounds on the derivatives of the perturbation terms. Use of a computer algebra system⁶ allows us to write

$$\frac{\partial r}{\partial L_{\nu}} = \frac{G_{\nu}^2 r + 2e_{\nu}^2 r^2 - G_{\nu}^4}{L_{\nu} e_{\nu}^2 r} \quad \frac{\partial \varphi}{\partial L_{\nu}} = -\frac{G_{\nu}(G_{\nu}^2 + r) \sin(u_{\nu})}{r^2 e_{\nu}}$$

For $\mu = 0.001$, $J_0 = 1.8$, and $G \in [1.67, 1.81]$, then $G_{\nu} = G - 2.8\mu \leq 1.81$ and $e_{\nu} \in [0.5, 1.03]$. Furthermore, $L_{\nu} \geq 1.6$ everywhere on the energy surface. These estimates let us write

$$\begin{aligned} \left| \frac{\partial r}{\partial L_{\nu}} \right| &= \left| \frac{G_{\nu}^2 r + 2e_{\nu}^2 r^2 - G_{\nu}^4}{L_{\nu} e_{\nu}^2 r} \right| \leq \frac{1.81^2 r + 2 \cdot 1.03^2 r^2 + 1.81^4}{1.6 \cdot 0.5^2 r} \\ \left| \frac{\partial \varphi}{\partial L_{\nu}} \right| &= \left| -\frac{G_{\nu}(G_{\nu}^2 + r) \sin(u_{\nu})}{r^2 e_{\nu}} \right| \leq \frac{1.81(1.81^2 + r)}{r^2 0.5} \end{aligned}$$

Note the upper bounds are at most $O(r)$. Hence

⁶Actually the computer algebra system is used to generate these quantities in Delaunay variables. Since $\mathcal{D}_{\nu} = \mathcal{D} \circ (P_{\varphi} \mapsto P_{\varphi} - \nu)$, then as algebraic expressions they are the same as in Delaunay. However the variables themselves have different values.

$$\left| \frac{\partial \Delta H}{\partial L_\nu} \right| \leq \left(\left| \frac{\partial \Delta H}{\partial r} \right| \right)^+ \left| \left(\frac{\partial r}{\partial L_\nu} \right) \right| + \left(\left| \frac{\partial \Delta H}{\partial \varphi} \right| \right)^+ \left| \left(\frac{\partial \varphi}{\partial L_\nu} \right) \right| = O\left(\frac{\mu}{r^3}\right)$$

Using a computer algebra system, it is not hard to show the upper bound is in fact strictly decreasing in r and the upper bound of $r > 1.6$ yields that $\left| \frac{\partial \Delta H}{\partial L_\nu} \right| \leq 0.105512$. Additionally the CAS can explicitly show the upper bound is less than $\frac{433\mu}{r^3}$ for $r \geq 1.6$ \square

Lemma 7.2. *If $J_0 = 1.8$, $\mu = 10^{-3}$, and $G \in [1.67, 1.81]$ then $\left| \frac{\partial \Delta H}{\partial G_\nu} \right| \leq 0.025$.*

Proof: In [GK1], we showed $\left| \frac{\partial \Delta H}{\partial G} \right| \leq 0.025$. This bound still holds since $\frac{\partial \Delta H}{\partial G_\nu} = \frac{\partial \Delta H}{\partial G} \frac{\partial G}{\partial G_\nu} = \frac{\partial \Delta H}{\partial G}$ and $\frac{\partial G_\nu}{\partial G} = 1$ since $G_\nu = G - \nu$. \square

Lemma 7.3. *If $J_0 = 1.8$, $\mu = 10^{-3}$, and $G \in [1.66, 1.81]$, then $\left(\frac{\partial L_\nu}{\partial G} \right) \geq 2.85283$.*

For the 2BP(SC), $L = (2J_0 - 2G)^{-\frac{1}{2}}$, so $\left(\frac{\partial L}{\partial G} \right) = 2(2J_0 - 2G)^{-\frac{3}{2}} \geq 6.74937$ for $J_0 = 1.8$ and $G \geq 1.66$.

Proof of Lemma 7.3: Write the RCP3BP Hamiltonian in ADDV and use L_ν as implicit function of the other variables.

$$(J_0 + \nu) = \frac{1}{2(L_\nu(J_0, \ell_\nu, G_\nu, g_\nu))^2} + G_\nu + \Delta H\left(L_\nu(J_0, \ell_\nu, G_\nu, g_\nu), \ell_\nu, G_\nu, g_\nu\right).$$

Take the derivative and solve for $\frac{\partial L_\nu}{\partial G_\nu}$ to get

$$(7) \quad \frac{\partial L_\nu}{\partial G_\nu} = \frac{1 - \frac{\partial \Delta H}{\partial G_\nu}}{L_\nu^{-3} + \frac{\partial \Delta H}{\partial L_\nu}}$$

From Lemmas 7.1 and 7.2 and the fact that $L_\nu \geq 1.6$, it follows that

$$\left(\frac{\partial L_\nu}{\partial G_\nu} \right)^{-1} \leq \frac{L_\nu^{-3} + 0.0802431}{1 - 0.025} \leq 0.350529$$

Hence the claim follows. \square

Lemma 7.4. *For $\mu = 0.001$, $r \geq 1.5$, $G \in [1.6, 1.81]$,*

$$\left| \left(\frac{\partial^2 \Delta H}{\partial \ell \partial G} \right) \right| \leq \frac{21}{r^{7/2}(1-e)^{3/2}} + \frac{35|P_r|}{r^{5/2}(1-e)^{5/2}}$$

Proof: Starting with

$$\frac{\partial \Delta H}{\partial \ell} = \left(\frac{\partial \Delta H}{\partial r} \right) \left(\frac{\partial r}{\partial \ell} \right) + \left(\frac{\partial \Delta H}{\partial \varphi} \right) \left(\frac{\partial \varphi}{\partial \ell} \right)$$

compute

$$\begin{aligned} \left(\frac{\partial^2 \Delta H}{\partial \ell \partial G} \right) &= \left(\frac{\partial \Delta H}{\partial r} \right) \left(\frac{\partial^2 r}{\partial \ell \partial G} \right) + \left(\frac{\partial r}{\partial \ell} \right) \left(\frac{\partial^2 \Delta H}{\partial r^2} \frac{\partial r}{\partial G} + \frac{\partial^2 \Delta H}{\partial r \partial \varphi} \frac{\partial \varphi}{\partial G} \right) + \\ &\quad \left(\frac{\partial \Delta H}{\partial \varphi} \right) \left(\frac{\partial^2 \varphi}{\partial \ell \partial G} \right) + \left(\frac{\partial \varphi}{\partial \ell} \right) \left(\frac{\partial^2 \Delta H}{\partial r \partial \varphi} \frac{\partial r}{\partial G} + \frac{\partial^2 \Delta H}{\partial \varphi^2} \frac{\partial \varphi}{\partial G} \right) \end{aligned}$$

We shall estimate each term. It is helpful to know the following conversions between polar and Delaunay. These formulas were found with the aid of a computer in some cases.

$$\begin{aligned}
r &= L^2 (1 - e \cos(u)) & P_r &= \frac{Le \sin(u)}{r} \\
\frac{\partial r}{\partial G} &= \frac{G(G^2 - r)}{re^2} & \frac{\partial \varphi}{\partial G} &= \frac{(G^2 + r)P_r}{re^2} \\
\frac{\partial r}{\partial \ell} &= L^3 P_r & \frac{\partial \varphi}{\partial \ell} &= -\frac{GL^3}{r^2} \\
\frac{\partial^2 r}{\partial \ell \partial G} &= \frac{(e^2 - 1)GL^5 P_r}{e^2 r^2} & \frac{\partial^2 \varphi}{\partial \ell} &= -\frac{L(2G^4 L^2 + e^2 L^2 r^2 + G^2(4r^2 - 6L^2 r))}{e^2 r^4}
\end{aligned}$$

For each term, we seek only an upper bound in absolute value. In general we desire as large a power of r in the denominator as possible, and as small a power of $(1 - e)$ in the denominator as possible.

$$\begin{aligned}
\left| \left(\frac{\partial \Delta H}{\partial r} \right) \left(\frac{\partial^2 r}{\partial \ell \partial G} \right) \right| &\leq \frac{15\mu (1 - e^2)GL^5 P_r}{r^4 e^2 r^2} \leq \frac{1.81L^5 P_r}{0.5^2 r^6} \leq \frac{14P_r}{(1 - e)^{5/2} r^{7/2}} \\
\left| \left(\frac{\partial r}{\partial \ell} \right) \left(\frac{\partial^2 \Delta H}{\partial r^2} \right) \left(\frac{\partial r}{\partial G} \right) \right| &\leq L^3 P_r \leq \frac{108\mu Gr + G^3}{r^5 re^2} \leq \frac{108\mu L^3(1.81r + 1.81^3)P_r}{r^{60.5^2}} \\
&\leq \frac{2.5L^3 P_r}{r^5} \leq \frac{2.5P_r}{r^{7/2}(1 - e)^{3/2}} \\
\left| \left(\frac{\partial r}{\partial \ell} \right) \left(\frac{\partial^2 \Delta H}{\partial r \partial \varphi} \right) \left(\frac{\partial \varphi}{\partial G} \right) \right| &\leq L^3 P_r \frac{319\mu (G^2 + r)P_r}{r^4 re^2} \leq \frac{319\mu L^3(1.81^2 + r)P_r}{0.5^2 r^5} \\
&\leq \frac{4.1L^3 P_r}{r^4} \leq \frac{4.1P_r}{r^{5/2}(1 - e)^{3/2}} \\
\left| \left(\frac{\partial \varphi}{\partial \ell} \right) \left(\frac{\partial^2 \Delta H}{\partial r \partial \varphi} \right) \left(\frac{\partial r}{\partial G} \right) \right| &\leq \frac{GL^3}{r^2} \frac{319\mu G(G^2 + r)}{r^4 re^2} \leq \frac{1.81L^3 319\mu 1.81(1.81^2 + r)}{0.5^2 r^7} \\
&\leq \frac{14L^3}{r^6} \leq \frac{14}{r^{9/2}(1 - e)^{3/2}} \\
\left| \left(\frac{\partial \varphi}{\partial \ell} \right) \left(\frac{\partial^2 \Delta H}{\partial \varphi^2} \right) \left(\frac{\partial \varphi}{\partial G} \right) \right| &\leq \frac{GL^3}{r^2} \leq \frac{719\mu (G^2 + r)P_r}{r^5 re^2} \leq \frac{1.81L^3 P_r(1.81^2 + r)719\mu}{r^{80.5^2}} \\
&\leq \frac{17L^3 P_r}{r^7} \leq \frac{17P_r}{r^{11/2}(1 - e)^{3/2}}
\end{aligned}$$

$$\begin{aligned}
\left| \left(\frac{\partial \Delta H}{\partial \varphi} \right) \left(\frac{\partial^2 \varphi}{\partial \ell \partial G} \right) \right| &\leq \frac{39\mu L(2G^4 L^2 + e^2 L^2 r^2 + G^2(4r^2 + 6L^2 r))}{r^3 e^2 r^4} \\
&\leq \frac{39\mu L(2 \cdot 1.81^4 L^2 + 1.01^2 L^2 r^2 + 1.81^2(4r^2 + 6L^2 r))}{0.5^2 r^7} \\
&\leq \frac{4L^3}{r^7} + \frac{4L^3}{r^6} + \frac{3L}{r^5} + \frac{L^3}{r^5} \\
&\leq \frac{4}{r^{11/2}(1-e)^{3/2}} + \frac{4}{r^{9/2}(1-e)^{3/2}} + \frac{3}{r^{9/2}(1-e)^{1/2}} + \frac{1}{r^{7/2}(1-e)^{3/2}}
\end{aligned}$$

