On Hausdorff dimension of oscillatory motions in three body problems

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Abstract

We show that for the Sitnikov example and for the restricted planar circular 3–body problem the set of *oscillatory motions* often has maximal Hausdorff dimension. Also, we construct Newhouse domains for both problems.

1   Introduction and the statement of main results

One of the most famous results by Poincare is on non-integrability of the three body problem. The key part of the proof is construction of the homoclinic picture. This picture was at the origin of "chaos theory" over a century ago. Later the results by Birkhoff and Smale gave a deep insight to the dynamics associated with the homoclinic picture and constructed a wide range of chaotic orbits. This, however, does not come close to providing a description of a typical orbit. In the context of abstract smooth dynamical systems Newhouse found domains (open sets) in the space of dynamical systems having many surprising phenomena that are generic! It includes a shocking counter-example to Thom's conjecture on generic finiteness of the number of coexisting attracting periodic points (see [N2]). One can also show that a generic map there has an arbitrary ahead growth of the number of periodic points along a subsequence (see [K]). Nowadays the former phenomenon is called *Newhouse phenomena* and domains with such properties *Newhouse domains*. These two examples give a glimpse how unbelievably complex and persistently changing the homoclinic picture can be (see also [GoK] for further strange generic properties in those domains).

In the present paper we show that Newhouse phenomena actually exist for certain three body problems (namely, for the Sitnikov problem and for the restricted planar circular three body problem). This leads to a package of highly surprising dynamical properties there. From historical perspective applying a variety of the deep techniques developed in dynamics during the last century to the context of the celestial mechanics brings us back to the original motivation: three body problem.

The classical 3–body problem consists in studying the dynamics of 3 point masses in the plane or in the 3-dimensional space mutually attracted under Newton gravitation: Let q_1, q_2, q_3 be point masses in $\mathbb{R}^d$ for $d = 2$ or 3 with masses m_1, m_2, m_3 respectively.

$$\ddot{q}_i = \sum_{j \neq i} m_j \frac{q_j - q_i}{|q_j - q_i|^3}, \qquad q_i \in \mathbb{R}^d, \quad i = 1, 2, 3.$$

Denote r_k the vector from q_i to q_j with $i \neq k, j \neq k, i < j$. One possible direction is to study qualitative behavior of bodies as time tends to infinity either in the future or in the past. Chazy [Ch] gave a classification of all possible types of asymptotic motions:

Theorem 1. *(Chazy, 1922 (see also [AKN])) Every solution of the three-body problem belongs to one of the following seven classes:*

- $\mathcal{H}^+$ *(hyperbolic):* $|r_k| \to \infty$, $|\dot{r}_k| \to c_k > 0$ *as* $t \to_+$;

- $\mathcal{H}E_k^+$ *(hyperbolic-parabolic):* $|r_k| \to \infty$, $|\dot{r}_k| \to 0, |\dot{r}_i| \to c_i > 0$ $(i \neq k)$;

- $\mathcal{H}E_k^+$ *(hyperbolic-elliptic):* $|r_k| \to \infty, |\dot{r}_i| \to c_i > 0$ $(i \neq k)$, $\sup_{t \geq t_0} |r_k| < \infty$;

- $\mathcal{P}E_k^+$ *(hyperbolic-elliptic):* $|r_k| \to \infty, |\dot{r}_i| \to 0$ $(i \neq k)$, $\sup_{t \geq t_0} |r_k| < \infty$;

- $\mathcal{P}^+$ *(parabolic)* $|r_k| \to \infty$, $|\dot{r}_k| \to 0$,

- B^+ *(bounded):* $\sup_{t \geq t_0} |r_k| < \infty$;

- OS *(oscillatory):* $\limsup_{t \to \infty} \max_k |r_k| = \infty$, $\liminf_{t \to \infty} \max_k |r_k| < \infty$;

Examples of the first six types were known to Chazy. The existence of oscillatory motions was proved Sitnikov [Si] in 1959. Properties of the set of these motions is the central subject of the present paper.

Recall that energy of the three-body problem

$$ H = \sum_k \frac{m_k |\dot{q}_k|^2}{2} - \sum_{i<j} \frac{m_i \, m_j}{|q_i - q_j|} $$

is preserved along the solutions. Notice that for positive energy only $\mathcal{H}^{\pm}$ and $\mathcal{H}E^{\pm}$ are possible and for negative energy only $\mathcal{H}E^{\pm}$, $B^{\pm}, OS^{\pm}$ are possible. It turns out that all logically possible intersections of the past and the future final motion do exist as solutions of the three-body problem. In the beginning of the last century there was a heated discussion whether there is a symmetry of final type in the past and in the future. In one of Chazy's papers (1929, see also [AKN]) a false assertion was stated that in the three-body problem the two final types in the past and the future coincide. It was disproved rigorously by Sitnikov in 1953. Moreover, for all of them, but one, it is know whether they form a set of initial conditions of positive or zero Lebesgue measure. Here are two tables from a famous paper by Alexeev [Ae, Af]. Letting bodies to be far from each other we see that there is a qualitative difference if energy H is positive or negative. Indeed, potential energy (the second term in energy H) goes to zero. So it is natural to distinguish two cases:

Positive energy $H > 0$		
	H^+	HE_i^+
H^-	Lagrange, 1772 (isolated examples); Chazy, 1922 Measure > 0	PARTIAL CAPTURE Measure > 0 Shmidt (numerical examples), 1947; Sitnikov (qualitative methods), 1953
HE_j^-	COMPLETE DISPERSAL Measure > 0	$i = j$ Measure > 0 Birkhoff, 1927 $i \neq j$ EXCHANGE, Measure > 0 Bekker (numerical examples), 1920; Alexeev (qualitative methods), 1956

Negative energy $H < 0$			
	HE_i^+	B^+	OS^+
HE_j^-	$i = j$ Measure > 0 Birkhoff, 1927 Exchange $i \neq j$ Measure > 0 Bekker, 1920 (numerical examples); Alexeev, 1956; (qualitative methods)	COMPLETE CAPTURE Measure $= 0$ Chazy,1929 & Merman,1954; Littlewood, 1952; Alexeev, 1968; $\neq \emptyset$	Measure $= 0$ Chazy,1929 & Merman,1954; Alexeev, 1968 $\neq \emptyset$
B^-	PARTIAL DISPERSAL $\neq \emptyset$ Measure $= 0$	Euler, 1772; Lagrange, 1772 Poincare, 1892 (isolated examples); Arnold, 1963	Littlewood, 1952 Measure $= 0$ Alexeev, 1968 $\neq \emptyset$
OS^-	$\neq \emptyset$ Measure $= 0$	$\neq \emptyset$ Measure $= 0$	Sitnikov, 1959, $\neq \emptyset$ Measure $=?$

The only major open problem is

Is Lebegue measure of the set of oscillatory motions positive?

Arnold in the conference in the honor of 70-th anniversary of Alexeev called this *the central*

problem of the celestial mechanics. The conjecture, which probably[1] goes back to Kolmogorov and stated in the paper of Alexeev [Ae, Af], is that Lebesgue measure of oscillatory motions is zero.

The 3–body problem is called *restricted* if one of the bodies has mass zero and the other two are strictly positive. Historically, the first example of oscillatory motions is due to Sitnikov [Si] for the restricted spatial three-body problem. In the pioneering work [Af, Ae] Alexeev not only extended the Sitnikov example to the spatial (unrestricted) 3–body problem, but also found important use of hyperbolic dynamics for the 3–body problem. Later Moser [Mo] gave a conceptually transparent proof of existence of oscillatory motions for the Sitnikov problem interpreting homoclinic intersections using symbolic dynamics. This paved a road to a varity of applications of hyperbolic dynamics to the three–body problem.

Existence of oscillatory motions for the planar three-body problem had to wait for another decade. Investigation of planar oscillatory motions was initiated by Llibre–Simo [LS] who proved their existence for the restricted planar circular 3–body problem (see also [MP] on the subject of evaluation of the Melnikov integral). In [X] Xia simplified their proof. An attempt to study oscillatory motions for the restricted planar elliptic three–body problem was made in [Xi2][2], and for the planar three–body problem in [Xi3] (see also [Bak]). Splitting of invariant manifolds formed by future (resp. past) parabolic motions is studied in details in [MP].

In this paper we investigate how large is the set *oscillatory motions* and show that for the Sitnikov example and the restricted planar circular 3–body problem this set often has maximal Hausdorff dimension. This certainly does not support Kolmogorov's conjecture, but is rather a counterpart of it. Now we discuss the Sitnikov example followed by the restricted planar circular 3–body problem and state our results.

1.1 The Sitnikov example

Consider two point masses q_1 and q_2 of equal mass $m_1 = m_2 = 1/2$. Suppose they move on the plane so that the center of mass is at the origin. Assume that their orbits are elliptic of eccentricity $e > 0$ and period 2π. We shall treat e as parameter. Consider a third massless point q_3 moving along the z-axis. Due to symmetry if an initial condition and velocity belong to the z-axis, then the whole orbit of q_3 also belongs to the z-axis. Denote by $(t, z(t), \dot{z}(t))$ an orbit of q_3, where the time $t \pmod{2\pi}$ determines position of primaries. Denote $r(t) = r_e(t)$ distance of primaries to the origin. Then the equation of motion of the massless body is of the form

$$\ddot{z} = -\frac{z}{\sqrt{z^2 + r^2(t)}} \tag{1}$$

and the corresponding Hamiltonian is of the form

$$H(t, z, \dot{z}) = \frac{\dot{z}^2}{2} - \frac{1}{\sqrt{z^2 + r^2(t)}}.$$

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[1]We are not sure because in in the English version [Ae] Alexeev refers to Kolmogorov, however, in the French one [Af] Kolmogorov's name is surprisingly omitted

[2]In this paper the author relies on a C^1 Inclination Lemma p. 235 referring to [R1]. That paper, in turn, states Theorem E p. 372 required for the proof of a C^1 Inclination Lemma. In the proof of Theorem E the author refers to the "standard" (hyperbolic) λ-lemma. This is not applicable for a degenerate saddle, see section 3.5.
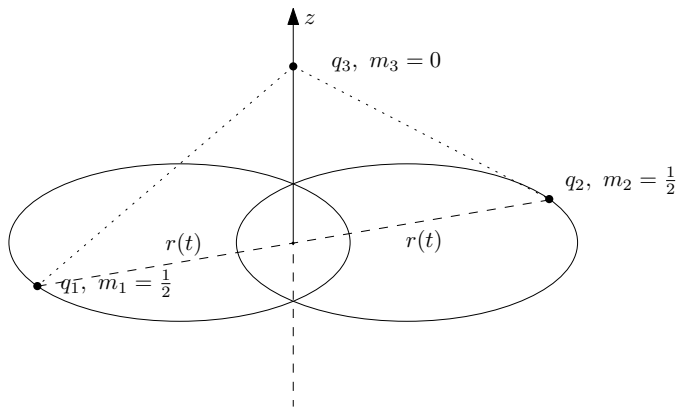
Figure 1: Sitnikov problem.

Theorem 2. *There is an open set $\mathcal{N} \subset (0,1)$, $0 \in \overline{\mathcal{N}}$, of values of eccentricity e such that for a Baire generic $e \in \mathcal{N}$ the set of oscillatory motions has Hausdorff dimension 3.*

1.2 The restricted planar circular 3–body problem (RPC3BP)

Consider the restricted planar circular 3–body problem. Namely, consider two massive bodies, the *primaries*, performing uniform circular motion about their center of mass. Normalizing the masses of the primaries so that their masses sum to one, we obtain primaries of mass μ and $1 - \mu$ respectively, where $0 < \mu < 1$ is the *mass ratio*. In addition, we chose coordinates so that the center of mass of the system is located at the origin, and we normalize the period of the circular motion to 2π. By entering into a frame which rotates with the primaries, we can choose rectangular coordinates (x, y) so that the primaries are fixed at $(1 - \mu, 0)$ and $(-\mu, 0)$, respectively. Finally, we introduce a third massless body P into the system, so that it does not effect the primaries. RPC3BP investigates how P moves.

The distance of P to the primaries is given by $d_1(x, y) = [(x - (1 - \mu))^2 + y^2]^{1/2}$ and $d_2(x, y) = [(x - \mu)^2 + y^2]^{1/2}$. The standard formula for the *Jacobi constant C*, the only integral for RPC3BP, is given by

$$C_\mu(x, y, \dot{x}, \dot{y}) = x^2 + y^2 + \frac{2\mu}{d_1} + \frac{2(1 - \mu)}{d_2} - (\dot{x}^2 + \dot{y}^2). \tag{2}$$

Here is the main result for the RPC3BP. Denote by RPC3BP(μ, C) the RPC3BP with mass ratio μ restricted to the energy surface $\Pi_C = \{C_\mu(x, y, \dot{x}, \dot{y}) = C\}$. We shall treat both μ and C as parameters.

Theorem 3. *A) For any C large enough there is an open set of mass ratios $\mathcal{N}_C \subset (0,1)$ such that for a residual subset $\mathcal{R} \subset \mathcal{N}$ and for any $(\mu, C) \in \mathcal{R}$ in the 3-dimensional energy surface Π_C the set of oscillatory motions of RPC3BP(μ, C) has Hausdorff dimension 3;*

B) For any $\mu \in (0,1)$ there is an open set $\mathcal{N}_\mu$ such that for a Baire generic $C \in \mathcal{N}_\mu$ in the 3-dimensional energy surface C the set of oscillatory motions of RPC3BP(μ, C) has Hausdorff dimension 3.
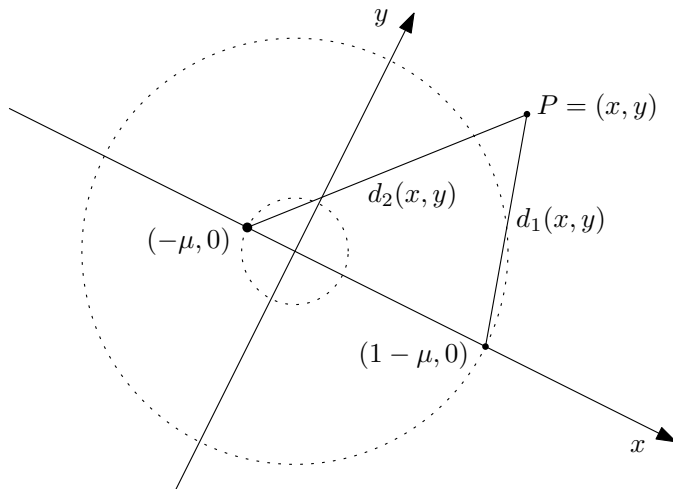
Figure 2: The restricted planar circular 3–body problem.

Remark 1.1. *Our technique could also be applied to the 3–body problem on the line [LS1, SX], but we do not elaborate on it here.*

In what follows the following motions are also of importance. A motion of the massless body is called *future (resp. past) parabolic* if the body escapes to infinity with vanishing speed as time tends to $+\infty$ (resp. $-\infty$).

1.3   Reduction to area-preserving maps

A natural way to reduce the Sitnikov example to a 2-dimensional Poincare map is as follows. Define

$$f_e : (z, \dot{z}) \mapsto (z', \dot{z}') \qquad (z, \dot{z}) \in \mathbb{R}^2 \tag{3}$$

where a trajectory of (1) with initial condition $(0, z, \dot{z})$ at time 2π is located at $(2\pi, z', \dot{z}')$. Since equations of motion are Hamiltonian this map is *area-preserving*.

There are many way to define a Poincare map for the RPC3BP(μ, C) with $C \geq 2\sqrt{2}$. Let's pick one. Consider the polar coordinates (r, φ) on the (x, y)-plane and let (P_r, P_φ) be their symplectic conjugate. Write the Hamiltonian of the RPC3BP in these coordinates:

$$H(r, P_r, \varphi, P_\varphi) = \frac{P_r^2}{2} + \frac{P_\varphi^2}{2r^2} - \frac{1}{r} - P_\varphi + \left(\frac{1}{r} - \frac{\mu}{d_1} - \frac{1-\mu}{d_2} \right) =: H_0 + \Delta H, \tag{4}$$

where d_1 and d_2 are the distances to the primaries as above (2),

$$
\begin{aligned}
d_1 &= \left(r^2 - 2(1-\mu)r \cos\varphi + (1-\mu)^2 \right)^{1/2} \\
d_2 &= \left(r^2 + 2\mu r \cos\varphi + \mu^2 \right)^{1/2},
\end{aligned}
$$

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P_r (resp. P_φ) is the variable conjugate to r (resp. φ). In other words, $P_r = \dot{r}$ and P_φ is the angular momentum. One can rewrite the Jacobi constant in the polar coordinates.

Since the Jacobi constant is the first integral of this problem, there is a 3-dimensional 'energy' surface $\Pi_C = \{C = C_\mu(r, \varphi, P_r, P_\varphi)\}$. It turns out that for $C > 2\sqrt{2}$ by the implicit function one can express $P_\varphi = P_\varphi(r, \varphi, P_r, C)$ on Π_C and consider a 3-dimensional differential equation on (r, φ, P_r). On a "large" open set $\dot{\varphi} = 1 - P_\varphi/r^2 > 0$ and $\varphi(t)$ is strictly monotone. Choose a 2-dimensional surface $S = \{\varphi = 0\} \subset \Pi_C$ and a Poincare return map

$$f_{\mu,C} : (r, P_r) \mapsto (r', P_r'), \tag{5}$$

where a trajectory of the RPC3BP with an initial condition $(r, 0, P_r, P_\varphi(r, 0, P_r, C))$ that passes through $(r', 2\pi, P_r', P_\varphi(r', 2\pi, P_r', C))$. This gives rise to *an area-preserving map $f_{\mu,C} : U \to \mathbb{R}^2$* defined on an open set $U \subset \mathbb{R}^2$.

1.4   Newhouse domains for area-preserving maps

We say that a saddle periodic point p of an area-preserving map f exhibits an *homoclinic tangency* (HT) if stable and unstable manifolds $W^s(p)$ and $W^u(p)$ of p respectively have a point of tangency. We say that f has an HT if some of its saddle points has an HT. Denote by $\mathcal{HT}$ the closure of the set of area-preserving maps with HT. It seems that appearance of an HT is of codimension 1 phenomenon and can be destroyed for an individual saddle. Astonishingly it turned out that $\mathcal{HT}$ has nonempty interior. This phenomenon was discovered first for dissipative 2-dimensional maps by Newhouse [N1, N2, N3]. It took over two decades to extend it to area-preserving setting. This was done by Duarte [Du1].

Call an open set with a dense subset of maps with an HT a *Newhouse domain*. One of main results if this paper is a proof of existence of Newhouse domains for the Sitnikov example and the RPC3BP. We shall also prove that

Theorem 4. *Let $\{f_e\}_{0<e<1}$ be the family of maps (3). Then there is a Newhouse domain $\mathcal{N} \subset (0, 1)$, i.e. for a dense set of e in $\mathcal{N}$ the Poincare map f_e has an HT.*

Theorem 5. *Let $\{f_{\mu,C}\}$ be the family of maps (5). Then*
A) for any C large enough there is a Newhouse domain $\mathcal{N}_C \subset (0, 1)$, i.e. for a dense set of μ in $\mathcal{N}_C$ the Poincare map $f_{\mu,C}$ has an HT.
B) for any $\mu \in (0, 1)$ there is a Newhouse domain $\mathcal{N}_\mu$ in a neighborhood of infinity, i.e. for a dense set of C in $\mathcal{N}_\mu$ the Poincare map $f_{\mu,C}$ has an HT.

Robinson [R2], using ideas of Newhouse, showed that for a generic 1-parameter unfolding a homoclinic tangency there are Newhouse domains on the parameter line. In a sense we prove a similar statement. Namely, we show that the above 1-parameter families are non-degenerate and Newhouse domains occur on the parameter line, not in infinite dimensional space of mappings. The proofs of these two theorems are based on similar results on conservative homoclinic bifurcations from [Du3, Du4].

1.5   Scheme of the proof

Since the construction consists of many involved steps, we provide here a very sketchy structure of the proof.

Step 1. Using McGehee coordinates one can show that infinity can be represented as a degenerate saddle (we denote it by O_∞) of some explicit form. Stable and unstable manifolds of this saddle are smooth, and correspond to parabolic motions. Oscillatory motions therefore correspond to the orbits that contain the degenerate saddle in their ω-limit set together with some other points. This step is standard, see [Mo].

Step 2. Zero value of the parameter (e in Sitnikov problem, μ in the RC3BP) corresponds to the integrable case, and in McGehee coordinates stable and unstable manifolds of the degenerate saddle coincide. The splitting of these manifolds for small values of the parameter is described by the corresponding Melnikov function. For the Sitnikov problem the Melnikov function was explicitly calculated in [GP], and for RPC3BP we derive the form of the Melnikov function from [MP]. In particular, this implies existence of the transverse homoclinic points for the degenerate saddle O_∞.

Step 3. We study the dynamics near the degenerate saddle O_∞ (which represents infinity in McGehee coordinates). Namely, we show that in spite of the fact that an analog of inclination lemma does not hold for the degenerate saddle, iterates of a transversal to he stable manifold accumulate in C^2 metric to the unstable manifold away from singularity (Theorems 6 and 7), establish quantitative version of the cone condition (see (21) and (22)), study the shape of the images of the transversal (see (24)). Finally, we study the dependence of these images on the parameter of the system (see (25)).

First we establish those properties for a simplified system (i.e. neglecting higher order terms), and then check that for small values of the parameter the neglected terms do not change the established results.

Step 4. Using the form of the Melnikov function and the properties of the dynamics near the degenerate saddle we construct a sequence of parameters of the system $e_n \to 0$ (or $\mu_n \to 0$ for RC3BP) such that the invariant manifolds $W^u_{e_n}(O_\infty)$ and $W^s_{e_n}(O_\infty)$ have a point of quadratic tangency that unfolds transversally with the change of parameter (Theorem 15).

Step 5. We construct a sequence of the hyperbolic periodic points converging to a point of transverse intersection of $W^u(O_\infty)$ and $W^s(O_\infty)$, homoclinically related to O_∞, and have quadratic homoclinic tangencies that unfold generically with the parameter (Proposition 8.2).

Step 6. Unfolding of a quadratic homoclinic tangency associated to a hyperbolic saddle give birth to a hyperbolic horseshoe Λ of Hausdorff dimension arbitrarily close to 2, see Theorem 16. Existence of such sets was proven by Gorodetski [Go] using a previous work of Duarte [Du2, Du3, Du4]. Besides, Λ exhibits persistent homoclinic tangencies, and in the case of the Sitnikov problem this proves Theorem 4 (respectively, Theorem 5 in the case of the RPC3BP). Degenerate saddle O_∞ is homoclinically related to Λ.

Step 7. Next we construct a transitive locally maximal invariant compact set $\Lambda^\#$ that contains both horseshoe Λ of large Hausdorff dimension and the degenerate saddle O_∞, Theorem 10. Then we check that the classical Manning-McCluskey result on the relation between the entropy, Lyapunov exponents, and Hausdorff dimension of a measure supported on a horseshoe [MM] also holds for the "non-hyperbolic horseshoe" $\Lambda^\#$, see Theorem 12.

Step 8. Using the thermodynamics formalism and Manning-McCluskey result we show that Hausdorff dimensions of $\Lambda^\#$ intersected with a stable (unstable) manifold are not less than the corresponding Hausdorff dimensions for the horseshoe Λ, hence close to 1, see Proposition 7.4.

Step 9. For uniformly hyperbolic sets the holonomy map along stable (unstable) manifolds is Holder continuous with Holder exponent arbitrarily close to 1 (see [PV]). We prove that the same statement holds away from the degenerate saddle for the holonomies in $\Lambda^\#$, Proposition 7.5. This

implies that the set of points whose ω-limit set contains both O_∞ and some other points (this corresponds to the set of initial conditions of oscillatory motions) has Hausdorff dimension close to 2, see Theorem 9. Standard genericity arguments show that for a residual set of parameters in some interval oscillatory motions form a set of maximal possible Hausdorff dimension, see Section 8.4, therefore completing the proof.

2   McGehee coordinates

2.1   McGehee coordinates for the Sitnikov problem

For the Sitnikov problem equations of motion are

$$\ddot{z} + \frac{z}{(z^2 + r_e^2(t))^{3/2}} = 0, \quad r = \frac{1}{2}(1 - e\cos t) + O(e^2). \tag{6}$$

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FROM THE OLD FILE:

McGehee transformation

$$z = \frac{2}{q^2} \quad \dot{z} = -p, \quad ds = 4q^{-3}dt, \quad q = \sqrt{2/q}.$$

New equations of motion become

$$\begin{cases} \dfrac{dq}{ds} = p \\ \dfrac{dp}{ds} = q\left(1 + \dfrac{q^4}{4}r^2\right)^{-3/2} \\ \dfrac{dt}{ds} = \dfrac{4}{q^3}. \end{cases} \tag{7}$$

The invariant manifolds have the following form

$$q = \chi(p, t) = p(1 + a_4 p^4 + a_7(t)p^7 + \cdots)$$

and

$$q = \chi(-p, -t) = -p(1 + a_4 p^4 - a_7(t)p^7 + \cdots),$$

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where $a_4 = (32\pi)^{-1} \int_0^{2\pi} r_e^2(t) dt$. This is shown in Moser [Mo] ch.6.2 and provides us location of the invariant manifolds. Time rescaling is highly degenerate as $(x, y) \to 0$. This degeneracy is serious enough to prevent us from obtaining information about (6) using (9). We make a change of coordinates such that the invariant manifolds become coordinate axis and compute differential equation after such a change. To derive such an equation it is convenient to look for d/ds derivatives.

Normalizing coordinate change

$$x = \frac{1}{4}(q - \chi(-p, -t)) = \frac{1}{4}(p + q) + \frac{a_4}{4}p^5 - \frac{a_7(-t)p^8}{4} + \cdots$$

$$y = \frac{1}{4}(q - \chi(p, t)) = \frac{1}{4}(q - p) - \frac{a_4}{4}p^5 - \frac{a_7(t)p^8}{4} + \cdots$$

Direct differentiation gives

$$\frac{dx}{ds} = \frac{1}{4}(p + q(1 + \frac{q^4}{4}r^2)^{-3/2}) +$$

$$5\frac{a^4}{4}p^4 q(1 + \frac{q^4}{4}r^2)^{-3/2} - \frac{a_7(-t)}{4}8p^7 q(1 + \frac{q^4}{4}r^2)^{-3/2} + \cdots =$$

$$= \frac{1}{4}(p + q - \frac{3}{8}q^5 r^2) + O(q^9 p^4) + \frac{5a_4}{4}p^4 q -$$

$$- \frac{15}{8}a_4 p^4 q^5 r^2 - 2a_7(-t)p^7 q + \cdots =$$

$$= \frac{1}{4}(p + q - 6a_4 q^5) + \frac{5}{4}a_4 p^4 q + O_8.$$

$$\frac{dx}{ds} = \frac{1}{4}(1 - 6a_4 q^4)(p(1 + 6a_4 q^4) + q(1 + 5a_4 p^4) + O_8) =$$

$$= \frac{1}{4}((p + q) + 6a_4 q^4 p - 6a_4 pq^4 + 5a_4 p^4 q - 6a_4 q^5 + O_8)$$

Divide the RHS by $x = \frac{1}{4}(q + p + a_4 p^5 + \cdots)$. We get

$$= x + \frac{1}{4}(-a_4 p^5 + 5a_4 p^4 q - 6a_4 q^5) + O_8$$

Dividing $\frac{1}{4}(-a_4 p^5 + 5a_4 p^4 q - 6a_4 q^5)$ by x and neglecting higher order terms gives

$$a_4(6qp^3 - q^2 p^2 + q^3 p - p^4 - 6q^4).$$

$$\frac{dy}{ds} = \frac{1}{4}(p - q(1 + \frac{q^4}{4}r^2)^{-3/2}) - 5\frac{a^4}{4}p^4 q + \cdots =$$

$$= \frac{1}{4}(p - q + \frac{3}{8}q^5 r^2) - \frac{5a_4}{4}p^4 q + \cdots =$$

$$= \frac{1}{4}(-q + p + a_4 p^5 - a_4 p^5 + 6a_4 q^5 - 5a_4 p^4 q + \cdots) =$$

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$$= -y + \frac{1}{4}(-a_4 p^5 + 6a_4 q^5 - 5a_4 p^4 q + \cdots) =$$

$$= -y(1 - a_4 p^4 - 6a_4 q^3 p - 6a_4 p^2 q^2 - 6a_4 p q^3 - 6a_4 q^4 + \cdots)$$

After substituting $q = 2(x + y)$ and $p + a_4 p^5 = 2(x - y)$. Neglecting higher order terms we get

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{(x+y)^3 x}{2} \ (1 + 16a_4(6(x^2 - y^2)(x^2 + 3y^2) - 6(x+y)^4 - (x-y)^4)) \\ \dfrac{dy}{ds} = -\dfrac{(x+y)^3 y}{2} \ (1 - 16a_4(6(x^2 - y^2)(3x^2 + y^2) + 6(x+y)^4 + (x-y)^4)). \end{cases}$$

Open brackets, rescale time by a factor of two and get

$$\begin{cases} \dfrac{dx}{dt} = (x+y)^3 x \ (1 + O_4(x, y)) \\ \dfrac{dy}{dt} = -(x+y)^3 y \ (1 + O_4(x, y)), \end{cases} \tag{8}$$

where $O_4(x, y)$ denotes terms of order 4 and higher in x and y. [3]

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After McGehee's transformation

$$z = \frac{2}{q^2} \quad \dot{z} = -p, \quad ds = 4q^{-3} dt,$$

the new equations of motion become

$$\begin{cases} \dfrac{dq}{ds} = p \\ \dfrac{dp}{ds} = q \left(1 + \dfrac{q^4}{4} r^2\right)^{-3/2} \\ \dfrac{dt}{ds} = \dfrac{4}{q^3}. \end{cases} \tag{9}$$

By McGehee's theorem [McG] this equation has a topological saddle point at the origin and the invariant manifolds that are analytic away from the origin. According to Moser [Mo] ch III, sect 2.b) these manifolds have the following form

$$q = \chi(p, t) = p + a_4 p^5 + a_8(t) p^9 ...$$

and

$$q = \chi(-p, -t) = -p - a_4 p^5 - a_8(t) p^9 ...,$$

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[3]One could compute the leading terms of O_4 in the first and the second line $P_4(x, y) = 16a_4(x^4 + 20x^3 y + 30x^2 y^2 + 20xy^3 + 25y^4)$ and $Q_4(x, y) = 16a_4(y^4 + 20y^3 x + 30x^2 y^2 + 20yx^3 + 25x^4)$ resp.

where a_4 is some nonzero constant and $a_8(t)$ is a time-periodic function. Expressions for the invariant manifolds show that for

$$x = \frac{1}{4}(q - \chi(-p, -t)) = \frac{1}{4}(p + q) + \frac{a_4}{4}p^5 + \frac{a_8(t)}{4}p^9 \cdots$$

$$y = \frac{1}{4}(q - \chi(p, t)) = \frac{1}{4}(q - p) - \frac{a_4}{4}p^5 - \frac{a_8(t)}{4}p^9 \cdots$$

they are represented by x and y coordinate axis.

