Conservative homoclinic bifurcations and some applications

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Abstract

We study generic unfoldings of homoclinic tangencies of two dimensional area preserving diffeomorphisms (conservative Newhouse phenomena) and show that they give rise to invariant hyperbolic sets of arbitrary large Hausdorff dimension. As applications, we discuss the size of stochastic layer of standard map, and the Hausdorff dimension of invariant hyperbolic sets for certain restricted three body problems. We avoid involved technical details and only concentrate on the ideas involved into the proof of the presented results.

1 Introduction

In the case of dissipative dynamical systems homoclinic bifurcations were intensively investigated; some of the dynamical phenomena that appear after a bifurcation in this case are persistent tangencies and infinite number of sinks (Newhose phenomena [N1], [N2], [N3]), strange attractors (Mora, Viana [MV]), arbitrarily degenerate periodic points of arbitrary high periods (Gonchenko, Shilnikov, Turaev [GST]), and superexponential growth of periodic orbits (Kaloshin [K]).

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The conservative (area preserving) case is known to be more complicated. For example, it took over two decades to prove an analog of Newhouse results for area preserving surface diffeomorphisms (Duarte [Du1], [Du2], Gonchenko, Shilnikov [GS]). For the case of C^1 maps see also [N4], but here we will be interested in the case of higher smoothness.

By $\mathrm{Diff}^r(M^2, \mathrm{Leb})$, $0 < r \leq \infty$, we denote the space of C^r-diffeomorphisms of a two–dimensional Riemannian manifold M^2 that preserve the natural Lebesque measure on M^2. Let a diffeomorphism $f \in \mathrm{Diff}^\infty(M^2, Leb)$ have a quadratic homoclinic tangency associated to some hyperbolic fixed point P. Here is a zoo on known phenomena that appear as a result of a two-dimensional conservative homoclinic bifurcation:

- **Henon map in the renormalization limit.** An appropriately chosen and properly rescaled return map near a point of homoclinic tangency can be arbitrarily C^r-close to an area preserving Henon family $H_a(x, y) = (y, -x + a - y^2)$, where a can be arbitrary [MR],[GS].

- **Elliptic periodic points.** A small perturbation of f may have an elliptic periodic point near a point of homoclinic tangency. One of the ways to prove it formally is to consider the aforementioned renormalization limit and to observe that the limit map H_a has an elliptic fixed point for some values of a.

- **Hyperbolic sets with persistent tangencies.** Locally maximal hyperbolic sets exhibiting persistent tangencies of leaves of stable and unstable foliation can be born as a result of an unfolding of a homoclinic tangency [Du2]. A one-parameter version of this result is now also available [Du4].

- **Infinitely many coexisting elliptic periodic points (conservative Newhouse phenomena).** Duarte [Du1], [Du4] and Gonchenko, Shilnikov [GS] showed that near f there exists an open set $\mathcal{U} \subset \mathrm{Diff}^\infty(M^2, Leb)$ such that a generic diffeomorphism from $\mathcal{U}$ has infinitely many coexisting elliptic periodic points. Recently Duarte [Du4] proved also a one-parameter version of this result, therefore making it possible to apply it in many concrete finite-parameter families. Usually open sets of diffeomorphisms with persistent homoclinic tangency are called *Newhouse domains*.[1]

[1] One of the results we establish is existence of Newhouse domains for certain three body problems. See Section 4.4

- **Elliptic periodic points of arbitrarily high order of degeneracy.** Maps with infinitely many elliptic periodic orbits of every order of degeneracy are dense in the Newhouse regions in space of two-dimensional area-preserving analytic maps [GST].

- **Tangencies of arbitrary high order.** Area preserving surface diffeomorphisms with homoclinic tangencies of arbitrarily high orders are dense in the Newhouse regions [GST].

- **Universal maps.** Gonchenko, Shilnikov, Turaev [GST] showed that near homoclinic tangency one can approximate any ahead given dynamics in the following sense: Every area preserving diffeomorphism of a two-dimensional disc can be C^r–approximated by a diffeomorphism, which arises as an appropriately chosen and properly rescaled return map near a point of homoclinic tangency. In other words, every dynamical phenomenon which is generic for some open set of symplectic diffeomorphisms of a two-dimensional disc can be encountered arbitrarily close to any area preserving two-dimensional map exhibiting a homoclinic tangency. It is hard to verify appearance of this phenomenon in a concrete finite–parameter family. However, this serves as a great illustration of complexity of dynamics of unfoldings of a homoclinic tangency.

The purpose of this paper is to add another animal to this zoo, namely:

- **Hyperbolic sets of large Hausdorff dimension.** Locally maximal hyperbolic sets of Hausdorff dimension arbitrary close to two appear after a generic one-parameter unfolding of a homoclinic tangency; see Section 2 for the formal statement.

Besides, we discuss two applications of this phenomenon. Namely, in Section 3 we prove that stochastic layer (the set of orbits with non-zero Lyapunov exponents) of the Taylor-Chirikov standard map has full Hausdorff dimension for large generic values of the parameter. In Section 4 we construct invariant hyperbolic sets of Hausdorff dimension arbitrary close to two in the Sitnikov problem and in the restricted planar circular three body problem for many parameter values.

While proof in Section 3 are complete, Sections 2 and 4 present only outline of the proof. A complete account is fairly involved and will appear elsewhere.

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2 Hyperbolic sets of large Hausdorff dimension

Several famous long standing conjectures discuss the measure of certain invariant sets of some dynamical system (see introduction to Sections 3 and 4). Any set of positive Lebesgue measure has Hausdorff dimension which is equal to the dimension of the ambient manifold. Therefore it is reasonable to ask whether those invariant sets indeed have full Hausdorff dimension.

Downarowicz and Newhouse [DN] proved that there is a residual subset $\mathcal{R}$ of the space of C^r-diffeomorphisms of a compact two dimensional manifold M such that if $f \in \mathcal{R}$ and f has a homoclinic tangency, then f has compact invariant topologically transitive sets of Hausdorff dimension two. In their proof they used results by Gonchenko, Shilnikov and Turaev [GST] to create degererate saddle-nodes. Therefore this approach can not be generalized to the conservative case, and also does not allow the result to be formulated for generic finite–parameter families of diffeomorphisms.

In conservative setting Newhouse [N6] proved that in $\mathrm{Diff}^1(M^2, Leb)$ there is a residual subset of maps such that every homoclinic class[2] for each of those maps has Hausdorff dimension 2. Later Arnaud, Bonatti and Crovisier [BC], [ABC] essentially improved that result and showed that in the space of C^1 symplectic maps the residual subset consists of the transitive maps that have only one homoclinic class (the whole manifold). Notice that due to KAM theory this result can not be extended to higher smoothness.

In this section we show that a generic one parameter area-preserving homoclinic bifurcation always give birth to a compact invariant topologically transitive set of Hausdorff dimension two. This set is the closure of the union of a countable sequence of hyperbolic sets of Hausdorff dimension arbitrary close to two.