Adding all the terms with P_r 's we find

$$\begin{aligned}
\frac{14P_r}{(1-e)^{5/2} r^{7/2}} + \frac{2.5P_r}{r^{7/2}(1-e)^{3/2}} + \frac{17P_r}{r^{11/2}(1-e)^{3/2}} + \frac{4.1P_r}{r^{5/2}(1-e)^{3/2}} \\
\leq \frac{17 + 17r^2 + 5r^3 + 1.01(17 + 3r^2 + 5r^3)}{r^{11/2}(1-e)^{5/2}} \leq \frac{35P_r}{r^{5/2}(1-e)^{5/2}}
\end{aligned}$$

Adding all the terms without P_r 's we find

$$\begin{aligned}
\frac{4}{r^{11/2}(1-e)^{3/2}} + \frac{4}{r^{9/2}(1-e)^{3/2}} + \frac{3}{r^{9/2}(1-e)^{1/2}} + \frac{1}{r^{7/2}(1-e)^{3/2}} + \frac{14}{r^{9/2}(1-e)^{3/2}} \\
\leq \frac{4 + 27r + r^2}{r^{11/2}(1-e)^{3/2}} \leq \frac{21}{r^{7/2}(1-e)^{3/2}}
\end{aligned}$$

□

8. APPENDIX: ESTIMATES ON CHANGE IN ANGULAR MOMENTUM

In [GK1], Appendix 11 we prove the following lemma on change in angular momentum.

Lemma 8.1. *Assume $\mu = 10^{-3}$, $J_0 = 1.8$, and $P_\varphi(t) \in [1.66, 1.81]$ (i.e. $e(t) \in [0.48, 1.04]$) for a sufficiently long time interval. Then*

- *When approaching a perihelion from the previous apohelion (or from infinity), angular momentum does not change by more than 0.0215298μ provided $r \geq 5$.*
- *When approaching a perihelion from the previous apohelion (or from infinity), angular momentum does not change by more than 4.44885μ .*
- *Angular momentum won't change by more than 1.444μ after a full revolution around the sun.*

The proof of the lemma is constructive and produces the function $\rho(r)$ used in numerous locations in this document. Define ρ to be

$$\begin{aligned}
(8) \quad |\Delta P_\varphi(t_0, t_1)| &\leq \rho(r) := \frac{1}{1 - \frac{M}{r^2}} \left((|\Delta H|)^+(r) + \int_r^\infty (|\frac{\partial \Delta H}{\partial r}|)^+ dr \right) \\
&= \frac{2\mu(1-\mu)r}{(r^2 - M)(r - 1 + \mu)(r + \mu)}
\end{aligned}$$

provided that the radius is decreasing from $r(t_0) = r_0$ to $r(t_1) = r$. In [GK1] we justified using $M = (\max P_\varphi) = 1.81$ for RCP3BP($10^{-3}, 1.8$).

9. APPENDIX: PROOF OF THEOREM 3.1

Now using the bounds in appendices 7 and 8 we prove Theorem 3.1.

Proof of Theorem 3.1: On the energy surface $\mathcal{S}^{out}(J_0)$, one can explicitly solve for P_r .

$$(9) \quad P_r(J_0, r, \varphi, P_\varphi) = \pm \sqrt{-2J_0 + 2P_\varphi - \frac{P_\varphi^2}{r^2} + \frac{2}{r} - 2\Delta H(r, \varphi)}$$

It is desirable to bound this expression independently of φ . Furthermore for $\mu > 0$, P_φ is no longer identically constant and it is desirable to bound this quantity independently of fluctuations in P_φ .

Suppose \mathcal{P}_μ^+ is a forward separatrix for RCP3BP(μ, J_0). (The proof in the backward case is analogous.) Then \mathcal{P}_μ^+ has a parameterization $(r, \varphi, P_r, P_\varphi)(t)$ for $t > 0$. Our goal is bound the P_r component. By definition of parabolic escape, $P_r(t) \rightarrow 0$ as $t \rightarrow \infty$. It follows from (9) that $P_\varphi(t) \rightarrow J_0$ as $t \rightarrow \infty$. It then follows from the definition of ρ in section 8 that $P_\varphi(t) \leq J_0 + \rho(r(t))$ for $t \geq 0$.

Since we are on the energy surface $\mathcal{S}^{out}(J_0)$, we can use (9) to compute

$$|P_r(J_0, r, \varphi, P_\varphi)| \leq \sqrt{-2J_0 + 2(J_0 + \rho(r)) - \frac{(J_0 + \rho(r))^2}{r^2} + \frac{2}{r} + 2(|\Delta H|^+)(r)}$$

This follows from extremizing ΔH using the bounds in section 7, and the fact that as a function $P_r(J_0, r, \varphi, P_\varphi)$ is increasing in the variable P_φ in the outer Hill region, so it suffices to replace P_φ by $J_0 + \rho(r)$ to obtain an upper bound.

Claim: There exists $\nu > 0$ such that for $r > 1 + \mu$ we have

$$\sqrt{2\rho(r) - \frac{(J_0 + \rho(r))^2}{r^2} + \frac{2}{r} + 2(|\Delta H|^+)(r)} \leq \sqrt{\frac{2}{r} - \frac{(J_0 - \nu)^2}{r^2}}.$$

Proof of Claim: The expressions $\rho(r)$ and $(|\Delta H|^+)(r)$ are $O(\frac{\mu}{r^3})$ (see appendix 7 and bound (8)). Since $\rho(r) \geq 0$, then replacing $-\frac{(J_0 + \rho(r))^2}{r^2}$ by $-\frac{(J_0 - c)^2}{r^2}$ for some $c > 0$ has the effect of increasing the terms under the radical by a factor of $O(\frac{1}{r^2})$ which dominates the $O(\frac{\mu}{r^3})$ terms $2\rho(r) + 2(|\Delta H|^+)$. Hence there exists some smallest positive c , which we denote ν for which the claimed the bound holds. \square

By Lemma 1.1 we know that radius of perihelion is bounded from below by $r = \frac{J_0^2}{2} - 8\mu$. Due to monotonicity we have

$$\nu = J_0 - \sqrt{(J_0 + \rho(r))^2 - 2r^2((|\Delta H|^+)(r) + \rho(r))} \Big|_{r=\frac{J_0^2}{2}-8\mu}.$$

We note that right hand side of the inequality in the claim also arises by solving the equation $H_\nu(r, P_r, J_0) = 0$ for P_r , where H_ν is defined in (3). This quantity, also given in (4), parameterizes the homoclinic loop for H_ν . Hence the separatrices for RCP3BP(μ, J_0) are contained inside of this loop.

To be explicit, let us use the parameters $\mu = 0.001$ and $J_0 = 1.8$. If P_r is radial velocity of a separatrix parameterized by $(r, \varphi, P_\varphi)(t)$ on $\mathcal{S}^{out}(1.8)$, one can prove

$$\begin{aligned} |P_r(1.8, r, \varphi, P_\varphi)(t)| &\leq \sqrt{-2 \cdot 1.8 + 2 \cdot P_\varphi - \frac{P_\varphi^2}{r^2} + \frac{2}{r} - 2\Delta H(r, \varphi)} \\ &\leq \sqrt{\frac{2}{r} - \frac{(1.8 - 2.8\mu)^2}{r^2}} \end{aligned}$$

Hence we have successfully enclosed the RCP3BP(10^{-3} , 1.8) separatrices inside the homoclinic loop for H_ν with $\nu \leq 2.8\mu$. \square

10. APPENDIX: AN ANALYSIS OF THE VARIATIONAL EQUATIONS FOR THE RCP3BP

In this section we analyze the equations of first variation for the RCP3BP. Consider the following time dependent matrix $A(t)$ which depends on the flow of the RCP3BP at time t :

$$A(t) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ \frac{-2P_\varphi}{r^3} & 0 & 0 & \frac{1}{r^2} \\ \frac{-3P_\varphi^2}{r^4} + \frac{2}{r^3} - \frac{\partial^2 \Delta H}{\partial r^2} & -\frac{\partial^2 \Delta H}{\partial r \partial \varphi} & 0 & \frac{2P_\varphi}{r^3} \\ -\frac{\partial^2 \Delta H}{\partial r \partial \varphi} & -\frac{\partial^2 \Delta H}{\partial \varphi^2} & 0 & 0 \end{pmatrix}$$

Let

$$X(t) = \begin{pmatrix} \frac{\partial r}{\partial r_0} & \frac{\partial r}{\partial \varphi_0} & \frac{\partial r}{\partial P_{r_0}} & \frac{\partial r}{\partial P_{\varphi_0}} \\ \frac{\partial \varphi}{\partial r_0} & \frac{\partial \varphi}{\partial \varphi_0} & \frac{\partial \varphi}{\partial P_{r_0}} & \frac{\partial \varphi}{\partial P_{\varphi_0}} \\ \frac{\partial P_r}{\partial r_0} & \frac{\partial P_r}{\partial \varphi_0} & \frac{\partial P_r}{\partial P_{r_0}} & \frac{\partial P_r}{\partial P_{\varphi_0}} \\ \frac{\partial P_\varphi}{\partial r_0} & \frac{\partial P_\varphi}{\partial \varphi_0} & \frac{\partial P_\varphi}{\partial P_{r_0}} & \frac{\partial P_\varphi}{\partial P_{\varphi_0}} \end{pmatrix}.$$