In order to simplify notation we introduce the following class of functions. For any pair of positive integers $n < m$ denote $O_{n,m}(x, y, t)$, $x, y \geq 0$ the class of C^∞ functions $f(x, y, t)$ periodic in t and such that $f(x, y, t) = f_1(x, y) + f_2(x, y, t)$ and

$$\lambda^{-n} f_1(\lambda x, \lambda y), \quad \lambda^{-m} f_2(\lambda x, \lambda y, t), \quad \lambda^{-m} \partial_t f_2(\lambda x, \lambda y, t), \quad \lambda^{-m} \partial_t^2 f_2(\lambda x, \lambda y, t)$$

stay uniformly bounded for all t as $\lambda \to 0$. It will also be convenient to write "$O_{n,\infty}$" in the case of a function independent of t. Notice that if $f \in O_{n+1,m+1}$, then $f/(ax + by) \in O_{n,m}$ when $a, b > 0$. This is a convenient way to write remainders in the derivations below. In the next statement we collect some properties of this class of functions.

Lemma 2.1. *The introduced class of functions has the following properties:*

1. *If $f \in O_{n+1,m+1}$, then $\partial f \in O_{n,m}$, where partial drivative is with respect to x or y;*

2. *If $f \in O_{n,m}$, then $(ax + by)f \in O_{n+1,m+1}$ for any $a, b \in \mathbb{R}$;*

3. *If $f \in O_{n,m}$ and $g \in O_{k,p}$, then $fg \in O_{n+k,m+p}$;*

4. *If $f \in O_{n+1,m+1}$, then $f/(ax + by) \in O_{n,m}$ for any $a, b > 0$;*

5. *If $f \in O_{n,m}$, then for any $\kappa > 0$ there is $\delta_0 = \delta_0(\kappa) > 0$ such that $|f(x, y, t)| < \kappa$ for any t and $0 \leq x, y \leq \delta_0$.*

In order to shorten the lengthy formulas appearing below for a nonzero number $A \in \mathbb{R}$ and a function in $O_{n,m}$ we denote by $A_{n,m} = A + O_{n,m}$ the sum of the two.

Remark 2.2. *Warning: If $f \in O_{n+1,m+1}$, then f/x or f/y does not necessarily belongs to $O_{n,m}$. Pick $f = y^{n+1}$ and divide it by x. It is not longer smooth for $x, y \geq 0$.*

In these terms we can rewrite (9) in the following way. There is a function $P \in O_{5,9}$ such that $q = 2(x + y)$ and $p = 2(x - y) + P(x, y, t)$ and in xy-coordinates (9) can be written as

$$\begin{cases} \dfrac{dx}{dt} = x(x + y)^3(1 + O_{4,8}) \\ \dfrac{dy}{dt} = -y(x + y)^3(1 + O_{4,8}) \\ \dfrac{ds}{dt} = 2(x + y)^3. \end{cases} \qquad (10)$$

With respect to s time we have a saddle

$$\begin{cases} \dfrac{dx}{ds} = x(1 + O_{4,8}) \\ \dfrac{dy}{ds} = -y(1 + O_{4,8}). \end{cases} \qquad (11)$$

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In order to obtain a similar expression for the RPC3BP we discuss the Kepler problem first. Then we come back to RPC3BP and notice that it is a small perturbation of the Kepler problem for small mass ratio μ.

2.2 The Kepler problem (2BP) and the polar coordinates

Recall that the polar coordinates for the Kepler problem are formed by the polar coordinates (r, φ) and the momenta (P_r, P_φ) conjugate to (r, φ) resp. The Hamiltonian for the RPC3BP in rotating[4] polar coordinates $(r, \varphi, P_r, P_\varphi)$ is

$$H_0(r, \varphi, P_r, P_\varphi) = \frac{P_r^2}{2} + \frac{P_\varphi^2}{2r^2} - P_\varphi - \frac{1}{r}.$$

Notice that
— angular momentum P_φ is the first integral;
— levels sets of H_0 on the (r, P_r)-plane coincide with trajectories of the Kepler problem;
— for $H_0 < 0$ levels sets are compact and, therefore, the corresponding trajectories are periodic;
— motions of the Kepler problem are conic sections (see e.g. [AKN]);
— the last two imply that each of these periodic trajectories form an ellipse on the inertial plane of motion $\mathbb{R}^2 \setminus \{0\}$;
— in the inertial coordinate system the Hamiltonian becomes $H_0 + P_\varphi$ and both future and past parabolic motions coincide and correspond to the curve $\{H_0 + P_\varphi = 0\}$.

2.3 McGehee coordinates for the Restricted Planar Circular 3 Body Problem

The Hamiltonian for RPC3BP in rotating polar coordinates $(r, \varphi, P_r, P_\varphi)$ is given by (4). So the equations of motion become

$$\begin{cases} \dot{r} &= P_r \\ \dot{P}_r &= P_\varphi^2 r^{-3} - r^{-2} - \partial_r \Delta H \\ \dot{\varphi} &= -1 + P_\varphi r^{-2} \\ \dot{P}_\varphi &= -\partial_\varphi \Delta H. \end{cases} \tag{12}$$

Directly one could prove the following bounds on the perturbation term ΔH and its derivatives satisfy

Lemma 2.3. *[GaK] For any $r > 1$*

$$\max_\varphi |\Delta H(r, \varphi)| \leq \frac{\mu}{r^2(r-1)}, \quad \max_\varphi |\partial_r \Delta H(r, \varphi)| \leq \frac{\mu}{r^2(r-1)^2}. \tag{13}$$

Notice that in the rotating polar coordinates H is μ-close to the Kepler Hamiltonian H_0. So nearly parabolic motions belong to a neighborhood of $\{H_0 + P_\varphi = 0\}$. *The motions we investigate belong to such the neighborhood $-0.1 < H + P_\varphi < 0.1$ for μ small enough.* We could replace 0.1 by a smaller number decreasing μ in return.

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[4]we need to consider rotating frame because it fits to RPC3BP very well.

In Theorem 3 we assume that $C > 4$. Thus, $H_0 + P_\varphi = 2C > 4$ on such an energy surface and we can express P_φ as an implicit function. Indeed, (4) leads to

$$\frac{P_\varphi^2}{r^2} - 2P_\varphi - \frac{2}{r} + 2\Delta H + P_r^2 = C.$$

and closest return to the origin $r \geq C^2/8 - O(\mu) > 1.9$ and $2P_\varphi = C + O(\mu)$ for nearly parabolic motions. Thus, we can remove equation for P_φ and use the implicit function.

Introduce $u = r^{-1/2}$ along with a function $d_r(u, \varphi) = u^{-3}\partial_r \Delta H(u^{-2}, \varphi)$ which are well defined for $u \geq 0$ and $\phi \in \mathbb{T}$. By lemma 2.3 we have that $d_r(u, \varphi)$ has u-zero of order 5 at $u = 0$. Plug u into the equations of motion (12):

$$\begin{cases} \dot{u} &= -\frac{1}{2}P_r\, u^3 \\ \dot{P}_r &= -u^3(u + P_\varphi^2\, u^3 + d_r(u, \varphi)) \\ \dot{\varphi} &= -1 + P_\varphi\, u^4. \end{cases} \tag{14}$$

By McGehee's theorem [McG] this equation has a topological saddle point at the origin and the invariant manifolds that are analytic away from the origin. We would like to write this system in the form similar to (10) and (11).

Introduce new time s and let $x = u + P_r/\sqrt{2} - h - g$, $y = u - P_r/\sqrt{2} - h + g$ for some functions $h, g \in O_3$. Then $2u = x + y + 2h$ and and $\sqrt{2}\,P_r = x - y + 2g$. Then for a proper choice of h and g in O_3 the equations on x and y can be written in the form

$$\begin{cases} \dot{x} &= \dfrac{x(x + y + 2h)^3}{\sqrt{2}}(1 + O_{3,7}) \\ \dot{y} &= -\dfrac{y(x + y + 2h)^3}{\sqrt{2}}(1 + O_{3,7}) \\ \dot{\varphi} &= -1 + O_{4,\infty}. \end{cases} \tag{15}$$

In a fixed but small neighborhood of the origin introduce a new time s given by $ds/d\varphi = 2^{-3/2}(-1 + O_3)(x + y + h)^3$ and equation become

$$\begin{cases} \dfrac{dx}{ds} &= x(1 + O_{3,7}) \\ \dfrac{dy}{ds} &= -y(1 + O_{3,7}). \end{cases} \tag{16}$$

3   Dynamics at infinity

In order to study C^1 and C^2 dynamics at infinity of the Sitnikov and the RPC3BP we start with a class of differential equations on $(x, y) \in \mathbb{R}^2$ which contains both.

First we add one differential equation on λ, coupled with the first two, so that it describes evolution of slopes $\lambda(\cdot)$ of certain class of curves on $\mathbb{R}^2$ carried under evolution of the first two equations. This gives us information about C^1-dynamics.

Finally, we add one more differential equation on μ, coupled with the first three, so that it describes evolution of quantity $\mu(\cdot)$ related to curvature of some curves on $\mathbb{R}^2$ carried under evolution of the first two equations. This gives us information about C^2-dynamics sufficient for the proof.
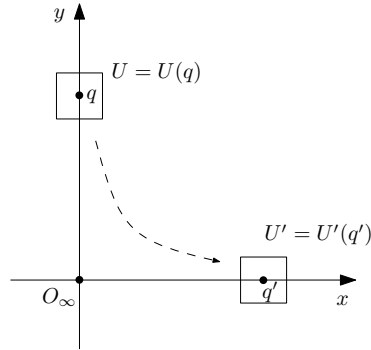
Figure 3: Dynamics near infinity.

Remark 3.1. *Below studying evolution we do not use specific form of O_3-terms in differential equations on (x, y). This will allow us to use these results for other types of three body problems.*

3.1 Evolution of slopes and quasi-curvatures near degenerate saddles

We start with an equation in the form which covers (11) and the system (16) obtained for restricted circular three body problem.

$$\begin{cases} \dfrac{dx}{dt} = (x + y)^3\, x\, (1 + O_3(x, y)) \\ \dfrac{dy}{dt} = -(x + y)^3\, y\, (1 + O_3(x, y)). \end{cases} \tag{17}$$

We shall study the class of differential equations of the type (17), where the exact form of remainder in O_3 turns out to be irrelevant for your analysis. We abbreviate $O_n(x, y, t)$ by O_n to keep size of formulas down. Notice that each O_n appearing in equation (11) and (16) are smooth in e and μ respectively.

3.2 Derivation of equations for slopes and quasi-curvature for evolving curves

Equation in variations is the following:

$$\begin{pmatrix} dx \\ dy \end{pmatrix}^{\cdot} = (x + y)^2 \begin{pmatrix} (4_3 x + 1_3 y) & 3_3 x \\ -3_3 y & -(4_3 y + 1_3 x) \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

To construct homoclinic tangencies and saddle periodic points for the time 2π-map of such an equation we need to analyze evolution of tangent directions. If $\lambda = \frac{dy}{dx}$ then

$$\lambda^{\cdot} = \frac{(dy)^{\cdot} dx - (dx)^{\cdot} dy}{(dx)^2} = -(x + y)^2 [3_3 y + 5_3(x + y)\lambda + 3_3 x \lambda^2].$$

Therefore the equation in 1-jets is the following:

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$$\begin{cases} \dot{x} = x(x+y)^3\,1_3 \\ \dot{y} = -y(x+y)^3\,1_3 \\ \dot{\lambda} = -(x+y)^2[3_3y + 5_3(x+y)\lambda + 3_3x\lambda^2] \end{cases}$$

In order to obtain an equation in 2-jets, set $\mu = \frac{d\lambda}{dx}$, an equation in second variations is

$$\begin{pmatrix} dx \\ dy \\ d\lambda \end{pmatrix}^{\!\!\cdot} = (x+y)^2 \times A \begin{pmatrix} dx \\ dy \\ d\lambda \end{pmatrix}, \tag{18}$$

where A is 3×3 matrix. Here is a computation of entries of this matrix. Partial derivatives in x are

$$\partial_x\left[x(x+y)^3 1_3\right] = (x+y)^3 1_3 + 3x(x+y)^2 1_3 + x(x+y)^3 O_2 = (x+y)^2(4_3x + 1_3y).$$

$$\partial_x\left[-y(x+y)^3 1_3\right] = -3y(x+y)^2 1_3 - y(x+y)^3 O_2 = -(x+y)^2 \cdot 3_3y.$$

$$\partial_x\left[-(x+y)^2\left(3_3y + 5_3(x+y)\lambda + 3_3x\lambda^2\right)\right] =$$

$$= (x+y)((-6_3y - 10_3(x+y)\lambda - 6_3x\lambda^2) - (x+y)(yO_2 + 5_3\lambda + (x+y)\lambda O_2 + 3_3\lambda^2 + x\lambda^2 O_2)) =$$

$$= (x+y)(-6_3y - 15_3(x+y)\lambda - (9_3x + 3_3y)\lambda^2)$$

Partial derivatives in y are

$$\partial_y\left[x(x+y)^3 1_3\right] = 3x(x+y)^2 1_3 + x(x+y)^3 O_2 = (x+y)^2 \cdot 3x.$$

$$\partial_y\left[-y(x+y)^3 1_3\right] = -3y(x+y)^2 1_3 - (x+y)^3 1_3 - y(x+y)^3 O_2 = -(x+y)^2(1_3x + 4_3y).$$

$$\partial_y\left[-(x+y)^2\left(3_3y + (5_3x + 5_3y)\lambda + 3_3x\lambda^2\right)\right] =$$

$$= (x+y)\left[(-6_3y - 10_3(x+y)\lambda - 6_3x\lambda^2) - (x+y)(3_3 + yO_2 + 5_3\lambda + (x+y)\lambda O_2 + x\lambda^2 O_2)\right] =$$

$$= (x+y)\left[-9_3y - 3_3x - 15_3(x+y)\lambda - 6_3x\lambda^2\right]$$

Partial derivatives in λ are easy and are included below. We obtain the matrix A whose entries are written column by column. The first column of this matrix has the form

$$\begin{pmatrix} 4_3x + 1_3y \\ -3_3y \\ -\dfrac{6_3y + (15_3x + 15_3y)\lambda + (9_3x + 3_3y)\lambda^2}{x+y} \end{pmatrix}$$

The second column has the form

$$\begin{pmatrix} 3_3x \\ -(4_3y + 1_3x) \\ -\dfrac{(3_3x + 9_3y) + (15_3x + 15_3y)\lambda + 6_3x\lambda^2}{x+y} \end{pmatrix}$$

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The third column has the form

$$\begin{pmatrix} 0 \\ 0 \\ -(5_3 x + 5_3 y) - 6_3 x \lambda \end{pmatrix}$$

Therefore,

$$\dot{\mu} = \frac{(d\lambda)^{\cdot} dx - (dx)^{\cdot} d\lambda}{(dx)^2} =$$

$$= (x+y)(-6_3 y - 15_3(x+y)\lambda - (9_3 x + 3_3 y)\lambda^2)) + (x+y)(-9_3 y - 3_3 x - 15(x+y)\lambda - 6_3 x\lambda^2)\lambda -$$

$$- (x+y)^2(5_3(x+y) + 6_3 x\lambda)\mu - (x+y)^2((4_3 x + 1_3 y)\mu + 3_3 x\lambda\mu =$$

$$= -3(x+y)(2_3 y + (6_3 x + 8_3 y)\lambda + (8_3 x + 6_3 y)\lambda^2 + 2_3 x\lambda^3 + (3_3 x + 2_3 y + 3_3 x\lambda)(x+y)\mu)$$

Finally we get the following system:

$$\begin{cases} \dot{x} = x(x+y)^3 \, 1_3 \\ \dot{y} = -y(x+y)^3 \, 1_3 \\ \dot{\lambda} = -(x+y)^2[3_3 y + 5_3(x+y)\lambda + 3_3 x\lambda^2] \\ \dot{\mu} = -3(x+y)[2_3 y + (6_3 x + 8_3 y)\lambda + (8_3 x + 6_3 y)\lambda^2 + 2_3 x\lambda^3 + \\ \phantom{\dot{\mu} = -3(x+y)[} + (3_3 x + 2_3 y + 3_3 x\lambda)(x+y)\mu] \end{cases} \qquad (19)$$

The first two equations describe evolution of position, the third — evolution of slope, and the forth — of the second derivative. It is natural to call $\lambda(x,y)$ *slope of evolving curve*. Denote by $\mu(x,y)$ *second derivative of evolving curve*. In the calculations below first we omit O_3 terms. Then we show that the arguments presented below also apply to the full system with O_3 terms. We call the system without O_3 terms *the simplified system*.

In order to construct a horseshoe near the saddle point at the origin and have some information of stable and unstable leaves of it we need to analyze the system (19). Consider a small $e > 0$. By Moser [Mo] we know that stable and unstable manifolds of the origin cross transversally. Denote by $X = (1,0)$ and $Y = (0,1)$ two point of these crossings.

Let U_X (resp. U_Y) be a small neighborhood of X (resp. Y). Consider a small curve $\gamma = \{(x, y(x)), x \in [0, \delta_0]\}$ for some small $\delta_0 > 0$ such that $y'(x)$ is well defined for all $x \in [0, \delta_0]$, $\gamma \subset U'$, and the crossing $y(0)$ is the stable manifold near q'. Recall that $f_e : (x,y) \to (x', y')$ denotes the Poincare map associated to the equation (17). Denote $\pi_y : (x,y) \to y$ the natural projection and ϕ_t the time t map of the flow.

Notice that for large enough n the image $f_e^n(\gamma)$ intersects U. Denote by γ_n such an intersection. We shall prove that it is still a graph over the x-axis. Then it is naturally parametrized by x-coordinate. Define $(x_t(x), y_t(x)) = \phi_t(x, y(x))$ for $t > 0$ as long as $x_t(x) < A$.

Remark 3.2. *In what follows we pick a small $\kappa > 0$ and assume that we study dynamics in such a neighborhood of O that all O_3 terms are bounded in absolute value by κ. In above notations we pick a and b small enough.*

3.3 Statement of main results on dynamics at infinity

Introduce notations: $X = (x,y) \in \mathbb{R}^2$, $\overline{X} = (x,y,\lambda) \in \mathbb{R}^3$, $\overline{\overline{X}} = (x,y,\lambda,\mu) \in \mathbb{R}^4$. Let $\overline{\overline{\phi}}_t$ be the time t map of the equation (19). Naturally $\overline{\phi}_t$ be the time t map of first three equations

(19), which is well defined because the first three equations describe dynamics in the space of 1-jets, and are independent of μ. Finally, ϕ_t denote the time t map of first two equations of (19). Denote

$$\overline{\overline{X}}_t = (x_t, y_t, \lambda_t, \mu_t) = \overline{\overline{\phi}}_t X,$$

$$\overline{X}_t = (x_t, y_t, \lambda_t, \mu_t) = \overline{\phi}_t X.$$

$$X_t = (x_t, y_t, \lambda_t, \mu_t) = \phi_t X.$$

It turns out that for different purposes we need statements about evolution of X_t, $\overline{X}_t$, and $\overline{\overline{X}}_t$. Such an evolution is not always defined for all time. Let $\gamma = \{(x, y(x)), \ x \in [0, \delta]\}$ be a smooth curve. Denote $\overline{\gamma} = \{(x, y(x), \lambda(x)), \ x \in [0, \delta]\}$, where $\lambda(x) = y'(x)$. Denote $\overline{\overline{\gamma}} = \{(x, y(x), \lambda(x), \mu(x)), \ x \in [0, \delta]\}$, where $\mu(x) = y''(x)$.

Assume that the flow ϕ_t preserves a smooth area form $a(x, y)dx \wedge dy$ with $a(x, y)$ being well defined for $x \geq 0, y \geq 0, \ (x, y) \neq 0$.

Theorem 6. *With notations above for any $\lambda^*, \mu^* > 0$ and small $\delta > 0$ there is $s_0 = s_0(\lambda^*) > 0$ and $C = C(\delta, \lambda^*) > 1$ such that for any γ with $\max|\lambda(x)| < \lambda^*$ we have the following estimates. Then for large N intersection of the image $f^N \gamma \cap U$ is a graph of a function $y_N(x)$. Denote $\lambda_N(x) = y'_N(x)$. Then the following properties hold*

$$0 < -\lambda_N(x) \leq Cy_N(x) \tag{20}$$

$$\frac{1}{Cx^{2.5}} \leq |df^N(x)v_x| \leq \frac{C}{x^{2.5}}, \quad \text{where } v_x \text{ is a tangent to } \gamma. \tag{21}$$

Moreover, for any two unit vectors $v', v'' \in T_{(x,y)}U'$ with slopes $|\lambda'|, |\lambda''| < \lambda^*$ the following holds. Let $N = N(x, y)$ be the number of iterates of the Poincare map f to get to U, i.e. $f^N(x, y) \in U$. Then

$$\frac{x^5}{C} \leq \angle(df^N(x, y)(v'), df^N(x, y)(v'')) \leq Cx^5. \tag{22}$$

In the case γ is also C^2 smooth, denote $\mu(x) = y''(x)$ and $f^N(\gamma) = \gamma_N$ and $\overline{\overline{\gamma}}_N = \{(x, y_N(x), \lambda_N(x), \mu_N(x))\}$. If $\max|\mu(x)| < \mu^*$ then for all large enough N we have

$$0 < \mu_N(x) < Cy_N(x). \tag{23}$$

Denote $\Delta = \{(x, y) : \ x = y, 0 < x < \delta\}$ for small δ as above.

Theorem 7. *(C^2 behavior of the images of the diagonal) For large enough N the intersection $f_e^N(\Delta) \cap U$ is a graph of a function $y_N(x, e)$ such that*

$$\begin{aligned} 0 < \quad &\partial_x^2 y_N(x, e) = \mu_N(x) < Cy_N(x, e), \\ 0 < - &\partial_x y_N(x, e) = -\lambda_N(x) < Cy_N(x, e). \end{aligned} \tag{24}$$

We also have

$$\left|\frac{d}{de}y_N(x, e)\right| \leq Cy_N(x, e). \tag{25}$$

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Remark 3.3. *For the convenience of the reader we list here the statements that use the properties from Theorems 6 and 7. Property (20) is used in Theorem 10 and Proposition 8.1, (21) in the proof of Theorem 10, Lemmas 7.6 and 7.7, and Proposition 8.1; (22) is needed to show (21), (23) is used in Propositions 8.1 and 8.2, (24) in Theorem 15, (25) in Theorem 15, Propositions 8.1 and 8.2.*

Remark 3.4. *Under conditions and in notations of Theorem 6 there are $C^* = C^*(x)$ and $C' = C'(x)$ independent of N such that for large enough N*

$$|\lambda_N(x) - C^* y_N(x)| < C y_N^2(x), \tag{26}$$

$$|\mu_N(x) - C' y_N(x)| < C y_N^2(x). \tag{27}$$

The same statement holds under conditions and in notations of Theorem 7, i.e. when $y_N(x)$ is an image of the diagonal Δ.

Remark 3.5. *It is interesting to compare the dynamics near the degenerate saddle O_∞ that we study in this section, and the well known dynamical properties of a linear hyperbolic saddle. This comparison is not directly needed for the proof, but rather highlights the difficulties that we had to face. Notice that topologically there is **no** difference. Indeed, due to [Mor] there is a continuous conjugacy between these local dynamical systems. We, however, need C^2 analysis, and in smooth category these systems are drastically different. Here we list several dynamical properties that confirm this statement.*

1. Expansion rates of the maps along the saddle ares different.

*2. Inclination lemma does **not** hold for the degenerate saddle.*

3. Transition times are certainly very much different.

4. Corresponding dynamics in 1- and 2-jets is essentially non-linear and is quite non-trivial in the case of the degenerate saddle. In the case of a linear hyperbolic saddle the dynamics in 2-jets is linear and hyperbolic, and is easy to describe.

5. For a non-linear hyperbolic saddle normal forms (smooth and in some cases analytic) or even linearizing coordinates that essentially simplify the picture are available. In the case of the degenerate saddle even a model case (see Section 4) is highly non-trivial if one is interested in C^2 or even C^1 dynamical properties.

4   Evolution of curves in the simplified system

In order to study the behavior of solutions of the system (19) we will first neglect some higher order terms and obtain the required properties for the simplified system. After that we show that the neglected terms do not change the obtained results.

After we rescale the time by $ds = (x + y)^3 \, dt$ and neglect some higher order terms in the

system (19), we get the following *simplified model*:

$$\begin{cases} \dfrac{dx}{ds} = x \\ \dfrac{dy}{ds} = -y \\ \dfrac{d\lambda}{ds} = \dfrac{-3y}{x+y} - 5\lambda - \dfrac{3x}{x+y}\lambda^2 \\ \dfrac{d\mu}{ds} = \dfrac{-3}{(x+y)^2}\left[2y + (6x+8y)\lambda + (8x+6y)\lambda^2 + 2x\lambda^3 + (x+y)(3x+2y+3x\lambda)\mu\right] \end{cases} \qquad (28)$$

In this section we prove analogs of Theorems 6 and 7 for the dynamics defined by the simplified system (28).

4.1 Dynamics of 1-jets for the simplified system

First of all let us understand the asymptotic behavior of λ under the dynamics defined by (28). We need to consider only the first three equations of the system:

$$\begin{cases} \dfrac{dx}{ds} = x \\ \dfrac{dy}{ds} = -y \\ \dfrac{d\lambda}{ds} = \dfrac{-3y}{x+y} - 5\lambda - \dfrac{3x}{x+y}\lambda^2 \end{cases} \qquad (29)$$

The right hand side of the equation

$$\frac{d\lambda}{ds} = \frac{-3y}{x+y} - 5\lambda - \frac{3x}{x+y}\lambda^2$$

is a quadratic polynomial in λ, and can be rewritten as

$$\frac{d\lambda}{ds} = -\frac{3x}{x+y}\left(\lambda - \lambda_0(x,y)\right)\left(\lambda - \lambda_1(x,y)\right). \qquad (30)$$

Definition 4.1. *A family of intervals* $[\lambda_-(x,y),\lambda_+(x,y)]$ *is called* a family of absorbing intervals *or simply* absorbing intervals *if any solution* $(x(s),y(s),\lambda(s))$ *of the system (29) satisfies the the following conditions:*

$$\begin{aligned} \text{If} \quad \lambda(s) = \lambda_-(x(s),y(s)), \qquad &then \qquad \frac{d\lambda}{ds}(s) > \frac{d}{ds}\lambda_-(x(s),y(s)), \quad and \\ \text{if} \quad \lambda(s) = \lambda_+(x(s),y(s)), \qquad &then \qquad \frac{d\lambda}{ds}(s) < \frac{d}{ds}\lambda_+(x(s),y(s)). \end{aligned} \qquad (31)$$

Notice that if for some s_0 the value $\lambda(s_0)$ gets into an absorbing interval then it will stay there for all $s > s_0$.

In order to provide the explicit expressions for $\lambda_0(x,y)$ and $\lambda_1(x,y)$ it is convenient to introduce an intermediate variable $\tau := \frac{y}{x}$. Also, denote
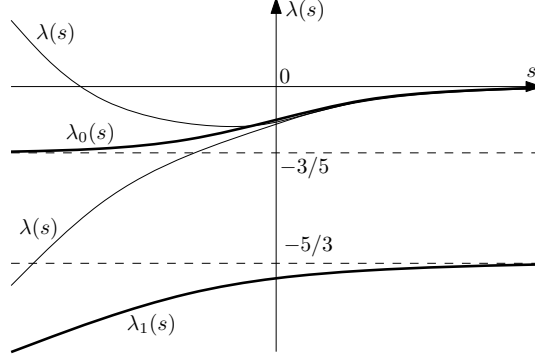
$$P(\tau) = (1+\tau)^2 - 1.44\tau. \qquad (32)$$

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Figure 4: Dynamics of slopes $\lambda(s)$ and $\lambda_{0,1}(x(s), y(s))$.

Then we have

$$\frac{d\lambda}{ds} = -\frac{3\lambda^2}{1+\tau} - 5\lambda - \frac{3\tau}{1+\tau} = -\frac{3}{1+\tau}\left(\lambda - \lambda_1(\tau)\right)\left(\lambda - \lambda_0(\tau)\right), \tag{33}$$

where

$$\lambda_{0,1}(\tau) = -\frac{5}{6}(1+\tau) \pm \frac{5}{6}\sqrt{(1+\tau)^2 - 1.44\tau} = -\frac{5}{6}(1+\tau) \pm \frac{5}{6}\sqrt{P(\tau)}. \tag{34}$$

In the next statement we establish the existence of an absorbing interval whose size tends to zero as $y \to 0$.

Lemma 4.2. *The family of intervals*

$$[\lambda_-(x,y), \lambda_+(x,y)] := \left[2\lambda_0(\tau), \frac{9}{10}\lambda_0(\tau)\right]$$

is an absorbing family.