[2]Let P be a hyperbolic saddle of a diffeomorphism f. A homoclinic class $H(P, f)$ is a closure of the union of all the transversal homoclinic points of P.

2.1 The area preserving Henon family

First of all we consider area preserving Henon family. For $a = -1$ this map has a degenerate fixed point at $(x, y) = (-1, 1)$. We construct invariant hyperbolic sets of large Hausdorff dimension for a slightly larger than -1 near this fixed point. Later we use the renormalization results to reduce the case of a generic unfolding of an area preserving surface diffeomorphism with a homoclinic tangency to this construction.

Theorem 1. *Consider the family of area preserving Henon maps*

$$H_a : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ -x + a - y^2 \end{pmatrix}. \tag{1}$$

There is a (piecewise continuous) family of sets Λ_a, $a \in [-1, -1 + \varepsilon]$ for some $\varepsilon > 0$, such that the following properties hold.

1. *The set Λ_a is a locally maximal hyperbolic set of the map H_a;*

2. *The set Λ_a contains a saddle fixed point of the map H_a;*

3. *The set Λ_a has an open and closed (in Λ_a) subset $\widetilde{\Lambda}_a$ such that the first return map for $\widetilde{\Lambda}_a$ is a two-component Smale horseshoe;*

4. *Hausdorff dimension $dim_H \widetilde{\Lambda}_a \to 2$ as $a \to -1$.*

A similar statement holds also for any generic one parameter unfolding of an extremal periodic point (see [Du1] for a formal definition) as soon as the form of the splitting of separatrices can be established (see [G1, GL] for the relevant results on splitting of separatrices).

Theorem 1 can be considered as an improvement of Lemma A from [Du4], where Duarte proves that area preserving Henon maps have hyperbolic sets of large "left-right thickness" (see [Du4, Mo] for a definition) for values of a slightly larger than -1.

Sketch of the proof of Theorem 1.

Step 1. Change of coordinates and rescaling. Up to the change of parameter and coordinates there exists only one one-parameter area preserving quadratic family with some conditions on the fixed points (Henon family), see [H], [F]. In particular, we can consider the family

$$F_\varepsilon : (x, y) \mapsto (x + y - x^2 + \varepsilon,\ y - x^2 + \varepsilon) \tag{2}$$

instead of (1). In this form it is a partial case of a so called *generalized standard family*, and it was considered in [G1].

An affine change of coordinates conjugates $\{F_\varepsilon\}$ with the family of maps

$$(u, v) \mapsto (u, v) + \delta(v, 2u - u^2) + \delta^2(2u - u^2, 0), \tag{3}$$

where $\delta = \varepsilon^{\frac{1}{4}}$. This is a family of maps close to identity. For each of these maps the origin is a saddle with eigenvalues

$$\lambda_1 = 1 + \delta^2 + \sqrt{\delta^4 + 2\delta^2} = 1 + \sqrt{2}\delta + O(\delta^2) > 1,$$

$$\lambda_2 = \lambda_1^{-1} = 1 + \delta^2 - \sqrt{\delta^4 + 2\delta^2} = 1 - \sqrt{2}\delta + O(\delta^2) < 1.$$

Set $h = \log \lambda_1$. By definition $h = \sqrt{2}\delta + O(\delta^2)$, and δ can be expressed as a nice function of h. We parameterize the maps of the family (3) by h, and denote the family (3) by $\mathfrak{F}_h$.

Step 2. Gelfreich normal form and splitting of separatrices for Henon family. The family $\mathfrak{F}_h$ is closely related to the conservative vector field

$$\begin{cases} \dot{x} = y, \\ \dot{y} = 2x - x^2. \end{cases} \tag{4}$$

Namely, due to Theorems A and A' from [FS1] (see also Proposition 5.1 from [FS]) the separatrix phase curve of the vector field (4) (let us denote it by σ) gives a good approximation of some finite pieces of $W^s(0,0)$ and $W^u(0,0)$. Denote by $\widetilde{\sigma}$ a finite segment of separatrix σ.

The restriction of the map $\mathfrak{F}_h$ on the local unstable separatrix $W_h^u(0)$ is conjugated with a multiplication $\xi \mapsto \lambda\xi$, $\xi \in (\mathbb{R}, 0)$. Call a parameter t on $W_h^u(0)$ *standard* if it is obtained by a substitution of e^t instead of ξ into the conjugating function. Such a parametrization is defined up to a substitution $t \mapsto t + const$.

Theorem 4 in [G3] claims that there is an area preserving real analytic change of coordinates Ψ_h that conjugate the map $\mathfrak{F}_h$ in a neighborhood of $\widetilde{\sigma}$ with the shift $(t, E) \mapsto (t + h, E)$, $\Psi_h^{-1}(W_h^u) = \{E = 0\}$, and t gives a standard parametrization of the unstable manifold. Moreover, from [G1], [G2] it follows that in these normalizing coordinates stable manifold $\Psi_h^{-1}(W_h^s)$ can be represented as a graph of a real-analytic h-periodic function $\Theta(t)$,

$$\Theta(t) = 8\sqrt{2}|\,\Theta_1|\,h^{-6}e^{-2\pi^2/h}\sin\frac{2\pi t}{h} + O(h^{-5}e^{-2\pi^2/h}).$$

Also, Gelfreich and Sauzin [GS] proved that $|\,\Theta_1| \neq 0$ (see also [Ch], where some numerical results are described).

Step 3. Birkhoff normal form and construction of a horseshoe. Recall that the real analytic area preserving diffeomorphism of a two dimensional domain in a neighborhood of a saddle with eigenvalues (λ, λ^{-1}) by an analytic change of coordinate can be reduced to the Birkhoff normal form ([S], see also [SM]):

$$N(x,y) = (\Delta(xy)x, \Delta^{-1}(xy)y), \tag{5}$$

where $\Delta(xy) = \lambda + a_1xy + a_2(xy)^2 + \ldots$ is analytic. From [FS], [Du2], [Du4] it follows that for an analytic one-parameter family of maps the change of coordinates and the function Δ depend analytically on the parameter. Together with the description of the splitting of separatrices this allows not only to construct the horseshoes for $\mathfrak{F}_h$ using the transversal homoclinic points, but also to estimate some quantitative characteristics of these horseshoes. Namely, dynamics in a neighborhood of the saddle is controlled by the Birkhoff normal form, dynamics and geometry in a neighborhood of $\widetilde{\sigma}$ is described by the Gelfreich normal form and the form of the splitting of separatrices, and all the transitions and changes of coordinates have uniformly bounded distortions.

Step 4. Estimates of left- and right- thicknesses for the constructed horseshoes. Topologically a constructed horseshoe K is a product of a "stable" and "unstable" Cantor sets K^s and K^u. Moreover, Hausdorff dimension $\dim_H K = \dim_H K^s + \dim_H K^u$, see [MM], [PV]. Therefore, we can consider each of the Cantor sets separately. We will first estimate left- and right- thicknesses of K^s and K^u.