Then the equations of variation are given by a time-dependent linear ODE:

$$(10) \quad \dot{X} = A(t)X$$

The equations of variation tell us how a vector $v = (v_1, v_2, v_3, v_4)$ is transported under the flow in the tangent space. Since we are working on an energy surface $\mathcal{S}(J_0)$, we must consider vectors which are tangent to the energy surface. Differentiating $H(r, \varphi, P_r, P_\varphi) = -J_0$, we see that vectors must satisfy the constraint:

$$\left(\frac{\partial H}{\partial r}\right)v_1 + \left(\frac{\partial H}{\partial \varphi}\right)v_2 + \left(\frac{\partial H}{\partial P_r}\right)v_3 + \left(\frac{\partial H}{\partial P_\varphi}\right)v_4 = 0$$

Using this constraint on the initial conditions, we can solve a reduced linear system involving only 3 of the 4 variables in each column of X . Additionally knowledge of solutions involving 3 of the column vectors may be used to obtain solutions to the fourth. It shall be made clear in context which variable is being implicitly defined. In all cases, we call the equations of variation with the flow on $\mathcal{S}(J_0)$ and tangent vectors satisfying (10) the *reduced equations of variation*. It is not hard to show if an initial tangent vector satisfies (10), then it satisfies (10) for all time under the full flow. Hence when convenient, we shall use the full system with the understanding the the initial conditions are taken tangent to the energy surface.

Theorem 10.1. *Consider RCP3BP($10^{-3}, 1.8$) and suppose the initial conditions for the flow of the are at the apohelion. If $\left(\frac{\partial r}{\partial P_{\varphi_0}}, \frac{\partial \varphi}{\partial P_{\varphi_0}}, \frac{\partial P_{\varphi}}{\partial P_{\varphi_0}}\right)(0) = (0, 0, 1)$ and the remaining initial condition satisfies (10) then*

- (1) $\left|\left(\frac{\partial P_{\varphi}}{\partial P_{\varphi_0}}\right)(t) - 1\right| \leq 0.000268671$ for $t \in [0, t_5]$ where t_5 is the first positive time such that $r(t_5) = 5$.
- (2) $\left(\frac{\partial P_{\varphi}}{\partial P_{\varphi_0}}\right)(t) \in [0.12, 1.79]$ for $t \in [0, T]$ where T is the first positive time that the comet is at the perihelion.

Note that for r large $A(t) \approx (0)$ so we expect the system to behave roughly like $X(t) = X(0)$. Hence Theorem 10.1 says that far enough from the sun, the variational equations don't vary too much.

Consider the decomposition of the configuration space into kick and outside regions. When $\mu = 10^{-3}$, for $r \geq 5$ one can show that $|\Delta H| \leq 10^{-5}$. Call the region $\{r \geq 5\}$ the *outside region* since the comet is practically outside the range of influence of Jupiter. Call the region $\{r \leq 5\}$ the *kick region* as the comet's orbital parameters are perturbed (or kicked) more in this region. In the outside region, the variational equations behave very much like those of the 2BP(SC).

The theorem is broken down into a series of lemmas. First a series of rough estimates is established in the outside region where the perturbation term is small. These estimates may be done by hand. These rough estimates are then used to produce refined estimates of the behavior in the outside region. This shall yield the first claim of theorem. The second claim is proved with the assistance of a computer since the perturbation terms have a much stronger influence in the kick region and by hand estimates are insufficient to obtain good estimates.

Lemma 10.2. *Consider RCP3BP($10^{-3}, 1.8$) and suppose the initial conditions for the flow are at the apohelion. If $\left(\frac{\partial r}{\partial P_{\varphi_0}}, \frac{\partial \varphi}{\partial P_{\varphi_0}}, \frac{\partial P_{\varphi}}{\partial P_{\varphi_0}}\right)(0) = (0, 0, 1)$ and the remaining initial condition satisfies (10), then*

$$\left|\left(\frac{\partial r}{\partial P_{\varphi_0}}, \frac{\partial \varphi}{\partial P_{\varphi_0}}, \frac{\partial P_{\varphi}}{\partial P_{\varphi_0}}\right)(t) - (0, 0, 1)\right| \leq 1.31926$$

for $t \in [0, t_5]$ where t_5 is the first positive time such that $r(t_5) = 5$, and where the norm considered is the $p = 1$ norm.

Proof of Lemma 10.2: First we shall reduce the 2-degree-of-freedom Hamiltonian for the RCP3BP in polar to a 1.5-degree-of-freedom time periodic system. Then we analyze the corresponding reduced equations of variation.

Let us begin by defining a time-rescaled Hamiltonian

$$H_{J_0} = -J_0 + P_r$$

where H given by (2) is the Hamiltonian for RCP3BP in rotating polar coordinates. This Hamiltonian arises by considering the energy reduction procedure e.g [A]. Examining the

equations of motion,

$$\begin{aligned}\frac{\partial H_{J_0}}{\partial P_r} &= 1 & \frac{\partial H_{J_0}}{\partial P_\varphi} &= \frac{\partial H}{\partial P_\varphi} \\ \frac{\partial H_{J_0}}{\partial r} &= \frac{\partial H}{\partial r} & \frac{\partial H_{J_0}}{\partial \varphi} &= \frac{\partial H}{\partial \varphi}\end{aligned}$$

we notice that flows of H on $\mathcal{S}(J_0) = \{H = -J_0\}$ and flows on H_{J_0} are the same up to rescaling of time, except at apohelion/perihelion points where $P_r = \dot{r}$ is zero. Furthermore $P_r = P_r(J_0, r, \varphi, P_\varphi)$ is explicitly given by (9) on $\mathcal{S}(J_0)$. The time rescaling for H_{J_0} is given by $r(t) \mapsto t$. Notice that this map is monotonic except at apohelions/perihelions and this is why the rescaling is not defined at those points.

We now consider the equations of variations for the rescaled system. Let $v = \left(\frac{\partial r}{\partial P_{\varphi_0}}, \frac{\partial \varphi}{\partial P_{\varphi_0}}, \frac{\partial P_r}{\partial P_{\varphi_0}}, \frac{\partial P_\varphi}{\partial P_{\varphi_0}} \right) (0)$. Notice that P_r is given implicitly by (9), $v_3 = \frac{\partial P_r}{\partial P_{\varphi_0}}$ is given implicitly by (10), and r serves as the time variable. Hence it suffices to consider only the 2×2 system

$$(11) \quad \left(\frac{d}{dr} \right) \begin{pmatrix} \frac{\partial \varphi}{\partial P_{\varphi_0}} \\ \frac{\partial P_\varphi}{\partial P_{\varphi_0}} \end{pmatrix} = \frac{1}{P_r} \begin{pmatrix} 0 & \frac{1}{r^2} \\ -\frac{\partial^2 \Delta H}{\partial \varphi^2} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \varphi}{\partial P_{\varphi_0}} \\ \frac{\partial P_\varphi}{\partial P_{\varphi_0}} \end{pmatrix}$$

Let $\tilde{A}(r)$ denote the reduced coefficient matrix above, and let $x = \left(\frac{\partial \varphi}{\partial P_{\varphi_0}}, \frac{\partial P_\varphi}{\partial P_{\varphi_0}} \right)$.

Suppose $r_1 \leq r_0$ and $P_r = P_r(J_0, r, \varphi, P_\varphi)$ is nonzero for $r \in (r_1, r_0)$. It follows from the standard theory of linear ODEs that for $r \in [r_1, r_0]$, the solutions to the reduced variational equations are of the form

$$x(r) = x(r_0) + \int_{r_0}^r \tilde{A}(s)x(s)ds.$$

We rewrite this as

$$x(r) - x(r_0) = \int_{r_0}^r \tilde{A}(s)x(r_0)ds + \int_{r_0}^r \tilde{A}(s)(x(s) - x(r_0))ds.$$

Letting $|\cdot|$ denote the p -norm for $p = 1$, we have

$$|x(r) - x(r_0)| \leq \int_{r_1}^{r_0} |\tilde{A}(s)x(r_0)|ds + \int_r^{r_0} |\tilde{A}(s)||x(s) - x(r_0)|ds.$$

Note we the first term on the right is independent of r (since we have used the upper bound $\int_{r_1}^{r_0} |\tilde{A}(s)x(r_0)|ds \geq \int_r^{r_0} |\tilde{A}(s)x(r_0)|ds$), hence we can apply Gronwall's Inequality and obtain

$$|x(r) - x(r_0)| \leq |x(r_0)| \cdot \int_{r_1}^{r_0} |\tilde{A}(s)|ds \cdot \exp \left(\int_{r_1}^{r_0} |\tilde{A}(s)|ds \right).$$

Hence to obtain a concrete estimation, it suffices to estimate $\int_{r_1}^{r_0} |\tilde{A}(s)|ds$. This is done in Lemma 10.5 (proved below) which provides a function $a(r_1)$ to uniformly bound $\int_{r_1}^{r_0} |\tilde{A}(s)|ds$ over all $P_\varphi \in [1.7, 1.81]$. The estimates from the lemma yield

Claim: $\int_{r_1}^{r_0} |\tilde{A}(s)| ds \leq a(5) = 0.673031$

See Lemma 10.5 for an explicit form of $a(r_1)$. The claim is believable since far from the Sun, the terms in the matrix \tilde{A} goto zero at a rate of at most $O(\frac{1}{r^2})$ so integration with respect to r should produce a convergent quantity. The claim yields

$$|x(r) - x(r_0)| \leq 1.31926|x(r_0)|$$

□

The following Lemma establishes claim (1) of Theorem 10.1.

Lemma 10.3. *Consider RCP3BP(10⁻³, 1.8) and suppose the initial conditions for the flow are at the apohelion. If $(\frac{\partial r}{\partial P_{\varphi_0}}, \frac{\partial \varphi}{\partial P_{\varphi_0}}, \frac{\partial P_{\varphi}}{\partial P_{\varphi_0}})(0) = (0, 0, 1)$ and the remaining initial condition satisfies (10), then*

$$|\left(\frac{\partial P_{\varphi}}{\partial P_{\varphi_0}}\right)(t) - 1| \leq 0.268671\mu$$

for $t \in [0, t_5]$ where t_5 is the first positive time such that $r(t_5) = 5$.