Lemma 4.3. *(λ-absorbing interval) For any $\lambda^* > 0$ there is $s_0 = s_0(\lambda^*) > 0$ such that for any C^1-smooth curve $\overline{\gamma} = \{(x, y(x), \lambda(x) = y'(x)), \ x \in [0, \delta]\}, \gamma \subset U$ with small $\delta > 0$ and $\max |\lambda(x)| < \lambda^*$ we have the following. Let $\{(x(s), y(s), \lambda(s))\}$ be the image of some point $\{(x, y, \lambda)\} \in \overline{\gamma}$ under the flow (29). Then for any $s > s_0$*

$$\lambda(s) \in \left[2\lambda_0(\tau(s)), \frac{9}{10}\lambda_0(\tau(s))\right] =: [\lambda_-(x(s), y(s)), \lambda_+(x(s), y(s))]$$

Lemma 4.4. *(λ-absorbing interval for the diagonal) There are $s_0 > 0$ and small $\delta > 0$ such that for any point $(x, y, \lambda), x = y, \lambda = 1$, for any $s > s_0$ the image $\{(x(s), y(s), \lambda(s))\}$ under the flow (29) is such that*

$$\lambda(s) \in \left[2\lambda_0(\tau(s)), \frac{9}{10}\lambda_0(\tau(s))\right] =: [\lambda_-(x(s), y(s)), \lambda_+(x(s), y(s))]$$

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Remark 4.5. *Lemma 4.3 is needed to prove Theorem 6, and Lemma 4.4 is needed to prove (24) in Theorem 7.*

Before we start the proof of Lemma 4.3 and Lemma 4.4 we need to obtain some details on behavior of the function $\lambda_0(x, y)$.

Proposition 4.6. *The following properties hold:*

(i) *For any $\tau > 0$ we have $0.8 \leq \frac{\sqrt{P(\tau)}}{1+\tau} < 1$;*

(ii) $\lim_{\tau \to +\infty} \lambda_0(\tau) = -0.6$;

(iii) $\lim_{\tau \to 0} \frac{\lambda_0(\tau)}{\tau} = -0.6$;

(iv) $\frac{d}{ds}\lambda_0(s) = -\frac{\sqrt{P(\tau)} - \tau + 1}{\sqrt{P(\tau)}} \lambda_0(s)$;

(v) $-\frac{1.28 \lambda_0(s)}{\sqrt{P(\tau)}} < \frac{d}{ds}\lambda_0(s) < -\frac{2\lambda_0(s)}{\sqrt{P(\tau)}}$;

(vi) $\lim_{\tau \to 0} \frac{\frac{d}{ds}\lambda_0}{\lambda_0} = -2$.

Proof of Proposition 4.6. Here is the proof of the part (i). We have

$$\frac{\sqrt{P(\tau)}}{1+\tau} = \frac{\sqrt{(1+\tau)^2 - 1.44\tau}}{1+\tau} = \sqrt{1 - \frac{1.44\tau}{(1+\tau)^2}} < 1.$$

On the other hand, $\max_{\tau > 0} \frac{\tau}{(1+\tau)^2} = \frac{1}{4}$, hence

$$\frac{\sqrt{P(\tau)}}{1+\tau} = \sqrt{1 - \frac{1.44\tau}{(1+\tau)^2}} \geq \sqrt{1 - \frac{1.44}{4}} = 0.8$$

Now let us show that (ii) and (iii) hold. One has

$$\lambda_0(\tau) = -\frac{5}{6}(1+\tau) + \frac{5}{6}\sqrt{(1+\tau)^2 - 1.44\tau} = -\frac{1.2\tau}{(1+\tau) + \sqrt{(1+\tau)^2 - 1.44\tau}} \to -0.6 \text{ as } \tau \to +\infty,$$

and

$$\frac{\lambda_0(\tau)}{\tau} = -\frac{1.2}{(1+\tau) + \sqrt{(1+\tau)^2 - 1.44\tau}} \to -0.6 \text{ as } \tau \to 0.$$

Here is how the formula (iv) can be justified. We have

$$\frac{d}{ds}\lambda_0(s) = \frac{d}{d\tau}\lambda_0(\tau)\frac{d\tau}{ds} = -2\tau\frac{d}{d\tau}\lambda_0(\tau)$$

At the same time we have

$$\frac{\frac{d}{d\tau}\lambda_0(\tau)}{\lambda_0(\tau)} = \frac{d}{d\tau}\left(\ln|\lambda_0(\tau)|\right) =$$

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$$= \frac{d}{d\tau}\left(\ln\frac{1.2\tau}{(1+\tau)+\sqrt{(1+\tau)^2-1.44\tau}}\right) = \frac{d}{d\tau}\left(\ln(1.2\tau)-\ln(1+\tau+\sqrt{P(\tau)})\right) =$$

$$= \frac{1}{\tau} - \frac{\frac{d}{d\tau}(1+\tau+\sqrt{P(\tau)})}{1+\tau+\sqrt{P(\tau)}} = \frac{1}{\tau} - \frac{1+\frac{\tau+0.28}{\sqrt{P(\tau)}}}{1+\tau+\sqrt{P(\tau)}} = \frac{\sqrt{P(\tau)}-\tau+1}{2\tau\sqrt{P(\tau)}}.$$

Hence

$$\frac{d}{ds}\lambda_0(s) = -2\tau\frac{\sqrt{P(\tau)}-\tau+1}{2\tau\sqrt{P(\tau)}}\lambda_0(\tau) = -\frac{\sqrt{P(\tau)}-\tau+1}{\sqrt{P(\tau)}}\lambda_0(s).$$

Since $(1+\tau)^2 - 1.44\tau < (1+\tau)^2$ for any $\tau > 0$, we have

$$\sqrt{(1+\tau)^2-1.44\tau}-\tau < 1, \quad \text{or} \quad \sqrt{P(\tau)}-\tau+1 < 2.$$

On the other hand,

$$\sqrt{P(\tau)}-\tau+1 = \frac{2.56\tau}{\sqrt{P(\tau)}-1+\tau} = \frac{2.56}{\sqrt{\left(1+\frac{1}{\tau}\right)^2-\frac{1.44}{\tau}}-\frac{1}{\tau}+1} > \frac{2.56}{2} = 1.28$$

Therefore, $1.28 < \sqrt{P(\tau)}-\tau+1 < 2$, and so (iv) implies (v). Finally,

$$\lim_{\tau\to 0}\frac{\frac{d}{ds}\lambda_0}{\lambda_0} = \lim_{\tau\to 0}\left(-\frac{\sqrt{P(\tau)}-\tau+1}{\sqrt{P(\tau)}}\right) = -2, \quad \text{so } (vi) \text{ holds.}$$

$\square$

Now we are prepared to start the proof of Lemma 4.2.

Proof of Lemma 4.2. Let us show that the interval $\left[2\lambda_0(\tau(s)), \frac{9}{10}\lambda_0(\tau(s))\right]$ is an absorbing interval.

Rewrite equation (33) in the form

$$\frac{d}{ds}\lambda(s) = -\frac{3[(\lambda-\lambda_0(\tau(s)))-(\lambda_1(\tau(s))-\lambda_0(\tau(s)))]}{1+\tau(s)}[\lambda-\lambda_0(\tau(s))] =$$

$$= \left(5\frac{\sqrt{\tau^2+0.56\tau+1}}{1+\tau} + \frac{3(\lambda-\lambda_0(\tau(s)))}{1+\tau(s)}\right)[\lambda_0(\tau(s))-\lambda].$$

Now we can check whether the inequalities (31) hold. Plug in the upper bound $\lambda = \frac{9}{10}\lambda_0(\tau(s))$. We have

$$\frac{d}{ds}\lambda(s)\bigg|_{\lambda=\frac{9}{10}\lambda_0(\tau(s))} = -\frac{3}{1+\tau(s)}\left(\frac{9}{10}\lambda_0(\tau(s))-\lambda_1(\tau(s))\right)\left(\frac{9}{10}\lambda_0(\tau(s))-\lambda_0(\tau(s))\right) < 0.$$

In order to have

$$\frac{d}{ds}\lambda(s)\bigg|_{\lambda=\frac{9}{10}\lambda_0(\tau(s))} < \frac{d}{ds}\left[\frac{9}{10}\lambda_0(\tau(s))\right]$$

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it suffices to have the right-hand side positive. By Proposition 4.6 (v) the latter is bounded by $-1.152\lambda_0(\tau(s))/\sqrt{P(\tau(s))}$ from below and, therefore, is strictly positive.

Plug in the lower bound $\lambda = 2\lambda_0(\tau(s))$ now. We have

$$\frac{d}{ds}\lambda(s)\mid_{\lambda=2\lambda_0(\tau(s))} = -\frac{3[(\lambda_0(\tau(s)) - \lambda_1(\tau(s))) + \lambda_0(\tau(s))]}{1+\tau(s)}\lambda_0(\tau(s)) =$$

$$= -\left(\frac{5\sqrt{P(\tau(s))}}{1+\tau(s)} + \frac{3\lambda_0(\tau(s))}{1+\tau(s)}\right)\lambda_0(\tau(s)).$$

We need

$$\frac{d}{ds}\lambda(s)\mid_{\lambda=2\lambda_0(\tau(s))} > 2\frac{d}{ds}\lambda_0(\tau(s)).$$

By Proposition 4.6 (v) the latter is upper bounded by $-4\,\lambda_0(\tau(s))/\sqrt{P(\tau(s))}$. Thus, for the above inequality it suffices to have

$$\frac{d}{ds}\lambda(s)\mid_{\lambda=2\lambda_0(\tau(s))} > -4\lambda_0(\tau(s))/\sqrt{P(\tau(s))},$$

which in turn is implied by

$$\frac{5\sqrt{P(\tau(s))}}{1+\tau(s)} + \frac{3\lambda_0(\tau(s))}{1+\tau(s)} > \frac{4}{\sqrt{P(\tau(s))}}.$$

Therefore it is enough to show that for all $\tau > 0$

$$5P(\tau) > 4(1+\tau) - 3\lambda_0(\tau)\sqrt{P(\tau)}. \tag{35}$$

If one substitute the expressions for $P(\tau)$ and $\lambda_0(\tau)$ (see (32) and (35)) then direct calculations show that (35) is equivalent to the inequality

$$225\tau^4 - 13\tau^3 + 170.84\tau^2 - 33.4\tau + 24 > 0. \tag{36}$$

One can easily check that for all τ

$$225\tau^2 - 13\tau + 0.84 > 0 \quad \text{and} \quad 170\tau^2 - 33.4\tau + 24 > 0.$$

This implies (36) and, hence, (35). This completes the proof of Lemma 4.2. $\qquad\square$

Proof of Lemma 4.3. Let us now show that the condition $\max|\lambda(x)| < \lambda^*$ implies that a vectors tangent to the curve γ will enter this absorbing interval $[\lambda_-(x(s), y(s)), \lambda_+(x(s), y(s))] = [2\lambda_0(\tau(s)), \frac{9}{10}\lambda_0(\tau(s))]$ within a finite time. For large τ (we can guarantee that by choosing small enough δ) we have $\lambda_0(\tau(s)) \simeq -0.6$, see Proposition 4.6 (ii). Also, in this case we have

$$\frac{d\lambda}{ds} = -\frac{3\lambda^2}{1+\tau} - 5\lambda - \frac{3\tau}{1+\tau} \simeq -5\lambda - 3 = -5(\lambda + 0.6).$$

Therefore, since $|\lambda(0)|$ is bounded by λ^*, $\lambda(s)$ within a finite time will enter a small neighborhood of $\lambda_0(\tau(s)) \simeq -0.6$, and, hence, the absorbing interval $[2\lambda_0(\tau(s)), \frac{9}{10}\lambda_0(\tau(s))]$ that contains -0.6 as an internal point for large values of τ.

This completes the proof of Lemma 4.3. $\qquad\square$

Proof of Lemma 4.4. Notice that for $\tau < 1$ we have

$$\lambda_0 - \lambda_1 = \frac{5}{3}\sqrt{P(\tau)} = \frac{5}{3}\sqrt{1 + 0.56\tau + \tau^2} > \frac{5}{3}.$$

This implies that if $\lambda > \lambda_0$ then $\lambda - \lambda_1 > \frac{5}{3}$. Therefore

$$\frac{d(\lambda - \lambda_0)}{ds} = -\frac{3}{1+\tau}(\lambda - \lambda_1)(\lambda - \lambda_0) - \frac{d\lambda_0}{ds} \le -\frac{3}{2}\frac{5}{3}(\lambda - \lambda_0) = -\frac{5}{2}(\lambda - \lambda_0),$$

and $|\lambda(s) - \lambda_0(s)| \le |\lambda(0) - \lambda_0(0)|e^{-2.5s} = \left|1 + \frac{1.2\tau}{\sqrt{P}+1+\tau}\,|_{\tau=1}\right|e^{-2.5s} = \frac{4}{3}e^{-2.5s}$. On the other hand for $\tau < 1$ we also have

$$\left|\frac{1}{10}\lambda_0\right| = \frac{1}{10}\frac{1.2\tau}{\sqrt{P}+1+\tau} > \frac{1}{30}\tau = \frac{1}{30}e^{-2s},$$

hence in a finite time the distance between $\lambda(s)$ and $\lambda_0(s)$ becomes smaller than $\left|\frac{1}{10}\lambda_0\right|$, and $\lambda(s)$ enters the absorbing interval. This proves Lemma 4.4. $\qquad\square$

The next statement can be interpreted as an analog of a "cone condition" in a neighborhood of the degenerate saddle.

Lemma 4.7. *(stretching lemma) For any $\lambda^* > 0$ there exist $0 < c = c(\lambda^*) < 1$ and $\delta = \delta(\lambda^*) > 0$ with the following property.*
a) Let $\overline{X}' = (x, y, \lambda')$ and $\overline{X}'' = (x, y, \lambda'')$, $|\lambda'|, |\lambda''| \le \lambda^*$, be two initial conditions with $X = (x, y) \in U'$ and $0 < x < \delta$. Then at any moment of time s such that $X_s \in U$ we have

$$cx^5 \le |\lambda'(s) - \lambda''(s)| \le \frac{x^5}{c}.$$

b) Let $(x, y) \in U'$ and $v = (v_x, v_y)$ satisfy $0 < x < \delta$ and $|v_x| > \lambda^*|v_y|$. Let $N(x, y)$ be the number of iterates of the Poincare map f to get to U, i.e. $f^N(x, y) \in U$. Then

$$cx^{-2.5} \le |df^N(x)\,v| \le \frac{x^{-2.5}}{c}.$$

Proof of Lemma 4.7. Let us prove the part *a)* first. Notice that both functions $\lambda'(s)$ and $\lambda''(s)$ satisfy the same equation

$$\frac{d\lambda}{ds} = -\frac{3\lambda^2}{1+\tau} - 5\lambda - \frac{3\tau}{1+\tau}$$

but have different initial conditions $\lambda'(0)$ and $\lambda''(0)$. Denote $\Delta(s) = \lambda'(s) - \lambda''(s)$. Then

$$\frac{d\Delta}{ds} = -\frac{3}{1+\tau}((\lambda')^2 - (\lambda'')^2) - 5(\lambda' - \lambda'') = -\left(5 + \frac{3}{1+\tau}(\lambda' + \lambda'')\right)\Delta$$

Let $\widetilde{\Delta}(s)$ be a solution of the equation $\dfrac{d\widetilde{\Delta}}{ds} = -5\widetilde{\Delta}$ with the initial condition $\widetilde{\Delta}(0) = \Delta(0)$. Then

$$\widetilde{\Delta}(2T_0) = \exp(-10T_0)\Delta(0) = (\exp(-2T_0))^5\Delta(0) = (x(0))^5\Delta(0). \tag{37}$$

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On the other hand

$$\frac{d}{ds}\left(\frac{\Delta(s)}{\widetilde{\Delta}(s)}\right) = -\frac{3(\lambda' + \lambda'')}{1 + \tau}\left(\frac{\Delta(s)}{\widetilde{\Delta}(s)}\right), \qquad \frac{\Delta(0)}{\widetilde{\Delta}(0)} = 1.$$

Therefore

$$\frac{\Delta(2T_0)}{\widetilde{\Delta}(2T_0)} = \exp\left(-\int_0^{2T_0}\frac{3(\lambda' + \lambda'')}{1 + \tau}ds\right)$$

Notice that the integral $\int_0^{2T_0}\frac{\lambda'+\lambda''}{1+\tau}ds$ is uniformly bounded. Indeed, let s_0 be as in Lemma 4.3. It is enough to show that $\int_{s_0}^{2T_0}\frac{\lambda'+\lambda''}{1+\tau}ds$ is uniformly bounded. For $s > s_0$ due to Lemma 4.3 $\lambda', \lambda'' \in [2\lambda_0, \frac{9}{10}\lambda_0]$, hence $\lambda' + \lambda'' \in [4\lambda_0, 1.8\lambda_0]$. This implies that $\int_{s_0}^{T_0}\frac{\lambda'+\lambda''}{1+\tau}ds$ is uniformly bounded, and $\tau(s)$ grows exponentially fast as s decreases from T_0 to s_0. Also, $\int_{T_0}^{2T_0}\frac{\lambda'+\lambda''}{1+\tau}ds$ is uniformly bounded since $\frac{1}{1+\tau} < \frac{1}{2}$ for $\tau > 1$, and $|\lambda' + \lambda''|$ is majorated by $4|\lambda_0|$, and due to Proposition 4.6 *(iii)* $|\lambda_0| \sim \tau$ as $\tau \to 0$, hence $|\lambda_0(s)|$ decreases exponentially fast as s changes from T_0 to $2T_0$. This implies that the ratio $\frac{\Delta(2T_0)}{\widetilde{\Delta}(2T_0)}$ is uniformly bounded, and together with (37) this proves the part *a)* of Lemma 4.7.

Now notice that the time $N = 2T_0 + O(1)$ map has to preserve a smooth area-form. Since at initial point $y(0) \approx 1$ and at the final point $x(T_0 + O(1)) \approx 1$, these points are away rom infinity, hence the ratio of densities of the invariant form at these points is uniformly bounded. The differential of the time N map at the initial point is hyperbolic, and if $v = (v_x, v_y)$ is such that $|v_x| > \lambda^*|v_y|$ then it must be expanded. Since the determinant of this map is also uniformly bounded, an expansion rate is of order of the inverse of the root square of the contraction in the unit bundle. This proves the part *b)* of Lemma 4.7.

$\square$

4.2 Dynamics of 2-jets for the simplified system

Now we present behavior of convexity of curves. It is useful to keep in mind that this dynamics depends on slope so we incorporate slope λ into the model. Recall that due to above discussion of evolution of slope it is natural to assume that $\lambda(s)$ is bounded and becomes small when s is close to $2T_0$.

Lemma 4.8. *For any $\lambda^* > 0$ and $\mu^* > 0$ there exist a constant $C = C(\lambda^*, \mu^*) > 1$ and neighborhoods $U(q), U'(q')$ such that for any 2-jet $\overline{\overline{X}} = (x, y, \lambda, \mu)$ satisfying $|\lambda| \leq \lambda^*$, $|\mu| \leq \mu^*$, $(x, y) \in U(q)$, and for any moment of time s such that $X_s \in U'(q')$ we have $\overline{\overline{X}}_s = (x_s, y_s, \lambda_s, \mu_s)$, where*

$$0 < \mu_s < Cy_s.$$

Notice that Lemma 4.8 for integer values of s proves (23).
Lemma 4.8 follows from Lemma 4.17 and Lemma 4.19.

Lemma 4.9. *Suppose that the image $(x_s, y_s, \lambda_s, \mu_s)$ of the 2-jet $(x_0, y_0, \lambda_0, \mu_0)$, $x_0 = y_0$, $\lambda_0 = 1, \mu_0 = 0$, under the flow (28) is such that $(x_s, y_s) \in U'$. If $x_0 = y_0$ is small enough then*

$$0 < \mu_s < Cy_s.$$

These lemmas follow from Lemma 4.19 proven below.

Once again, notice that Lemma 4.9 for integer values of s proves (24).

Substituting $y = \tau x$ into the last equation of (28) directly gives the following:

Lemma 4.10. *The function $\mu(s)$ along a solution satisfies the equation*

$$\frac{d\mu}{ds} = -d(\tau(s), \lambda)\mu - \frac{B(\tau(s), \lambda)}{x + y},$$

where

$$B(\tau(s), \lambda) = \frac{6\tau}{1 + \tau} + \frac{18 + 24\tau + (24 + 18\tau)\lambda}{1 + \tau}\lambda + \frac{6\lambda^3}{1 + \tau}$$

and

$$d(\tau(s), \lambda) = \frac{9 + 6\tau}{1 + \tau} + \frac{9\lambda}{1 + \tau}.$$

Denote also $b(x(s), y(s), \lambda) = \frac{B(\tau(s), \lambda)}{x + y}$.

4.2.1 Upper and lower bounds for $d(s)$.

Recall that $x(s) = x_0 e^s$ and $\tau(s) = \tau_0 e^{-2s}$ with $\tau(0) = O(1)/x(0) = O(\exp(2T_0))$, $\tau(T_0) = O(1)$, $x(T_0) = O(\exp(-T_0))$, and $\tau(2T_0) = O(\exp(-2T_0))$, $x(2T_0) = O(1)$. Denote $S = T_0$; we have

$$b(s, \lambda, S) := 6\left(\frac{\tau(s)}{(1 + \tau(s))^2 x(s)} + \frac{3 + 4\tau(s) + (4 + 3\tau(s))\lambda(s)}{(1 + \tau(s))^2 x(s)}\lambda(s) + \frac{\lambda^3(s)}{(1 + \tau(s))^2 x(s)}\right)$$

and

$$d(s, \lambda) := \frac{9 + 6\tau(s)}{1 + \tau(s)} + \frac{9\lambda(s)}{1 + \tau(s)}.$$

Lemma 4.11. *If λ belongs to the absorbing interval $\left[2\lambda_0(\tau(s)), \frac{9}{10}\lambda_0(\tau(s))\right]$ then*

$$3 \le d(s, \lambda) \le 9.$$

Proof of Lemma 4.11. The upper bound is almost trivial. Indeed, if $\lambda \in \left[2\lambda_0(\tau(s)), \frac{9}{10}\lambda_0(\tau(s))\right]$ then $\lambda < 0$, and

$$d(s, \lambda) = \frac{9 + 6\tau(s)}{1 + \tau(s)} + \frac{9\lambda(s)}{1 + \tau(s)} = 6 + \frac{3}{1 + \tau} + \frac{9\lambda}{1 + \tau} \le 6 + \frac{3}{1 + \tau} \le 9$$

for any $\tau > 0$ and $\lambda < 0$.

In order to prove the lower bound one needs to show that

$$d(s, \lambda) = \frac{9 + 6\tau(s)}{1 + \tau(s)} + \frac{9\lambda(s)}{1 + \tau(s)} \ge 3,$$

which is equivalent to

$$6 + 3\tau(s) \ge -9\lambda(\tau(s)). \tag{38}$$

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We will consider separately the cases when $\tau < 1$, when $1 \leq \tau \leq 1.6$, and when $\tau > 1$.

If $\tau < 1$ then $\lambda_0(\tau) \geq \lambda_0(1) = -\left.\frac{1.2\tau}{\sqrt{P(\tau)+1+\tau}}\right|_{\tau=1} = -\frac{1}{3}$. Since $\lambda \in \left[2\lambda_0(\tau(s)), \frac{9}{10}\lambda_0(\tau(s))\right]$, we have $\lambda(\tau) \geq 2\lambda_0(\tau) \geq -\frac{2}{3}$. This implies that

$$6 + 3\tau > 6 = -9 \cdot \left(-\frac{2}{3}\right) \geq -9\lambda(\tau),$$

that is, (38) holds for $\tau < 1$.

If $1 \leq \tau \leq 1.6$ then $6 + 3\tau \geq 9$. Also we have $\lambda(\tau) \geq \lambda_0(1.6) = -\left.\frac{1.2\tau}{\sqrt{P(\tau)+1+\tau}}\right|_{\tau=1.6} \approx -0.40756$, and therefore

$$6 + 3\tau \geq 9 > -9 \cdot 2 \cdot (-0.40756\ldots) = -9 \cdot 2\lambda_0(1.6) \geq -9\lambda(\tau).$$

Finally, consider the case when $\tau > 1.6$. Notice that for any $\tau > 0$ we have $\lambda_0(\tau) > -0.6$, and for any $\lambda \in \left[2\lambda_0(\tau(s)), \frac{9}{10}\lambda_0(\tau(s))\right]$ we have

$$-9\lambda(\tau) < -18\lambda_0(\tau) < 10.8$$

On the other hand, for $\tau > 1.6$ we have

$$6 + 3\tau > 6 + 3 \cdot 1.6 = 10.8,$$

hence (38) holds. This completes the proof of Lemma 4.11. $\qquad\square$

For $\tau \in (0,1)$ the estimates given in Lemma 4.11 can be improved.

Lemma 4.12. *If λ belongs to the absorbing interval $\left[2\lambda_0(\tau(s)), \frac{9}{10}\lambda_0(\tau(s))\right]$ and $\tau \in (0,1)$ then*

$$4 \leq d(s, \lambda) \leq 9.$$

Proof of Lemma 4.12. We have

$$d(s,\lambda) = \frac{9 + 6\tau(s)}{1 + \tau(s)} + \frac{9\lambda(s)}{1 + \tau(s)} \geq \frac{9 + \tau + 2\lambda_0(\tau)}{1 + \tau} = \frac{9 + \tau + 2\left[-\frac{5}{6}(1+\tau) + \frac{5}{6}\sqrt{P(\tau)}\right]}{1+\tau} =$$

$$= \frac{8}{1+\tau} + 1 - \frac{5}{3} + \frac{5}{3}\frac{\sqrt{P(\tau)}}{1+\tau} \geq \frac{8}{1+\tau} - \frac{2}{3} + \frac{5}{3} \cdot 0.8 = \frac{8}{1+\tau} + \frac{2}{3} > 4$$

if $\tau \in (0,1)$. $\qquad\square$

4.2.2 Upper and lower bound for $B(\tau(s), \lambda)$.

We will need the following statements.

Proposition 4.13. *If λ belongs to the absorbing interval $\left[2\lambda_0(\tau(s)), \frac{9}{10}\lambda_0(\tau(s))\right]$ then*

$$-42\tau\frac{1+\lambda}{1+\tau} \leq B(\tau(s), \lambda) \leq -3\tau\frac{1+\lambda}{1+\tau}$$

Corollary 4.14. *There is a constant $C^* > 0$ such that for any $\lambda \in \left[2\lambda_0(\tau(s)), \frac{9}{10}\lambda_0(\tau(s))\right]$ we have $|B(\tau, \lambda(\tau))| < C^*$.*

The main part of the proof of Proposition 4.13 is the following estimate on $B(\tau(s), \lambda)$.

Lemma 4.15. *If λ belongs to the absorbing interval $\left[2\lambda_0(\tau(s)), \frac{9}{10}\lambda_0(\tau(s))\right]$ then*

$$-7\tau \le \tau(1 + 3\lambda) + \lambda(3 + \lambda) \le -0.5\tau.$$

Proof of Lemma 4.15. Let us first check that for $\lambda \in \left[2\lambda_0(\tau(s)), \frac{9}{10}\lambda_0(\tau(s))\right]$, where $\lambda_0 = -\frac{1.2\tau}{\sqrt{P(\tau)}+1+\tau}$ and $P(\tau) = (1 + \tau)^2 - 1.44\tau$, we have

$$\tau(1 + 3\lambda) + \lambda(3 + \lambda) \le -0.5\tau, \quad \text{or}$$

$$R_\tau(\lambda) \equiv \lambda^2 + 3(\tau + 1)\lambda + 1.5\tau < 0.$$

Since $R_\tau(\lambda)$ is a quadratic polynomial, it is enough to check that $R_\tau(2\lambda_0) < 0$ and $R_\tau(0.9\lambda_0) < 0$. We have

$$R_\tau(2\lambda_0) = \left(\frac{2.4\tau}{\sqrt{P(\tau)}+1+\tau}\right)^2 - 3(\tau+1)\left(\frac{2.4\tau}{\sqrt{P(\tau)}+1+\tau}\right) + 1.5\tau =$$

$$= \frac{3\tau}{(\sqrt{P(\tau)}+1+\tau)^2(1+\tau)^2}\left[1.92\frac{\tau}{(1+\tau)^2} - 2.4\left(\frac{\sqrt{P(\tau)}}{1+\tau}+1\right) + 0.5\left(\frac{\sqrt{P(\tau)}}{1+\tau}+1\right)^2\right]$$

Notice that $\max_{\tau>0}\frac{\tau}{(\tau+1)^2} = \frac{1}{4}$, and due to Proposition 4.6 *(i)* we have $0.8 \le \frac{\sqrt{P(\tau)}}{1+\tau} \le 1$ for $\tau > 0$. This implies that

$$1.92\frac{\tau}{(1+\tau)^2} - 2.4\left(\frac{\sqrt{P(\tau)}}{1+\tau}+1\right) + 0.5\left(\frac{\sqrt{P(\tau)}}{1+\tau}+1\right)^2 \le$$

$$\le 1.92\frac{1}{4} - 2.4(0.8+1) + 0.5(1+1)^2 = 0.48 - 4.32 + 2 < 0,$$

hence $R_\tau(2\lambda_0) < 0$.

Now let us check that $R_\tau(0.9\lambda_0) < 0$. We have

$$R_\tau(0.9\lambda_0) = \left(\frac{1.08\tau}{\sqrt{P(\tau)}+1+\tau}\right)^2 - 3(\tau+1)\left(\frac{1.08\tau}{\sqrt{P(\tau)}+1+\tau}\right) + 1.5\tau =$$

$$= \frac{3\tau}{(\sqrt{P(\tau)}+1+\tau)^2(1+\tau)^2}\left[0.3888\frac{\tau}{(1+\tau)^2} - 1.08\left(\frac{\sqrt{P(\tau)}}{1+\tau}+1\right) + 0.5\left(\frac{\sqrt{P(\tau)}}{1+\tau}+1\right)^2\right].$$

Since $\max_{x\in[1.8,2]}(-1.08x + 0.5x^2) = (-1.08x + 0.5x^2)|_{x=2} = -0.16$, we have

$$0.3888\frac{\tau}{(1+\tau)^2} - 1.08\left(\frac{\sqrt{P(\tau)}}{1+\tau}+1\right) + 0.5\left(\frac{\sqrt{P(\tau)}}{1+\tau}+1\right)^2 \le \frac{1}{4}0.3888 - 0.16 < 0,$$

hence $R_\tau(0.9\lambda_0) < 0$. This proves the upper bound in Lemma 4.15.