Let I be a finite closed interval, and $\psi_1, \psi_2 : I \to I$ be strictly monotonous contracting maps, $\psi_1(I) \cap \psi_2(I) = \emptyset$. Denote $I_1 = \psi(I) \cup \psi_2(I)$, and set $I_{n+1} = \psi(I_n) \cup \psi_2(I_n)$. Then $C = \cap_{n\in\mathbb{N}} I_n$ is a Cantor set.

To define left- and right- thickness we consider the gaps in the Cantor set C. A *gap* of C is a bounded component of the complement $\mathbb{R}\backslash C$. The gaps of C are ordered in the following way. A bounded component U of I_1 is a gap of order zero (see Figure

1). A bounded component U' of I_{n+1}, which is not a gap of order less or equal to $n-1$, is a gap of order n. For example, U' on the Figure is a gap of order one. It is straightforward to check that every gap of C is a gap of some finite order.

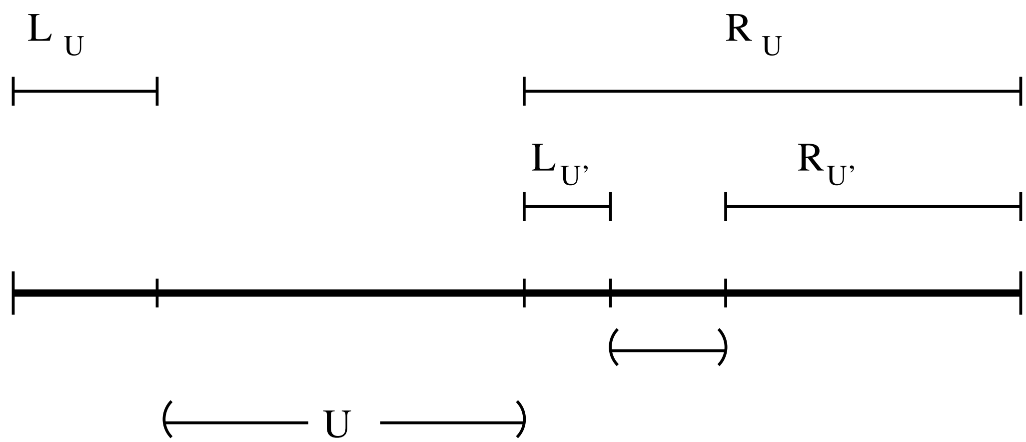

Figure 1:

Given a gap U of C of order n, we denote by L_U, respectively R_U, the component of I_{n+1} that is left, respectively right, adjacent to U. The greatest lower bounds

$$\tau_L(C) = \inf \left\{ \frac{|L_U|}{|U|} : U \text{ is a gap of } C \right\}$$

$$\tau_R(C) = \inf \left\{ \frac{|R_U|}{|U|} : U \text{ is a gap of } C \right\}$$

are respectively called the *left* and *right thickness* of C. For more details on the left and right thickness see [Du2] and [Mo]. See also [PT] for a more standard definition and properties of the thickness of a Cantor set.

Fix any small constant $\nu > 0$. Using Birkhoff normal form and the description of the splitting of separatrices, the construction of the Cantor set K can be carried over in such a way that $\tau_L(K^s) \sim h^{-1}$ and $\tau_R(K^s) \sim h^{\nu}$ as $h \to 0$.

Step 5. Relation between one-sided thicknesses and Hausdorff dimension of a Cantor set. We will use the following

Proposition 1. *Denote by τ_L and τ_R the left and right thicknesses of a Cantor set*

$C \subset \mathbb{R}$. Then Hausdorff dimension

$$\dim_H C > \max\left(\frac{\log\left(1+\frac{\tau_R}{1+\tau_L}\right)}{\log\left(1+\frac{1+\tau_R}{\tau_L}\right)}, \quad \frac{\log\left(1+\frac{\tau_L}{1+\tau_R}\right)}{\log\left(1+\frac{1+\tau_L}{\tau_R}\right)} \right).$$

In our case this implies that Hausdorff dimension $\dim_H K^s > \frac{1}{1+\nu}$ if h is small enough. Therefore $\dim_H K > \frac{2}{1+\nu}$. Since ν could be chosen arbitrary small, this proves Theorem 1. $\square$

2.2 Conservative homoclinic bifurcations and hyperbolic sets of large Hausdorff dimension

In order to construct transitive invariant sets of full Hausdorff dimension we use the notion of a *homoclinic class.*

Definition 1. *Let P be a hyperbolic saddle of a diffeomorphism f. A homoclinic class $H(P, f)$ is a closure of the union of all the transversal homoclinic points of P.*

It is known that $H(P, f)$ is a transitive invariant set of f, see [N5]. Moreover, consider all basic sets (locally maximal transitive hyperbolic sets) that contain the saddle P. A homoclinic class $H(P, f)$ is a smallest closed invariant set that contains all of them.

Theorem 2. *Let $f_0 \in \mathrm{Diff}^\infty(M^2, Leb)$ have an orbit $\mathcal{O}$ of quadratic homoclinic tangencies associated to some hyperbolic fixed point P_0, and $\{f_\mu\}$ be a generic unfolding of f_0 in $\mathrm{Diff}^\infty(M^2, Leb)$. Then for any $\delta > 0$ there is an open set $\mathcal{U} \subseteq \mathbb{R}^1$, $0 \in \overline{\mathcal{U}}$, such that the following holds:*

(1) *for every $\mu \in \mathcal{U}$ the map f_μ has a basic set Δ_μ that contains the unique fixed point P_μ near P_0, exhibits persistent homoclinic tangencies, and Hausdorff dimension*

$$\dim_H \Delta_\mu > 2 - \delta;$$

(2) *there is a dense subset $\mathcal{D} \subseteq \mathcal{U}$ such that for every $\mu \in \mathcal{D}$ the map f_μ has a homoclinic tangency of the fixed point P_μ;*

(3) *there is a residual subset* $\mathcal{R} \subseteq \mathcal{U}$ *such that for every* $\mu \in \mathcal{R}$

(3.1) *the homoclinic class* $H(P_\mu, f_\mu)$ *is accumulated by* f_μ*'s generic elliptic points,*

(3.2) *the homoclinic class* $H(P_\mu, f_\mu)$ *contains hyperbolic sets of Hausdorff dimension arbitrary close to 2; in particular,* $\dim_H H(P_\mu, f_\mu) = 2$.

Sketch of proof of Theorem 2.

Step 1. A sequence of bifurcation values $\mu_n \to 0$ *with quadratic homoclinic tangencies.* A generic one parameter family of diffeomorphisms unfolding a quadratic homoclinic tangency does not have isolated bifurcation values of the parameter, e.g. see [PT]. Therefore we can choose a sequence of parameters $\{\mu_n\}$, $\mu_n \to 0$, such that f_{μ_n} has a quadratic homoclinic tangency and a transversal homoclinic points.