It should be noted that in the 2BP(SC), $(\frac{\partial P_{\varphi}}{\partial P_{\varphi_0}})(t) \equiv 1$ for all t . Hence the above lemma is in some sense a measure of nonintegrability in the outside region in a similiar vein to Lemma 8.1 which estimates the change in P_{φ} in the outside region to be at most 0.0215298μ . The order of magnitude larger change in the equations of variation is to be expected since these equations are more sensitive to instabilities than the original equations of motion.

Proof of Lemma 10.3: Use the same reductions as in Lemma 10.2. The equations of motion for $(\frac{\partial P_{\varphi}}{\partial P_{\varphi_0}})(t)$ are

$$\left(\frac{d}{dr}\right)\left(\frac{\partial P_{\varphi}}{\partial P_{\varphi_0}}\right) = -P_r^{-1}\left(\frac{\partial^2 \Delta H}{\partial \varphi^2}\right)\left(\frac{\partial \varphi}{\partial P_{\varphi_0}}\right)$$

From Lemma 10.2 we have that $|\left(\frac{\partial \varphi}{\partial P_{\varphi_0}}\right)| \leq 1.31926$. Using the bound $a_2(r_1)$ from Lemma 10.5 then have that

$$|\left(\frac{\partial P_{\varphi}}{\partial P_{\varphi_0}}\right) - 1| \leq \int_{r_1}^{r_0} |P_r^{-1}\left(\frac{\partial^2 \Delta H}{\partial \varphi^2}\right)| 1.31926 |dr| \leq a_2(5) \cdot 1.31926 = 0.000268671$$

□

Now that we have generated bounds in the outside region for equations of variation, we use computer assistance to obtain bounds in the kick region where the perturbation terms are larger and the above analysis is insufficient to produce useful bounds. This lemma will establish claim (2) in the theorem.

Lemma 10.4. *Consider RCP3BP(10⁻³, 1.8) and suppose the initial conditions for the flow are at the apohelion and furthermore suppose $(\frac{\partial r}{\partial P_{\varphi_0}}, \frac{\partial \varphi}{\partial P_{\varphi_0}}, \frac{\partial P_{\varphi}}{\partial P_{\varphi_0}})(0) = (0, 0, 1)$ and the remaining initial condition satisfies (10). Then*

$$\left(\frac{\partial P_{\varphi}}{\partial P_{\varphi_0}}\right)(t) \in [0.12, 1.79]$$

where $t \in [t_5, T]$ where t_5 is the first positive time such that $r(t_5) = 5$ and T first positive time there is a perihelion.

Proof: Let $(v_1, v_2, v_4) = \left(\frac{\partial r}{\partial P_{\varphi_0}}, \frac{\partial \varphi}{\partial P_{\varphi_0}}, \frac{\partial P_{\varphi}}{\partial P_{\varphi_0}} \right) (t_5)$. We know from Lemmas 10.2 and 10.3 that $v_2 \in [-1.31927, 1.31927]$ and $v_4 \in 1 + [-0.000268671, 0.000268671]$; by assumption $v_1 = 0$. We may use (10) to solve for the remaining initial condition v_3 as an interval. To do this, we also use $r = 5$, $\varphi \in [0, 2\pi]$, $P_{\varphi} \in P_{\varphi}(0) + [-\rho(5), \rho(5)]$ and P_r implicitly defined by (9). This produces a vector of intervals $v := (v_1, v_2, v_3, v_4)$ which contain the solution in the tangent space at the time $t = t_5$.

We can use the CAPD rigorous numerical integrator to integrate the flow of the RCP3BP and its associated variational equations. CAPD can transport intervals of initial conditions. See appendix 14. We use it to transport the interval vector v in the tangent space from $r = 5$ to the perihelion for all initial conditions in the base space. To do this we use a 5th order intervalized Taylor Method with step size $\Delta t = 0.1$ with initial conditions φ direction divided into box sizes of 0.1 and initial conditions in the P_{φ} direction divided into box sizes of 0.0001. CAPD flows the box of initial conditions until all points inside it has passed through the perihelion. It records $\left(\frac{\partial P_{\varphi}}{\partial P_{\varphi_0}} \right) (t) \in [0.12, 1.79]$. \square

Remark: The bound produced is not nearly optimal. Non-rigorous numerics indicate that $\left(\frac{\partial P_{\varphi}}{\partial P_{\varphi_0}} \right) (t) \in [0.9, 1.1]$.

We now justify the technical estimates found in the above lemmas.

Lemma 10.5. For $\mu = 10^{-3}$, $J_0 = 1.8$, $P_{\varphi} \in [1.7, 1.81]$, $r_1 \geq 5$, r_0 an apohelion radius (possibly infinite), then there exists a function $a(r_1)$ so that $\int_{r_1}^{r_0} |\tilde{A}(s)| ds \leq a(r_1)$.

Proof of Lemma 10.5: In [GK1] we introduced the idea of extreme 2BPs. By considering the most angular momentum could change in the RCP3P over the whole outside region $r > 5$ for a specific set of initial conditions, we constructed two body problems whose behaviors enclosed (in the sense of interval arithmetic) that of the RCP3BP which spawned it. See e.g, Lemma 8 which estimates change in angular momentum for a single trajectory.

We use the machinery of extreme 2BPs now. Unlike [GK1], we do not need careful estimates for each trajectory; instead we need uniform upper bounds on quantities for an entire class of trajectories on $\mathcal{S}(1.8)$ with $P_{\varphi} \in [1.7, 1.81]$. It suffices to compute uniform upper bounds on all 2BP trajectories with $P_{\varphi} \in [1.7 - \rho(5), 1.81 + \rho(5)]$.

Notice that every entry in the matrix \tilde{A} is of the form $\frac{f(r, \varphi, P_{\varphi})}{P_r}$ for some function f where r enters with a power of at most -2 . In appendix 13, a method of evaluating such integrals **in closed form** is established. The fact that there is a closed form means that a computer algebra system can obtain rigorous upper bounds on the integrals; no numerical integration is required. We define the following quantities:

$$a_1(r_1) = \max \left(\sup_{P_\varphi \in [1.7, 1.8]} \lim_{x \rightarrow r^+} \frac{I_{-1}(r^-, r^+, r_1, x)}{\sqrt{2(1.8 - P_\varphi)}}, \sup_{P_\varphi \in [1.8, 1.81]} \lim_{x \rightarrow \infty} I_{-1}^{hyp}(r^-, N, M, x, r_1) \right)$$

$$a_2(r_1) = 2\mu \max \left(\sup_{P_\varphi \in [1.7, 1.8]} \lim_{x \rightarrow r^+} \frac{I_{-3}(r^-, r^+, r_1, x)}{\sqrt{2(1.8 - P_\varphi)}}, \sup_{P_\varphi \in [1.8, 1.81]} \lim_{x \rightarrow \infty} I_{-3}^{hyp}(r^-, N, M, x, r_1) \right)$$

where r^+, r^-, N, M are given in section 13; r^\pm are formulas for the apohelion and perihelion of a 2BP. The bounds a_i are easily computable with a computer algebra system.

It follows from section 7 for $r \geq 5$ that $|\frac{\partial^2 \Delta H}{\partial \varphi^2}| \leq \frac{2\mu}{r^4}$. Hence

$$\int_{r_1}^{r_0} \left| \frac{1}{r^2 P_r} \right| dr \leq a_1(r_1) \quad \int_{r_1}^{r_0} \left| P_r^{-1} \frac{\partial^2 \Delta H}{\partial \varphi^2} \right| dr \leq a_2(r_1)$$

Since we have used the $p = 1$ norm to bound the matrix A , then the corresponding norm induced on the matrix is taking maximum of the absolute value of the sum of columns. Hence the function $a(r_1) = \max(a_1(r_1), a_2(r_1))$. □

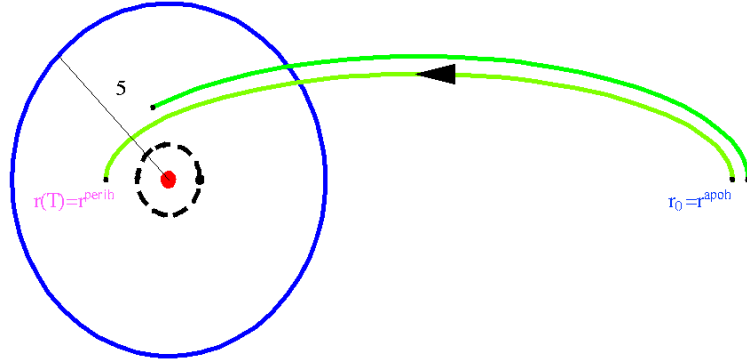
11. A SUFFICIENT CONDITION FOR TWIST IN THE RCP3BP

In section 4 we argued that in terms of the motions of the comet, twisting says that high eccentricity comets revolve around the sun more slowly than their low eccentricity counterparts. In this section, this idea is rephrased in a rigorous fashion to give a meaning to twisting in polar coordinates. This done by noting that on a fixed energy surface when the comet is at the apohelion, increasing eccentricity corresponds to increasing the semimajor axis of the ellipse of motion. By Kepler's Laws, increasing the semi-major axis corresponds to increasing the period of revolution. Let us make this rigorous now.

Definition: For a given set of initial conditions $(r, \varphi, P_r, P_\varphi)(0) = (r_0, \varphi_0, 0, P_{\varphi_0})$, with $r_0 > J_0^2$ such that the comet is at the apohelion, consider the motion of the comet starting at the apohelion and moving towards the perihelion. Define $T(r_0, \varphi_0, P_{\varphi_0})$ be the smallest positive time such that $P_r(T(r_0, \varphi_0, P_{\varphi_0})) = 0$, i.e. the time to the next perihelion. Call T the *one-half period* of the comet. A formulation of twist in polar coordinates is given in the following theorem.