Let us now show that for $\lambda \in \left[2\lambda_0(\tau(s)), \frac{9}{10}\lambda_0(\tau(s))\right]$ we have $-7\tau \le \tau(1+3\lambda) + \lambda(3+\lambda)$, or

$$Q_\tau(\lambda) = \lambda^2 + 3(\tau+1)\lambda + 8\tau > 0.$$

Due to our assumptions $\lambda = C\lambda_0$ for some $C \in [0.9, 2]$. Therefore

$$Q_\tau(\lambda) = Q_\tau(C\lambda_0) = C^2 \left(\frac{1.2\tau}{\sqrt{P(\tau)}+1+\tau}\right)^2 - 3(\tau+1)\frac{1.2\tau}{\sqrt{P(\tau)}+1+\tau}C + 8\tau =$$

$$= \frac{4\tau}{(\sqrt{P(\tau)}+1+\tau)^2(\tau+1)^2}\left[0.36\frac{\tau}{(1+\tau)^2}C^2 - 0.9\left(\frac{\sqrt{P(\tau)}}{1+\tau}+1\right)C + 2\left(\frac{\sqrt{P(\tau)}}{1+\tau}+1\right)^2\right] \ge$$

$$\ge = \frac{4\tau}{(\sqrt{P(\tau)}+1+\tau)^2(\tau+1)^2}\left[-0.9\left(\frac{\sqrt{P(\tau)}}{1+\tau}+1\right)C + 2\left(\frac{\sqrt{P(\tau)}}{1+\tau}+1\right)^2\right] \ge$$

$$\ge \frac{4\tau}{(\sqrt{P(\tau)}+1+\tau)^2(\tau+1)^2}(-0.9 \cdot 2 \cdot 2 + 2 \cdot 1.8^2) > 0$$

for any $C \in [0.9, 2]$. This completes the proof of Lemma 4.15. $\qquad\square$

Now we are ready to prove Proposition 4.13.

Proof of Proposition 4.13. We have

$$B(\tau(s), \lambda) = 6\frac{1+\lambda(s)}{1+\tau(s)}\left(\tau(s) + 3\lambda(s)\tau(s) + 3\lambda(s) + \lambda^2(s)\right) =$$

$$= 6\frac{1+\lambda(s)}{1+\tau(s)}\left(\tau(s)(1+3\lambda(s)) + \lambda(s)(3+\lambda(s))\right).$$

Now the required estimates follow directly from Lemma 4.15. $\qquad\square$

4.2.3 Evolution of 2-jets

In the following statement we show that solutions of the equation $\frac{d\mu}{ds} = -d(s)\mu - b(s)$ cannot grow above a uniform upper bound on solutions of the equation $\frac{d\bar\mu}{ds} = -b(s)$.

Lemma 4.16. *Denote by $\bar\mu(\mu_0, \bar s, T_0)$ the value at $s = T_0$ of the solution of equation*

$$\frac{d\bar\mu}{ds} = -b(s) \quad \text{with the initial condition} \quad \bar\mu(\bar s) = \mu_0.$$

Set $M = \sup_{\bar s \in [0, T_0], |\mu_0| \le \mu^*} |\bar\mu(\mu_0, \bar s, T_0)|$. Then $|\mu(T_0)| \le M$, where $\mu(s)$ is a solution of the equation $\frac{d\mu}{ds} = -d(s)\mu - b(s)$, $|\mu(0)| \le \mu^*$.

Proof of Lemma 4.16. From the definition of M it is clear that $M \geq \mu^*$. If $|\mu(T_0)| \leq \mu^*$ then there is nothing to prove. Suppose that $\mu(T_0) > \mu^* > 0$. Denote

$$\hat{s} = \sup\{s \in [0, T_0] \mid |\mu(s)| \leq \mu^*\} < T_0.$$

Then $\mu(s) > \mu^* > 0$ and $d(s)\mu(s) > 0$ for all $s \in (\hat{s}, T_0]$. This implies that $-b(s) > -d(s)\mu - b(s)$ for $s \in (\hat{s}, T_0]$, hence $\bar{\mu}(\mu_0, \bar{s}, T_0) > \mu(T_0) > 0$.

Similarly, if $\mu(T_0) < -\mu^* < 0$ then $\mu(s) < \mu^* < 0$ for all $s \in (\hat{s}, T_0]$, and $d(s)\mu(s) < 0$, hence $-b(s) < -d(s)\mu - b(s)$ for $s \in (\hat{s}, T_0]$. So $\bar{\mu}(-\mu^*, \hat{s}, T_0) < \mu(T_0) < 0$.

In any case,

$$|\mu(T_0)| < \sup_{\bar{s} \in [0,T_0], |\mu_0| \leq \mu^*} |\bar{\mu}(\mu_0, \bar{s}, T_0)| = M.$$

$\square$

It is clear that the upper bound M given by Lemma 4.16 can be large if T_0 is large (or, equivalently, if we are taking the solution that starts very close to the y-axis). The next statement provides an explicit estimate of that upper bound in terms of T_0.

Lemma 4.17. *If T_0 is large enough then $M < 2C^* e^{T_0}$, where C^* is an upper bound on $B(s, \lambda)$ provided by Corollary 4.14.*

Proof of Lemma 4.17. Due to Corollary 4.14 we have $|B(s, \lambda(s))| \leq C^*$, so $|b(s)| = \left| \frac{B(s, \lambda(s))}{x(s) + y(s)} \right| \leq \frac{C^*}{x(s) + y(s)}$, therefore the solution of the equation $\frac{d\bar{\mu}}{ds} = -b(s)$, $|\bar{\mu}(\bar{s})| \leq \mu^*$, can be estimated

$$|\bar{\mu}(T_0)| \leq \mu^* + \int_0^{T_0} \frac{C^*}{x(s) + y(s)} ds = \mu^* + \int_0^{T_0} \frac{C^*}{e^{s-2T_0} + e^{-s}} ds <$$

$$< \mu^* + C^* \int_0^{T_0} \frac{ds}{e^{-s}} = \mu^* + C^*(e^{T_0} - 1) < \mu^* + C^* e^{T_0}.$$

Hence, if T_0 is large enough, $|\bar{\mu}(T_0)| < \mu^* + C^* e^{T_0} < 2C^* e^{T_0}$. $\square$

Lemma 4.18. *There are constants $C_1 \geq C_2 \geq 0$ such that for $s \in [T_0, 2T_0]$ (i.e. for $\tau \leq 1$) we have*

$$-C_1 e^{4T_0 - 3s} \leq b(s) \leq -C_2 e^{4T_0 - 3s}.$$

Proof of Lemma 4.18. Notice first that if $\tau \leq 1$ then $\lambda_0(\tau) \in \left[-\frac{1}{3}, 0\right)$. Indeed,

$$\lambda_0|_{\tau=1} = -\frac{5}{6} \cdot 2 + \frac{5}{6}\sqrt{2^2 - 1.44} = -\frac{5}{3} + \frac{5}{6} \cdot 1.6 = -\frac{1}{3},$$

and $\lambda_0(1) < \lambda_0(\tau) < 0$ if $\tau < 1$. This implies that if $\lambda \in [2\lambda_0, \frac{9}{10}\lambda_0]$ then $\lambda \in \left[-\frac{2}{3}, 0\right)$. Therefore for $\tau \leq 1$ we have

$$\frac{1+\lambda}{1+\tau} \in \left[\frac{1 - \frac{2}{3}}{2}, 1\right) = \left[\frac{1}{3}, 1\right).$$

Proposition 4.13 implies now that for $\tau \leq 1$ we have $-42\tau \leq B(\lambda, \tau) \leq -\frac{1}{2}\tau$. Since $x(s) = e^{s-2T_0}$, $y(s) = e^{-s}$, and $\tau = \frac{y(s)}{x(s)} = e^{2T_0 - 2s}$, we have $b(s) = \frac{B(\lambda, s)}{x(s) + y(s)} = \frac{B(\lambda, s)}{e^{s-2T_0} + e^{-s}}$, and therefore

$$-42 \frac{e^{2T_0 - 2s}}{e^{s-2T_0} + e^{-s}} \leq b(s) \leq -\frac{1}{2} \frac{e^{2T_0 - 2s}}{e^{s-2T_0} + e^{-s}}.$$

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Since $\tau \leq 1$, we have $s \in [T_0, 2T_0]$, and $e^{s-2T_0} \geq e^{-s}$. Hence

$$-42e^{4T_0-3s} = -42\frac{e^{2T_0-2s}}{e^{s-2T_0}} \leq -42\frac{e^{2T_0-2s}}{e^{s-2T_0} + e^{-s}} \leq b(s) \leq$$

$$\leq -\frac{1}{2}\frac{e^{2T_0-2s}}{e^{s-2T_0} + e^{-s}} \leq -\frac{1}{2}\frac{e^{2T_0-2s}}{2e^{s-2T_0}} = -\frac{1}{4}e^{4T_0-3s}.$$

$\square$

Lemma 4.19. *Given constants C^*, C_1, C_2, there is a constant $C > 0$ such that for all large enough T_0 the following holds. Suppose $\mu(s)$, $s \in [T_0, 2T_0]$ is a solution of the equation*

$$\frac{d\mu}{ds} = -d(s)\mu - b(s), \quad |\mu(T_0)| \leq 2C^* e^{T_0}, \tag{39}$$

and the coefficients $d(s), b(s)$ satisfy the following estimates for $s \in [T_0, 2T_0]$:

$$4 \leq d(s) \leq 9, \quad and \quad b(s) \in [-C_1 e^{4T_0-3s}, -C_2 e^{4T_0-3s}].$$

Then $\mu(2T_0) \in (0, Ce^{-2T_0})$.

Proof of Lemma 4.19. First of all, notice that if $C = \max(C_1, C_2, 2C^*)$ then $[0, Ce^{4T_0-3s}]$ is an absorbing interval. Indeed,

$$(-d\mu - b)|_{\mu=0} = -b > 0, \quad and \quad (-d\mu - b)|_{\mu=Ce^{4T_0-3s}} = -dCe^{4T_0-3s} - b \leq$$

$$\leq -dCe^{4T_0-3s} + C_1 e^{4T_0-3s} \leq (-4C + C_1)e^{4T_0-3s} \leq -3dCe^{4T_0-3s} = \frac{d}{ds}(dCe^{4T_0-3s}).$$

Let us show that if $|\mu(T_0)| < 2C^* e^{T_0}$ then $\mu(s)$ will enter he absorbing interval $[0, Ce^{4T_0-3s}]$. If $\mu(T_0) \geq 0$ then there is nothing to prove, so let us assume that $\mu(T_0) < 0$. In order to do so in the following statement we will compare the behavior of $\mu(s)$ with solution of a differential equation that represents a "worst case scenario".

Lemma 4.20. *Consider $\mu(s)$ - the solution of the equation (51), and $\tilde{\mu}(s)$ - the solution of the equation*

$$\frac{d\tilde{\mu}}{ds} = -4\tilde{\mu} + C_2 e^{4T_0-3s} \quad with \ the \ same \ initial \ condition \quad \tilde{\mu}(T_0) = \mu(T_0). \tag{40}$$

If for each $s \in [T_0, s^*]$ we have $\mu(s) < 0$ then $\tilde{\mu}(s) \leq \mu(s) < 0$ for $s \in [T_0, s^*]$.

Proof of Lemma 4.20. Set $\nu(s) = \mu(s) - \tilde{\mu}(s)$. Then $\nu(T_0) = 0$, and

$$\frac{d\nu}{ds} = (-d(s)\mu - b(s)) - (-4\tilde{\mu} - C_2 e^{4T_0-3s}) = -4\nu + (4 - d(s))\mu + (-b(s) - C_2 e^{4T_0-3s}).$$

Therefore, $\frac{d\nu}{ds}|_{\nu=0} = (4 - d(s))\mu + (-b(s) - C_2 e^{4T_0-3s}) > 0$ if $\mu(s) < 0$. This implies that $\nu(s) \geq 0$ for $s \in [T_0, s^*]$, which proves Lemma 4.20. $\square$

But the equation (40) can be solved explicitly. Namely, we have

$$\tilde{\mu}(s) = \mu(T_0)e^{4T_0-4s} + C_2\left(e^{4T_0-3s} - e^{5T_0-4s}\right) = e^{4T_0-4s}\left(\mu(T_0) + C_2(e^s - e^{T_0})\right).$$

If $\tilde{\mu}(T_0) = \mu(T_0) < 0$ then $\tilde{\mu}(s_0) = 0$ at $s_0 = \ln\left(e^{T_0} - \frac{\mu(T_0)}{C_2}\right)$. Since $|\mu(T_0)| \le 2C^* e^{T_0}$, we have $s_0 \le T_0 + \ln\left(1 + \frac{2C^*}{C_2}\right)$. Together with Lemma 4.20 this implies that if T_0 is large enough, $\mu(s)$ will enter the absorbing interval $[0, Ce^{4T_0-3s}]$ at some moment $s_1 > T_0$ where $T_0 < s_1 \le s_0 \le T_0 + \ln\left(1 + \frac{2C^*}{C_2}\right) < 2T_0$. For all $s > s_1$ we have $\mu(s) \in [0, Ce^{4T_0-3s}]$, and, in particular, $\mu(2T_0) \in (0, Ce^{-2T_0})$. Proof of Lemma 4.19 is complete. $\qquad\square$

5   Evolution of curves in the general case

Here we study the general case, without the simplifying assumptions made in Section 4. Namely, we describe the system

$$\begin{cases} \dot{x} = x(x+y)^3 \, 1_3 \\ \dot{y} = -y(x+y)^3 \, 1_3 \\ \dot{\lambda} = -(x+y)^2[3_3 y + 5_3(x+y)\lambda + 3_3 x\lambda^2] \\ \dot{\mu} = -3(x+y)[2_3 y + (6_3 x + 8_3 y)\lambda + (8_3 x + 6_3 y)\lambda^2 + 2_3 x\lambda^3 + \\ \phantom{\dot{\mu} = -3(x+y)[} + (3_3 x + 2_3 y + 3_3 x\lambda)(x+y)\mu] \end{cases} \tag{41}$$

and prove Theorems 6 and 7 in this general case.

5.1   Construction of globally absorbing intervals of λ's and dynamics of slopes in the general case.

Let us start with the system that describe the dynamics of 1-jets (i.e. of the slopes of evolving curves). After time change the first three equations of (41) turn into the following system:

$$\begin{cases} \frac{dx}{ds} = x \, 1_3 \\ \frac{dy}{ds} = -y \\ \frac{d\lambda}{ds} = -\frac{1}{x+y}[3_3 y + 5_3(x+y)\lambda + 3_3 x\lambda^2] \end{cases} \tag{42}$$

The last equation can be represented as

$$\frac{d\lambda}{ds} = -\frac{1}{x+y}[3_3 y + 5_3(x+y)\lambda + 3_3 x\lambda^2] =$$

$$= -\frac{3_3}{1+\tau}\left[\lambda^2 + \left(\frac{5}{3}\right)_3(1+\tau)\lambda + 1_3\tau\right] = -\frac{3_3}{1+\tau}[\lambda^2 + b(1+\tau)\lambda + c\tau],$$

where $b = b(x,y) = \left(\frac{5}{3}\right)_3$ and $c = c(x,y) = 1_3$.

Therefore we can rewrite it as

$$\frac{d\lambda}{ds} = -\frac{3_3}{1+\tau}(\lambda - \lambda_0^g)(\lambda - \lambda_1^g),$$

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where

$$\lambda_0^g = -\frac{1}{2}b(1+\tau) + \frac{b}{2}\sqrt{(1+\tau)^2 - \frac{4c\tau}{b^2}},$$

$$\lambda_1^g = -\frac{1}{2}b(1+\tau) - \frac{b}{2}\sqrt{(1+\tau)^2 - \frac{4c\tau}{b^2}}.$$

It will be convenient to denote $P_g = P_g(x, y, \tau) = (1+\tau)^2 - \frac{4c\tau}{b^2}$.

A family of absorbing intervals for the system (42) can be defined exactly in the same way it was given by Definition 4.1 for the system (29).

Lemma 5.1. *The family of intervals*

$$[\lambda_-(x,y), \lambda_+(x,y)] := \left[2\lambda_0^g(x,y), \frac{9}{10}\lambda_0^g(x,y)\right]$$

is an absorbing family for the system (42).

Proof of Lemma 5.1. We need to show that

$$\frac{d\lambda}{ds}(s)|_{\lambda=2\lambda_0^g} > 2\frac{d}{ds}\lambda_0^g(x(s), y(s)) \tag{43}$$

and

$$\frac{d\lambda}{ds}(s)|_{\lambda=\frac{9}{10}\lambda_0^g} < \frac{d}{ds}\left(\frac{9}{10}\lambda_0^g(x(s), y(s))\right). \tag{44}$$

We will need the following statement.

Lemma 5.2. *We have*

$$\frac{d}{ds}\lambda_0^g = \frac{2_3c\tau}{b\sqrt{P_g}} \cdot \frac{\sqrt{P_g} - \tau + 1}{\sqrt{P_g} + \tau + 1} + \frac{\tau}{\tau + 1}O_3$$

Proof of Lemma 5.2. Indeed,

$$\frac{d}{ds}\lambda_0^g = \frac{\partial\lambda_0^g}{\partial\tau}\frac{d\tau}{ds} + \frac{\partial\lambda_0^g}{\partial b}\frac{db}{ds} + \frac{\partial\lambda_0^g}{\partial c}\frac{dc}{ds},$$

where $\frac{d\tau}{ds} = -2_3\tau$, $\frac{db}{ds} = O_3$, and $\frac{dc}{ds} = O_3$. We have

$$\frac{\partial\lambda_0^g}{\partial\tau} = -\frac{b}{2} + \frac{b}{2}\cdot\frac{2(1+\tau) - \frac{4c}{b^2}}{2\sqrt{P_g}} = -\frac{b}{2}\left[1 - \frac{1+\tau - \frac{2c}{b^2}}{\sqrt{P_g}}\right] = -\frac{b}{2}\cdot\frac{\sqrt{P_g} - 1 - \tau + \frac{2c}{b^2}}{\sqrt{P_g}} =$$

$$= -\frac{b}{2}\left[\frac{1}{\sqrt{P_g}}\cdot\frac{P_g - (1+\tau)^2}{\sqrt{P_g} + \tau + 1} + \frac{2c}{b^2\sqrt{P_g}}\right] = -\frac{c}{b\sqrt{P_g}}\left[1 - \frac{2\tau}{\sqrt{P_g} + \tau + 1}\right] = -\frac{c}{b\sqrt{P_g}}\cdot\frac{\sqrt{P_g} - \tau + 1}{\sqrt{P_g} + \tau + 1}$$

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Also we have

$$\frac{\partial \lambda_0^g}{\partial b} = -\frac{1}{2}(1+\tau) + \frac{1}{2}\sqrt{P_g} + \frac{b}{2}\cdot\frac{2\cdot\frac{4c\tau}{b^3}}{2\sqrt{P_g}} =$$

$$= -\frac{1}{2}(\sqrt{P_g}-\tau-1) + \frac{2c\tau}{b^2\sqrt{P_g}} = -\frac{1}{2}\cdot\frac{P_g-(\tau+1)^2}{\sqrt{P_g}+\tau+1} + \frac{2c\tau}{b^2\sqrt{P_g}} = \frac{2c\tau}{b^2}\left(\frac{1}{\sqrt{P_g}+\tau+1} + \frac{1}{\sqrt{P_g}}\right)$$

and

$$\frac{\partial \lambda_0^g}{\partial c} = -\frac{b}{2}\cdot\frac{\frac{4\tau}{b^2}}{2\sqrt{P_g}} = -\frac{\tau}{b\sqrt{P_g}}$$

Combining these formulas together we get

$$\frac{d}{ds}\lambda_0^g = \frac{2_3 c\tau}{b\sqrt{P_g}}\cdot\frac{\sqrt{P_g}-\tau+1}{\sqrt{P_g}+\tau+1} + \frac{2c\tau}{b^2}\left(\frac{1}{\sqrt{P_g}+\tau+1} + \frac{1}{\sqrt{P_g}}\right)O_3 - \frac{\tau}{b\sqrt{P_g}}O_3 =$$

$$= \frac{2_3 c\tau}{b\sqrt{P_g}}\cdot\frac{\sqrt{P_g}-\tau+1}{\sqrt{P_g}+\tau+1} + \frac{\tau}{\tau+1}O_3$$

POYASNIT"!!! Nuzhen analog Proposition 4.6.   □

Let us now show that

$$\frac{d\lambda}{ds}(s)|_{\lambda=\frac{9}{10}\lambda_0^g} < \frac{d}{ds}\left(\frac{9}{10}\lambda_0^g(x(s),y(s))\right). \tag{45}$$

We have

$$\frac{d\lambda}{ds}(s)|_{\lambda=\frac{9}{10}\lambda_0^g} = -\frac{3_3}{1+\tau}\left(\frac{9}{10}\lambda_0^g - \lambda_0^g\right)\left(\frac{9}{10}\lambda_0^g - \lambda_1^g\right) =$$

$$= \frac{3_3}{1+\tau}\cdot\frac{\lambda_0^g}{10}\left(\frac{9}{10}\left[-\frac{1}{2}b(1+\tau) + \frac{b}{2}\sqrt{P}\right] + \frac{1}{2}b(1+\tau) + \frac{b}{2}\sqrt{P_g}\right) =$$

$$= \frac{3_3 b}{2(1+\tau)}\cdot\frac{\lambda_0^g}{10}\left(\frac{9}{10}(-(1+\tau) + \sqrt{P_g}) + (1+\tau) + \sqrt{P_g}\right) =$$

$$= \frac{3_3 b\lambda_0^g}{20(1+\tau)}\left(\frac{1}{10}(1+\tau) + \frac{19}{10}\sqrt{P_g}\right) = \frac{3_3 b\lambda_0^g}{200(1+\tau)}(1+\tau+19\sqrt{P_g})$$

Notice that

$$\lambda_0^g = -\frac{b}{2}(1+\tau-\sqrt{P_g}) = \frac{b}{2}\cdot\frac{P_g-(1+\tau)^2}{\sqrt{P_g}+1+\tau} = \frac{b}{2}\cdot\frac{-\frac{4\tau}{b^2}}{\sqrt{P_g}+1+\tau} = -\frac{2c\tau}{b(\sqrt{P_g}+1+\tau)},$$

therefore we have

$$\frac{d\lambda}{ds}(s)|_{\lambda=\frac{9}{10}\lambda_0^g} = -\frac{3_3 c\tau}{100(1+\tau)}\cdot\frac{19\sqrt{P_g}+1+\tau}{\sqrt{P_g}+1+\tau}$$

On the other hand, we have

$$\frac{d}{ds}\left(\frac{9}{10}\lambda_0^g\right) = \frac{9}{10} \cdot \frac{2_3 c}{b\sqrt{P_g}} \frac{\sqrt{P_g}+1-\tau}{\sqrt{P_g}+1+\tau} + \frac{\tau}{1+\tau}O_3,$$

so we need to show that

$$-\frac{3_3 c\tau}{100(1+\tau)} \cdot \frac{19\sqrt{P_g}+1+\tau}{\sqrt{P_g}+1+\tau} < \frac{2_3 c}{b\sqrt{P_g}} \frac{\sqrt{P_g}+1-\tau}{\sqrt{P_g}+1+\tau} + \frac{\tau}{1+\tau}O_3 \qquad (46)$$

which is equivalent to

$$-\left(\frac{3}{100}\right)_3 \frac{1}{1+\tau} \frac{19\sqrt{P_g}+1+\tau}{\sqrt{P_g}+1+\tau} < \left(\frac{9}{5}\right)_3\left(\frac{3}{5}\right)_3 \frac{1}{\sqrt{P_g}} \frac{\sqrt{P_g}+1-\tau}{\sqrt{P_g}+1+\tau} + \frac{1}{1+\tau}O_3$$

or to

$$-\left(\frac{1}{4}\right)_3 \frac{1}{1+\tau}(19\sqrt{P_g}+1+\tau) < 9_3\frac{\sqrt{P_g}+1-\tau}{\sqrt{P_g}} + \frac{\sqrt{P_g}+1+\tau}{1+\tau}O_3$$

Since functions $\frac{19\sqrt{P_g}+1+\tau}{1+\tau}$, $\frac{\sqrt{P_g}-\tau+1}{\sqrt{P_g}}$, and $\frac{\sqrt{P_g}+1+\tau}{1+\tau}$ are bounded, this follows from

$$-\frac{1}{4} \cdot \frac{19\sqrt{P_g}+1+\tau}{1+\tau} < 9\frac{\sqrt{P_g}+1-\tau}{\sqrt{P_g}} + O_3 \qquad (47)$$

In order to prove this let us consider the difference

$$9\frac{\sqrt{P_g}+1-\tau}{\sqrt{P_g}} - \left(-\frac{1}{4} \cdot \frac{19\sqrt{P_g}+1+\tau}{1+\tau}\right) = 9 - 9\frac{\tau}{\sqrt{P_g}} + \frac{9}{\sqrt{P_g}} + \frac{1}{4} \cdot \frac{19\sqrt{P_g}}{1+\tau} + \frac{1}{4} >$$

$$> 9 - 9\frac{\tau}{\sqrt{1+\tau^2+0.55\tau}} + \frac{9}{\sqrt{1+\tau^2+0.57\tau}} + \frac{1}{4} \cdot \frac{19\sqrt{1+\tau^2+0.55\tau}}{1\tau} + \frac{1}{4} >$$

$$> \frac{9}{\sqrt{1+\tau^2+0.57\tau}} + \frac{1}{4} \cdot \frac{19\sqrt{1+\tau^2+0.55\tau}}{1\tau} + \frac{1}{4} > \frac{1}{4} > O_3$$

This proves (47) and hence (46) and (45).

Now let us show that

$$\frac{d\lambda}{ds}(s)|_{\lambda=2\lambda_0^g} > 2\frac{d}{ds}\lambda_0^g(x(s),y(s))$$

We have:

$$\frac{d\lambda}{ds}(s)|_{\lambda=2\lambda_0^g} = -\frac{3_3}{1+\tau}(2\lambda_0^g-\lambda_0^g)(2\lambda_0^g-\lambda_1^g) =$$

$$= -\frac{3_3\lambda_0^g}{1+\tau}\left[2\left(-\frac{1}{2}b(1+\tau)+\frac{b}{2}\sqrt{P_g}\right)+\frac{1}{2}b(1+\tau)+\frac{b}{2}\sqrt{P_g}\right] = -\frac{3_3\lambda_0^g}{1+\tau}\frac{b}{2}\left[-(1+\tau)+3\sqrt{P_g}\right] =$$

$$= -\frac{3_3}{1+\tau} \cdot \frac{b}{2}(3\sqrt{P_g}-1-\tau)\left(-\frac{2c\tau}{b(\sqrt{P_g}+1+\tau)}\right) = \frac{3_3 c\tau}{1+\tau} \cdot \frac{3\sqrt{P_g}-1-\tau}{\sqrt{P_g}+1+\tau}$$

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At the same time we have

$$2\frac{d}{ds}\lambda_0^g = 4_3\frac{c\tau}{b\sqrt{P_g}}\frac{\sqrt{P_g}+1-\tau}{\sqrt{P_g}+1+\tau} + \frac{\tau}{1+\tau}O_3$$

So we need to show that

$$\frac{3_3 c\tau}{1+\tau}\cdot\frac{3\sqrt{P_g}-1-\tau}{\sqrt{P_g}+1+\tau} > 4_3\frac{c\tau}{b\sqrt{P_g}}\frac{\sqrt{P_g}+1-\tau}{\sqrt{P_g}+1+\tau} + \frac{\tau}{1+\tau}O_3 \tag{48}$$

This follows from

$$\frac{3_3}{1+\tau}\cdot\frac{3\sqrt{P_g}-1-\tau}{\sqrt{P_g}+1+\tau} > \frac{4_3}{b\sqrt{P_g}}\cdot\frac{\sqrt{P_g}+1-\tau}{\sqrt{P_g}+1+\tau} + \frac{1}{1+\tau}O_3$$

which is equivalent to

$$\frac{3_3}{1+\tau}(3\sqrt{P_g}-1-\tau) > \frac{4_3}{b\sqrt{P_g}}(\sqrt{P_g}+1-\tau) + \frac{\sqrt{P_g}+1+\tau}{1+\tau}O_3 \tag{49}$$

Since the functions $\frac{3\sqrt{P_g}-1-\tau}{1+\tau}$, $\frac{\sqrt{P_g}+1-\tau}{\sqrt{P_g}}$, and $\frac{\sqrt{P_g}+1+\tau}{1+\tau}$ are bounded, (49) is equivalent to

$$\frac{3}{1+\tau}(3\sqrt{P_g}-1-\tau) > \frac{12}{5}\frac{1}{\sqrt{P_g}}(\sqrt{P_g}+1-\tau) + O_3$$

Let us estimate the difference

$$\frac{3}{1+\tau}(3\sqrt{P_g}-1-\tau) - \frac{12}{5}\frac{1}{\sqrt{P_g}}(\sqrt{P_g}+1-\tau) = \frac{9\sqrt{P_g}}{1+\tau} - 3 - \frac{12}{5} + \frac{12}{5}\frac{\tau-1}{\sqrt{P_g}} =$$

$$= -\frac{27}{5} + \frac{1}{5}\cdot\frac{45P_g+12(\tau^2-1)}{(1+\tau)\sqrt{P_g}} = -\frac{27}{5} + \frac{1}{5}\cdot\frac{45(\tau^2+0.56_3\tau+1)+12(\tau^2-1)}{(1+\tau)\sqrt{P_g}} =$$

$$= -\frac{27}{5} + \frac{1}{5}\cdot\frac{57\tau^2+25.2_3\tau+33}{(1+\tau)\sqrt{P_g}} > \frac{1}{5}\cdot\frac{57\tau^2+25\tau+33}{(1+\tau)\sqrt{P_g}} - \frac{27}{5} =$$

$$= \frac{(57\tau^2+25\tau+33)-27(1+\tau)\sqrt{P_g}}{5(1+\tau)\sqrt{P_g}} = \frac{(57\tau^2+25\tau+33)^2-27^2(1+\tau)^2P_g}{5(1+\tau)\sqrt{P_g}((57\tau^2+25\tau+33)+27(1+\tau)\sqrt{P_g})} >$$

$$> \frac{57^2\tau^4+2\cdot24\cdot57\tau^3+(24^2+2\cdot57\cdot33)\tau^2+2\cdot33\cdot24\tau+33^2-27^2(1+2\tau+\tau^2)(1+0.57\tau+\tau^2)}{5(1+\tau)\sqrt{P_g}((57\tau^2+25\tau+33)+27(1+\tau)\sqrt{P_g})} =$$

$$= \frac{3249\tau^4+2736\tau^3+4338\tau^2+1584\tau+1089-729(\tau^4+2.57\tau^3+3.14\tau^2+2.57\tau+1)}{5(1+\tau)\sqrt{P_g}((57\tau^2+25\tau+33)+27(1+\tau)\sqrt{P_g})} =$$

$$= \frac{2520\tau^4+862.47\tau^3+2048.94\tau^2-289.53\tau+360}{5(1+\tau)\sqrt{P_g}((57\tau^2+25\tau+33)+27(1+\tau)\sqrt{P_g})} >$$

$$> \frac{2520\tau^4+862.47\tau^3+2048.94\tau^2-289.53\tau+360}{5(1+\tau)\sqrt{\tau^2+0.57\tau+1}((57\tau^2+25\tau+33)+27(1+\tau)\sqrt{\tau^2+0.57\tau+1})} > const > 0$$

since the polynomial in the numerator does not have any positive zeros and the function has finite positive limits as $\tau \to 0$ and as $\tau \to \infty$. This proves (49) and hence (48) and (43).