Step 2. Appearance of invariant hyperbolic sets of large Hausdorff dimension. Using the renormalization technics by Mora-Romero [MR] an appropriately chosen and rescaled map near a homoclinic tangency is C^r–close to a Henon map H_a for any ahead chosen a. By Theorem 1 for a slightly larger -1 the map H_a has an invariant hyperbolic set $\tilde{\Lambda}_a$ of Hausdorff dimension close to 2 with persistent hyperbolic tangencies. By continuous dependence of Hausdorff dimension of a invariant hyperbolic set on a diffeomorphism [MM, PV] near each μ_n there is an open interval of parameters U_n such that for $\mu \in U_n$ the map f_μ has an invariant locally maximal transitive hyperbolic set Δ^*_μ which also has persistent homoclinic tangencies and with Hausdorff dimension greater than $2 - \delta$.

Step 3. Connecting the invariant set Λ^*_μ *with* P_μ. The hyperbolic saddle P_μ and the set Δ^*_μ are homoclinically related, see Lemma 2 from [Du1]. Therefore for every $\mu \in U_n$ there exists a basic set Δ_μ such that $P_\mu \in \Delta_\mu$ and $\Delta^*_\mu \subset \Delta_\mu$. Since Δ^*_μ has persistent homoclinic tangencies, so does Δ_μ. Also, $\dim_H \Delta_\mu \geq \dim_H \Delta^*_\mu > 2 - \delta$. This proves the part (1).

Step 4. Completion of proof of part (2). Since Λ^*_μ has persistent homoclinic tangencies standard arguments, see e.g. [PT], show that for a dense subset of parameters $D_n \subset U_n$ for each $\mu \in D_n$ we have that f_μ has a homoclinic tangency for the fixed point P_μ. This step completes the proof of the part (2).

Step 5. Construction of elliptic periodic points. Take any $\mu \in U_n$. If Q_μ is a transversal homoclinic point of the saddle P_μ then in can be continued for some intervals of parameters $I_Q \subseteq U_n$. Assume that $I_Q \subseteq U_n$ is a maximal subinterval of U_n where such a continuation is possible. All homoclinic points of P_μ for all values $\mu \in U_n$ generate countable number of such subintervals $\{I_s\}_{s\in\mathbb{N}}$ in U_n.

From [MR] it follows that for each I_s there exists a residual set $R_s \subseteq I_s$ of parameters such that for $\mu \in R_s$ the corresponding homoclinic point Q_μ is an accumulation point of elliptic periodic points of f_μ. Denote $\widetilde{R}_s = \left(U_n \backslash \overline{I_s}\right) \cup R_s$ – residual subset of U_n. Now set $\mathcal{R}_1 = \cap_{s\in\mathbb{N}} \widetilde{R}_s$ – also a residual subset in U_n. For $\mu \in \mathcal{R}_1$ every transversal homoclinic point of the saddle P_μ is an accumulation point of elliptic periodic points of f_μ, and this proves (3.1).

Step 6. Construction of a homoclinic class of full Hausdorff dimension. From Theorem 1 and [MR] it follows that for every $m \in \mathbb{N}$ there exists an open and dense subset $A_m \subset U_n$ such that for every $\mu \in A_m$ there exists a hyperbolic set Δ_μ^m such that $\dim_H \Delta_\mu^m > 2 - \frac{1}{m}$. From Lemma 2 from [Du1] it follows that P_μ and Δ_μ^m are homoclinically related. Therefore there exists a basic set $\widetilde{\Delta}_\mu$ such that $P_\mu \in \widetilde{\Delta}_\mu^m$ and $\Delta_\mu^m \subset \widetilde{\Delta}_\mu^m$. In particular, for $\mu \in \mathcal{R}_2 = \cap_{m\geq 1} A_m$ we have $\dim_H H(P_\mu, f_\mu) = 2$. Set $\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2$. This proves (3.2).

This completes the sketch of the proof of Theorem 2. $\square$

3 Standard map

The KAM theorem on the conservation of quasiperiodic motions in near-integrable Hamiltonian systems gave rise to the question on dynamical behavior in the regions where invariant tori are destroyed. In a more general form this question can be stated in the following way: "Can an analytic symplectic map have a chaotic component of positive measure and the Kolmogorov-Arnold-Moser (KAM) tori coexist? " Katok [Ka] gave a construction of a C^∞-smooth Bernoulli diffeomorphism on the two-dimensional disc which is equal to the identity on the boundary. One can perform a "smooth surgery" to combine this transformation with any other type of transformations. It proves that in principle quasiperiodic motions and the Bernoulli (chaotic) component can coexist in a smooth area preserving dynamical system. Since then several more or less artificial examples of coexistence of regular and chaotic component were suggested [Bu], [Do], [Li], [Pr], [W]. Nevertheless for "natural" examples (including those that appear in applications) the rigorous proof of positivity of the metric entropy (due to Pesin's theory [P] this is equivalent to the existence of positive measure set of orbits with non-zero Lyapunov exponents) is still missing. The simplest and most famous system where one would expect *mixed behavior* (KAM tori and orbits with non-zero Lyapunov exponents both have positive measure) is the Taylor-Chirikov standard map of the two–dimensional torus $\mathbb{T}^2$, given by

$$f_k(x,y) = (x + y + k\sin(2\pi x), y + k\sin(2\pi x)) \bmod \mathbb{Z}^2, \tag{6}$$

This family is also a model for numerous physical problems, e.g. see [C], [I], [SS].

Conjecture (*Sinai* [Sin]) *Is the metric entropy of f_k positive for some values of k? for positive measure of values of k? for all non-zero values of k?*

Currently even existence of at least one value of k with this property is not known. In the study of the standard family in the current context Duarte [Du3] proved the following important result:

Theorem A (Duarte, [Du3]). *There is a family of basic sets Λ_k of f_k such that:*

1. Λ_k is dynamically increasing, meaning for small $\varepsilon > 0$, $\Lambda_{k+\varepsilon}$ contains the continuation of Λ_k at parameter $k + \varepsilon$.

2. Hausdorff Dimension of Λ_k increases up to 2. For large k,

$$\dim_H(\Lambda_k) \geq 2\frac{\log 2}{\log\left(2 + \frac{9}{k^{1/3}}\right)}.$$

3. Λ_k fills in $\mathbb{T}^2 \ni (x,y)$, meaning that as k goes to ∞ the maximum distance of any point in $\mathbb{T}^2$ to Λ_k tends to 0. For large k, the set Λ_k is δ_k-dense on $\mathbb{T}^2$ for $\delta_k = \frac{4}{k^{1/3}}$.

Theorem B (Duarte, [Du3]). *There exists $k_0 > 0$ and a residual set $R \subseteq [k_0, \infty)$ such that for $k \in R$ the closure of the f_k's elliptic points contains Λ_k.*

Here we provide an improvement of Theorems A and B that claims, roughly speaking, that stochastic layer of the standard map has full Hausdorff dimension for large parameters from a residual set in the space of parameters.