Theorem 11.1. *There exists an $R > J_0^2$ such that for all $r \geq R$, $\frac{\partial T}{\partial r_0}|_{r_0=r} > 0$. In particular, for $\mu = 10^{-3}$, Jacobi constant $J_0 = 1.8$, and $R = 15$ (corresponding to $e > 0.8$) we have $\frac{\partial T}{\partial r_0} > 0$.*

Let us offer a heuristic proof of this result now. In the two body problem, Kepler's Third Law states that the one-half period $T = \pi a^{\frac{3}{2}}$ where the semi-major axis $a = \frac{r^{apoh} - r^{perih}}{2} \approx \frac{r^{apoh}}{2}$ for large eccentricities. Taking initial conditions to be $r_0 = r^{apoh} > J_0^2$, then $a \approx \frac{1}{2}r_0$. Computing $\frac{\partial T}{\partial r_0} = \frac{3}{4}\pi a^{1/2} > 0$ says that increasing the semi-major axis, i.e. increasing r^{apoh} , increases the half period of the comet. To prove this result for the RCP3BP, one must justify these approximations, as well as carefully account for the effects of the perturbation term.

FIGURE 12. The Effect of Increasing r_0 

Proof of Theorem 11.1 : Formally compute $\frac{\partial T}{\partial r_0}$.

$$\frac{\partial}{\partial r_0} \left(P_r(T(r_0, \varphi_0, P_{\varphi_0}), (r_0, \varphi_0, P_{\varphi_0})) \right) = \dot{P}_r(T) \cdot \frac{\partial T}{\partial r_0} + \frac{\partial P_r}{\partial r_0}(T) = 0$$

Solving for $\frac{\partial T}{\partial r_0}$ yields

$$(12) \quad \frac{\partial T}{\partial r_0} = -\frac{\frac{\partial P_r}{\partial r_0}(T)}{\dot{P}_r(T)}.$$

One must show for nearly parabolic comets that $\dot{P}_r(T) > 0$ so that equation (12) is well defined.

Lemma 11.2. *There exists an $e_0 = e_0(\mu, J_0)$ such that if $e(t) \geq e_0$ for $t \in [0, T]$ then $\dot{P}_r(T) > 0$ (where T is the half period). In particular $e_0(0.001, 1.8) \leq 0.13$.*

Proof: Start by noting

$$\dot{P}_r = \frac{P_\varphi^2}{r^3} - \frac{1}{r^2} - \frac{\partial \Delta H}{\partial r}$$

and $\frac{\partial \Delta H}{\partial r} = O(\frac{\mu}{r^4})$. Furthermore $|\frac{\partial \Delta H}{\partial r}| \leq (|\frac{\partial \Delta H}{\partial r}|)^+$ and $(|\frac{\partial \Delta H}{\partial r}|)^+ \leq 3\mu/(r^2(r-1)^2)$ (see appendix 7).

It suffices to show that at a perihelion we have

$$\frac{P_\varphi^2}{r^3} - \frac{1}{r^2} - \frac{3\mu}{r^2(r-1)^2} = \frac{1}{r^3} \left(P_\varphi^2 - r - \frac{3\mu r}{(r-1)^2} \right) > 0.$$

Recall relation from section 3.1. We have $e_\nu = \sqrt{1 - G_\nu^2/L_\nu^2}$, $P_\varphi = G_\nu + \nu$, and $r = L_\nu^2(1 - e_\nu(0) \cos u_\nu)$. Therefore, $r \geq L_\nu^2(1 - e_\nu(0)) = (G_\nu)^2/(1 + e_\nu(0))$. Plug the minimal value of $r = r^{perih} = (G_\nu)^2/(1 + e_\nu(0))$ in to get

$$P_\varphi^2 - r - \frac{3\mu r}{(r-1)^2} = \frac{e_\nu(0)P_\varphi^2 + \nu(2P_\varphi - \nu)}{1 + e_\nu(0)} - \frac{3\mu r}{(r-1)^2}.$$

By Lemma 1.1 we have $L_\nu^2 \geq r \geq \frac{J_0^2}{2} - 8\mu$. It is not hard to show that $e_\nu \leq e$ and that the difference is only ν -small. Since $J_0 + \nu = 1/(2L_\nu^2) + G_\nu$, these estimates show that if $e_\nu(0)$ is not too small, then $\dot{P}_r > 0$. Concrete numbers for $\mu = 0.001$ and $J_0 = 1.8$ are not difficult to obtain. \square

To complete the proof of Theorem 11.1 it remains to show that $\frac{\partial P_r}{\partial r_0}(T) < 0$.

Lemma 11.3. *Consider RCP3BP(10⁻³, 1.8) and suppose the initial conditions for the flow are at the apohelion. If $(\frac{\partial r}{\partial r_0}, \frac{\partial \varphi}{\partial r_0}, \frac{\partial P_r}{\partial r_0})(0) = (1, 0, 0)$ and the remaining initial condition satisfies (10). Then $\frac{\partial P_r}{\partial r_0}(T) < 0$ where T is the first positive time there is a perihelion.*

Proof: Note that

$$\frac{\partial P_r}{\partial r_0} = \left(\frac{\partial P_r}{\partial P_\varphi} \right) \left(\frac{\partial P_\varphi}{\partial P_{\varphi_0}} \right) \left(\frac{\partial P_{\varphi_0}}{\partial r_0} \right)$$

We know from the claim in Theorem 10.1 that $(\frac{\partial P_\varphi}{\partial P_{\varphi_0}})(T) > 0$. From differentiating formula (9), we obtain

$$\frac{\partial P_r}{\partial P_\varphi} = \frac{1 - \frac{P_\varphi}{r^2}}{P_r} < 0$$

since the denominator is positive for $r \geq 1.5$ and $P_\varphi \leq 1.81$ and $P_r \leq 0$ for $t \in [0, T]$.

On the energy surface $\mathcal{S}(J_0)$ we can solve to find

$$P_\varphi = r^2 - \sqrt{2r - 2J_0r^2 - P_r^2r^2 + r^4 - 2r^2\Delta H(r, \varphi)}$$

When $t = 0$, then $P_r = 0$, $r = r_+$ (the apohelion radius) and the expression simplifies. Compute the derivative of the simplified expression to find:

$$(13) \quad \frac{\partial P_{\varphi_0}}{\partial r_0} = \frac{-1 + 2r_+(J_0 - P_\varphi + \Delta H) + r_+^2 \frac{\partial \Delta H}{\partial r}}{r_+^2 - P_\varphi}.$$

For r large, the denominator is positive. This certainly holds at the apohelion. Note that formally, the apohelion satisfies

$$r_+ = \frac{1 + \sqrt{1 - P_\varphi^2(J_0 - P_\varphi + \Delta H)}}{2(J_0 - P_\varphi + \Delta H)}$$

Hence at the apohelion, we have that numerator of (13) simplifies to become

$\sqrt{1 - P_\varphi^2(J_0 - P_\varphi + \Delta H)} + r_+^2 \frac{\partial \Delta H}{\partial r}$. It not hard to show that the first term in this expression is at least 0.64 and second term is larger than -0.007 for $J_0 = 1.8$, $r \geq 1.5$, and $P_\varphi \geq 1.7$. Hence the numerator, and (13) are positive. It follows that $\frac{\partial P_r}{\partial r_0}(T)$ is negative. \square

12. APPLICATION OF THEOREM 5.2 TO THE RCP3BP

Here we show that Theorem 5.2 can be applied to the RCP3BP($10^{-3}, 1.8$). Let $H(\ell_\nu, L_\nu, t)$ be the energy-reduced Hamiltonian of the RCP3BP in algebraically deformed Delaunay variables. (See [A] sect 45 for a refresher on how to do energy reductions in our settings.) In the context of the RCP3BP, we wish to prove the existence of a coordinate system with concavity, i.e. $\partial_{L_s}^2 \partial_{L_s}^2 (H \circ \Psi^{-1})|_{\Psi(\mathcal{W})} < 0$. In the case of the 2BP(SC) this boils down to the statement that $\partial_{LL} H = -\frac{3}{L^4}$.

We start by fixing a section $\Sigma = \{\ell = \pi \pmod{2\pi}\}$, i.e. the apohelion surface. The main result of section 11 says $\frac{\partial T}{\partial r_0} > 0$ where T is the half period of comet and r_0 is an apohelion radius. This condition can be reformulated to say $\frac{\partial T}{\partial L_0} > 0$ by noting that

$$\begin{aligned} \frac{\partial T}{\partial L_0} &= \frac{\partial T}{\partial r_0} \frac{\partial r_0}{\partial L_0} \\ \frac{\partial r_0}{\partial L_0} &= \frac{2r_0}{L_0} + \frac{G_0^2}{L_0 e_0} > 0. \end{aligned}$$

The second identity follows from formulas for converting polar to Delaunay at the apohelion. Hence condition 4 of Definition 5.1 is satisfied⁷.

12.1. Domain of Definition for RCP3BP. Let us construct a domain of relevant solutions for the RCP3BP where condition 2 of definition 5.1 is satisfied. When enlarging the domain of definition via the algebraic deformation, we increased the number of solutions which have representations in action-angle variables. However not all of the points inside the homoclinic loop generated from H_ν are of interest. For example some of the points outside of the separatrices make one passage by the Sun-Jupiter system then escape the solar system. By the way ADDV are defined, it is possible solutions near the separatrix to flow out of the homoclinic loop where the coordinate system is not well defined. This is because the area inside of homoclinic loop for H_ν is not an invariant set for the flow induced by H_{Polar} , the Hamiltonian for the RCP3BP.

Definition 12.1. *An initial condition is inside parabolic if the following three conditions all hold (fig. 13):*

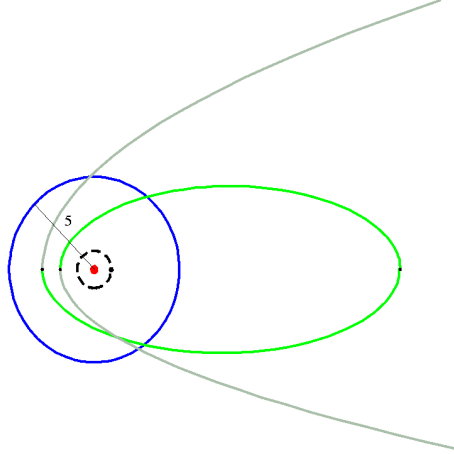
- If $\dot{r} > 0$, then there is $t > 0$ such that $\dot{r} = 0$, i.e. there is an apohelion in the future.
- If $\dot{r} < 0$, then there is $t < 0$ such that $\dot{r} = 0$, i.e. there is a perihelion in the past.
- For some finite times $t_- < 0 < t_+$ it holds that $|\psi(t_+) - \psi(t_-)| = 4\pi$.