This completes the proof of Lemma 5.1. $\qquad\square$

The following statement is the general case analog of Lemma 4.3.

Lemma 5.3. *(λ-absorbing interval) For any $\lambda^* > 0$ there is $s_0 = s_0(\lambda^*) > 0$ such that for any C^1-smooth curve $\overline{\gamma} = \{(x, y(x), \lambda(x) = y'(x)), \ x \in [0, \delta]\}, \gamma \subset U$ with small $\delta > 0$ and $\max |\lambda(x)| < \lambda^*$ we have the following. Let $\{(x(s), y(s), \lambda(s))\}$ be the image of some point $\{(x, y, \lambda)\} \in \overline{\gamma}$ under the flow (42). Then for any $s > s_0$*

$$\lambda(s) \in \left[2\lambda_0(\tau(s)), \frac{9}{10}\lambda_0(\tau(s)) \right] =: [\lambda_-(x(s), y(s)), \lambda_+(x(s), y(s))]$$

The proof of Lemma 5.3 is almost verbatim repetition of the proof of Lemma 4.3 and is left to the reader.

We will also need the analog of Lemma 4.4.

Lemma 5.4. *(λ-absorbing interval for the diagonal) There are $s_0 > 0$ and small $\delta > 0$ such that for any point $(x, y, \lambda), x = y, \lambda = 1$, for any $s > s_0$ the image $\{(x(s), y(s), \lambda(s))\}$ under the flow (42) is such that*

$$\lambda(s) \in \left[2\lambda_0(\tau(s)), \frac{9}{10}\lambda_0(\tau(s)) \right] =: [\lambda_-(x(s), y(s)), \lambda_+(x(s), y(s))]$$

Before to proceed with proofs of Lemma 5.4 let us establish some basic properties of the function λ_0^g.

Lemma 5.5. *The following properties hold:*

(i) $\lambda_0^g = -\frac{2c\tau}{b(\sqrt{P_g}+\tau+1)}$;

(ii) $\lambda_0^g|_{\tau=1} = -\left(\frac{1}{3}\right)_3$;

(iii) *If $\tau \le 1$ then $\sqrt{P_g} - \tau + 1 \ge 2.56_3 > 2.55$;*

(iv) *If $\tau \le 1$ then $\frac{d\lambda_0^g}{ds} \ge 2.48_3\tau > 2.47\tau > 0$.*

Proof of Lemma 5.5. The first two items follow from straightforward calculations. If $\tau \le 1$ then

$$\sqrt{P_g} - \tau + 1 = 1 + \frac{\tau^2 + 0.56_3\tau + 1 - \tau^2}{\sqrt{P_g}+\tau} = 1 + \frac{0.56_3\tau + 1}{\sqrt{P_g}+\tau} \ge 1 + \frac{1.56_3}{1} = 2.56_3 > 2.55$$

Also, if $\tau \le 1$ then we have

$$\frac{d\lambda_0^g}{ds} = \frac{2_3c\tau}{b\sqrt{P_g}} \cdot \frac{\sqrt{P_g}-\tau+1}{\sqrt{P_g}+\tau+1} + \frac{\tau}{1+\tau}O_3 = \tau\left[\left(\frac{6}{5}\right)_3 \cdot \frac{1}{\sqrt{P_g}} \cdot \frac{\sqrt{P_g}-\tau+1}{\sqrt{P_g}+\tau+1} + O_3 \right] \ge$$

$$\ge \tau\left[\left(\frac{6}{5}\right)_3 \cdot 1 \cdot \frac{2.56_3}{2} + O_3 \right] = 2.48_3\tau > 2.47\tau > 0$$

$\qquad\square$

Let us now prove Lemma 5.4.

Proof of Lemma 5.4. The proof is quite similar to the proof of Lemma 4.4. Notice that for $\tau \le 1$ we have

$$\lambda_0^g - \lambda_1 = b\sqrt{P_g} = \left(\frac{5}{3}\right)_3 \sqrt{1 + 0.56_3\tau + \tau^2} \ge \left(\frac{5}{3}\right)_3 > \frac{49}{30}.$$

This implies that if $\lambda > \lambda_0$ then $\lambda - \lambda_1 > \frac{49}{30}$, and (since $\frac{d\lambda_0^g}{ds} > 0$)

$$\frac{d(\lambda - \lambda_0^g)}{ds} = -\frac{3_3}{1 + \tau}(\lambda - \lambda_1)(\lambda - \lambda_0^g) - \frac{d\lambda_0^g}{ds} \le -\left(\frac{3}{2}\right)_3 \frac{49}{30}(\lambda - \lambda_0) < -2.4(\lambda - \lambda_0),$$

and $|\lambda(s) - \lambda_0(s)| \le |\lambda(0) - \lambda_0(0)|e^{-2.4s} = |1 - \lambda_0^g|_{\tau=1}|e^{-2.4s} = \left(\frac{4}{3}\right)_3 e^{-2.4s} < e^{-2.4s}$. On the other hand for $\tau < 1$ we also have (since $\frac{d\tau}{ds} = -2_3\tau$, and the value of τ on the diagonal is equal to 1)

$$\left|\frac{1}{10}\lambda_0^g\right| = \frac{1}{10}\frac{2c\tau}{b(\sqrt{P_g} + \tau + 1)} \ge \left(\frac{1}{30}\right)_3 \tau \ge \left(\frac{1}{30}\right)_3 e^{-2.1s},$$

hence in a finite time the distance between $\lambda(s)$ and $\lambda_0^g(s)$ becomes smaller than $\left|\frac{1}{10}\lambda_0^g\right|$, and $\lambda(s)$ enters the absorbing interval. This proves Lemma 5.4. $\qquad\square$

The next lemma is an analog of Lemma 4.7 for the general case.

Lemma 5.6. *(stretching lemma, general case) For any $\lambda^* > 0$ there exist $0 < c = c(\lambda^*) < 1$ and $\delta = \delta(\lambda^*) > 0$ with the following property.*

a) Let $\overline{X}' = (x, y, \lambda')$ and $\overline{X}'' = (x, y, \lambda'')$, $|\lambda'|, |\lambda''| \le \lambda^*$, be two initial conditions with $X = (x, y) \in U$ and $0 < x < \delta$. Then at any moment of time s such that $X_s \in U'$ we have

$$cx^5 \le |\lambda'(s) - \lambda''(s)| \le \frac{x^5}{c}.$$

b) Let $(x, y) \in U$ and $v = (v_x, v_y)$ satisfy $0 < x < \delta$ and $|v_x| > \lambda^*|v_y|$. Let $N(x, y)$ be the number of iterates of the Poincare map f to get to U', i.e. $f^N(x, y) \in U'$. Then

$$cx^{-2.5} \le |df^N(x)\, v| \le \frac{x^{-2.5}}{c}.$$

Proof of Lemma 5.6. First of all, notice that since $\frac{dy}{ds} = -y$ and $\frac{dx}{ds} = -1_3x$, we have that $y(s) = y(0)e^{-s}$ and $\frac{x(t)}{x(0)e^s}$ is uniformly bounded (and separated from zero), $s \in [0, 2T_0]$.

Since both functions $\lambda'(s)$ and $\lambda''(s)$ satisfy the same equation

$$\frac{d\lambda}{ds} = -\frac{\alpha\lambda^2}{1 + \tau} - \beta\lambda - \frac{\gamma\tau}{1 + \tau}, \quad \alpha = 3_3, \ \beta = 5_3, \ \gamma = 3_3,$$

but have different initial conditions $\lambda'(0)$ and $\lambda''(0)$. Denote $\Delta(s) = \lambda'(s) - \lambda''(s)$. Then

$$\frac{d\Delta}{ds} = -\frac{\alpha}{1 + \tau}((\lambda')^2 - (\lambda'')^2) - \beta(\lambda' - \lambda'') = -\left(\beta + \frac{\alpha}{1 + \tau}(\lambda' + \lambda'')\right)\Delta$$

Let $\widetilde{\Delta}(s)$ be a solution of the equation $\dfrac{d\widetilde{\Delta}}{ds} = -\beta\widetilde{\Delta}$ with the initial condition $\widetilde{\Delta}(0) = \Delta(0)$. Then

$$\widetilde{\Delta}(2T_0) \sim \exp(-10T_0)\Delta(0) = (\exp(-2T_0))^5\Delta(0) \sim (x(0))^5\Delta(0). \qquad (50)$$

On the other hand

$$\frac{d}{ds}\left(\frac{\Delta(s)}{\widetilde{\Delta}(s)}\right) = -\frac{\alpha(\lambda' + \lambda'')}{1+\tau}\left(\frac{\Delta(s)}{\widetilde{\Delta}(s)}\right), \qquad \frac{\Delta(0)}{\widetilde{\Delta}(0)} = 1.$$

Therefore

$$\frac{\Delta(2T_0)}{\widetilde{\Delta}(2T_0)} = \exp\left(-\int_0^{2T_0}\frac{\alpha(\lambda' + \lambda'')}{1+\tau}ds\right)$$

The integral $\int_0^{2T_0}\frac{\lambda' + \lambda''}{1+\tau}ds$ is uniformly bounded (due to the arguments similar to those in the proof of Lemma 4.7). This implies that the ratio $\frac{\Delta(2T_0)}{\widetilde{\Delta}(2T_0)}$ is uniformly bounded, and together with (50) this proves the part a) of Lemma 5.6.

The part b) of Lemma 5.6 follows from exactly the same arguments as the part b) of Lemma 4.7. $\qquad\square$

6   Melnikov function for the RPC3BP

In this section we extract the form of Melnikov function for the RPC3BP from the calculations of Martinez-Pinyol [MP] paper. Unfortunately, we can't give a direct reference as the authors consider the Restricted Planar Elliptic 3 Body Problem with the following additional restriction. If e is eccentricity of the primaries and P_ϕ is the angular momentum, then the main result (Theorem A, pg. 300) valid is eP_ϕ is sufficiently large. The RPC3BP, however, requires $e = 0$. This forces us to petentrate into details of the proof of Theorem A.

Recall that $\mu \in (0,1)$ denotes mass ratio. Let $\alpha \in [0, 2\pi)$ be angle of direction of the symmetry axis of a parabola of a parabolic motion with the OX axis, t — time so that $t(\mod 2\pi) = 0$ the smaller primary is at $(1 - \mu, 0)$ Denote by G the angular momentum.

Theorem 8. *[MP] The Melnikov function $M(G, \alpha, t)$ has the following form*

$$M(G, \alpha, t) = -\sqrt{\pi}\,\exp\left(-\frac{G^3}{3}\right)\frac{\sqrt{G}}{2^{3/2}}\left(\sin(\alpha - t) + O(1/G) + O(1/G^{3/2}) + O(\exp(-G/3))\right).$$

Proof To simplify life of the reader we first switch to the notations of [MP] and then extract a proof from [MP] and switch back to our notations.

In notations of [MP] we have $G \to \rho_0$ — angular momentum, $\alpha \to \alpha_0$ — direction of the symmetry axis of a parabola of a parabolic motion, $t \to \omega$ — angle giving position of primaries (for $e = 0$ this is just the angle of the smaller primary with the OX axis). Then $D(\rho_0, \alpha_0, \omega)$ denotes the Melnikov function.

To see relation between notations compare formula (5), pg. 304 with our formula (12). Notice that $u = x/\sqrt{2}$ and $G = \rho$.

After a long series of expansions and manipulations (see formula before (27), pg 316) the authors expand the Melnikov function $D(\phi_0, \alpha_0, \omega)$ into an infinite sum of simpler functions:

$$D(\phi_0, \alpha_0, \omega) = \sum_{m \geq 0} T^{(m)}(\phi_0, \alpha_0, \omega).$$

where functions $T^{(m)}$'s are given by formulas (41,42) pg. 233

$$T^{(0)}(\phi_0, \alpha_0, \omega) = -\exp\left(-\frac{\rho_0^3}{3}\right) \sin\omega \sum_{n \geq 1} \left(c_1^{2n,0} + c_{-1}^{2n,0}\right) \left(\frac{\pi}{2}\right) \frac{1}{2\rho_0^{n-1/2}} \frac{(2n-1)!!}{[(2n)!!]^2},$$

$$T^{(m)}(\phi_0, \alpha_0, \omega) = (-1)^m \sqrt{\pi} \exp\left(-\frac{\rho_0^3}{3}\right) \times$$

$$\times \sum_{n \geq n_1} \Delta_m^{(2n)} \frac{\rho_0^{m-n+1/2}}{2^{m+1/2}} \left\{ \sin(m\alpha_0 - \omega) c_1^{m+2n,m} \sum_{k=0}^{m} \frac{\xi_k^{(1)}(m,n)}{\rho_0^{3k/2}} + \right.$$

$$\left. + \sin(m\alpha_0 + \omega) c_{-1}^{m+2n,m} \sum_{k=0}^{m} \frac{\xi_k^{(2)}(m,n)}{\rho_0^{3k/2}} \right\},$$

where $n_1 = 1$ if $m = 1$, and $n_1 = 0$ if $m \geq 2$, and the ξ coefficients are defined by formulas (43,44) pg. 323

$$\xi_0^{(1,2)}(m,n) = \frac{\tilde{t}_m^m(\pm 1 + 1)}{(2m + 2n - 1)!!},$$

while for $1 \leq k \leq m$ we have

$$\xi_k^{(1,2)}(m,n) = \frac{\tilde{t}_{m-k}^m 2^k L_k}{(2m + 2n - k + 1)!!}$$

$$\left\{ \pm(2m + 2n - k + 1) + \frac{m^2(2m - 2k + 1) + (m + 2n + 1)(2m^2 + m - k)}{m(2m - 2k + 1)} \right\},$$

L_k is 1 if k i even and $\sqrt{\pi/2}$ if k is odd. Notice that compare to expanstions (27), pg 317, terms $\exp(-s\rho^3/3)$ with $s > 1$ have been neglected. This is the meaning of the commetary after (44), pg. 323.

By Lemma 2, pg. 314 if eccentricity of primaries $e = 0$ (our circular case), then $c_m^{n,m} = 1$ for $n \in \mathbb{Z}_+$, $n > 0$ and $c_k^{n,m} = 0$ for $n \in \mathbb{Z}_+$, $n > 0, k \neq m$. This implies that $c_{\pm s}^{2n,0} = 0$ and $T^{(0)} \equiv 0$. Similarly, $c_{\pm 1}^{m+2n,m} = 0$. Thus, $T^{(m)} \equiv 0$ for $m > 1$. We have that

$$D(\phi_0, \alpha_0, \omega) = T^{(1)}(\phi_0, \alpha_0, \omega) + O(\exp(-2\rho^3/3)).$$

By Lemma 2, pg. 314 we have $c_1^{2n+1,1} = 1$ and $c_{-1}^{2n+1,1} = 0$. Thus,

$$T^{(1)}(\phi_0, \alpha_0, \omega) = -\sqrt{\pi} \exp\left(-\frac{\rho_0^3}{3}\right) \sum_{n \geq n_1} \Delta_1^{(2n)} \frac{\rho_0^{-n+3/2}}{2^{3/2}} \sin(\alpha_0 - \omega) \sum_{k=0}^{1} \frac{\xi_k^{(1)}(m,n)}{\rho_0^{3k/2}}.$$

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At this point we need to extract $\xi_{0,1}^{(1)}(m,n)$ from section 7 [MP]. Coefficients $\tilde{t}_j^{(m)}$ are related to coefficients of the Chebyshev polynomials. See top of page 312 before formula (21). We have

$\tilde{t}_j^m = (-1)^{m-j} \, m \, 4^j \, \dfrac{(m+j-1)!}{(2j)!(m-j)!}$. Plug in $m = 1, j = 0, 1$ and get $\tilde{t}_1^1 = 2$ and $\tilde{t}_0^1 = -1$. Thus,

$$\xi_0^{(1,2)}(m,n) = \frac{4}{(2n+1)!!} \qquad \xi_1^{(1,2)}(m,n) = -\frac{\sqrt{2\pi}}{(2n+2)!!}\left(2n+2+\frac{2n+2}{1}\right) = -\frac{2\sqrt{2\pi}}{(2n)!!}.$$

We also need the last formula on pg 312

$$\Delta_m^{(2n)} = \frac{(2n-1)!!}{(2n)!!}\frac{(2m+2n-1)!!}{(2m+2n)!!}$$

Notice that for $n > 1$ there is an additional factor $1/\rho_0$ and for $k = 1$ such a factor $1/\rho_0^{3/2}$. Thus, the leading contribution comes from $n = 1$, $k - 0$. Combining all of the above parameters we have

$$-\sqrt{\pi}\,\exp\left(-\frac{\rho_0^3}{3}\right)\,\frac{\sqrt{\rho_0}}{2^{3/2}}\left(\sin(\alpha_0 - \omega)\,\frac{3}{4}\cdot\frac{4}{3} + O(1/\rho_0) + O(1/\rho_0^{3/2}) + O(\exp(-\rho_0/3))\right).$$

6.0.1 Evolution of 2-jets: general case

In the following statement we show that solutions of the equation $\frac{d\mu}{ds} = -d^g(s)\mu - b^g(s)$ cannot grow above a uniform upper bound on solutions of the equation $\frac{d\bar{\mu}}{ds} = -b^g(s)$. Recall that after time proper rescaling in (42) the function $\mu(s)$ obeys the following equation:

$$\dot{\mu} = -\frac{1}{x+y}\left[\frac{6_3\tau}{1+\tau} + \frac{18_3 + 24_3\tau + (24_3 + 18_3\tau)\lambda}{1+\tau}\lambda + \frac{12_3\lambda^3}{1+\lambda}\right] + \frac{9_3 + 6_3\tau}{1+\tau} + \frac{9_3\lambda}{1+\tau}.$$

We have the following analog of Lemma 4.10.

Lemma 6.1. *The function $\mu(s)$ along a solution satisfies the equation*

$$\frac{d\mu}{ds} = -d^g(\tau(s), \lambda; x(s), y(s))\mu - \frac{B^g(\tau(s), \lambda; x(s), y(s))}{x(s) + y(s)},$$

where

$$B^g(\tau, \lambda; x, y) = \frac{6_3\tau}{1+\tau} + \frac{18_3 + 24_3\tau + (24_3 + 18_3\tau)\lambda}{1+\tau}\lambda + \frac{6_3\lambda^3}{1+\tau}$$

and

$$d^g(\tau, \lambda; x, y) = \frac{9_3 + 6_3\tau}{1+\tau} + \frac{9_3\lambda}{1+\tau}.$$

By analogy with the model case denote $b^g(\tau, \lambda; x, y) = \frac{B^g(\tau,\lambda)}{x+y}$. Namely,

$$b^g(\tau, \lambda; x, y) := 6\left(\frac{1_3\tau}{(1+\tau)^2 x} + \frac{3_3 + 4_3\tau + (4_3 + 3_3\tau)\lambda}{(1+\tau)^2 x(s)}\lambda + \frac{1_3\lambda^3}{(1+\tau)^2 x}\right)$$

We shall follow the same strategy as in the model case.

Lemma 4.8 and Lemma 4.9 will be proven in the general case. Namely, we prove analogs of Lemma 4.17 and Lemma 4.19 to prove the former and an analog of Lemma 4.19 for the latter one.

Now we obtain upper and lower bounds for $d^g(\tau, \lambda; x, y)$.

We have the following Lemmas analogous to Lemmas 4.11 and 4.12:

Lemma 6.2. *If λ belongs to the absorbing interval $\left[2\lambda_0(\tau), \frac{9}{10}\lambda_0(\tau)\right]$, then*

$$3_3 \leq d^g(\tau, \lambda; x, y) \leq 9_3.$$

Lemma 6.3. *If λ belongs to the absorbing interval $\left[2\lambda_0(\tau), \frac{9}{10}\lambda_0(\tau)\right]$ and $\tau \in (0,1)$, then*

$$4 \leq d^g(\tau, \lambda; x, y) \leq 9.$$

The proof of both lemmas is a repetition of the proofs of Lemmas 4.11 and 4.12.
Now we obtain upper and lower bounds for $B^g(\tau; x, y)$.

We need the following auxiliary

Proposition 6.4. *For any $\tau > 0$ we have $0.8_3 \leq \frac{\sqrt{P_g(\tau; x, y)}}{1+\tau} < 1$;*

Proof. We have

$$\frac{\sqrt{P_g(\tau; x, y)}}{1 + \tau} = \frac{\sqrt{(1+\tau)^2 - \frac{4c}{b^2}\tau}}{1 + \tau} = \sqrt{1 - \frac{4c\tau}{b^2(1+\tau)^2}} < 1.$$

On the other hand, $\max_{\tau > 0} \frac{\tau}{(1+\tau)^2} = \frac{1}{4}$, hence

$$\frac{\sqrt{P_g(\tau; x, y)}}{1 + \tau} = \sqrt{1 - \frac{4c\tau}{b^2(1+\tau)^2}} \geq \sqrt{1 - \frac{4c}{b^2}} = 0.8_3$$

$\square$

Lemma 6.5. *If λ belongs to the absorbing interval $\left[2\lambda_0^g(\tau; x, y), \frac{9}{10}\lambda_0^g(\tau; x, y)\right]$, then*

$$-7\tau \leq \tau(1_3 + 3_3\lambda) + \lambda(3_3 + 1_3\lambda) \leq -0.5\tau.$$

The proof follows the proof of Lemma 4.15 replacing application of Proposition 4.6 (i) with Proposition above. Notice that $\lambda_0 = -\frac{1.2\tau}{\sqrt{P(\tau)} + 1 + \tau}$ and $P(\tau) = (1+\tau)^2 - 1.44\tau$ in the general case becomes $\lambda_0^g = -\frac{2c\tau}{b(\sqrt{P_g(\tau; x, y)} + 1 + \tau)}$ and $P(\tau) = (1+\tau)^2 - \frac{4c}{b^2}\tau$.

Proposition 4.13 applies to the general case. The proof is exactly the same.
Corollary 4.14 also applies to the general case, namely, $|B^g(\tau, \lambda(\tau); x, y)| < C^*$ with the same proof.

Lemma 4.16 in the general case becomes

Lemma 6.6. *For any $\rho > 0$ and T_0 large enough we have $M < 2C^* e^{(1+\rho)T_0}$, where C^* is an upper bound on $|B(\tau, \lambda; x, y)|$ provided by Corollary 4.14.*

The difference in the proof is that $y(s) = \exp(-s)$ and $x(s) = \int_0^s 1_3\, x(t)\, dt$. Since $|1_3 - 1| = |O_3|$ can be chosen arbitrary small, it can be upper bounded by any predetermined $\rho > 0$. We have that

$$e^{(1-\rho)t} x(0) \le x(t) \le e^{(1+\rho)t} x(0) \qquad \text{for any } 0 \le t \le T_0.$$

Plugging in this bound in the formula for some $\bar{C} > 0$ independent of T_0 we get that in the domain of definition

$$\bar{C}^{-1} e^t\, x(0) \le x(t) \le \bar{C}\, e^t\, x(0).$$

This implies that if $2T_0$ is time of travel from $U(q)$ to $U'(q')$, then there is $\rho > 0$ independent of T_0 such that $(x(t), y(t))$ crosses the diagonal $x = y$ at time t^* such that $|T_0 - t^*| < \rho$.

Notice that Lemma 4.16 works in the general case. Namely, to any solution of the equation $\frac{d\mu}{ds} = -d^g(s)\mu - b^g(s)$ and $|\mu(0)| \le \mu^*$.

Lemma 4.17 also applies to the general case.

Here is an analog of Lemma 4.18.

Lemma 6.7. *There are constants $C_1 \ge C_2 \ge 0$ such that for s satisfying $\tau(s) = y(s)/x(s) \le 1$, i.e. $s \in [T_0 - \rho, 2T_0]$ for some ρ independent of T_0, we have*

$$-C_1\, e^{4T_0 - 3s} \le b^g(\tau(s); x(s), y(s)) \le -C_2\, e^{4T_0 - 3s}.$$

We follow the proof of Lemma 4.18. Here are the andjustment we need to take.

We have $\lambda_0^g|_{\tau=1} = -(\frac{1}{3})_3$. This implies that if $\lambda \in [2\lambda_0^g, \frac{9}{10}\lambda_0^g]$, then $\lambda \in \left[(-\frac{2}{3})_3, O_3\right)$. Therefore, for $\tau \le 1$ we have

$$\frac{1+\lambda}{1+\tau} \in \left[\frac{1 - \frac{2}{3}}{2}, 1\right) = \left[\frac{1}{3}, 1\right).$$

Proposition 4.13 implies now that for $\tau \le 1$ we have $-42\tau \le B(\lambda, \tau; x, y) \le -\frac{1}{2}\tau$. Since

$$\bar{C}^{-1} \exp(s - 2T_0) \le x(s) \le \bar{C} \exp(s - 2T_0), \qquad y(s) = e^{-s},$$

and

$$\bar{C}^{-1}\, e^{2T_0 - 2s} \le \tau(s) = \frac{y(s)}{x(s)} \le \bar{C}\, e^{2T_0 - 2s},$$

we have

$$\frac{B^g(\lambda, \tau; x, y)}{\bar{C}e^{s - 2T_0} + e^{-s}} \le b^g(\tau(s); x, y) \le \frac{B^g(\lambda, \tau; x, y)}{\bar{C}^{-1} e^{s - 2T_0} + e^{-s}},$$

and, therefore,

$$-C_2'\frac{e^{2T_0 - 2s}}{e^{s - 2T_0} + e^{-s}} \le b^g(s) \le -C_1'\frac{e^{2T_0 - 2s}}{e^{s - 2T_0} + e^{-s}}$$

for some C_1' and C_2'. Since $\tau \le 1$, we have $s \in [T_0 - \rho, 2T_0]$, and $e^{s - 2T_0} \ge e^{2\rho - s}$. Hence

$$-C_2 e^{4T_0 - 3s} = -C_2\frac{e^{2T_0 - 2s}}{e^{s - 2T_0}} \le -C_2\frac{e^{2T_0 - 2s}}{\bar{C}\, e^{s - 2T_0} + e^{-s}} \le b(s) \le$$

$$\le -C_1'\frac{e^{2T_0 - 2s}}{\bar{C}^{-1} e^{s - 2T_0} + e^{-s}} \le -C_1\frac{e^{2T_0 - 2s}}{4e^{s - 2T_0}} \le -C_1 e^{4T_0 - 3s}.$$

We need to modify Lemma 4.19 as follows.

Lemma 6.8. *For any $\rho > 0$, given constants C^*, there is a constant $C > 0$ such that for all large enough T_0 the following holds. Suppose $\mu(s)$, $s \in [T_0 - \rho, 2T_0]$ is a solution of the equation*

$$\frac{d\mu}{ds} = -d^g(s)\mu - b^g(s), \quad |\mu(T_0 - \rho)| \leq 2C^* e^{T_0}, \tag{51}$$

and the coefficients $d^g(s)$, $b^g(s)$ satisfy the following estimates for $s \in [T_0 - \rho, 2T_0]$:

$$4 \leq d^g(s) \leq 9, \quad and \ \ b^g(s) \in [-C_2 \, e^{4T_0 - 3s}, -C_1 \, e^{4T_0 - 3s}].$$

Then $\mu(2T_0) \in (0, Ce^{-2T_0})$.

The proof of Lemma 4.19 applies.

In the next two sections we use the properties of the local dynamics obtained above to construct a set of oscillatory motions of Hausdorff dimension close to two for an open set of parameters and to complete the proof of our main results.

7 Oscillatory motions born by hyperbolic dynamics

Existence of a hyperbolic set homoclinically, denoted Λ, related to O_∞ implies existence of oscillatory motions. Here we show that the Hausdorff dimension of the set of oscillatory motions in this case is not less than the Hausdorff dimension of the hyperbolic set Λ. Namely, we prove the following result.

Theorem 9. *Suppose that stable (unstable) manifold of O_∞ has a point of transverse intersection with an unstable (stable) manifold of some saddle periodic point of a zero-dimensional hyperbolic set Λ. Then*

$$\dim_H \left\{ q \in \mathbb{R}^2 \mid O_\infty \in \omega(q) \cap \alpha(q) \right\} \geq \dim_H \Lambda.$$

Here is the structure of this section (and the plan of proof of Theorem 9). In Subsection 7.1 we construct a locally maximal invariant transitive subset $\Lambda^\#$ that contains both the hyperbolic set Λ and the degenerate saddle O_∞. In Subsection 7.2 we prove that the classical result from [Ma] that relates entropy of a Borel measure supported on a hyperbolic set with Lyapunov exponent and Hausdorff dimension of a set of generic points can actually be applied to the constructed set $\Lambda^\#$ too. Then in Subsection 7.3 we use the results from [MT] to construct a measure supported on $\Lambda^\#$ that is weak close to an equilibrium measure supported on Λ (and has almost the same entropy and Lyapunov exponent). Application of results from [MM] and from Subsection 7.2 proves a "1-dimensional" version of Theorem 9. In order to get a "2-dimensional" result we follow the approach from [PV] and in Subsection 7.4 prove that the holonomy maps in $\Lambda^\#$ away from O_∞ are Hölder continuous with Hölder exponent arbitrary close to one. And, as we show in Subsection 7.5, together with "1-dimensional" result from Subsection 7.3 this implies Theorem 9.