Theorem 3. *There exists $k_0 > 0$ and a residual set $\mathcal{R} \in [k_0, +\infty)$ such that for every $k \in \mathcal{R}$ there exists an infinite sequence of transitive locally maximal hyperbolic sets of* *the map f_k*

$$\Lambda_k^{(0)} \subseteq \Lambda_k^{(1)} \subseteq \Lambda_k^{(2)} \subseteq \ldots \subseteq \Lambda_k^{(n)} \subseteq \ldots \tag{7}$$

that has the following properties:

1. *The set $\Lambda_k^{(0)} = \Lambda_k$, where the family of sets $\{\Lambda_k\}$ is described in Theorem A;*

2. *Hausdorff dimension $\mathrm{dim}_H \Lambda_k^{(n)} \to 2$ as $n \to \infty$;*

3. *$\Omega_k = \overline{\cup_{n\in\mathbb{N}} \Lambda_k^{(n)}}$ is a transitive invariant set of the map f_k, and $\mathrm{dim}_H \Omega_k = 2$;*

4. *for any $x \in \Omega_k, k \in \mathcal{R}$, and any $\varepsilon > 0$ Hausdorff dimension*

$$\mathrm{dim}_H B_\varepsilon(x) \cap \Omega_k = \mathrm{dim}_H \Omega_k = 2,$$

where $B_\varepsilon(x)$ is an open ball of radius ε centered at x;

5. *Each point of Ω_k is an accumulation point of elliptic islands of the map f_k.*

For an open set of parameters our construction provides invariant hyperbolic sets of Hausdorff dimension arbitrarily close to 2.

Theorem 4. *There exists $k_0 > 0$ such that for any $\delta > 0$ there exists an open and dense subset $U \in [k_0, +\infty)$ such that for every $k \in U$ the map f_k has an invariant hyperbolic set of Hausdorff dimension greater than $2 - \delta$.*

Notice that these results give a partial explanation of the difficulties that we encounter studying the standard family. Indeed, one of the possible approaches is to consider an invariant hyperbolic set in the stochastic layer and to try to extend the hyperbolic behavior to a larger part of the phase space through homoclinic bifurcations. Unavoidably Newhouse domains (see [N3], [R] for dissipative case, and [Du1], [Du2], [Du4] for the conservative case) associated with absence of hyperbolicity appear after small change of the parameter. If the Hausdorff dimension of the initial hyperbolic set is less than one, then the measure of the set of parameters that correspond to Newhouse domains is small and has zero density at the critical value, see [NP], [PT1]. For the case when the Hausdorff dimension of the hyperbolic set is slightly bigger than one, similar result was recently obtained by Palis and Yoccoz [PY], and the proof is astonishingly involved. They also conjectured that analogous property holds for an initial hyperbolic set of any Hausdorff dimension, but the proof would require even more technical and complicated considerations. Here is what Palis and Yoccoz [PY] wrote:

"Of course, we expect the same to be true for all cases $0 < \dim_H(\Lambda) < 2$. For that, it seems to us that our methods need to be considerably sharpened: we have to study deeper the dynamical recurrence of points near tangencies of higher order (cubic, quartic, ...) between stable and unstable curves. We also hope that the ideas introduced in the present paper might be useful in broader contexts. In the horizon lies the famous question whether for the standard family of area preserving maps one can find sets of positive Lebesgue probability in parameter space such that the corresponding maps display non-zero Lyapunov exponents in sets of positive Lebesgue probability in phase space."

Theorems 3 and 4 show that in order to understand the dynamics of the stochastic layer of the standard map one has to face these difficulties.

Proof of Theorem 3 and Theorem 4. First of all we reduce Theorem 3 to the following proposition. Denote by $\mathcal{N}(N) = (n_1, \ldots, n_N)$ an N-tuple with $n_i \in \mathbb{N}$.

Proposition 2. *There exists $k_0 > 0$ such that for each $N \in \mathbb{N}$ there is a family of finite open intervals $\mathcal{U}_{\mathcal{N}(N)} \subseteq [k_0, +\infty)$ indexed by N-tuples $\mathcal{N}(N) = (n_1, \ldots, n_N)$ satisfying the following properties:*

- **U1)** *For pair of tuples $\mathcal{N}(N) \neq \mathcal{N}'(N)$ intervals $\mathcal{U}_{\mathcal{N}(N)}$ and $\mathcal{U}_{\mathcal{N}'(N)}$ are disjoint.*
- **U2)** *For any tuple $\mathcal{N}(N+1) = (\mathcal{N}(N), n_{N+1})$ we have $\mathcal{U}_{\mathcal{N}(N+1)} \subseteq \mathcal{U}_{\mathcal{N}(N)}$.*
- **U3)** *The union $\cup_{n_1 \in} \mathcal{U}_{n_1}$ is dense in $[k_0, +\infty)$, and for each $N \in \mathbb{N}$ the union $\cup_{j \in \mathbb{N}} \mathcal{U}_{(\mathcal{N}(N), j)}$ is dense in $\mathcal{U}_{\mathcal{N}(N)}$.*
- **U4)** *Every diffeomorphism f_k, $k \in \mathcal{U}_{\mathcal{N}(N)}$, has a sequence of invariant basic sets*

  $$\Lambda_k^{(n_1)} \subseteq \Lambda_k^{(n_1, n_2)} \subseteq \ldots \subseteq \Lambda_k^{\mathcal{N}(N)},$$

  and $\Lambda_k^{\mathcal{N}(N)}$ depends continuously on $k \in \mathcal{U}_{\mathcal{N}(N)}$.
- **U5)** *$\Lambda_k \subseteq \Lambda_k^{(n_1)}$ for each $n_1 \in \mathbb{N}$ and $k \in \mathcal{U}_{n_1}$, where Λ_k is a hyperbolic set from Theorem A.*

U6) $\dim_H \Lambda_k^{\mathcal{N}(N)} > 2 - 1/N$.

U7) *For any point* $x \in \Lambda_k^{\mathcal{N}(N)}$ *there exists an elliptic periodic point* p_x *of* f_k *such that* $dist(p_x, x) < 1/N$.

Theorems 3 and 4 follow from Proposition 2. Indeed, set $\mathbf{U}_N = \cup_{\mathcal{N}(N)} \mathcal{U}_{\mathcal{N}(N)}$. Due to **U3)** the set $\mathbf{U}_N$ is dense in $[k_0, +\infty)$. Therefore $\mathcal{R} = \cap_{N \in \mathbb{N}} \mathbf{U}_N$ is a residual subset of $[k_0, +\infty)$. Properties **U1)** and **U2)** imply that for each $k \in \mathcal{R}$ the value k belongs to each element of the uniquely defined nested sequence of intervals

$$\mathcal{U}_{n_1} \supseteq \mathcal{U}_{n_1,n_2} \supseteq \ldots \supseteq \mathcal{U}_{\mathcal{N}(N)} \supseteq \ldots$$

Therefore for $k \in \mathcal{R}$ the sequence of basic sets

$$\Lambda_k \subseteq \Lambda_k^{(n_1)} \subseteq \Lambda_k^{(n_1,n_2)} \subseteq \ldots \subseteq \Lambda_k^{\mathcal{N}(N)} \subseteq \ldots$$

is defined such that Hausdorff dimension $\dim_H \Lambda_k^{\mathcal{N}(N)} > 2 - 1/N$. Since k is fixed now, redenote $\Lambda_k^N = \Lambda_k^{\mathcal{N}(N)}$. Items 1. and 2. of Theorem 3 follows from **U5)** and **U6)**.