Denote the class of inside parabolic initial conditions for Jacobi constant J_0 by $IP(J_0)$. Notice that hyperbolic or parabolic orbits satisfy $|\psi(t_+) - \psi(t_-)| < 2\pi$. If an orbit of RCP3BP goes to directed infinity in the past and in the future makes one loop around the sun before going to directed infinity, then $|\psi(t_+) - \psi(t_-)|$ can be close to 2π .

Inside parabolic curves are considered since their representations in algebraically deformed Delaunay variables have the action variable L_ν finite for at least one revolution around the sun. Unfortunately, not all of $IP(J_0)$ is contained inside the new homoclinic loop since it is

⁷Note that due to negative convexity, the sign of the derivative is reversed

FIGURE 13. A trajectory with inside parabolic initial conditions

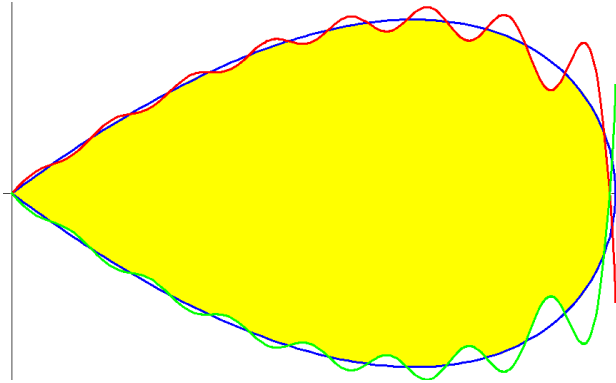


still possible for a separatrix to leave the new homoclinic loop. We ignore these points as they have too high eccentricity to matter. The initial conditions we care about are those inside parabolic motions which are inside of the homoclinic loop for H_ν (see fig. 14). Let

$$\Omega(J_0, \mu) = IP(J_0) \cap \{P_\varphi \leq J_0, \varphi \in \mathbb{S}^1, r_*^{perih} \leq r, |P_r| \leq \sqrt{\frac{2}{r} - \frac{(P_\varphi - \nu)^2}{r^2} + 2(J_0 - P_\varphi)}\}$$

where r_*^{perih} is the smallest perihelion radius allowed in the deformed homoclinic loop. This is implicitly defined by $0 = H_\nu(r_*^{perih}, 0, 0, J_0)$. (See eqn. (3) for definition of H_ν .)

FIGURE 14. Domain of Definition



We take $\mathcal{W} := \Omega(J_0, \mu) \cap \{e \geq 0.46\}$ so the apohelion's radii are at least $r^{apoh} = 5$ and the results of [GK1] can be applied. Condition 2 of Definition 5.1 follows by construction since points in \mathcal{W} are in the class of inside parabolic motions (see section 12.1). We must still verify conditions 1 and 3 of Definition 5.1 for trajectories in \mathcal{W} .

12.2. Verification of condition (1). To show condition (1) of Definition 5.1, note that for every inside parabolic orbit starting at a perihelion there is an apohelion for some $0 < t^*$. Let $(\ell_\nu^*, L_\nu^*, t^*)$ be such an apohelion point. Let $P_\varphi(t)$ be angular momentum along the orbit. Lemma 8.1 of appendix 8 tells us that

$$|P_\varphi(t) - P_\varphi(0)| \leq 4.5\mu \text{ for } 0 \leq t \leq t^*$$

and by the relations in section 3.2.1 this implies that $L_\nu(T) - L_\nu(0)$ is finite (since $2\nu > 4.5\mu$), where T is the return time to the section $\Sigma = \{\ell = \pi \pmod{2\pi}\}$. Moreover the bound is uniform in any compact region $\{L_\nu \leq \text{const}\}$. This implies that return time T is finite and condition (1) holds.

12.3. Bounds on angle of twist in the RCP3BP. In this section, condition (3) of Definition 5.1 is verified. We shall work with the reduced RCP3BP Hamiltonian in $H = H(L_\nu, \ell_\nu, s)$ where $s = g_\nu$ is the rescaled time and G_ν is an implicit function of J_0 and the other algebraically deformed Delaunay variables. Recall that the angle of twist $\eta(t) = \eta(t; L_\nu(0), 0, s)$ from the vertical is given by the formula

$$(14) \quad \tan(\eta(t)) = \frac{\left(\frac{\partial \ell_\nu(t)}{\partial L_\nu(0)}\right)}{\left(\frac{\partial L_\nu(t)}{\partial L_\nu(0)}\right)}$$

where the initial conditions are taken to be $(0, L_\nu(0), s) \in \Sigma = \{\ell_\nu = \pi \pmod{2\pi}\}$. (See [MF] for abstract statement about angle of twist.) We prove

Lemma 12.2. *Consider RCP3BP($10^{-3}, 1.8$). For all $(0, L_\nu(0), s) \in \Sigma = \{\ell_\nu = \pi \pmod{2\pi}\}$ for $0 \leq t \leq T(L_\nu(0), s)$ the angle of twist $\eta(t) \in [-\kappa, \kappa]$ for some $\kappa < \frac{\pi}{2}$ where T is the return time to Σ .*

First we offer a heuristic explanation of why the angle of twist from the vertical is uniformly bounded away from $\frac{\pi}{2}$ for the 2BP(SC). Note that the angle η of twist from the vertical in Delaunay variables can be explicitly computed for the 2BP(SC). It is given by

$$\tan(\eta) = \frac{\left(\frac{\partial \ell}{\partial L_0}\right)(t)}{\left(\frac{\partial L}{\partial L_0}\right)(t)} = -\frac{3t}{L_0^4}$$

Hence $\eta = \arctan(-\frac{3t}{L_0^4})$ and this function is decreasing as a function of t for $t \geq 0$. Recall the period of the 2BP(SC) is $T = 2\pi L^3$. Then for $t \in [0, T]$, the minimum angle is $\arctan(-\frac{6\pi}{L_0})$. For $L_0 \geq 1.86$ (i.e. when $r^{apoh} > 5$ on $\mathcal{S}(1.8)$) it follows that $0 \geq \eta(t) \geq -1.47244 > -\frac{\pi}{2}$. The case for $t \leq 0$ is symmetric in the 2BP(SC).

Proof of Lemma 12.2: Upon examination of formula (14) we see that the angle of twist rotates by more than an angle of $\frac{\pi}{2}$ from the vertical if at some point in the flow $|\tan \eta| = \infty$. This happens if and only if the numerator becomes infinite or the denominator becomes zero. Hence it suffices to have a uniform lower bound on $\frac{\partial L_\nu}{\partial L_\nu(0)}$ and a uniform upper bound on $\frac{\partial \ell_\nu}{\partial L_\nu(0)}$ over one full period for all trajectories considered.

We remark that condition (3) requires bounds in both forward and backward time for an individual trajectory. By condition (2) every point on the section Σ has a preimage and an

image of Σ and bounding the angle of twist uniformly for all trajectories forward in time is enough to also bound the angle of twist uniformly for trajectories in reverse time. Hence it suffices to generate uniform bounds over all trajectories which are approaching the sun from an apohelion in forward time.

The method of proof is to convert from algebraically deformed Delaunay variables into polar, then use bounds on equations in polar coordinates. Bounds are computed with the assistance of a computer in the kick region, and with some by-hand calculations in the outside region.

It is easier to make calculations using $G = P_\varphi$ since this variable can be computed using polar coordinates. For a fixed energy surface $\mathcal{S}(J_0)$, we recall that $G_\nu = G_\nu(J_0, L_\nu, \ell_\nu, s) \approx J_0 - \frac{1}{2L_\nu^2}$ is implicitly defined. At the same we may also think of $L_\nu = L_\nu(J_0, \ell_\nu, G_\nu, s) \approx \frac{1}{\sqrt{2(J_0 - G_\nu)}}$. It follows that the formula for angle of twist can be written as

$$(15) \quad \begin{aligned} \frac{\left(\frac{\partial \ell_\nu}{\partial L_\nu(0)}\right)}{\left(\frac{\partial L_\nu}{\partial L_\nu(0)}\right)} &= \frac{\left(\frac{\partial \ell_\nu}{\partial G_\nu(0)}\right)\left(\frac{\partial G_\nu(0)}{\partial L_\nu(0)}\right)}{\left(\frac{\partial L_\nu}{\partial G_\nu}\right)\left(\frac{\partial G_\nu}{\partial L_\nu(0)}\right)\left(\frac{\partial G_\nu(0)}{\partial L_\nu(0)}\right)} = \frac{\left(\frac{\partial \ell_\nu}{\partial G_\nu(0)}\right)}{\left(\frac{\partial G_\nu}{\partial L_\nu(0)}\right)\left(\frac{\partial L_\nu}{\partial G_\nu}\right)} = \frac{\left(\frac{\partial \ell_\nu}{\partial G(0)}\right)\left(\frac{\partial G(0)}{\partial G_\nu(0)}\right)}{\left(\frac{\partial L_\nu}{\partial G_\nu}\right)\left(\frac{\partial G_\nu}{\partial G}\right)\left(\frac{\partial G}{\partial G(0)}\right)\left(\frac{\partial G(0)}{\partial G_\nu(0)}\right)} \\ &= \frac{\frac{\partial \ell_\nu}{\partial G(0)}}{\left(\frac{\partial G}{\partial G(0)}\right)\left(\frac{\partial G_\nu}{\partial G}\right)\left(\frac{\partial L_\nu}{\partial G_\nu}\right)} = \frac{\left(\frac{\partial \ell_\nu}{\partial G_\nu(0)}\right)\left(\frac{\partial G_\nu(0)}{\partial G(0)}\right)\left(\frac{\partial G_\nu}{\partial L_\nu}\right)}{\left(\frac{\partial G}{\partial G(0)}\right)\left(\frac{\partial G_\nu}{\partial G}\right)} \end{aligned}$$

Since $G_\nu = G - \nu$, then $\frac{\partial G_\nu}{\partial G} = \frac{\partial G_\nu(0)}{\partial G(0)} = 1$. By Lemma 7.3, $\left|\left(\frac{\partial G_\nu}{\partial L_\nu}\right)\right| \leq 0.350529$. By Theorem 10.1, $\left(\frac{\partial G(t)}{\partial G(0)}\right) \in [0.12, 1.79]$. It follows that the denominator does not goto zero.