7.1 Enlargement of a hyperbolic set homoclinically related to O_∞ to include O_∞

Suppose that Δ is an invariant compact set of $f : M^2 \to M^2$. Denote

$$W^s_\beta(x) = \{y \in M^2 \mid \mathrm{dist}(f^k(x), f^k(y)) \leq \beta \text{ for all } k \geq 0\},$$

$$W^u_\beta(x) = \{y \in M^2 \mid \operatorname{dist}(f^{-k}(x), f^{-k}(y)) \leq \beta \text{ for all } k \geq 0\}.$$

Definition 7.1. *The set Δ has a local product structure if for any sufficiently small $\beta > 0$ there exists $\delta > 0$ such that the following holds. If $\operatorname{dist}(x, y) < \delta$ then $W^s_\beta(x) \cap W^u_\beta(y)$ consists of exactly one point $z \equiv [x, y] \in \Delta$, and the map $(x, y) \to [x, y]$ is continuous.*

If the points $x, y \in \Delta$ are close enough, then for all points $x' \in W^u_\beta$ that are close enough to x a local holonomy map h along stable leaves is defined as $h(x') = W^s_\beta(x') \cap W^u_\beta(y)$. Similarly a local holonomy map along unstable leaves can be defined.

Suppose that Λ is a topologically zero-dimensional locally maximal transitive hyperbolic set appearing for the Poincare map f_e and $p_1, p_2 \in \Lambda$. Assume that $W^u(p_1)$ has a point Y of transverse intersection with $W^s(O_\infty)$, and $W^s(p_2)$ has a point X of transverse intersection with $W^u(O_\infty)$. Fix a small neighborhood $U(\Lambda)$. Denote by $\mathcal{O}(X)$ the whole orbit of a point X.

Denote $S = \Lambda \cup O_\infty \cup \mathcal{O}(X) \cup \mathcal{O}(Y)$.

Theorem 10. *There exist $\beta > 0$ and a neighborhood $U(S)$ such that the following holds.*

1. The set $\Lambda^\# = \cap_{n \in \mathbb{Z}} f^n_e(U(S))$ is a compact zero-dimensional transitive locally maximal invariant set with a local product structure;

2. The set $\Lambda^\#$ is homeomorphic to a Cantor set and the restriction of f_e to $\Lambda^\#$ is topologically conjugate to a topologically mixing subshift of finite type;

3. For each $x \in \Lambda^\#$ local stable and unstable sets are smooth curves;

4. For $y \in \Lambda^\# \cap U(\Lambda)$ the local invariant manifold $W^u_\beta(x)$ (respectively, $W^s_\beta(x)$) converges in C^1 topology to $W^u_\beta(y)$ (respectively, $W^s_\beta(y)$) as $x \to y$.

Proof of Theorem 10. It is a fairly standard fact (e.g. see [Mo]) that for small enough $U(S)$ the set $\Lambda^\# = \cap_{n \in \mathbb{Z}} f^n_e(U(S))$ is a compact zero-dimensional transitive locally maximal invariant Cantor set with a local product structure, and the restriction of f_e to $\Lambda^\#$ is topologically conjugate to a topological Markov chain, so we do not elaborate on that here. Any transitive topological Markov chain that has a fixed point must be topologically mixing. Indeed, due to the Spectral Decomposition Theorem some power of a transitive topological Markov chain is a disjoint union of several topologically mixing basic sets, and the original map permutes these sets cyclically. Hence if the original map is not topologically transitive, it cannot have any fixed points. This justifies the second claim in Theorem 10.

In order to show that local stable and unstable sets of $\Lambda^\#$ are smooth, let us first recall how a similar statement can be proved for a locally maximal hyperbolic set Λ (see [HP]; we follow the exposition presented in Section 6 of [KaH]). One can choose a system of charts of uniform size centered at each point of Λ in such a way that the map in each chart can be represented as a C^1 small perturbation of a hyperbolic linear map with one contracting and one expanding direction (that correspond to stable and unstable subspaces in the hyperbolic splitting). These restrictions can be extended to the maps of the plane that coincide with the restrictions in some neighborhood of the origin, and are linear hyperbolic maps away from some larger neighborhood of the origin, see Lemma 6.2.7 in [KaH]. Now in order to study the behavior of the map along a given orbit inside of the hyperbolic set one can consider a bi-infinite sequence of maps of the plane, and the usual graph transform technics imply existence of the sequence of local invariant manifolds, and their continuous (with respect to C^1 metric) dependence on initial point, see Theorems 6.2.8 and 6.4.9 from [KaH].

Now we will do the same for orbits of the set $\Lambda^\#$, but as soon as the orbit enters the rectangle Π_0, we will apply $f^{kN_0}_e$, see Figure 5, where k is chosen in such a way that $f^{kN_0}_e$ sends our orbit to

a neighborhood of the point X. Due to Theorem 6 conditions (20, 21) the map $f_e^{kN_0}$ in this case satisfies the cone condition (sends horizontal cones to horizontal cones, and stretches the vectors within the cones, possibly very strongly), and therefore the corresponding graph transform must be contracting. This implies last two claims of the statement of Theorem 10 for the points of $\Lambda^{\#} \cap U(\Lambda)$ that do not belong to $W^u(O_\infty)$ (for the points in $\Lambda^{\#} \cap U(\Lambda) \cap W^u(O_\infty)$ these statements follow directly from Theorem 6).
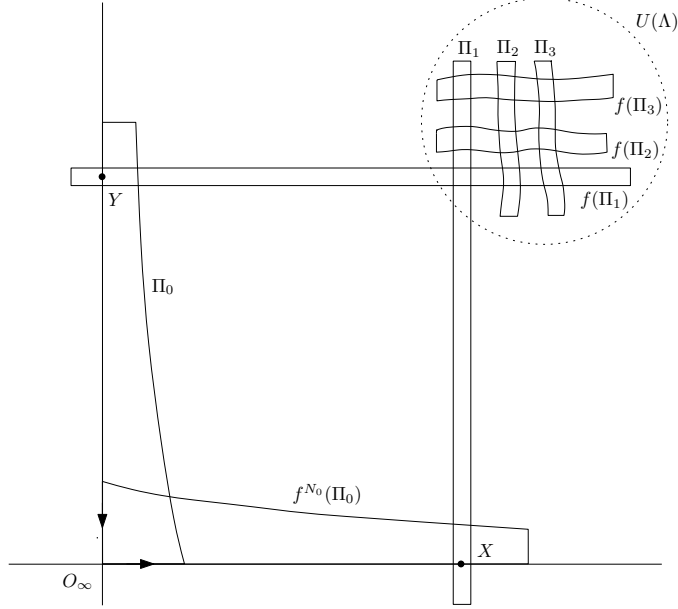

Figure 5: Construction of the set $\Lambda^{\#}$.

$\square$

Remark 7.2. *Notice that in the last claim we cannot replace $U(\Lambda)$ by $U(S)$. In fact, an analog of C^1 Inclination Lemma does* **not** *hold in a neighborhood of the degenerate saddle in our system, as one can see from Lemma 4.3 and Proposition 4.6, (ii). This shows that technical difficulties we experience in the present proof of the main result are not superficial.*

7.2 Manning's Hausdorff dimension formula

The classical result by Manning states the following.

Theorem 11 ([Ma]). *Let $f : M \to M$ be a C^1 Axiom A diffeomorphism of a surface M^2 preserving an ergodic Borel probability measure μ. Denote G_μ to be the set of future generic points for μ. Let Ω_1 be the basic set of f for which $G_\mu \subset W^s(\Omega_1)$. Then the Hausdorff dimension, denoted by δ_μ, of $G_\mu \cap W_{loc}^u(x)$ is independent of $x \in \Omega_1$. If χ_μ denotes the positive Lyapunov exponent corresponding to h_μ of f is given by the formula $h_\mu = \delta_\mu \chi_\mu$.*

In this section we prove the following generalization.

Theorem 12. *Suppose $f \in \mathrm{Diff}^1(M^2)$ has a locally maximal set $\Lambda^{\#}$ such that local stable and unstable manifolds of points in $\Lambda^{\#}$ are smooth curves, and $f|_{\Lambda^{\#}}$ is topologically conjugate to a subshift of finite type. Suppose that μ is a Borel ergodic probability measure whose supp $\mu = \Lambda^{\#}$. Suppose also that the function $\phi^u(x) = -\log \|Df_x|_{E_x^u}\|$ is continuous. Then $h_\mu = \delta\lambda$, where δ is a Hausdorff dimension of the set of future generic points of μ on a local unstable manifold, and λ is a positive Lyapunov exponent of μ.*

The proof is an almost verbatim repetition of the proof from [Ma], and we provide it here for completeness.

Proof of Theorem 12. We will need the following definition of a topological entropy $h(\mathcal{F}, Y)$ for a possibly non-compact subset Y of a compact space X and a continuous map $\mathcal{F} : X \to X$ due to Bowen [Bo1]. Let $\mathcal{A}$ be a finite cover of X and write $E \prec \mathcal{A}$ is contained in some member of $\mathcal{A}$. Write $n_{\mathcal{A}}(E)$, or simply $n(E)$, for the largest non-negative integer such that

$$\mathcal{F}^k E \prec \mathcal{A} \quad \text{for } 0 \le k < n_{\mathcal{A}}(E).$$

If $\mathcal{E} = \{E_1, E_2, \dots\}$ has union containing Y, set

$$D_{\mathcal{A}}(\mathcal{E}, \lambda) = \sum_{i=1}^{\infty} \exp\left(-\lambda n_{\mathcal{A}}(E_i)\right).$$

Define $m_{\mathcal{A}, \lambda}$ by

$$m_{\mathcal{A}, \lambda} = \lim_{\varepsilon \to 0} \inf \left\{ D_{\mathcal{A}}(\mathcal{E}, \lambda) : \ \mathcal{E} = \{E_1, E_2, \dots\}, \ \cup_{i=1}^{\infty} E_i \supset Y \text{ and } \exp(-n_{\mathcal{A}}(E_i)) < \varepsilon \text{ for each } i\}.$$

Then define

$$h_{\mathcal{A}}(\mathcal{F}, Y) = \inf\{\lambda : \ m_{\mathcal{A}, \lambda}(Y) = 0\}$$

and

$$h(\mathcal{F}, Y) = \sup_{\mathcal{A}} h_{\mathcal{A}}(\mathcal{F}, Y).$$

With this definition topological entropy resembles Hausdorff dimension, where the diameter of a set replaced by the length of time for which images remain finer than a given cover. Hausdorff dimension $HD(Y)$ is defined, for example by

$$HD(Y) = \inf\{\lambda : \ m_\lambda(Y) = 0\},$$

where

$$m_\lambda(Y) = \lim_{\varepsilon \to 0} \inf \left\{ \sum_{i=1}^{\infty} (\mathrm{diam} E_i)^\lambda : \ \cup_{i=1}^{\infty} E_i \supset Y \text{ and } \mathrm{diam}\, E_i < \varepsilon \text{ for each } i \right\}.$$

Let W be any closed interval on any $W^u(X)$ so W is compact. Provided ε is small, $\mathcal{A}$ is so fine that W crosses each A from $\mathcal{A}$ at most once. Define

$$G_{\mu, r} = \{x \in G_\mu : \ \frac{1}{m}\left| \sum_{i=0}^{m-1} \phi^{(u)}(f^i x) + \lambda \right| \le \varepsilon \quad \forall m \ge r\}$$

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and note that $G_\mu = \cup_{r=0}^\infty G_{\mu,r}$, since μ is ergodic and $\lambda = -\mu(\phi^{(u)})$.

Given $\varepsilon > 0$ let $\mathcal{A}$ be covering of $\Lambda^\#$ by open rectangles on each of which $\phi^{(u)}$ varies at most by ε. Let $l > 0$ be such that for any subset of $\Lambda^\#$ is contained in the set of $\mathcal{A}$ if it is small enough of its intersection with any local stable or unstable manifold to be less that l. Consider an interval W in any unstable manifold and choose m so large that

$$f^m W \cap W^s(x, l/2) \neq \emptyset \text{ for every } x \in \Lambda^\#.$$

For each r the Hausdorff dimension of $f^m W \cap G_{\mu,r}$ is at most δ so we can take a fine cover $\mathcal{U}_r$ of $f^m W \cap G_{\mu,r}$ by open intervals in $f^m W$ satisfying

$$\sum_{U \in \mathcal{U}_r} (\operatorname{diam} U)^{\delta+\varepsilon} < 2^{-r}.$$

For each $U \in \mathcal{U}_r$ define U^* as $\cup_{x \in U \cap \Lambda^\#} W^s(x, l/2)$. Then $f^n U^*$ is contained in some element of $\mathcal{A}$ so long as diam $f^n U < l$. The ratio of diam $f^n U$ and diam U is $\|Df^n_y|_{E^u_y}\|$ for some $y \in U$ and, since the orbit of y from time 0 to $n(U^*)$ is close to that of a point in $U \cap G_{\mu,r}$ and $\mathcal{U}_r$ can be chosen fine enough so that $n(U^*) > r$, we have

$$\|Df^n_y|_{E^u_y}\| \leq \exp[(\lambda + 2\varepsilon)n(U^*)].$$

Thus,

$$l \leq \operatorname{diam} U \cdot \exp[(\lambda + 2\varepsilon)n(U^*)].$$

This implies

$$\sum \exp[-(\lambda+\varepsilon)(\delta+2\varepsilon)n(U^*)] \leq l^{-(\delta+\varepsilon)} \sum (\operatorname{diam} U)^{\delta+\varepsilon} < l^{-(\delta+\varepsilon)} 2^{-r}.$$

Every point of $G_\mu \cap \Lambda^\#$ is in $W^s(x, l/2)$ for some $x \in f^m W \cap G_{\mu,r}$ for all sufficiently large r. By combining all U^* obtained from the covers $\mathcal{U}_r$ for all $r > q$ we obtain a cover $\mathcal{U}_q^*$ of $G_\mu \cap \Lambda^\#$ with

$$\sum \exp[-(\lambda+\varepsilon)(\delta+2\varepsilon)n(U^*)] \leq l^{-(\delta+\varepsilon)} 2^{-r}.$$

Therefore,

$$h_\mathcal{A} \leq (\lambda + \varepsilon)(\delta + 2\varepsilon).$$

For open covers $h_\mathcal{A} \geq h_\mathcal{B}$ when $\mathcal{A}$ refines $\mathcal{B}$ so that $h = \sup_\mathcal{A} h_\mathcal{A}$ is actually a limit of $h_\mathcal{A}$ as the mesh of $\mathcal{A}$ tends to zero. Thus,

$$h \leq (\lambda + \varepsilon)(\delta + 2\varepsilon).$$

Since ε could be taken arbitrary small, this implies that $h \leq \lambda\delta$.

On the other hand, given $\varepsilon > 0$ an open cover $\mathcal{A}$ of $\Lambda^\#$ by rectangles small enough so that $\phi^{(u)}$, which is continuous on the compact set $\Lambda^\#$, varies by at most ε on the region enclosed by each rectangle. Here

$$W^u_{\varepsilon'}(\Lambda^\#) = \cup_{x \in \Lambda^\#} W^u(x, \varepsilon'),$$

where $W^u(x, \varepsilon') = \{y \in M \mid d(f^j y, f^i x) \leq \varepsilon' \text{ for } j \leq 0\}$. Then for any $\alpha > 0$ there is a finite open cover $\mathcal{U}$ of G_μ with

$$D_\mathcal{A}(\mathcal{U}, h + \varepsilon) = \sum_{U \in \mathcal{U}} \exp(-n_\mathcal{A}(U))(h + \varepsilon) < \alpha.$$

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Fix r. Provided α is small enough we have $n(U) \geq r$ for each U in $\mathcal{U}$. The mean value theorem gives

$$\mathrm{diam}(W \cap G_{\mu,r} \cap U) = \mathrm{diam} f^{n(U)}(W \cap G_{\mu,r} \cap U)/\|Df_y^{n(U)}|_{E_y^u}\| \leq \mathrm{mesh}_u \mathcal{A}/\|Df_y^{n(U)}|_{E_y^u}\|$$

for some y in the convex hull in W of $(W \cap G_{\mu,r} \cap U)$. Here $\mathrm{mesh}_u \mathcal{A}$ denotes the longest interval of an unstable manifold in an element of $\mathcal{A}$. Since the orbit of y from time 0 to n keeps the same element of $\mathcal{A}$ as that of some point of $G_{\mu,r}$ we have

$$\|Df_y^{n(U)}|_{E_y^u}\| \geq \exp[(\lambda - 2\varepsilon)n(U)].$$

Thus the cover

$$\mathcal{U}' = \{W \cap G_{\mu,r} \cap U \; ; \; U \in \mathcal{U}\}$$

has

$$\sum_{U' \in \mathcal{U}'} (\mathrm{diam}\, U')^{(h+\varepsilon)/(\lambda-2\varepsilon)} \leq (\mathrm{mesh}_u \mathcal{A})^{(h+\varepsilon)/(\lambda-2\varepsilon)} \sum_{U \in \mathcal{U}} \exp[-(h+\varepsilon)n(U)] <$$

$$< \alpha(\mathrm{mesh}_u \mathcal{A})^{(h+\varepsilon)/(\lambda-2\varepsilon)}.$$

Now $\mathcal{U}'$ can be made as fine as required by making $\mathcal{U}$ fine. Thus the Hausdorff dimension of $W \cap G_{\mu,r}$ is at most $(h+\varepsilon)/(\lambda-2\varepsilon)$. By taking a countable union over r and then letting $\varepsilon \to 0$ we obtain

$$\delta_W \leq h/\lambda$$

as required, where δ_W is the Hausdorff dimension of $W \cap G_\mu$. $\qquad \square$

7.3 Thermodynamic formalism and equilibrium measures

We shall use the following result due to Morita and Tanaka [MT].

Let $d \geq 2$ be a positive integer and let $S = \{1, 2, \ldots, d\}$ be a finite set endowed with the discrete topology. Let M be $d \times d$ zero-one matrix. Consider a subshift of finite type (Σ_M^+, σ_M). Given $\theta \in (0,1)$, we define a metric $d_\theta : \Sigma_M^+ \to \mathbb{R}$ such that for distinct elements ω and ω' in Σ_M^+, $d_\theta(\omega, \omega') = \theta^n$, where n is the smallest non-negative integer such that $\omega_n \neq \omega'_n$.

Let $F_\theta(\Sigma_M^+ \to \mathbb{R})$ be the Banach space consisting of all $\mathbb{R}$-valued Lipschitz-continuous functions with respect to the metric d_θ, endowed with the norm defined by $\|f\|_\theta = \|f\|_\infty + [f]_\theta$, where $\|f\|_\infty$ is the supremum norm $\sup_{\omega \in \Sigma_M^+} |f(\omega)|$ and $[f]_\theta = \max_{i \in S} [f]_{\theta,i}$, with

$$[f]_{\theta,i} = \sup\{|f(\omega) - f(\omega_0)|/d_\theta(\omega, \omega_0) : \omega_0 = \omega'_0 = i, \omega \neq \omega'\}.$$

Now let us formulate our problem. Let $A = (A(ij))$ and $B = (B(ij))$ be $d \times d$ zero-one matrices that satisfy the following conditions.

(A.1) There exists a positive integer n_0 such that $A^{n_0} > 0$, i.e. the subshift (Σ_A^+, σ_A) is topologically mixing.

(A.2) $B(ij) = 1$ implies $A(ij) = 1$.

(A.3) The set Σ_B^+ is not empty and, if Σ^+ is the maximal σ–invariant subset of Σ_B^+, then the subshift $(\Sigma^+, \sigma|_{\Sigma^+})$ is topologically mixing.

Note that there exists a subset $S_0 = \{k_1, k_2, \ldots, k_{d'}\}$ of S such that $\Sigma^+ = \Sigma_C^+$, where C is a $d' \times d'$ zero-one matrix indexed by the set S_0, given by $C = (B(ij))$, $i, j \in S_0$. Thus, in what

follows, we may assume without loss of generality that $S_0 = \{1, 2, \ldots, d'\}$ and $C = (C(ij))_{1 \leq i,j \leq d'}$ with $C(ij) = B(ij)$. It is obvious that the mixing condition (A.3) is equivalent to the existence of a positive integer n_1 such that $C^{n_1} > 0$. Next, we introduce a family $\{\Phi(\alpha, \cdot)\}$ of potential functions on Σ_A^+ parameterized by $\alpha > 0$, which enables us to formulate both the potential perturbation and the spatial one simultaneously. Let $N = \cap_{ij: \ B(ij)=0}[ij]^A$. We consider a family $\{\Phi(\alpha, \cdot) = \phi(\alpha, \cdot) + \psi(\alpha, \cdot)\chi_N : \alpha > 0\}$ of potential functions that satisfy the following conditions, where χ_N denotes the indicator of the set N.

(B.1) For each $\alpha > 0, \phi(\alpha, \cdot) \in F_\theta(\Sigma_A \to \mathbb{R})$. Moreover, there exists a positive number C_1 and a function $\phi \in F_\theta(\Sigma_A^+ \to \mathbb{R})$ such that $\sup_{\alpha>0}[\phi(\alpha, \cdot)]_\theta \leq C_1$ and $\|\phi(\alpha, \cdot) - \phi\|_\infty \to 0$ as $\alpha \to \infty$.

(B.2) For each $\omega \in N$, $\psi(\alpha, \omega) \to -\infty$ as $\alpha \to \infty$.

(B.3) For each $\alpha > 0$, $\psi(\alpha, \cdot) \in F_\theta(\Sigma_A^+ \to \mathbb{R})$ and there exists a positive number C_2 such that $\sup_{\alpha>0}[\psi(\alpha, \cdot)]_\theta \leq C_2$.

Theorem 13 (Theorem 1.1 from [MT]). *Assume that the conditions (A.1)–(A.3) and (B.1)– (B.3) are satised. Then, as $\alpha \to \infty$, the Gibbs measure of the potential $\phi(\alpha, \cdot)$ on (Σ_A^+, σ_A) converges weakly to a Borel probability measure supported on Σ_C^+ and, if we regard the limit measure as the measure on Σ_C^+, it coincides with the Gibbs measure μ_{ϕ_C} of the potential ϕ_C on Σ_C^+, where $\phi_C = \phi|_{\Sigma_C^+}$. In particular, we obtain $\lim_{\alpha \to \infty} P(\sigma_A, \mu_{\Phi(\alpha, \cdot)}) = P(\sigma_C, \phi_C)$ and $\lim_{\alpha \to \infty} h(\sigma_A, \mu_{\Phi(\alpha, \cdot)}) = h(\sigma_C, \mu_{\phi_C})$.*

We use this result to construct a measure supported on $\Lambda^\#$ that is weak-$*$ close to an equilibrium measure supported on Λ, and has almost the same entropy and Lyapunov exponent.

Proposition 7.3. *Suppose we are in the setting of Theorem 10. Let μ be the equilibrium measure on the hyperbolic set Λ that corresponds to the potential $\phi^u = -\log\|Df_x|_{E_x^u}\|$. For arbitrary small $\varepsilon > 0$ there exists an ergodic Borel invariant measure ν of $f|_{\Lambda^\#}$ such that*

1) $\operatorname{supp} \nu = \Lambda^\#$,

2) $|h_\nu(f|_{\Lambda^\#}) - h_\mu(f|_{\Lambda^\#})| < \varepsilon$,

3) $|\lambda_\nu(f|_{\Lambda^\#}) - \lambda_\mu(f|_{\Lambda^\#})| < \varepsilon$.

Proof of Proposition 7.3. Theorem 13 is formulated for one-sided shifts, but the same statement holds for two-sided shifts as well, as can be established using standard technics. Namely, for any potential ϕ there exists a homologous potential $\widetilde{\phi}$ such that $\widetilde{\phi}(\omega)$ depends only on $\{\omega_i\}_{i=0}^\infty$ and equilibrium measures for ϕ and for $\widetilde{\phi}$ coincide (e.g. see Lemmas 1.5 and 1.6 from [Bo2]). Therefore we can apply it to the family of potentials

$$\Phi(\alpha, x) = \phi(\alpha, x) - \alpha\chi_{\Lambda^\# \cap \Pi_0}(x),$$

where Π_0 is an additional element for the Markov partition (see Figure 5) that was used in the proof of Theorem 10 to construct $\Lambda^\#$. This gives a family of measures ν_α on $\Lambda^\#$ that weak-$*$ converges to the equilibrium measure μ and such that $h_{\nu_\alpha}(f|_{\Lambda^\#}) \to h_\mu(f|_{\Lambda^\#})$ as $\alpha \to \infty$. Since $\lambda_{\nu_\alpha}(f|_{\Lambda^\#}) = -\int \phi^u d\nu_\alpha$, $\lambda_\mu(f|_{\Lambda^\#}) = -\int \phi^u d\mu$, $\nu_\alpha \to \mu$ as $\alpha \to \infty$, and ϕ^u is continuous, we also have that $\lambda_{\nu_\alpha}(f|_{\Lambda^\#}) \to \lambda_\mu(f|_{\Lambda^\#})$. Therefore one can take ν_α for large enough α as a required measure. $\square$

Now we are ready to prove the following "1-dimensional" version of Theorem 9.

Proposition 7.4. *Suppose that stable (unstable) manifold of O_∞ has a point of transverse intersection with an unstable (stable) manifold of some saddle periodic point of a hyperbolic set Λ. Then*

$$\dim_H \{x \in W^u(O_\infty) \mid O_\infty \in \omega(x)\} \geq h_\Lambda^u, \text{ and } \dim_H \{x \in W^s(O_\infty) \mid O_\infty \in \alpha(x)\} \geq h_\Lambda^s.$$

Proof of Proposition 7.4. It is enough to prove the first inequality. For any ergodic measure ν, $\mathrm{supp}\,\nu = \Lambda^\#$, we have

$$\{x \in W^u_{loc}(O_\infty) \mid x \text{ is future generic for } \nu\} \subseteq \{x \in W^u(O_\infty) \mid O_\infty \in \omega(x)\}. \tag{52}$$

The classical result by McCluskey–Manning states the following.

Theorem 14 ([MM]). *Let Λ be a basic set of a C^1 Axiom A diffeomorphism $f : M^2 \to M^2$ with*

$$T_\Lambda M = E^u \oplus E^s$$

is a splitting into 1-dimensional bundles and $\phi^u(x) : W^u(\Lambda) \to \mathbb{R}$ given by

$$\phi^u = -\log \|Df_x|_{E^u_x}\|.$$

The Hausdorff dimension of $W^u(x) \cap \Lambda$ is given by the unique δ for which

$$P_{f|_\Lambda}(\delta\phi^u) = 0,$$

is independent of $x \in \Lambda$, and $0 \leq \delta \leq 1$.

Consider the unique equilibrium measure μ on Λ with respect to the potential $\delta\phi^u$, where $\delta = \dim_H W^u(x) \cap \Lambda$. Then by definition of the equilibrium measure $h_\mu + \delta\mu(\phi^u) = 0$. Since $\lambda_\mu = -\mu(\phi^u)$ is the positive Lyapunov exponent of f with respect to measure μ, we have $h_\Lambda^u = \frac{h_\mu}{\lambda_\mu}$. Fix arbitrarily small $\beta > 0$. Apply Proposition 7.3 with $\varepsilon = \frac{\beta\lambda_\mu h_\mu}{\lambda_\mu + (1-\beta)h_\mu}$. This gives an ergodic measure ν, $\mathrm{supp}\,\nu = \Lambda^\#$, such that

$$\dim_H \{x \in W^u_{loc}(O_\infty) \mid x \text{ is future generic for } \nu\} = \frac{h_\nu}{\lambda_\nu} \geq \frac{h_\mu - \varepsilon}{\lambda_\mu + \varepsilon} \geq \frac{h_\mu}{\lambda_\mu}(1 - \beta) = h_\Lambda^u(1 - \beta).$$

Since β could be taken arbitrarily small, together with (52) this implies that

$$\dim_H \{x \in W^u(O_\infty) \mid O_\infty \in \omega(x)\} \geq h_\Lambda^u.$$

$\square$

7.4 Hölder continuity of the holonomy maps

Here we prove the following addendum to Theorem 10. Recall that f_e denotes the Poincare map of the Sitnikov problem defined in (3).

Proposition 7.5. *Given $\gamma \in (0,1)$, there exist $\beta > 0, \delta > 0, C_\gamma > 0$, and a neighborhood $U(S)$ such that the following holds. Let $\Lambda^\# = \cap_{n\in\mathbb{Z}} f^n(U(S))$ be a maximal invariant set in $U(S)$ (whose properties were described in Theorem 10). Then for any $x, y \in \Lambda^\# \cap U(\Lambda)$ with $\mathrm{dist}(x,y) < \delta$ the local holonomy maps $W^u_\beta(x) \to W^u_\beta(y)$ and $W^s_\beta(x) \to W^s_\beta(y)$ are Hölder continuous with exponent γ and constant C_γ.*

In the proof we will use the following statements.

Let O_∞ be the saddle at infinity. Let Λ be a locally maximal uniformly hyperbolic set homoclinically related to O_∞. Let $X \in W^u(O_\infty) \cap W^s(\Lambda)$ and $Y \in W^s(O_\infty) \cap W^u(\Lambda)$. Let U_Y and U_X be neighborhoods of Y and X resp. Let γ', γ'' be two curves in $W^s(\Lambda)$ near X transversal to $W^u(O_\infty)$. Let N_0 be such that for each $N > N_0$ both sets $\gamma'_N = f^{-N}(\gamma')$ and $\gamma''_N = f^{-N}(\gamma'')$ intersects the neighborhood U_Y. Denote by Π_N curvilinear rectangle formed by the boundary of U_Y and curves γ'_N and γ''_N (see Figures 5, **??**). Let $\Lambda^\#$ be as in Theorem 10.