The closure of the union of a nested sequence of transitive sets is transitive, so property 3. follows.

For a locally maximal transitive invariant set of a surface diffeomorphism the Hausdorff dimension of the set is equal to the Hausdorff dimension of any open subset of this set, see [MM]. This implies the property 4. for the sets $\Omega_k, k \in \mathcal{R}$.

Finally, property 5. follows directly from **U7)**.

In order to prove Theorem 4 one just need to consider the family of basic sets $\Lambda_k^{\mathcal{N}(N)}$ defined for $k \in \mathbf{U}_N$ for large enough N. Then Proposition 2 itself can be reduced to the following

Lemma 3. *Given* $k^* \in (k_0, +\infty)$, $\varepsilon > 0$ *and* $\delta > 0$, *there exists a finite open interval* $V \subset (k^* - \varepsilon, k^*)$ *such that for all* $k \in V$ *the map* f_k *has a basic set* Λ_k^* *such that*

1) Λ_k^* *depends continuously on* $k \in V$;

2) $\Lambda_k^* \supseteq \Lambda_k$, *where* Λ_k *is a basic set from theorem A;*

3) Hausdorff dimension $\dim_H \Lambda_k^* > 2 - \delta$, *4) For any point $x \in \Lambda_k^*$ there exists an elliptic periodic point p_x of f_k such that $dist(p_x, x) < \delta$.*

Indeed, let us show how to construct the intervals $\mathcal{U}_{n_1}$ and the sets $\Lambda_k^{(n_1)}$. Let $\{k_l\}_{l\in\mathbb{N}}$ be a dense set of points in $(k_0, +\infty)$. Apply Lemma 3 to each $k^* = k_l$, $l \in \mathbb{N}$, for $\delta = \delta_1$, $\varepsilon = \varepsilon_l < \frac{1}{l}$. That gives a sequence of open intervals $\{V_l\}_{l\in\mathbb{N}}$. Since the sequence $\{k_l\}_{l\in\mathbb{N}}$ is dense in $(k_0, +\infty)$ and $\varepsilon_l \to 0$, intervals $\{V_l\}$ are dense in $(k_0, +\infty)$.

Take $\mathcal{U}_1 = V_1$. If $\mathcal{U}_1, \ldots, \mathcal{U}_t$ are constructed, take V_s – the first interval in the sequence $\{V_l\}_{l\in\mathbb{N}}$ that is not contained in $\overline{\cup_{n_1=1}^t \mathcal{U}_{n_1}}$. Then $V_s \backslash \overline{\cup_{n_1=1}^t \mathcal{U}_{n_1}}$ is a finite union of K open intervals. Take those intervals as $\mathcal{U}_{t+1}, \ldots, \mathcal{U}_{t+K}$, and continue in the same way. This gives a sequence of a disjoint intervals $\{\mathcal{U}_{n_1}\}_{n_1\in\mathbb{N}}$ with desired properties.

Now, assume that intervals $\{\mathcal{U}_{\mathcal{N}(N)}\}$ are constructed. Take one of the intervals $\mathcal{U}_{\mathcal{N}(N)}$. The set $\Lambda_k^{\mathcal{N}(N)}$ exhibits persistent tangencies, as the following result by Duarte claims:

Theorem C (Duarte, [Du3]). *There exists $k_0 > 0$ such that given any $k \geq k_0$ and any periodic point $P \in \Lambda_k$, the set of parameters $k' \geq k$ at which the invariant manifolds $W^s(P(k'))$ and $W^u(P(k'))$ generically unfold a quadratic tangency is dense in $[k, +\infty)$.*

Recall that $P(k')$ denotes the continuation of the periodic saddle P at parameter k'.

Therefore, application of Theorem 2 gives a dense sequence of intervals $\{V_{\mathcal{N}(N),l}\}_{l\in\mathbb{N}}$ in $\mathcal{U}_{\mathcal{N}(N)}$ such that for each $k \in V_{\mathcal{N}(N),l}$ the map f_k has a basic set Δ_k such that Hausdorff dimension $\dim_H \Delta_k > 2 - \dfrac{1}{N+1}$ and $\Delta_k \cap \Lambda_k^{\mathcal{N}(N)} \neq \emptyset$.

The following lemma is a standard statement from hyperbolic dynamics.

Lemma 4. *Let Δ_1 and Δ_2 be two basic sets of a diffeomorphism $f : M^2 \to M^2$ of a surface M^2. Suppose that $\Delta_1 \cap \Delta_2 \neq \emptyset$. Then there is a basic set $\Delta_3 \subseteq M^2$ such that $\Delta_1 \cup \Delta_2 \subseteq \Delta_3$.*

Apply Lemma 4 to Δ_k and $\Lambda_k^{\mathcal{N}(N)}$, and denote by $\widetilde{\Lambda}_k^{\mathcal{N}(N)} \supset \Delta_k \cup \Lambda_k^{\mathcal{N}(N)}$ the corresponding basic set. The set $\widetilde{\Lambda}_k^{\mathcal{N}(N)}$ also has persistent tangencies. The unfolding of a homoclinic tangency creates elliptic periodic orbits which shadow the orbit of homoclinic tangencies. The creation of these generic elliptic points can be seen from the renormalization at conservative homoclinic tangencies, see [MR]. Shrinking $V_{(\mathcal{N}(N),l)}$ if necessary we can guarantee that $\widetilde{\Lambda}_k^{\mathcal{N}(N)}$ can be δ_{N+1}-accumulated by elliptic periodic points. Now the same procedure that we applied above to intervals $\{V_l\}$ gives a collection of disjoint intervals $\{\mathcal{U}_{(\mathcal{N}(N),n_{N+1})}\}_{n_{N+1}\in\mathbb{N}}$ in $\mathcal{U}_{\mathcal{N}(N)}$. For any $k \in \mathcal{U}_{\mathcal{N}(N),n_{N+1}} \subset V_{(\mathcal{N}(N),l)}$ we take $\Lambda_k^{(\mathcal{N}(N),n_{N+1})} = \widetilde{\Lambda}_k^{\mathcal{N}(N)}$. Now all the properties in Proposition 2 are satisfied.