Let us consider $\left(\frac{\partial \ell_\nu}{\partial G(0)}\right)$. Since inside parabolic motions have well defined algebraically deformed Delaunay variables, then along a trajectory the equations of variation are well defined. Since we work over the class of inside parabolic motions, a finite return time to an apohelion is guaranteed (see e.g. the previous section where we showed condition (1)). Since the segment of trajectory is finite, then there is a uniform bound on $\left(\frac{\partial \ell_\nu}{\partial G(0)}\right)$ over the entire revolution. For any compact subset of initial conditions, there is a uniform bound on $\left(\frac{\partial \ell_\nu}{\partial G(0)}\right)$ for all trajectories with initial conditions in that subset. It follows that the angle of twist cannot become $\frac{\pi}{2}$ by the numerator going to infinity.

We now seek a finite uniform upper bound for the numerator *over all inside parabolic motions*.

Claim: $\left|\left(\frac{\partial \ell_\nu}{\partial G_\nu(0)}\right)\left(\frac{\partial G_\nu}{\partial L_\nu}\right)\right| \leq C < \infty$ for all inside parabolic motions.

The rest of section is dedicated to proving this claim.

Proof of Claim:

First notice that

$$\frac{\partial \ell_\nu(t)}{\partial G_\nu(0)} = \int_0^t \frac{\partial \dot{\ell}_\nu}{\partial G_\nu(0)} ds = \int_0^t \frac{\partial \dot{\ell}_\nu}{\partial G_\nu} \frac{\partial G_\nu}{\partial G_\nu(0)} ds$$

Since finite bounds on $\frac{\partial G_\nu}{\partial G_0} = \frac{\partial G}{\partial G_0} \in [0.12, 1.79]$ are known over a full period by Theorem 10.1, then it suffices to come up with bounds on $\int_0^t \frac{\partial \dot{\ell}_\nu}{\partial G_\nu} ds$.

From the equations of motion,

$$\dot{\ell}_\nu = L_\nu^{-3} + \partial_{\ell_\nu} \Delta H(L_\nu, \ell_\nu, G_\nu, g_\nu)$$

Notice on $\mathcal{S}(J_0)$ that $L_\nu = \frac{1}{\sqrt{2(J_0 + \nu - G_\nu + \Delta H)}}$, so

$$\begin{aligned} \dot{\ell}_\nu &= (2(J_0 + \nu - G_\nu + \Delta H))^{3/2} + \partial_{\ell_\nu} \Delta H \\ \frac{\partial \dot{\ell}_\nu}{\partial G_\nu} &= -3(2(J_0 + \nu - G_\nu + \Delta H))^{1/2} + \partial_{\ell_\nu G_\nu} \Delta H \end{aligned}$$

It follows that

$$\frac{\partial \ell_\nu}{\partial G_\nu}(t) = \int_0^t (-3(2(J_0 + \nu - G_\nu + \Delta H))^{1/2} + \partial_{\ell_\nu G_\nu} \Delta H) ds$$

We analyze each term in the integral separately.

12.3.1. *Analysis of the 2BP Part.* We seek bounds on $(\frac{\partial G_\nu}{\partial L_\nu}) \int_0^t -3(2(J_0 + \nu - G_\nu + \Delta H))^{1/2} dt$ for $t \in [0, T]$ where T is the time to perihelion and where initial conditions are started at the apohelion. This is the dominant term in the numerator of (15). For the 2BP, $(\frac{\partial G}{\partial L}) \int_0^T -3(2(J_0 - G))^{1/2} dt = -6\pi L^{-1}$. For the RCP3BP analysis, bounds on ΔH as well as the bounds on the change of $G = P_\varphi$ over the flow need to be included.

Let us begin by first noticing that

$$P_r = P_r(J_0, r, \varphi, G) = \frac{1}{r} \sqrt{2(J_0 - G + \Delta H)(r - r_-)(r_+ - r)}$$

where $r_\pm = r_\pm(J_0, r, \varphi, G)$ are the apohelion and perihelion respectively. Then

$$\int_0^T (2(J_0 + \nu - G_\nu + \Delta H))^{1/2} dt = \int_0^T \frac{r P_r}{\sqrt{(r - r_-)(r_+ - r)}} dt = \int_{r_-}^{r_+} \frac{r}{\sqrt{(r - r_-)(r_+ - r)}} dr$$

where $P_r = P_r(J_0 + \nu, r, \varphi, G_\nu)$ and $r_\pm = r_\pm(J_0 + \nu, r, \varphi, G_\nu)$. Notice this is of the form I_1 as in appendix 13. Computing the integral $I_1(r_-, r_+, r_-, r_+)$ yields

$$\int_0^T (2(J_0 + \nu - G_\nu + \Delta H))^{1/2} dt = \frac{\pi}{2}(r_+ + r_-)$$

Note that for the RCP3BP, r_\pm are difficult to compute exactly since the time to reach the apohelion/perihelion is a dynamical quantity that depends on the flow. However in light of formula (17) for r_\pm , these quantities are easy to bound by bounding $G_\nu = G - \nu$ over the

half-period. The quantity ν is known and formula (8) for ρ tells us how to bound G over the half-period. From (17), it is not hard to show that

$$(r_- + r_+)(J_0, r, \varphi, P_\varphi) = \frac{\sqrt{1 - 2P_\varphi^2(J_0 - P_\varphi + \Delta H)}}{2J_0 - 2P_\varphi + 2\Delta H}$$

So

$$\frac{\sqrt{1 - 2(G_\nu(0) - \rho(1.6))^2(J_0 + \nu - (G_\nu(0) - \rho(1.6)) + (|\Delta H|)^+(1.6))}}{2J_0 + \nu - 2G_\nu(0) + 2\rho(1.6) + 2(|\Delta H|)^+(1.6)} \leq (r_- + r_+) \leq \frac{\sqrt{1 - 2(G_\nu(0) + \rho(1.6))^2(J_0 + \nu - (G_\nu(0) + \rho(1.6)) - (|\Delta H|)^+(1.6))}}{2J_0 + \nu - 2G_\nu(0) - 2\rho(1.6) - 2(|\Delta H|)^+(1.6)}$$

It is not hard to show that the numerators are bounded. In fact for $J_0 = 1.8, \mu = 0.001, \nu = 2.8\mu, P_\varphi \in [1.66, 1.81], r \geq 1.6$, the numerator is contained in the interval $[0.239131, 1.04326]$.

Notice from formula (7) that

$$\frac{\partial G_\nu}{\partial L_\nu} = \frac{L_\nu^{-3} + \frac{\partial \Delta H}{\partial L_\nu}}{1 - \frac{\partial \Delta H}{\partial G_\nu}}$$

From Lemma 7.2 we observe that $1 - \frac{\partial \Delta H}{\partial G_\nu} \in [1 - 0.025, 1 + 0.025]$. Hence it suffices to obtain finite bounds for $L_\nu^{-3} + \frac{\partial \Delta H}{\partial L_\nu}$.

Notice that on $\mathcal{S}(1.8)$,

$$L_\nu^{-3} = (2(1.8 + \nu - G_\nu + \Delta H))^{3/2} \in [(2(1.8 + \nu - G_\nu(0) - \rho(1.6) - (|\Delta H|)^+(1.6)))^{3/2}, (2(1.8 + \nu - G_\nu(0) + \rho(1.6) + (|\Delta H|)^+(1.6)))^{3/2}]$$

From Lemma 7.1, we have

$$\left| \frac{\partial \Delta H}{\partial L_\nu} \right| \leq \frac{433\mu}{r^3} \leq \frac{433\mu}{L_\nu^6(1 - e_\nu)^3}$$

Now on $\mathcal{S}(J_0)$ that

$$e_\nu = \sqrt{1 - (G_\nu)^2(J_0 + \nu - G_\nu)}$$

Hence for $J_0 = 1.8, \mu = 0.001, \nu = 2.8\mu, G \leq 1.8 + 2\mu$ then $e_\nu = 0.997391 < 1$. (For values of $G > 1.8 + 2\mu$ comets do not have apohelions since even if angular momentum decreased maximally, it would still remain above 1.8. See Lemma 8.1.) Hence it is possible to get a uniform upper bound on $(1 - e_\nu)^{-3}$ for inside parabolic motions.

Hence

$$\left| \frac{\partial \Delta H}{\partial L_\nu} \right| \in \frac{433\mu}{(1 - e_\nu)^3} [(2(J_0 - G_\nu(0) - \rho(1.6) - (|\Delta H|)^+(1.6)))^3, (2(J_0 - G_\nu(0) + \rho(1.6) + (|\Delta H|)^+(1.6)))^3]$$

It follows that inside parabolic motions with $J_0 = 1.8, \mu = 0.001, G \in [1.7, 1.81], r \geq 1.6$ that

$$\begin{aligned} \left| \int_0^T (2(J_0 + \nu - G_\nu + \Delta H))^{1/2} dt \cdot \frac{\partial G_\nu}{\partial L_\nu} \right| &\leq C_1 \frac{(2(J_0 + \nu - G_\nu(0) + \rho(1.6) + (|\Delta H|^+(1.6))))^{3/2}}{2J_0 + 2\nu - 2G_\nu(0) - 2\rho(1.6) - 2(|\Delta H|^+(1.6))} \\ &\leq C_2 (2(J_0 + \nu - G_\nu(0) + \rho(1.6) + (|\Delta H|^+(1.6))))^{1/2} \\ &< \infty \end{aligned}$$

where $C_1, C_2 < \infty$ are constants. We have used that

$$\left| \frac{1.8 + \nu - G_\nu(0) + \rho(1.6) + (|\Delta H|^+(1.6))}{1.8 + \nu - G_\nu(0) - \rho(1.6) - (|\Delta H|^+(1.6))} \right| < \infty$$

since for inside parabolic motions, $1.8 + \nu - G_\nu(0) - \rho(1.6) - (|\Delta H|^+(1.6)) > 0.5\mu$.

12.3.2. *Analysis of the perturbation term.* In this subsection, we analyze $(\frac{\partial G_\nu}{\partial L_\nu}) \int_0^t (\partial_{L_\nu} G_\nu \Delta H) ds$. For the 2BP, this term is zero, and in general is $O(\frac{\mu}{L_\nu^3})$ small, so we don't expect too it to contribute much.