Lemma 7.6. *For any $\eta \in (0,1)$ there are neighborhoods U_X and U_Y such that the following holds. For any N such that $\Pi_N \subset U_Y$ and any curves $\nu' \subset W^u(p') \cap U_Y$ and $\nu'' \subset W^u(p'') \cap U_Y$ that belong to the same rectangle Π_N (see Figure new-one) denote*

$$A = \frac{|f_e^N(\nu')|}{|\nu'|}, \quad B = \frac{|f_e^N(\nu'')|}{|\nu''|}.$$

Then

$$A > B^\eta, \quad B > A^\eta.$$
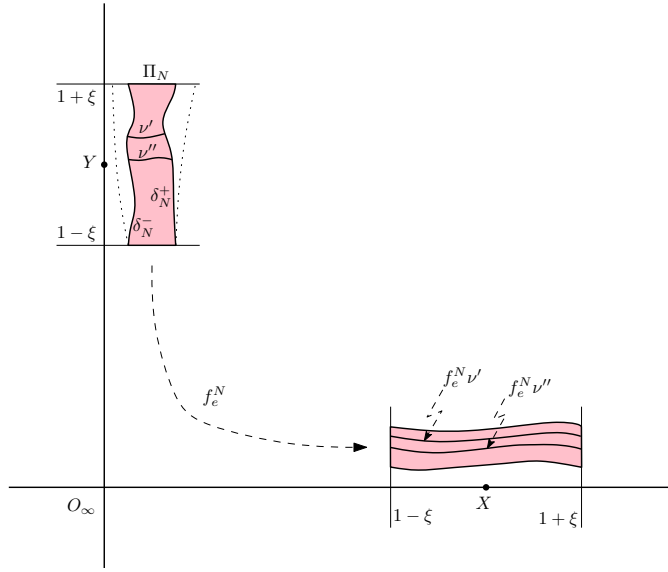
Figure 6: Curves ν' and ν'' in Lemma 7.6.

Proof of Lemma 7.6. Right and left boundary of Π_N can be considered as graphs of some functions δ_N^+ and $\delta_N^- : [1 - \xi, 1 + \xi] \to \mathbb{R}_x$. Consider integral curves $x_1(y)$ and $x_2(y)$ of the direction fields given by $\frac{dx}{dy} = Cx$ and $\frac{dx}{dy} = -Cx$ with the initial condition $x_{1,2}(1 - \xi) = \delta_N^+(1 - \xi)$. Due to Theorem 6 condition (20) the graph of the function δ^+ is located between these integral curves, and since

$$x_1(y) = \delta_N^+(1 - \xi)e^{C(1-\xi+y)}, \quad x_2(y) = \delta_N^+(1 - \xi)e^{C(1-\xi-y)} \quad \text{for} \ \ y \in [1 - \xi, 1 + \xi]$$

we have that $1 \leq \frac{\max \delta_N^+}{\min \delta_N^+} \leq e^{2C\xi}$. The same construction can be applied to δ_N^-.

On the other hand, Theorem 6 condition (21) also implies that

$$2\xi \frac{1}{C}(\delta_N^-(1-\xi))^{2.5} \leq \delta_N^+(1-\xi) - \delta_N^-(1-\xi) \leq 3\xi C(\delta_N^+(1-\xi))^{2.5}, \quad \text{and hence}$$

$$1 + 2\xi \frac{1}{C}(\delta_N^-(1-\xi))^{1.5} \leq \frac{\delta_N^+(1-\xi)}{\delta_N^-(1-\xi)} \leq 1 + 3\xi C(\delta_N^+(1-\xi))^{1.5}.$$

This implies that $1 \leq \frac{\max \delta_N^+}{\min \delta_N^-}$ is bounded by some constant independent of N. Theorem 6 condition (21) implies in this case that f_e^N must expand the curves with asymptotically the same rate, namely,

$$|f_e^N(\nu')| = O\left(\frac{|\nu'|}{x^\alpha}\right), \quad |f_e^N(\nu'')| = O\left(\frac{|\nu''|}{x^\alpha}\right),$$

where x is an x-coordinate of an arbitrary point in Π_N. For a given $\eta \in (0,1)$ one can choose U_X and U_Y small enough (which implies that the expansion $\sim x^{-\alpha}$ is large enough) to guarantee that

$$\frac{|f_e^N(\nu')|}{|\nu'|} > \left(\frac{|f_e^N(\nu'')|}{|\nu''|}\right)^\eta \quad \text{and} \quad \frac{|f_e^N(\nu'')|}{|\nu''|} > \left(\frac{|f_e^N(\nu')|}{|\nu'|}\right)^\eta.$$

$\square$
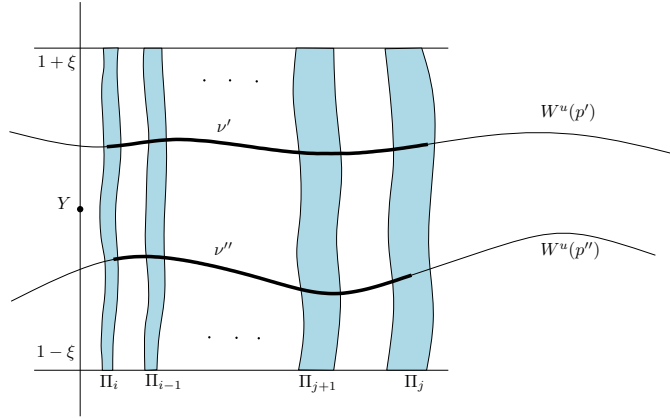
Figure 7: Curves ν' and ν'' in Lemma 7.7.

Lemma 7.7. *There is a constant $C > 1$ such that the following holds. Let p' and p'' belong to $\Lambda^\#$ and such that $W^u(p')$ and $W^u(p'')$ intersect $W^s(O_\infty)$ in U_Y. Let $\nu' \subset W^u(p') \cap U_Y$ and $\nu'' \subset W^u(p'') \cap U_Y$. For any $i \neq j$ and curves ν' and ν'' such that their endpoints belong to Π_i and Π_j (see Figure new-two) one has*

$$C^{-1} \leq \frac{|\nu'|}{|\nu''|} \leq C$$

Proof of Lemma 7.7. Assume first that $i = j+1$. Then using the same arguments as in the proof of the previous lemma one can show that the distance between Π_i and Π_j is of order $O(x_j^\alpha)$, where x_j is an x-coordinate of an arbitrary point from Π_j, and the horizontal size of these rectangles is also of order $O(x_j^\alpha)$. This implies that $|\nu'| = O(x_j^\alpha)$ and $|\nu''| = O(x_j^\alpha)$, and therefore there is a constant C independent of j such that $C^{-1} \leq \dfrac{|\nu'|}{|\nu''|} \leq C$.

Now let us consider the case when $i > j + 1$. Split the curve ν' into the disjoint union of curves $\nu' = \nu'_i \cup \nu'_{i-1} \cup \ldots \cup \nu'_{j+1}$ in such a way that ν'_k connects a point from Π_k with a point from Π_{k+1}. Consider a similar splitting $\nu'' = \nu''_i \cup \nu''_{i-1} \cup \ldots \cup \nu''_{j+1}$ for ν''. Then the argument above shows that $C^{-1} \leq \dfrac{|\nu'_k|}{|\nu''_k|} \leq C$ for $k = i, i-1, \ldots, j+1$. Hence

$$C^{-1} \leq \frac{\sum_{k=j+1}^{i} |\nu'_k|}{\sum_{k=j+1}^{i} |\nu''_k|} = \frac{|\nu'|}{|\nu''|} \leq C.$$

$\square$

Proof of Proposition 7.5. We will prove that the holonomy along stable leaves is Hölder continuous with exponent γ. One gets the proof for the holonomy along unstable leaves by time reversing.

We need to find $\varepsilon > 0$ and $C_\gamma > 0$ such that if

$$x_2 \in W_\varepsilon^u(x_1) \cap W_\varepsilon^s(y_1) \quad \text{and} \quad y_2 \in W_\varepsilon^u(y_1) \cap W_\varepsilon^s(x_1)$$

then

$$\text{dist}(x_1, x_2) \leq C_\gamma (\text{dist}(y_1, y_2))^\gamma.$$

Lemma 7.8. *Suppose that for some positive numbers $d_x, d_y, K_x, K_y, \beta, C, A, B,$ and $\gamma \in (0,1)$ the following inequalities hold:*

$$K_y d_y B \leq \beta, \quad A > B^\gamma, \quad K_x A d_x \leq C K_y B d_y.$$

Then

$$d_x \leq \left(C \frac{K_y^\gamma \beta^{1-\gamma}}{K_x} \right) d_y^\gamma.$$

Proof of Lemma 7.8.

$$d_x = d_x A A^{-1} \leq \frac{C}{K_x} K_y d_y B B^{-\gamma} = \frac{C}{K_x} K_y d_y B (B d_y d_y^{-1})^{-\gamma} =$$

$$= \left(C \frac{K_y}{K_x} \right) (d_y B)^{1-\gamma} d_y^\gamma = \left(C \frac{K_y^\gamma \beta^{1-\gamma}}{K_x} \right) d_y^\gamma$$

$\square$

Theorem 10 also implies the following lemma.

Lemma 7.9. *Suppose the neighborhood $U(\Lambda)$ is small enough. Then there are $\lambda^* > 1$ and $N^* \in \mathbb{N}$ such that for every $x \in \Lambda^\# \cap U(\Lambda)$*

$$\|Df_e^{N^*}|_{E^u}(x)\| \geq \lambda^* > 1.$$

Moreover, for a given $\gamma \in (0,1)$ there exists $\beta > 0$ such that for any $x, y \in \Lambda^\# \cap U(\Lambda)$, $\mathrm{dist}(x,y) < \beta$ we have

$$\|Df_e^{N^*}|_{E^u}(x)\| > \|Df_e^{N^*}|_{E^u}(y)\|^\gamma, \quad \|Df_e^{N^*}|_{E^u}(y)\| > \|Df_e^{N^*}|_{E^u}(x)\|^\gamma.$$

We will also need the following statement.

Lemma 7.10. *For any $\beta > 0$ and $\beta' \in (0, \beta)$ there are $C > 0$ and $\varepsilon > 0$ such that the following holds. Suppose $x_1, x_2, y_1, y_2 \in \Lambda^\#$ are such that*

$$x_1 \in W_\beta^u(x_2), \; y_1 \in W_\beta^u(y_2), \; y_1 \in W_\beta^s(x_1), \; y_2 \in W_\beta^s(x_2),$$

and

$$\mathrm{dist}(x_1, y_1) < \varepsilon, \; \mathrm{dist}(x_2, y_2) < \varepsilon, \; \mathrm{dist}(y_1, y_2) > \beta'.$$

Then

$$C^{-1}\mathrm{dist}(y_1, y_2) \leq \mathrm{dist}(x_1, x_2) \leq C\mathrm{dist}(y_1, y_2).$$

Proof of Lemma 7.10. Indeed, if ε is small enough, we have (here $d_y = \mathrm{dist}(y_1, y_2)$, $d_x = \mathrm{dist}(x_2, y_2)$)

$$\left(1 - \frac{2\varepsilon}{\beta'}\right) d_y \leq \left(1 - \frac{2\varepsilon}{d_y}\right) d_y = d_y - 2\varepsilon \leq d_x \leq d_y + 2\varepsilon = \left(1 + \frac{2\varepsilon}{d_y}\right) d_y \leq \left(1 + \frac{2\varepsilon}{\beta'}\right) d_y.$$

$\square$

Let the neighborhood $U(\Lambda)$ be as small as needed for Lemma 7.9. Suppose that $\widetilde{N} \in \mathbb{N}$ is such that $f_e^{\widetilde{N}}(X) \in U(\Lambda)$ and $f_e^{-\widetilde{N}}(Y) \in U(\Lambda)$. Set $L = \max(\|f_e\|_{C^1}, \|f_e^{-1}\|_{C^1})$ and denote $K = L^{4(\widetilde{N}+N^*)}$, where N^* is the same as in Lemma 7.9. Let us choose $A^* > 1$ so large that $(A^*)^{\frac{1-\gamma}{2}} > K$.

Choose neighborhoods $U(X)$ and $U(Y)$ so small that the following holds:

a) $f_e^{\widetilde{N}}(U(X)) \subset U(\Lambda)$, and $f_e^{-\widetilde{N}}(U(Y)) \subset U(\Lambda)$;

b) If $p \in Y(Y)$, $f_e^n(p) \in U(X)$, and $v \in K^u(p)$ is a vector from unstable cone, then $\|Df_e^n(v)\| > A^*\|v\|$.

c) Set $\eta = \frac{1+\gamma}{2}$. If U_Y and U_X are neighborhoods for which Lemma 7.6 holds, then $U(X) \subset U_X$ and $U(Y) \subset U_Y$.

Denote $F = \{x_1, x_2, y_1, y_2\}$. Let k^* be the smallest iterate such that one of the following cases holds:

H1. $\mathrm{dist}(f_e^{k^*}(x_1), f_e^{k^*}(x_2)) > \beta'$, $\mathrm{dist}(f_e^{k^*}(y_1), f_e^{k^*}(y_2)) > \beta'$, and $f_e^{k^*}(F) \subset U(\Lambda)$;

H2. $f_e^{k^*}(F) \subset U(Y)$, and $f_e^{k^*}(x_1), \; f_e^{k^*}(y_1) \in \Pi_i$, $f_e^{k^*}(x_2), f_e^{k^*}(y_2) \in \Pi_j$ where $i \neq j$.

Consider the case **H1** first.

Suppose that the image of F comes l times near the degenerate saddle O_∞. Then for some numbers $\{n_i^-, n_i^+\}_{i=1}^l$, $\{m_i^-, m_i^+\}_{i=1}^{l+1}$ we have:

1. $0 = m_1^- < m_1^+ < n_1^- < n_1^+ < m_2^- < m_2^+ < \ldots < n_l^- < n_l^+ < m_{l+1}^- < m_{l+1}^+ = k^*$,

2. $n_i^- - m_i^+ \leq \widetilde{N}$, $m_{i+1}^- - n_i^+ \leq \widetilde{N}$ for any $i \in \{1, \ldots, l\}$,

3. If $m_i^- \leq n \leq m_i^+$ then $f_e^n(F) \subset U(\Lambda)$;

4. $f_e^{n_i^-}(F) \subset U(Y)$, $f_e^{n_i^+}(F) \subset U(X)$, and for $n_i^- < n < n_i^+$ the set $f_e^n(F)$ is in a neighborhood of $O_\infty \cup \{f_e^t(Y)\}_{t \geq 0} \cup \{f_e^{-t}(X)\}_{t \geq 0}$.

Suppose for a moment that $l = 0$. If $k^* = sN^* + r$, $r < N^*$, then

$$\mathrm{dist}(f_e^{k^*}(x_1), f_e^{k^*}(x_2)) = K_x A \mathrm{dist}(x_1, x_2), \quad \mathrm{dist}(f_e^{k^*}(y_1), f_e^{k^*}(y_2)) = K_y B \mathrm{dist}(y_1, y_2)$$

where

$$A = \frac{\mathrm{dist}(f_e^{sN^*}(x_1), f_e^{sN^*}(x_2))}{\mathrm{dist}(x_1, x_2)}, \quad B = \frac{\mathrm{dist}(f_e^{sN^*}(y_1), f_e^{sN^*}(y_2))}{\mathrm{dist}(y_1, y_2)},$$

and $L^{-N^*} \leq K_x, K_y \leq L^{N^*}$.

Now we can apply Lemmas 7.8, 7.9, and 7.10 to $d_x = \mathrm{dist}(x_1, x_2)$ and $d_y = \mathrm{dist}(y_1, y_2)$. We get

$$\mathrm{dist}(x_1, x_2) \leq C_\gamma (\mathrm{dist}(y_1, y_2))^\gamma, \quad C_\gamma = C L^{N^*(1+\gamma)} (\mathrm{diam}\, \Lambda^\#)^{1-\gamma}.$$

Now suppose that $l > 0$. Set $M_i = m_i^+ - m_i^-$, $i = 1, \ldots, l+1$. Then $M_i = s_i N^* + r_i$, $r_i < N^*$. To shorten notations denote $d_x = \mathrm{dist}(x_1, x_2)$, $d_x(n) = \mathrm{dist}(f_e^n(x_1), f_e^n(x_2))$, and $d_y = \mathrm{dist}(y_1, y_2)$, $d_y(n) = \mathrm{dist}(f_e^n(y_1), f_e^n(y_2))$.

Then

$$d_x(m_i^+) = K_{i,x} A_{i,\Lambda} d_x(m_i^-), \quad d_y(m_i^+) = K_{i,y} A_{i,\Lambda} d_y(m_i^-),$$

where

$$A_{i,\Lambda} = \frac{d_x(s_i N^* + m_i^-)}{d_x(m_i^-)}, \quad B_{i,\Lambda} = \frac{d_y(s_i N^* + m_i^-)}{d_y(m_i^-)},$$

and $L^{-N^*} \leq K_{i,x}, K_{i,y} \leq L^{N^*}$.

Also, set

$$A_{i,O_\infty} = \frac{d_x(n_i^+)}{d_x(n_i^-)}, \quad B_{i,O_\infty} = \frac{d_y(n_i^+)}{d_y(n_i^-)}, \quad i = 1, \ldots, l.$$

By the choice of the neighborhoods $U(X), U(Y)$ above we have $A_{i,O_\infty}, B_{i,O_\infty} > A^*$, and due to Lemma 7.6 we also have $A_{i,O_\infty} > B_{i,O_\infty}^{\frac{1+\gamma}{2}}$, $B_{i,O_\infty} > A_{i,O_\infty}^{\frac{1+\gamma}{2}}$.

Now we can write

$$\frac{d_x(k^*)}{d_x} = \frac{d_x(m_1^+)}{d_x} \prod_{i=1}^l \left[\frac{d_x(n_i^-)}{d_x(m_i^+)} \cdot \frac{d_x(n_i^+)}{d_x(n_i^-)} \cdot \frac{d_x(m_{i+1}^-)}{d_x(n_i^+)} \cdot \frac{d_x(m_{i+1}^+)}{d_x(m_{i+1}^-)} \right] =$$

$$= K_{1,x} A_{1,\Lambda} \prod_{i=1}^{l} \left[\frac{d_x(n_i^-)}{d_x(m_i^+)} \cdot A_{i,O_\infty} \cdot \frac{d_x(m_{i+1}^-)}{d_x(n_i^+)} \cdot K_{i+1,x} A_{i+1,\Lambda} \right]$$

Notice that $L^{-\widetilde{N}} \leq \frac{d_x(n_i^-)}{d_x(m_i^+)} \leq L^{\widetilde{N}}$, and $L^{-\widetilde{N}} \leq \frac{d_x(m_{i+1}^-)}{d_x(n_i^+)} \leq L^{\widetilde{N}}$. Set

$$A = A_{1,\Lambda} \prod_{i=1}^{l} \left[\frac{d_x(n_i^-)}{d_x(m_i^+)} \cdot A_{i,O_\infty} \cdot \frac{d_x(m_{i+1}^-)}{d_x(n_i^+)} \cdot K_{i+1,x} A_{i+1,\Lambda} \right],$$

$$B = B_{1,\Lambda} \prod_{i=1}^{l} \left[\frac{d_y(n_i^-)}{d_y(m_i^+)} \cdot B_{i,O_\infty} \cdot \frac{d_y(m_{i+1}^-)}{d_y(n_i^+)} \cdot K_{i+1,y} B_{i+1,\Lambda} \right],$$

then $d_x(k^*) = K_{1,x} A d_x$, $d_y(k^*) = K_{1,y} B d_y$, and due to Lemma 7.10 and our choice of k^* we have

$$C^{-1} d_y(k^*) \leq d_x(k^*) \leq C d_y(k^*).$$

In order to apply Lemma 7.8 we need to check that $A > B^\gamma$. We have $A_{i,\Lambda} > B_{i,\Lambda}^\gamma$ from Lemma 7.9, and for each $i = 1, 2, \ldots, l$ we also have

$$\frac{d_x(n_i^-)}{d_x(m_i^+)} \cdot A_{i,O_\infty} \cdot \frac{d_x(m_{i+1}^-)}{d_x(n_i^+)} \cdot K_{i+1,x} > \left(\frac{d_x(n_i^-)}{d_x(m_i^+)} \cdot \frac{d_x(m_{i+1}^-)}{d_x(n_i^+)} \cdot K_{i+1,x} \right) B_{i,O_\infty}^{\frac{1+\gamma}{2}} =$$

$$= B_{i,O_\infty}^\gamma \cdot \left[B_{i,O_\infty}^{\frac{1-\gamma}{2}} \cdot \left(\frac{d_x(n_i^-)}{d_x(m_i^+)} \cdot \frac{d_x(m_{i+1}^-)}{d_x(n_i^+)} \cdot K_{i+1,x} \right) \right].$$

Since $B_{i,O_\infty}^{\frac{1-\gamma}{2}} > (A^*)^{\frac{1-\gamma}{2}} > K = L^{4(\widetilde{N}+N^*)}$, this implies that

$$\frac{d_x(n_i^-)}{d_x(m_i^+)} \cdot A_{i,O_\infty} \cdot \frac{d_x(m_{i+1}^-)}{d_x(n_i^+)} \cdot K_{i+1,x} > B_{i,O_\infty}^\gamma L^{2(\widetilde{N}+N^*)} > B_{i,O_\infty}^\gamma \left(\frac{d_y(n_i^-)}{d_y(m_i^+)} \cdot \frac{d_y(m_{i+1}^-)}{d_y(n_i^+)} \cdot K_{i+1,y} \right)^\gamma.$$

Taking product of these inequalities over $i = 1, \ldots, l$ we get $A > B^\gamma$. Now application of Lemma 7.8 gives

$$d_x \leq C_\gamma d_y^\gamma, \quad C_\gamma = C L^{N^*(1+\gamma)} (\operatorname{diam} \Lambda^\#)^{1-\gamma}.$$

Remark 7.11. *It looks like we could take C_γ independent of γ: $C_\gamma < C L^{2N^*} \operatorname{diam} \Lambda^\#$. But the value of β in Lemma 7.9 depends on γ, and therefore our estimates work for holonomy maps on a smaller interval as γ increases. In particular, if one needs to fix β and vary γ, then the local holonomy map is still Hölder continuous for any $\gamma \in (0,1)$ but C_γ will grow as γ approaches 1.*

Case **H2** is completely similar, one just need to use Lemma 7.7 instead of Lemma 7.10. $\square$

7.5 Obtaining a low bound on the Hausdorff dimension of oscillatory motions

Here we adapt some ideas from [PV] to our case and use Hölder continuity of holonomy maps and Theorem 7.4 to estimate the Hausdorff dimension of oscillatory motions.

Proof of Theorem 9. Fix any $\gamma \in (0,1)$. Pick any point $p \in \Lambda$, and consider the product space

$$\Delta = (W_\beta^u(p) \cap \Lambda^\#) \times (W_\beta^s(p) \cap \Lambda^\#).$$

Set

$$G^u = \left\{ x \in W_\beta^u(p) \mid O_\infty \in \omega(x) \right\} \text{ and } G^s = \left\{ x \in W_\beta^s(p) \mid O_\infty \in \alpha(x) \right\}.$$

Theorem 7.4 claims that $\dim_H(G^u) \geq h_\Lambda^u$ and $\dim_H(G^s) \geq h_\Lambda^s$. Due to [Mar],

$$\dim_H(G^s \times G^u) \geq \dim_H(G^s) + \dim_H(G^u).$$

Define the map $\Phi : \Delta \to \Lambda^\#$ by $\Phi(x,y) = W_{loc}^s(x) \cap W_{loc}^u(y)$. Then Φ is a homeomorphism onto a neighborhood of the point p in $\Lambda^\#$.

Proposition 7.12. *Both Φ and Φ^{-1} are Hölder continuous with exponent γ.*

Proof of Proposition 7.12. Let $w_1, w_2 \in \Delta$, $w_1 = (x_1, y_1)$, $w_2 = (x_2, y_2)$. By Proposition 7.5 we have

$$\text{dist}(\Phi(w_1), \Phi(w_2)) \leq \text{dist}(\Phi(x_1, y_1), \Phi(x_1, y_2)) + \text{dist}(\Phi(x_1, y_2), \Phi(x_2, y_2)) \leq$$

$$\leq C_\gamma (\text{dist}(x_1, x_2))^\gamma + C_\gamma (\text{dist}(y_1, y_2))^\gamma \leq 2C_\gamma \left(\max(\text{dist}(x_1, x_2), \text{dist}(y_1, y_2)) \right)^\gamma \leq 2C_\gamma (\text{dist}(w_1, w_2))^\gamma.$$

On the other hand, from uniform transversality of local stable and unstable manifolds of points from $\Lambda^\# \cap U(\Lambda)$ it follows that for some $k > 0$

$$\text{dist}(p_1, p_2) > k \max(\text{dist}(p_1, W_{loc}^u(p_1) \cap W_{loc}^s(p_2)), \text{dist}(W_{loc}^u(p_1) \cap W_{loc}^s(p_2), p_2))$$

for all $p_1, p_2 \in \Lambda^\# \cap U(\Lambda)$ that are close enough. From here and from Proposition 7.5 we have

$$\text{dist}(w_1, w_2) \leq 2 \max(\text{dist}(x_1, x_2), \text{dist}(y_1, y_2)) \leq$$

$$\leq 2 \max(C_\gamma \text{dist}(\Phi(x_1, y_1), \Phi(x_2, y_1))^\gamma, \ C_\gamma \text{dist}(\Phi(x_2, y_1), \Phi(x_2, y_2))^\gamma) \leq$$

$$\leq 2C_\gamma \max(\text{dist}(\Phi(x_1, y_1), \Phi(x_2, y_1)), \ \text{dist}(\Phi(x_2, y_1), \Phi(x_2, y_2)))^\gamma \leq 2C_\gamma k^{-\gamma} \text{dist}(\Phi(w_1), \Phi(w_2))^\gamma.$$

$\square$

In particular, Proposition 7.12 implies that

$$\gamma \dim_H(G^s \times G^u) \leq \dim_H(\Phi(G^s \times G^u)) \leq \gamma^{-1} \dim_H(G^s \times G^u).$$

Notice that if $x \in G^u$ and $y \in G^s$ then $\Phi(x,y) \in \{q \in \Lambda^\# \mid O_\infty \in \omega(q) \cap \alpha(q)\}$. Therefore $\Phi(G^s \times G^u) \subset \{q \in \Lambda^\# \mid O_\infty \in \omega(q) \cap \alpha(q)\}$, and we have

$$\dim_H\{q \in \Lambda^\# \mid O_\infty \in \omega(q) \cap \alpha(q)\} \geq \dim_H(\Phi(G^s \times G^u)) \geq$$

$$\geq \gamma \dim_H(G^s \times G^u) \geq \gamma (h_\Lambda^u + h_\Lambda^s) = \gamma \dim_H \Lambda.$$

Since γ could be chosen arbitrary close to one, this implies that

$$\dim_H\{q \in \Lambda^\# \mid O_\infty \in \omega(q) \cap \alpha(q)\} \geq \dim_H \Lambda.$$

This completes the proof of Theorem 9. $\square$

8 Newhouse domains in the Sitnikov problem and construction of a thick hyperbolic set

Here we show that for an open set of parameters f_e has a locally maximal hyperbolic set with persistent tangencies, homoclinically related to O_∞, of Hausdorff dimension close to two, and use it to complete the proof of the main results of the paper. In order to do so, we first use C^2-lambda lemma for dynamics near O_∞ (i.e. Theorem 7) and the explicit form of the Melnikov function for the Sitnikov problem from [GP] to prove (in Subsection 8.1) that there are arbitrary small values of the parameter e that correspond to a quadratic tangency between $W^s(O_\infty)$ and $W^u(O_\infty)$ that unfolds generically with e. Then in Subsection 8.2 we show that there are small values of e such that f_e has a hyperbolic saddle close to a homoclinic orbit of O_∞ whose certain finite parts of stable and unstable manifolds are C^2-close to the stable and unstable manifolds of O_∞. Unfolding of quadratic tangencies in two-dimensional conservative case leads to appearance of hyperbolic sets of Hausdorff dimension close to 2, see [Go], and we apply this fact to these saddles in Subsection 8.3. Finally, in Subsection 8.4 we apply the standard genericity arguments and the results of Section 7 to these hyperbolic sets of large Hausdorff dimension and prove the main results of the paper, Theorems 2 and 4.

8.1 Existence of quadratic tangencies between $W^u(O_\infty)$ and $W^s(O_\infty)$

Theorem 15. *(quadratic tangency) There exists a sequence $e_n \to 0$ as $n \to \infty$ such that $W^u(O_\infty)$ and $W^s(O_\infty)$ have a point of quadratic tangency at some q_n that unfolds generically as e changes, i.e. locally the hook of one separatrix moves with nonzero e-speed with respect to the other.*

Proof of Theorem 15. Due to Theorem 7 condition (24) for large N the image of the diagonal $\{x = y\}$ under f_e^N in the neighborhood U_X can be considered as a graph of some function $\Delta_N : [1 - \xi, 1 + \xi] \to \mathbb{R}_y^1$, and this function satisfies

$$-C\Delta(x) < \Delta_N'(x) < 0, \quad 0 < \Delta_N''(x) < C\Delta_N(x).$$

On the other hand, the explicit form of the Melnikov function for the Sitnikov problem was derived in [GP]. Set T so that $(T, 0) = f_e(1, 0)$, $T = T_0 + O(e)$, and denote by $W_e(x)$ the function whose graph in U_X represents a piece of stable manifold $W^s(O_\infty)$ in U_X. The results from [GP] claims that

$$W_e(x) = Ae\sin\left(\frac{2\pi}{T}(x - 1)\right) + O(e^2), \quad A > 0.$$

Suppose for a moment that $\Delta_N(x)$ and $W(x)$ have a point of tangency, and let us show that this tangency must be quadratic. Denote by x^* the x-coordinate of the point of tangency between $W_e(x)$ and $\Delta_N(x)$. Then x^* is a solution of the system

$$\begin{cases} \Delta_N(x) = W_e(x) = Ae\sin\left(\frac{2\pi}{T}(x - 1)\right) + O(e^2) \\ \Delta_N'(x) = W_e'(x) = Ae\frac{2\pi}{T}\cos\left(\frac{2\pi}{T}(x - 1)\right) + O(e^2) \end{cases}$$

This long with Theorem 7 conditions (24) implies that

$$-CAe\sin\left(\frac{2\pi}{T}(x^* - 1)\right) + O(e^2) = -C\Delta_N(x) < \Delta_N'(x) = Ae\frac{2\pi}{T}\cos\left(\frac{2\pi}{T}(x^* - 1)\right) + O(e^2),$$

and, therefore,

$$\tan\left(\frac{2\pi}{T}(x^*-1)\right) < -\frac{2\pi}{CT} + O(e), \quad \text{and} \quad \frac{\pi}{2} < \frac{2\pi}{T}(x^*-1) \leq \zeta < \pi$$

for some ζ independent of e. Then

$$W_e''(x^*) = -A\frac{4\pi^2}{T^2}e\sin\left(\frac{2\pi}{T}(x^*-1)\right) + O(e^2) < 0,$$

but $\Delta_N''(x^*) > 0$, and therefore the tangency at $(x^*, \Delta_N(x^*))$ is quadratic. This implies that the tangency between $W^s(O_\infty)$ and the diagonal $\{x = y\}$ is quadratic, and due to the symmetry of Sitnikov problem $W^s(O_\infty)$ and $W^u(O_\infty)$ also have a point of quadratic tangency.