Finally, Lemma 3 follows directly from Theorem 2 and Theorem C. $\square$

4 Hyperbolic sets of large Hausdorff dimension in the three body problems

Initially our interest in the conservative Newhouse phenomena was motivated by the fact that it appears in the three body problem. The classical three–body problem consists in studying the dynamics of 3 point masses in the plane or in the three-dimensional space mutually attracted under Newton gravitation. The three–body problem is called *restricted* if one of the bodies has mass zero and the other two are strictly positive. In the pioneering work [A] Alexeev found important use of hyperbolic dynamics for the three–body problem. He proved existence of the so called *oscillatory motions*. A motion of the three–body problem is called *oscillatory* if the limsup of the mutual distances is infinite and the liminf is finite. Existence of such motions was a long standing open problem. The first rigorous example of existence of such motions is due to Sitnikov [Si] for the restricted spacial three–body problem. Alexeev extended the Sitnikov example to the spatial three–body problem. Later Moser [M] gave a conceptually transparent proof of existence of oscillatory motions for the Sitnikov example interpreting homoclinic intersections. This paved a road to a variety of applications of hyperbolic dynamics to the three–body problem.

In this section we discuss the size of compact invariant hyperbolic sets in the Sitnikov example and the restricted planar circular three–body problem and show that these sets often has almost full Hausdorff dimension.

4.1 The Sitnikov example

Consider two point masses q_1 and q_2 of equal mass $m_1 = m_2 = 1/2$. Suppose they move on the plane so that the center of mass is at the origin. Assume that their orbits are elliptic of eccentricity $e > 0$ and period 2π. We shall treat e as parameter. Consider a third massless point q_3 moving along the z-axis. Due to symmetry if an initial condition and velocity belong to the z-axis, then the whole orbit of q_3 also belongs to the z-axis. Denote by $(t, z(t), \dot{z}(t))$ an orbit of q_3, where the time t (mod 2π) determines position of primaries. Denote $r(t) = r_e(t)$ distance of primaries to the origin. Then the equation of motion of the massless body has the form

$$\ddot{z} = -\frac{z}{\sqrt{z^2 + r^2(t)}} \tag{8}$$

and the corresponding Hamiltonian is time-periodic

$$H(z, Z, t) = \frac{Z^2}{2} - \frac{1}{\sqrt{z^2 + r^2(t)}},$$

where Z is the variable conjugate to z and coincides with velocity of z.

Theorem 5. *There is an open set $\mathcal{N} \subset (0,1)$ of values of eccentricity e and a residual subset $\mathcal{R} \subset \mathcal{N}$ such that for $e \in \mathcal{R}$ there are compact invariant hyperbolic sets of Hausdorff dimension arbitrary close to* 3*.*

4.2 The restricted planar circular three–body problem (RPC3BP)

Consider the restricted planar circular three–body problem. Namely, consider two massive bodies, called the *primaries*, performing uniform circular motion about their center of mass. Normalizing the masses of the primaries so that their masses sum to one, we obtain primaries of mass μ and $1-\mu$ respectively, where $0 < \mu < 1$ is called the *mass ratio*. In addition, we chose coordinates so that the center of mass of the system is located at the origin, and we normalize the period of the circular motion to 2π. By entering into a frame which rotates with the primaries, we can choose rectangular coordinates (x, y) so that the primaries are fixed at $(1-\mu, 0)$ and $(-\mu, 0)$, respectively. Finally, we introduce a third massless body P into the system, so that it does not effect the primaries. RPC3BP investigates how P moves.

The distance of P to the primaries is given by $d_1(x,y) = [(x-(1-\mu))^2 + y^2]^{1/2}$ and $d_2(x,y) = [(x-\mu)^2 + y^2]^{1/2}$. The standard formula for the *Jacobi constant* C, the only integral for RPC3BP, is given by

$$C_\mu(x,y,\dot{x},\dot{y}) = x^2 + y^2 + \frac{2\mu}{d_1} + \frac{2(1-\mu)}{d_2} - (\dot{x}^2 + \dot{y}^2). \tag{9}$$

Denote by RPC3BP(μ, C) the RPC3BP with mass ratio μ restricted to the energy surface $\Pi_C = \{C_\mu(x,y,\dot{x},\dot{y}) = C\}$. We shall treat both μ and C as parameters. Below we shall consider $C > 2\sqrt{2}$. Consider the set

$$\{(x,y):\ (x^2+y^2) + \frac{2\mu}{d_1} + \frac{2(1-\mu)}{d_2} \geq C\}.$$

Notice that this set defines the set of possible positions of P provides that its initial condition is on the energy surface Π_C. One could show that for $C > 2\sqrt{2}$ this set consists of three disjoint regions, called *Hill regions*: one surrounds the primary with mass $1-\mu$, another one, which is smaller, surrounds the other primary, and the last one occupies a complement to an open set which covers both primaries. The first one is called *the inner Hill region*, the second is *lunar Hill region*, and the last one is *outer Hill region*. Below we shall study only the outer Hill region.

Here is the main result for the RPC3BP.

Theorem 6. *A) For any $C > 2\sqrt{2}$ there is an open set of mass ratios $\mathcal{N}_C \subset (0,1)$ such that for a residual subset $\mathcal{R} \subset \mathcal{N}$ and for any $(\mu, C) \in \mathcal{R}$ in the three-dimensional energy surface Π_C there are compact invariant hyperbolic sets of RPC3BP(μ, C) of Hausdorff dimension arbitrary close to* 3*;*

B) For any $\mu \in (0,1)$ there is an open set $\mathcal{N}_\mu \subset (2\sqrt{2}, \infty)$ such that for a Baire generic $C \in \mathcal{N}_\mu$ in the three-dimensional energy surface C there are compact invariant hyperbolic sets of RPC3BP(μ, C) of Hausdorff dimension arbitrary close to 3.

Remark 1. *The minimal distance to the origin for a bounded orbit of the* 2*–body problem in terms of the Jacobi constant can be arbitrarily close to $C^2/8$. Therefore, for $C = 2\sqrt{2}$ such an orbit might pass nearly at unit distance to the origin. This might lead to a near collision with the primary of mass μ. We want to avoid that.*

Our technique could also be applied to the three–body problem on the line [LS1], [SX], but we do not elaborate on it here.

4.3 Reduction to area-preserving maps

A natural way to reduce the Sitnikov example to a 2-dimensional Poincare map is as follows. Define

$$f_e : (z, \dot{z}) \mapsto (z', \dot{z}') \qquad (z, \dot{z}) \in \mathbb{R}^2 \tag{10}$$

where a trajectory of (8) with initial condition $(0, z, \dot{z})$ at time 2π is located at $(2\pi, z', \dot{z}')$. Since equations of motion are Hamiltonian this map is *area-preserving.*

There are many way to define a Poincare map for the RPC3BP(μ, C) with $C \geq 2\sqrt{2}$. Let's pick one. Consider the polar coordinates (r, φ) on the (x, y)-plane and let (P_r, P_φ) be their symplectic conjugate. Write the Hamiltonian of the RPC3BP in these coordinates:

$$H(r, P_r, \varphi, P_\varphi) = \frac{P_r^2}{2} + \frac{P_\varphi^2}{2r^2} - \frac{1}{r} - P_\varphi + \left(\frac{1}{r} - \frac{\mu}{d_1} - \frac{1-\mu}{d_2}\right) =: H_0 + \Delta H,$$

where d_1 and d_2 are the distances to the primaries as above (9), P_r (resp. P_φ) is the variable conjugate to r (resp. φ). In other words, $P_r = \dot{r}$ and P_φ is the angular momentum. One can rewrite the Jacobi constant in the polar coordinates.