It follows⁸ from Lemma 7.4, we have for $\mu = 0.001, r > 1.5, G \in [1.7, 1.81]$,

$$\left| \left(\frac{\partial^2 \Delta H}{\partial \ell \partial G} \right) \right| \leq \frac{21}{r^{7/2}(1 - e_\nu)^{3/2}} + \frac{35|P_r|}{r^{5/2}(1 - e_\nu)^{5/2}}$$

Now

$$\begin{aligned} \int_0^T \left| \left(\frac{\partial^2 \Delta H}{\partial \ell \partial G} \right) \right| dt &\leq \int_0^T \frac{21}{r^{7/2}(1 - e_\nu)^{3/2}} + \frac{35|P_r|}{r^{5/2}(1 - e_\nu)^{5/2}} dt \\ &\leq \int_{r_-}^{r_+} \frac{21}{P_r r^3 (1 - e_\nu)^{3/2}} + \frac{35}{r^2 (1 - e_\nu)^{5/2}} dr \end{aligned}$$

The second term in the integral is easy to evaluate in closed form and is clearly bounded.

Now

$$\int_{r_-}^{r_+} \frac{1}{P_r r^3} dr = \int_{r_-}^{r_+} \frac{1}{r^2 \sqrt{2(J_0 - P_\varphi + \Delta H)(r - r_-)(r_+ - r)}} dr$$

This is of form I_{-2} from appendix 13. It not hard to show that for $r_\pm \geq 1.6$ that

$$I_{-2}(r_-, r_+, r_-, r_+) = \frac{\pi(r_- + r_+)}{2(r_- \cdot r_+)^{3/2}} \leq 1.3$$

It follows that for inside parabolic motions,

$$\int_0^T \left| \left(\frac{\partial^2 \Delta H}{\partial \ell \partial G} \right) \right| \leq \frac{21 \cdot 0.77}{(1 - e_\nu)^{3/2} \sqrt{2(J_0 + \nu - G_\nu - \rho(1.6) - (|\Delta H|^+(1.6)))}} + \frac{35 \cdot 1.6}{(1 - e_\nu)^{5/2}}$$

⁸Lemma 7.4 is stated in terms of Delaunay, however exactly the same bounds hold for ADDV. Simply put a ν on all the Delaunay terms in the estimates.

Now when we multiply the above upper bound by the upper bound for $\frac{\partial G_\nu}{\partial L_\nu}$, we get the result is of at most $C(2(J_0 + \nu - G_\nu - \rho(1.6) - (|\Delta H|)^+(1.6)))^{5/2}$ (for some C) which is bounded.

This completes the proof of the claim. Since we have bounded both the dominant and perturbation term from above, then

$$\left| \frac{\partial \ell_\nu}{\partial G} \frac{\partial G_\nu}{\partial L_\nu} \right| < C < \infty$$

Since there is uniform upper bound on the numerator of (15) and a uniform lower bound on the denominator over all inside parabolic motions, then it follows that there is a uniform bound away from $\frac{\pi}{2}$. \square

13. APPENDIX - TABLE OF INTEGRALS

Let us investigate some properties of the following commonly occurring integrals. At various times one encounters integrals of the form

$$\int_{t_0}^{t_1} r^k(t) dt = \int_{r_0}^{r_1} \frac{r^k}{\dot{r}} dr$$

For a 2BP with eccentricity less than one, this can be rewritten

$$(16) \quad F(J_0, r_0, r_1, P_\varphi, k) = \int_{r_0}^{r_1} \frac{r^{k+1} dr}{\sqrt{2(J_0 - P_\varphi)(r_+ - r)(r - r_-)}}$$

where r_\pm are the apohelion and perihelion radii given by

$$(17) \quad r_+ := \frac{1 + \sqrt{1 - 2(J_0 - P_\varphi)P_\varphi^2}}{2(J_0 - P_\varphi)} \quad r_- := \frac{P_\varphi^2}{1 + \sqrt{1 - 2(J_0 - P_\varphi)P_\varphi^2}}$$

Then to evaluate the integral (16) it suffices to know how to evaluate

$$I(a, b, c, d, k) := \int_c^d \frac{r^k dr}{\sqrt{(b-r)(r-a)}}$$

where in all cases, $a, b, c, d \geq 0$ and $a \leq c \leq r \leq d \leq b$. Specific forms are known for some k .

$$I_{-3}(a, b, c, d) := \left(\frac{\sqrt{ab}(b-x)(x-a)(2ab+3x(a+b)) + (3a^2+2ab+3b^2)x^2\sqrt{(b-x)(x-a)} \arctan\left(\frac{\sqrt{b(x-a)}}{\sqrt{a(b-x)}}\right)}{4(ab)^{\frac{5}{2}}x^2\sqrt{(b-x)(x-a)}} \right) \Big|_{x=c}^{x=d}$$

$$I_{-2}(a, b, c, d) := \left(\frac{\sqrt{ab}(b-x)(x-a) + (a+b)x\sqrt{(b-x)(x-a)} \arctan\left(\frac{\sqrt{b(x-a)}}{\sqrt{a(b-x)}}\right)}{(ab)^{\frac{3}{2}}x\sqrt{(b-x)(x-a)}} \right) \Big|_{x=c}^{x=d}$$

$$I_{-1}(a, b, c, d) := \left(\frac{\arctan\left(\frac{x(a+b)-2ab}{2\sqrt{ab(b-x)(x-a)}}\right)}{\sqrt{ab}} \right) \Big|_{x=c}^{x=d}$$

$$I_0(a, b, c, d) := \left(\arcsin\left(\frac{2x-b-a}{b-a}\right) \right) \Big|_{x=c}^{x=d}$$

$$I_1(a, b, c, d) := \left(\frac{a+b}{2} \arcsin\left(\frac{2x-a-b}{b-a}\right) - \frac{b-a}{2} \sqrt{1 - \left(\frac{2x-a-b}{b-a}\right)^2} \right) \Big|_{x=c}^{x=d}$$

For a 2BP with eccentricity greater than one the analogous formula is

$$(18) \quad \int_{r_0}^{r_1} \frac{r^k}{\dot{r}} dr = \int_{r_0}^{r_1} \frac{r^{k+1}}{\sqrt{(N + Mr)(r - r_-)}} dr$$

where $N := 1 + \sqrt{1 - 2P_\varphi^2(J_0 - P_\varphi)}$ and $M = 2(P_\varphi - J_0)$. Then to evaluate the integral (18) it suffices to know how to evaluate

$$I(a, b, c, d, k) := \int_c^d \frac{r^k dr}{\sqrt{(b + er)(r - a)}}$$

where in all cases, $a, b, c, d, e \geq 0$ and $a \leq c \leq r$. Specific forms are known for some k .

The following are the analogs of the integrals from above for hyperbolic motions.

$$\begin{aligned} I_{-\frac{3}{2}}^{hy p}(a, b, e, c, d) &= \left(\frac{\sqrt{ab}(b+ex)(x-a)(2ab-3x(ae-b)) - (3b^2-2abe+3a^2e^2)x^2\sqrt{(b+ex)(x-a)} \arctan\left(\frac{\sqrt{b(x-a)}}{\sqrt{a(b+ex)}}\right)}{4(ab)^{\frac{5}{2}}x^2\sqrt{(b+ex)(x-a)}} \right) \Big|_{x=c}^{x=d} \\ I_{-\frac{1}{2}}^{hy p}(a, b, e, c, d) &= \left(\frac{\sqrt{ab}(b+ex)(x-a) - (ae-b)x\sqrt{(b+ex)(x-a)} \arctan\left(\frac{\sqrt{b(x-a)}}{\sqrt{a(b+ex)}}\right)}{(ab)^{\frac{3}{2}}x\sqrt{(b+ex)(x-a)}} \right) \Big|_{x=c}^{x=d} \\ I_{-1}^{hy p}(a, b, e, c, d) &= \left(\frac{2 \arctan\left(\frac{\sqrt{b(x-a)}}{\sqrt{a(e x+b)}}\right)}{\sqrt{ab}} \right) \Big|_{x=c}^{x=d} \\ I_0^{hy p}(a, b, e, c, d) &= \left(\frac{2 \log(e\sqrt{x-a} + \sqrt{e(b+ex)})}{\sqrt{e}} \right) \Big|_{x=c}^{x=d} \\ I_1^{hy p}(a, b, e, c, d) &= \left(\frac{\sqrt{(ex+b)(x-a)}}{e} + \frac{ae-b}{e^{\frac{3}{2}}} \log(e\sqrt{x-a} + \sqrt{e(b+ex)}) \right) \Big|_{x=c}^{x=d} \end{aligned}$$

14. APPENDIX - COMPUTER ASSISTED PROOFS

In several key points of the proof, computers are used to bound perturbation terms or integrate the equations of motion in a mathematically rigorous fashion. Powerful tools, both theoretical and computational have been developed to handle these procedures. See [GK1] for an overview of the methods used. In summary, a procedure known as interval arithmetic is used to rigorously represent numbers on a computer and incorporate the rounding errors a computer makes into operations. Numbers are stored and manipulated as intervals whose endpoints are representable on a machine (see [KM] and [MZ] for an overview). With some care, it is possible to use interval arithmetic to formulate a mathematically rigorous numerical integrator. This done in [Z] and [WZ]. Mathematically, integrating with interval arithmetic is the equivalent of asking a computer to produce a verified ϵ -tube around a desired solution of an ODE.

We make heavy use of the CAPD (Computer Assisted Proofs in Dynamics) library to perform the rigorous numerical integration over short intervals of time, typically less than 50 time units. The CAPD package makes use of interval arithmetic to enclose numerical solutions of ODEs over short periods of time in rigorously verified ϵ -tubes. It is also capable of moving

small boxes of initial conditions under the flow. By covering a domain with many small intervals, CAPD can move the entire domain. CAPD can be obtained at capd.ii.uj.edu.pl. The CAPD library provides objects for intervals, vectors, matrices, maps, and integrators which can be included into C++ programs.

Mathematica was also used for its symbolic manipulation abilities, as well as its built in interval arithmetic. It allowed us to quickly verify many of the claims of enclosure in a non-rigorous fashion. Mathematica rigorously verified many of identities we used.

The programs written with CAPD and with Mathematica can be obtained online at www.math.umd.edu/~joeqi. The programs are packaged with a guide which gives explicit details on which programs carry out which parts of the proof, as well as information on obtaining libraries, compiling, running, and modifying the code for use on similar problems. Logs and outputs of some of the programs are also included due to the length of time needed to generate the data.

Most of the software ran continuously for two weeks, distributed over a cluster of 80 machines the fastest of which was a 3.4 GHz Pentium 4 with 2GB RAM and 120 GB HDD. It produced over 2GB of data. Each machine was running a variant of Linux with latest available build of CAPD and Mathematica 5.0 or better.

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