Let us now show that graphs of $W_e(x)$ and $\Delta_N(x)$ indeed have a point of tangency and that this tangency unfolds generically. We have

$$\frac{d}{de}(W_e(x)) = A\sin\left(\frac{2\pi}{T}(x-1)\right) + O(e), \quad A > 0,$$

and due to Theorem 7

$$\frac{d}{de}(\Delta_N(x)) \leq C\Delta_N(x,e) = O(e).$$

For $e = 0$ we have $W_e(x) \equiv 0$, and $\Delta_N(x) > 0$. Therefore if N is large enough for some e_N, $e_N \to 0$ as $N \to \infty$, graphs of $W_e(x)$ and $\Delta_N(x)$ must have a point of tangency and that this tangency unfolds generically. $\square$

8.2 Construction of a hyperbolic periodic saddle with a quadratic homoclinic tangency

Consider the part of $W_{e_n}^u(O_\infty)$ located between q_n^e and q_*^e, denoted $W_{e_n,[q_n,q^*]}^u$. Consider a nonself-intersecting tubular neighborhoods of $W_{[q_n,q^*]}^u$, denoted $T_{[q_n,q^*]}^u$. Denote by q_e^* the smooth deformation of the transverse homoclinic point q_* with respect to e, and by $W_{e,[q_n^e,q_*^e]}^u$ the connected component of the intersection of the unstable manifold of O_∞ with $T_{[q_n^e,q_*^e]}^u$ containing q_*^e.

Proposition 8.1. *For any e sufficiently close to e_n there is a sequence of hyperbolic periodic saddles $s_{e,l} \to q_*^e$ as $l \to \infty$ with the following property. Denote by $W_{[q_n^e,q_*^e],l}^u(s_{e,l})$ the connected component of the intersection of the unstable manifolds of $s_{e,l}$ with $T_{[q_n^e,q_*^e]}^u$ containing $s_{e,l}$. Then saddles $s_{e,l}$ and its manifolds $W_{[q_n^e,q_*^e],l}^u(s_{e,l})$ and $W_{e,[q_n^e,q_*^e]}^u$ depend analytically on e, moreover,*

$$W_{[q_n^e,q_*^e],l}^u(s_{e,l}) \to_{C^2} W_{e,[q_n^e,q_*^e]}^u \quad \text{as } l \to \infty. \tag{53}$$

Similar statement holds for stable parts.

Before we start the proof of Proposition 8.1, notice that it is in a sense an analog of Birkhoff-Smale Theorem on existence of infinitely many periodic orbits near transverse homoclinic point of a hyperbolic saddle, where we replace a hyperbolic saddle by our degenerate saddle at O_∞.
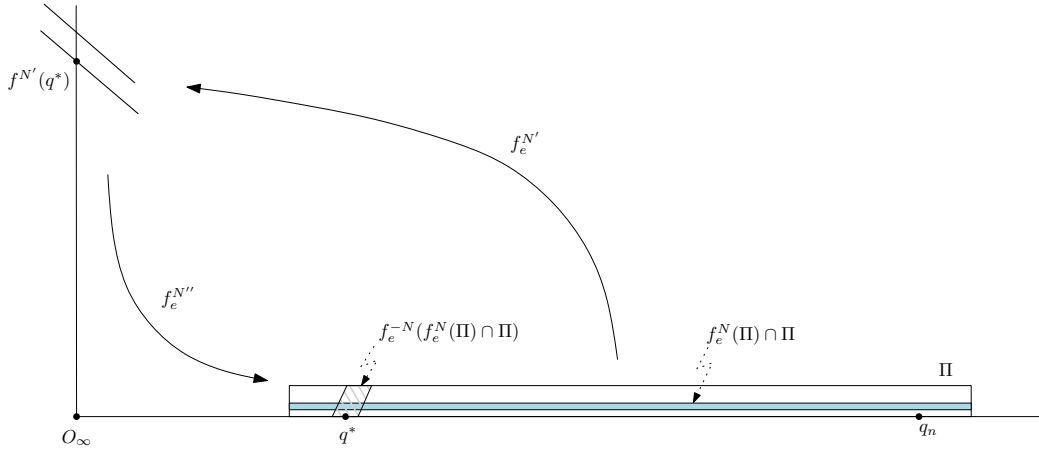
Figure 8: Creation of hyperbolic saddles near the homoclinic point of O_∞.

Proof of Proposition 8.1. Let us extend the neighborhood of the degenerate saddle where the flow has the form (10) along the unstable manifold of Q_∞ until it contains both q_*^e and q_n^e. Suppose that $N' \in \mathbb{N}$ is such that $f_e^{N'}(q_*^e)$ again belongs to the neighborhood of O_∞ where invariant manifolds are rectified. Fix a rectangle Π of a small height as shown on Fig.8.2.

Consider a horizontal cone field in Π,

$$K_{(x,y)} = \{\overline{v} \in T_{(x,y)}\mathbb{R}^2 \mid |v_y| \leq M|v_x| \} \qquad \text{for some } M > 0.$$

Take any curve $\gamma \in \Pi$ tangent to this cone field. Since q_*^e (and, hence, $f_e^{N'}(q_*^e)$) is a point of transversal intersection of $W^u(O_\infty)$ and $W^s(O_\infty)$, M and the hight of Π are taken small enough, then the image $f_e^{N'}(\gamma)$ is transversal to $W^s(O_\infty)$. Moreover, for some $\lambda^* > 0$ independent of γ the image $f_e^{N'}(\gamma)$ is a graph of a function $y(x)$ with $|y'(x)| < \lambda^*$. Therefore we can apply Theorem 6 to $f_e^{N'}(\gamma)$. This imply that for a large enough $N'' \in \mathbb{N}$ the image $f_e^{N''}(f_e^{N'}(\gamma)) \cap \Pi$ is again a curve tangent to to the cone field $\{K_{(x,y)}\}$, see (20), and every vector tangent to γ is expanding, see (21). Moreover, the curve $f_e^{N''}(f_e^{N'}(\gamma)) \cap \Pi$ is C^2-close to $W^u(O_\infty)$ if N'' is large enough, see Theorem 6 conditions (20) and (23). Therefore, the map $f_e^N : f_e^{-N}(f_e^N(\Pi) \cap \Pi) \to f_e^N(\Pi) \cap \Pi$, where $N = N' + N''$, is a non-linear hyperbolic rotation, and standard techniques of hyperbolic theory (see, for example, [IL]) imply that there exists a hyperbolic saddle point in $f_e^{-N}(f_e^N(\Pi) \cap \Pi) \cap (f_e^N(\Pi) \cap \Pi)$. Denote that saddle by $s_{e,N}$. The sequence of saddles $\{s_{e,N}\}$ satisfies the requirements of Proposition 8.1. $\qquad\square$

Fix n and parameter e_n from Proposition 8.1. Theorem 15 and Proposition 8.1 imply the following statement.

Proposition 8.2. *There exists a sequence $e_{k,n} \to e_n$ as $k \to \infty$ such that the map $f_{e_{k,n}}$ has a saddle periodic point s_k with the following properties:*

1) s_k is homoclinically related to O_∞ (i.e. $W^u(O_\infty)$ has a transversal intersection with $W^s(s_k)$, and $W^s(O_\infty)$ has a transversal intersection with $W^u(s_k)$);

2) $W^s(s_k)$ and $W^u(s_k)$ have a point of quadratic tangency that unfolds generically as e changes near e_n.

Proof of Proposition 8.2. Notice that for a given e_n there exists a small enough $\delta > 0$ such that the limit (53) is uniform in $e \in [e_n - \delta, e_n + \delta]$. Theorem 15 claims that the tangency between stable and unstable manifolds of O_∞ at q_n is quadratic. This implies that arbitrarily close to e_n there are parameter values $e_{k,n}$ such that stable and unstable manifolds of the saddles $s_{e_{k,n},l}$ have a point of quadratic tangency. Theorem 15 also claims that the quadratic tangency at q_n unfolds generically with e. Since the derivative (in e) of the distance between $W^u_{[q_n^e, q_*^e], l}(s_{e,l})$ and $W^u_{e,[q_n^e, q_*^e]}$ is of order of that distance (this follows from the same arguments that prove inequality (25) in Theorem 7), the quadratic tangency between $W^u(s_{e,l})$ and $W^s(s_{e,l})$ also unfolds generically. $\quad\square$

8.3 Conservative homoclinic bifurcations and hyperbolic sets of large Hausdorff dimension

In order to construct transitive invariant sets of full Hausdorff dimension we use the notion of a *homoclinic class.*

Let P be a hyperbolic saddle of a diffeomorphism f. A it homoclinic class $H(P, f)$ is a closure of the union of all the transversal homoclinic points of P.

It is known that $H(P, f)$ is a transitive invariant set of f, see [N5]. Moreover, consider all basic sets (locally maximal transitive hyperbolic sets) that contain the saddle P. A homoclinic class $H(P, f)$ is a smallest closed invariant set that contains all of them.

Theorem 16 ([Go]). *Let $f_0 \in \operatorname{Diff}^\infty(M^2, \operatorname{Leb})$ have an orbit $\mathcal{O}$ of quadratic homoclinic tangencies associated to some hyperbolic fixed point s_0, and $\{f_\mu\}$ be a generic unfolding of f_0 in $\operatorname{Diff}^\infty(M^2, \operatorname{Leb})$. Then for any $\delta > 0$ there is an open set $\mathcal{U} \subseteq \mathbb{R}^1$, $0 \in \overline{\mathcal{U}}$, such that the following holds:*

(1) for every $\mu \in \mathcal{U}$ the map f_μ has a basic set Δ_μ that contains the unique fixed point s_μ near s_0, exhibits persistent homoclinic tangencies, and Hausdorff dimension

$$\dim_H \Delta_\mu > 2 - \delta;$$

(2) there is a dense subset $\mathcal{D} \subseteq \mathcal{U}$ such that for every $\mu \in \mathcal{D}$ the map f_μ has a homoclinic tangency of the fixed point s_μ;

(3) there is a residual subset $\mathcal{R} \subseteq \mathcal{U}$ such that for every $\mu \in \mathcal{R}$

(3.1) the homoclinic class $H(P_\mu, f_\mu)$ is accumulated by f_μ's generic elliptic points,

(3.2) the homoclinic class $H(P_\mu, f_\mu)$ contains hyperbolic sets of Hausdorff dimension arbitrary close to 2; in particular, $\dim_H H(P_\mu, f_\mu) = 2$,

(3.3) $\dim_H \{x \in M \,|\, s_\mu \in \omega(x) \cap \alpha(x)\} = 2$.

In dissipative case Newhouse [N1] showed that near every surface diffeomorphism with a homoclinic tangency there are open sets (nowadays called *Newhouse domains*) of maps with persistence homoclinic tangencies. Moreover, in these open sets there are residual subsets of maps with infinitely many attracting periodic orbits. Later Robinson [R2] showed that this result can be formulated in terms of generic one parameter unfolding of a homoclinic tangency.

In area preserving case Duarte [Du1], [Du2], [Du4] showed that homoclinic tangencies also lead to similar phenomena, the role of sinks is played by elliptic points. Theorem 16 is a stronger version of the Duarte's result: we can control the Hausdorff dimension of the hyperbolic sets and homoclinic classes that appear in the construction.

8.4 Summing up

Let $\{f_e\}$ be the 2π-shift along the orbits of the Sitnikov problem (6). Denote by $OS(e) \subset \mathbb{R}^2$ the set of oscillatory motions of the Sitnikov problem (i.e. the set of initial conditions $(z, \dot{z}) \subset \mathbb{R}^2$ that correspond to an oscillatory motion of the Sitnikov problem). Here we construct an open set $\mathcal{N} \subset [0,1], 0 \in \overline{U}$, and a residual subset $R \subset \mathcal{N}$ such that for $e \in R$ we have $\dim_H OS(e) = 2$. Notice that this implies Theorem 2.

Due to Proposition 8.2 there exists a sequence of parameters $\{e_n\}_{n \in \mathbb{N}} \subset [0,1]$, $e_n \to 0$ as $n \to \infty$, such that f_{e_n} has a hyperbolic periodic saddle s_n that satisfies the following properties:

    a) in McGehee coordinates s_n is homoclinically related to O_∞;
    b) s_n has a quadratic homoclinic tangency that unfolds monotonically with e.

Theorem 16 implies that for each $n \in \mathbb{N}$ there exists an open set $\mathcal{N}_n \subset (\frac{1}{2}e_n, 2e_n)$ and a dense in $\mathcal{N}_n$ countable subset $D_n \subset \mathcal{N}_n$ such that for $e \in D_n$ the continuation of the saddle s_n has a quadratic homoclinic tangency that unfolds generically. Therefore $\mathcal{N} = \cup_n \mathcal{N}_n$ is a Newhouse domain for $\{f_e\}$. This proves Theorem 4.

Once again, Theorem 16 implies that for every $n, k \in \mathbb{N}$ there exists an open dense subset $V_{k,n} \subset \mathcal{N}_n$ such that for each $e \in V_{k,n}$ the map f_e has a hyperbolic set $\Lambda_{k,n}(e)$, $s_n(e) \in \Lambda_{k,n}(e)$, with $\dim_H \Lambda_{k,n}(e) > 2 - \frac{1}{k}$. Set $V_k = \cup_n V_{k,n}$, V_k is open and dense in $\mathcal{N}$. Theorem 9 implies that for $e \in V_k$ we have $\dim_H OS(e) > 2 - \frac{1}{k}$. Therefore for $e \in R = \cap_{k \in \mathbb{N}} V_k$ we have $\dim_H OS(e) = 2$. This proves Theorem 2.

9 Variation with respect to parameters

We start with the equations of motion (9). Then we need to perform a change of coordinates: $x = (p+q)/4 + a_4 p^5/4 + \cdots$ and $y = (q-p)/4 - a_4 p^5/4 + \cdots$. We get equation (10) which we reproduce here

$$\begin{cases} \dfrac{dx}{dt} = 2x(x+y+R_5^e)^3(1+Q_4^e) \\ \dfrac{dy}{dt} = -2y(x+y+R_5^e)^3(1+Q_4^e) \\ \dfrac{ds}{dt} = \dfrac{1}{2}(x+y+R_5^e)^3, \end{cases} \tag{54}$$

where $R_5^e \in O_5$ and $Q_4^e \in O_4$ and both depend smoothly on e. Consider the equation in variations with respect to e

$$\begin{cases} d_e \dfrac{dx}{dt} = 2x(x+y+R_5^e)^2 \left(3(1+Q_4^e)\partial_e R_5^e + \partial_e Q_4^e(x+y+R_5^e)\right) = 2x(x+y+R_5^e)^3 O_4^x \\ d_e \dfrac{dy}{dt} = -2y(x+y+R_5^e)^2 \left(3(1+Q_4^e)\partial_e R_5^e + \partial_e Q_4^e(x+y+R_5^e)\right) = -2y(x+y+R_5^e)^3 O_4^y. \end{cases} \tag{55}$$

After rescaling time we obtain the following system

$$\begin{cases} d_e \dfrac{dx}{ds} = xO_4^x \\ d_e \dfrac{dy}{ds} = -yO_4^y. \end{cases}$$

Now we need to upper estimate variation of the y-component. We know that $xy(s)$ being constant in the model case. Denote $h_0^2 = xy(0)$ and start with initial conditions $x(0) = y(0) = h_0$. Then for some $C > 0$ we have that

$$\left| d_e \frac{dy}{ds} \right| \leq C h_0^8 y^{-3}.$$

Substituting solution $y(s) = y(0) \exp(-s) = h_0 \exp(-s)$ we get $|d_e \frac{dy}{ds}| \leq C h_0^5 \exp(3s)$ and $|d_e\, y_T(x)| \leq 2\, C\, h_0$. Denote by $g(s)$ a solution of $\frac{dg}{ds} = C h_0 \exp(3s)$. Integrating this equation we have

$$g(s) = \frac{1}{3} C h_0^5 (\exp(3s) - 1) \leq \frac{1}{3} C h_0^2 \leq \frac{1}{3} C y(T).$$

for any s with $h_0^2 \geq \exp(-s)$.

Let $x(0) = y(0) = h_0 > 0$ and be small. In order to extend these arguments to the general case it suffices to prove that for some $C > 0$

$$|x(s) \exp(-s)|, \ |y(s) \exp(s)| < C$$

for all $s > 0$ such that $0 < x(s) < 1$. Prove it for $x(s)$. For $y(s)$ the proof is similar. Define an auxiliary function $\phi(s) = x(s) \exp(-s)$. It satisfies the equation $\frac{d\phi}{ds} = \exp(-s) O_4^\phi$. Since $x = 0$ is invariant, so is $\phi = 0$. Thus, $O_4^\phi = x O_3^\phi$ and the above equation becomes

$$\frac{d\phi}{ds} = \phi O_3^\phi.$$

Recall that we study the region $y \geq x \geq 0$. So we can bound the right hand-side by $\phi O_3^\phi \leq C \phi x^3$. Rescaling if necessary wlog we can assume that $C < 0.5$. Then $\frac{dx}{ds} \geq x/2$. Since $x(s)$ is positive and grows monotonically, there is a unique $T > 0$ such that $x(T) = 1$. For all $s \in [0, T]$ we have $x(s) \geq \exp \frac{1}{2}(T - s)$. This implies that $|O_3^\phi| \leq C \exp \frac{3}{2}(T - s)$. Using a formula for solutions we have

$$\phi(s) = \phi(0) \exp \left(\int_0^s O_3^\phi(x(\sigma), y(\sigma))\, d\sigma \right) \leq \phi(0) \exp \left(\frac{C}{1.5}(1 - \exp \frac{3}{2}(T - s)) \right).$$

This section should go after Theorem 15.

10 Construction of quadratic tangency

In this section we modify the proof of Theorem 15 to class of problems with the Melnikov function, denoted $M(x)$, satisfying the following condition:

For some $c, c' > 0$ if $M(x) > c$ and $M'(x) < -c'$, then we have $M''(x) > 0$. (56)

Recall that we denote absorbing intervals for slopes λ by $[\lambda_-(s), \lambda_+(s)]$ and for second derivative μ by $[\mu_-(s), \mu_+(s)]$. In Theorem 7 condition 24 we have that $\lambda_+(s) < \lambda_-(s) < 0 < \mu_-(s) < \mu_+(s)$ and $\lambda_\pm(s) \sim \mu_\pm(s) \sim y(s)$.

The Melnikov integral describe the shape of graphs of stable $W^s(O_\infty)$ and unstable $W^s(O_\infty)$ manifolds. Namely, the graph of $W^s(O_\infty)$ in U_X is given by $W_e^s(x) = eM(x) + O(e^2)$ for some analytic function M. The graph of $W^u(O_\infty)$ in U_Y is given by $W_e^u(y) = e\bar{M}(y) + O(e^2)$, where for some constant $C > 0$ we have $\bar{M}(x) = CM(x)$.

Consider an initial condition on the diagonal $x = y = h$, λ^*. Suppose that this initial condition is a point of primary tangency of $W^s(O_\infty)$ and $W^u(O_\infty)$. Namely, we start with two curves: γ^s being intersection of initial part of stable $W^s(O_\infty)$ with U_X and γ^u being intersection of initial part of stable $W^u(O_\infty)$ with U_Y. Then we flow both of them to the diagonal. Varying parameter e of the problem we create tangency and adjusting time pick it on the diagonal. Denote small parts of these curves near the tangency by γ_*^s and γ_*^u. Since the problem is reversible, i.e. reversing time does not change the class of differential equation we study, wlog assume that $\lambda^* \leq 1$. Due to Lemma 4.3 we have $\lambda^* \geq 2\lambda_0(1) = -2/3$. Let μ_*^s (resp. μ_*^u) be second derivative of γ_*^s (resp. γ_*^u) at the tangency. The fact that tangency is quadratic implied by the following separation lemma.

Lemma 10.1. *With the above notations start with a curve γ_* having two jet $(h, h, \lambda^*, -h^{-2})$ with small $h > 0$, then image of this curve*
    — under the forward flow to U_X has a quadratic tangent to γ^u and stays above it and
    — under the backward flow to U_Y has a quadratic tangent to γ^s and stays strictly to the right of this curve.

Indeed, it shows that second derivatives of γ_s and γ_u at tangency are distinct due to quadratic separation by γ_*.

Proof of Lemma 10.1: Consider the equation for μ:

$$\mu' = -\frac{B(\tau, s, \lambda)}{x + y} - d(\tau, s, \lambda)\mu.$$

We separately study forward and backward evolution of μ. Denote by T time $y(T) = 1$ or $T = -\ln h$.

Backward evolution: We divide backward evolution into two stages: $-2/3 < \lambda(s) \leq 1$ and $1 \leq \lambda(s)$. The second case does not exclude possibility of $\lambda(s)$ passing through infinity. Since $dx/dy = \widehat{\lambda}$-slopes of γ^s are small, slope $\lambda = dy/dx$ should cross 1. Denote $t^* < 0$ the time $\lambda(t^*) = 1$.

Consider the first time interval $[t^*, 0]$. For $\lambda \in [-2/3, 1]$ and $\tau \geq 1$ we have

$$B = 6(1 + \lambda)(\tau(1 + 3\lambda) + \lambda(3 + \lambda)) \leq 2(-2\tau - 14/9)$$

and

$$d = \frac{9 + 6\tau + 9\lambda}{1 + \tau} \geq \frac{6\tau + 3}{\tau + 1} \geq 3.$$

We also know that for some $C > 1$ we have

$$\frac{x(0)}{C} < x(s)\exp(s) < Cx(0) \text{ and } \frac{y(0)}{C} < y(s)\exp(-s) < Cy(0).$$

This implies that $\mu(s)$ with $\mu(0) = -h^{-2} \ll B(1,0,1)$ is monotonically increasing for $s \in [t^*, 0]$. Thus, $\mu(t^*) < -h^{-2}$.

In order to study the second interval we need to change coordinates and study evolution of curves $\{x = x(y) : y \in [y_-, y_+]\}$ with slope denoted $\widehat{\lambda} = dx/dy$ and second derivative $\widehat{\mu} = d^2x/dy^2$. Since the system of equation is reversible, backward evolution of $\widehat{\lambda}$ and $\widehat{\mu}$ obeys an ODE in the same class as evolution of λ and μ.

In order to switch from μ to $\widehat{\mu}$ recall that curvature of a parametrized curve $\gamma = \{(x(h), y(h)) : h \in [h_-, h_+]\}$ is given by

$$k(h) = \frac{x'y'' - y''x'}{((x')^2 + (y')^2)^{3/2}}.$$

Compute it using x-parametrization $h \equiv x$. We have

$$k = \frac{y''(x)}{(1 + (\frac{dy}{dx})^2)^{3/2}}$$

Compute it using y-parametrization $h \equiv y$. We have

$$k = \frac{x''(y)}{(1 + (\frac{dx}{dy})^2)^{3/2}}$$

In the case $dy/dx = 1$ we have $y''(x) = -x''(y)$. Thus, at $\lambda = 1$ we get $\mu = -\widehat{\mu}$.

Notice now that $\widehat{\mu}(t^*) > h^{-2}$ is above the absorbing interval $[\widehat{\mu}_-, \widehat{\mu}_+]$. Thus, at the time our orbit reaches U_X we have that $\widehat{\mu}$ exceeds the lower bound of the corresponding absorbing interval. Due to the sign condition (56) on the Melnikov function we have $\widehat{\mu}_- > 0 > e(M'(x) + O(e))$ and, as the result, the backward image of γ_* in U_Y is strictly to the right.

Forward evolution: It suffices to prove that after a certain time both λ and μ belong to their corresponding absorbing intervals $[\lambda_-(s), \lambda_+(s)]$ and $[\mu_-(s), \mu_+(s)]$. Due to the sign condition (56) on the Melnikov function we have that the forward image of γ_* in U_X is strictly higher γ^u and has a quadratic tangency with it.

By Lemma 4.13 we have

$$-3\tau\frac{1 + \lambda}{1 + \tau} \geq B(\lambda, \tau) - 42\tau\frac{1 + \lambda}{1 + \tau}$$

and by Lemma 4.10 the equation of evolution of μ has the form

$$\frac{d\mu}{ds} = -d(\tau(s), \lambda)\mu - \frac{B(\tau(s), \lambda)}{x + y},$$

In Lemma **??** we know that for some $s_0 > 0$ we have $\lambda(s_0)$ gets into the absorbing interval $[\lambda_-(s_0), \lambda_+(s_0)]$.

For $0 < \tau < 0.01$ we have $-0.03 < \lambda_0(\tau) < 0$. Thus, for $s \geq s_0$ we have $-0.06 < \lambda(s) < 0$ and

$$8.4 \geq d(\tau(s), \lambda) \geq 9.$$

Using these estimates we also have

$$-3.95\tau \geq B(\lambda, \tau) - 42\tau.$$

Since $B < 0$ and $B \ll -\mu$, at time s_0 we have $\mu(0) < \mu(s_0) \leq \exp(10s_0)\mu_0$.

Increasing s_0 if necessary assume $\tau(s_0) < 0.01$. Then for $s > s_0$ we have that $\lambda(s) \in [\lambda_-(s_0), \lambda_+(s_0)]$ and $\mu(s) \geq \exp(-8.4(s - s_0))\mu(s_0)$. Since $B \sum \tau(s) = \exp(-2s)$ and $\mu(s) \lesssim \exp(2T)exp(-8.4s)$, there is $s^* < T/3 = -\ln h/3$ with $\mu(s) > B(s)$. Then using arguments similar to the proof of Lemma **??** we show that $\mu(s^* + s') \in [\mu_-(s^* + s'), \mu_+(s^* + s')]$ for some uniformly bounded s'. $\square$

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11   Notations

f_e — the Poincare map of the Sitnikov problem (3);

$f_\mu = f_{\mu,C}$ — the Poincare map of the restricted planar circular 3–body problem (5);

$(p, q) = (-\dot{z}, \sqrt{\frac{2}{z}})$ — simplifying coordinates at "parabolic" infinity for the Sitnikov problem;

$(P_r, u) = \left(P_r, \sqrt{\frac{1}{r}}\right)$ — simplifying coordinates at "parabolic" infinity for the restricted planar circular 3–body problem;

$(x, y) = (\frac{1}{4}(p + q) + \cdots, \frac{1}{4}(q - p) + \cdots)$— coordinates straightening invariant manifolds $W^s(O_\infty)$ and $W^u(O_\infty)$ of O_∞;

$\lambda = \frac{dy}{dx}$ — slope and $\mu = \frac{d^2 y}{dx^2}$ — second derivative of curves, which are graphs of $y = f(x)$;

$X = (1, 0)$ and $Y = (0, 1)$ – points on the invariant manifolds $W^s(O_\infty)$ and $W^u(O_\infty)$ and $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ their orbits;

U_X (resp. U_Y) — a rectangular neighborhood of X (resp. Y) such that for some $\xi > 0$ its size in x (resp. y) direction is $[1 - \xi, 1 + \xi]$;

$\widehat{\lambda} = \frac{dx}{dy}$ — slope and $\widehat{\mu} = \frac{d^2 x}{dy^2}$— second derivative of curves, which are graphs of $x = f(y)$;

$[\lambda_-(s), \lambda_+(s)]$ — absorbing intervals of slopes or C^1–dynamics;

$[\mu_-(s), \mu_+(s)]$ — absorbing intervals of second derivatives or C^2–dynamics;

$\{e_n\}_n$ — a sequence of parameters $e_n \to 0$ such that f_{e_n} has a quadratic tangency at some q_n^e and q_e^* is a transverse homoclinic point;

For each e_n there is a sequence a saddle periodic points $\{s_{e,l}\}_l$ converging to q_e^*;

Λ — a locally maximal transitive hyperbolic set arising after an unfolding of a quadratic homoclinic tangency associated to a saddle s_l;

$\Lambda^\#$ — a locally maximal transitive set containing both Λ and O_∞. This is the set which contains oscillatory motions we study!

$S = \Lambda \cup O_\infty \cup \mathcal{O}(X) \cup \mathcal{O}(Y)$ — a set whose neighborhood contains $\Lambda^\#$ as a locally maximal transitive set;

$U(\Lambda)$ (resp. $U(S)$) — a neighborhood of Λ (resp. S);

β — size of local product structure of $\Lambda^\#$;

μ — invariant Borel probability measure supported on Λ;

ν — invariant Borel probability measure supported on $\Lambda^\#$;

h_μ and h_ν — entropies of μ and ν resp.

λ_μ and λ_ν — positive Lyapunov exponents of μ and ν resp.

$d \geq 2$ — positive integer, A, B are $d \times d$ matrices with 0 or 1 entries;

For a C^1 map f having a locally maximal set $\Lambda^\#$ with a local product structure we define $\phi^u(x) = \log \|Df_x|_{E_x^u}\|$;

(Σ_A^+, σ_A) and (Σ_B^+, σ_B) — topologically mixing sub-shifts of finite type;

μ — an invariant measure supported on Λ, which is Gibbs for the potential $\phi^u(x)$;

α — parameter going to infinity used to construct a new invariant measure $\nu_a l$ supported on $\Lambda^\#$ and in a weak* topology close to μ;

$\gamma > 0$ — exponent of Holder regularity of holonomy of invariant manifolds of $\Lambda^\#$;

Π_0^Λ — a curvilinear rectangle having O_∞ as a vertex and initial parts of $W^s(O_\infty)$ and $W^u(O_\infty)$ as sides;

Π_j^Λ — a curvilinear rectangle of a Markov partition of Λ;

Π_N — a curvilinear rectangle in U_Y such that after N iterates of f_e it is contained in U_X;

Δ_N — N-th image of the diagonal under f_e, i.e. under f_e^N.

$OS(e)$ — oscillatory motions of f_e of the Sitnikov problem with parameter e (see (1));

C — plays diverse roles as an internal variable to denote a constant in a proof.

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