Since the Jacobi constant is the first integral of this problem, there is a three-dimensional 'energy' surface $\Pi_C = \{C = C_\mu(r, \varphi, P_r, P_\varphi)\}$. It turns out that for $C > 2\sqrt{2}$ in the outer Hill region by the implicit function one can express $P_\varphi = P_\varphi(r, \varphi, P_r, C)$ on Π_C and consider a three-dimensional differential equation on (r, φ, P_r). On a "large" open set $\dot{\varphi} = 1 - P_\varphi/r^2 > 0$ and $\varphi(t)$ is strictly monotone. Choose a 2-dimensional surface $S = \{\varphi = 0\} \subset \Pi_C$ and a Poincare return map

$$f_{\mu,C} : (r, P_r) \mapsto (r', P_r'), \tag{11}$$

where a trajectory of the RPC3BP with initial condition $(r, 0, P_r, P_\varphi(r, 0, P_r, C))$ passes through $(r', 2\pi, P_r', P_\varphi(r', 2\pi, P_r', C))$. This gives rise to *an area-preserving map* $f_{\mu,C} : U \to \mathbb{R}^2$ defined on an open set $U \subset \mathbb{R}^2$.

4.4 Newhouse domains in the three body problems

Recall that a saddle periodic point p of an area-preserving map f exhibits an homoclinic tangency if stable and unstable manifolds $W^s(p)$ and $W^u(p)$ of p respectively have a point of tangency. We say that f has an homoclinic tangency if some of its saddle points has an homoclinic tangency. Call an open set with a dense subset of maps with an homoclinic tangency a *Newhouse domain.*

Theorem 7. *Let $\{f_e\}_{0<e<1}$ be the family of maps (10). Then there is a Newhouse domain $\mathcal{N} \subset (0,1)$, i.e. for a dense set of e in $\mathcal{N}$ the Poincare map f_e has an homoclinic tangency.*

Theorem 8. *Let $\{f_{\mu,C}\}$ be the family of maps (11). Then*

A) for any $C > 2\sqrt{2}$ there is a Newhouse domain $\mathcal{N}_C \subset (0,1)$, i.e. for a dense set of μ in $\mathcal{N}_C$ the Poincare map $f_{\mu,C}$ has an homoclinic tangency.

B) for any $\mu \in (0,1)$ there is a Newhouse domain $\mathcal{N}_\mu \subset (2\sqrt{2},+\infty)$, i.e. for a dense set of C in $\mathcal{N}_\mu$ the Poincare map $f_{\mu,C}$ has an homoclinic tangency.

Robinson [R], using ideas of Newhouse, showed that for a generic 1-parameter unfolding a homoclinic tangency there are Newhouse domains on the parameter line. In a sense we prove a similar statement for the two concrete conservative systems. Namely, we show that the above 1-parameter families are non-degenerate and Newhouse domains occur on the parameter line not in infinite dimensional space of mappings. The proofs of these two theorems are based on similar results for area-preserving Henon maps, see [Du4] or Theorem 1 above.

4.5 Plan of the proof of Theorem 5 and Theorem 7.

Proofs of both parts of Theorem 6 and Theorem 8 follow very similar strategy burdened by more involved technical details.

In what follows the following motions play a special role.

Definition 2. *A motion of the massless body is called future (resp. past) parabolic if the body escapes to infinity with vanishing speed as time tends to $+\infty$ (resp. $-\infty$).*

Plan of the proof of Theorem 5 and Theorem 7.

Recall that $f_e : (z,\dot{z}) \mapsto (z',\dot{z}')$ is the Poincare map (10).

The change of coordinates $(z, \dot{z}) \to (u = z^{-1/2}, v = \dot{z})$ sends $(z = \infty,\ \dot{z} = 0)$ to the origin and the origin $(u, v) = 0$ becomes a degenerate saddle fixed point. McGehee [McG] showed that its separatrices are smooth manifolds, denoted $W^s_e(0)$ and $W^u_e(0)$, correspond to parabolic motions. For $e = 0$ the manifolds coincide $W^s_0(0) = W^u_0(0)$ and form a separatrix loop given by $\frac{v^2}{2} - \frac{2u^2}{\sqrt{4+u^2}} = 0$. It turns out that for small positive e separatrices $W^s_e(0)$ and $W^u_e(0)$ intersect transversally. It follows from nondegeneracy of the Melnikov function proved by Moser [Mo]. Explicit form was calculated in [GP] (see a related paper Dankowicz–Holmes [DH]). Similar statement for the RPC3BP was proved by Llibre-Simo [LS1] and later using a different method by Xia ([X], sect. 3).

Step 1. Invariant cone field near degenerate saddle. Similarly to the results in [Mo] one can show that there exists an invariant cone filed in a neighborhood of the degenerate saddle. Moreover, differential of the transition from a point near $W^s_e(0)$ to a point near $W^u_e(0)$ through that neighborhood expands vectors from the invariant cones.

Step 2. Hyperbolic periodic saddles near parabolic motions. Similarly to the classical Poincare-Birkhoff Theorem, the cone condition in a neighborhood of the degenerate saddle and existence of transversal homoclinic points imply existence of hyperbolic saddle periodic points $\{p^m_e\}$ that accumulates to the homoclinic point. Compact parts of stable and unstable manifolds of $\{p^m_e\}$ are C^1 close to the corresponding pieces of $W^s_e(0)$ and $W^u_e(0)$.

Step 3. Appearance of a homoclinic tangency. Using the splitting of separatrices one can show that for there is a sequence of e_k monotonically decreasing to zero such that there is a quadratic tangency of $W^s_{e_k}(0)$ and $W^u_{e_k}(0)$. Moreover, unfolding of this tangency in e is nondegenerate. Since $W^s_e(p^m_{e_k})$ and $W^u_e(p^m_{e_k})$ are close to $W^s_{e_k}(0)$ and $W^u_{e_k}(0)$, there is also a sequence e'_k such that $W^s_e(p^m_{e'_k})$ and $W^u_e(p^m_{e'_k})$ have a quadratic tangency, and unfolding of this tangency in e is also nondegenerate.

Step 4. Generic unfolding of a homoclinic tangency and appearance of hyperbolic sets of large Hausdorff dimension. Application of Theorem 2 to the nondegenerate unfolding of a quadratic tangency between $W^s_e(p^m_{e'_k})$ and $W^u_e(p^m_{e'_k})$ immediately proves both Theorem 5 and Theorem 7. □

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