Arnol′d Diffusion in a Pendulum Lattice

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Abstract

The main model studied in this paper is a lattice of pendula with a nearest-neighbor coupling. If the coupling is weak, then the system is near-integrable and KAM tori fill most of the phase space. For all KAM trajectories the energy of each pendulum stays within a narrow band for all time. Still, we show that for an arbitrarily weak coupling of a certain localized type, the neighboring pendula can exchange energy. In fact, the energy can be transferred between the pendula in any prescribed way. © 2013 Wiley Periodicals, Inc.

1 Description of the Motion

We consider a system of pendula with a nearest-neighbor coupling:

$$\ddot{x}_i + \sin x_i = -\varepsilon \frac{\partial}{\partial x_i} \beta(x_{i-1}, x_i, x_{i+1}, \varepsilon), \quad i \in \mathbb{Z}, \tag{1.1}$$

where the interaction potential β is localized and will be defined later. This system can be written in the Hamiltonian form with $\mathbf{x} = \{x_i\}_{i \in \mathbb{Z}}$, $\mathbf{y} = \{y_i\}_{i \in \mathbb{Z}}$, x_i and $y_i \in \mathbb{R}$, with the Hamiltonian

$$\begin{aligned} H_\varepsilon(\mathbf{x}, \mathbf{y}) &= \sum_{i \in \mathbb{Z}} \frac{y_i^2}{2} + (-\cos x_i - 1) + \varepsilon \beta(x_{i-1}, x_i, x_{i+1}, \varepsilon) \\ &= \sum_{i \in \mathbb{Z}} \frac{y_i^2}{2} + V(x_i) + \varepsilon \beta(x_{i-1}, x_i, x_{i+1}, \varepsilon), \end{aligned} \tag{1.2}$$

where $V(x) = -\cos x - 1$ is the pendulum potential. For $\varepsilon = 0$ and each integer i the (x_i, y_i)-component forms a pendulum, whose phase portrait is on Figure 1.1.

The system is near-integrable for small ε, and most (in the sense of measure) of the system's phase space is taken up by invariant KAM tori. In particular, for most initial data the energy of each pendulum will stay close to its initial value

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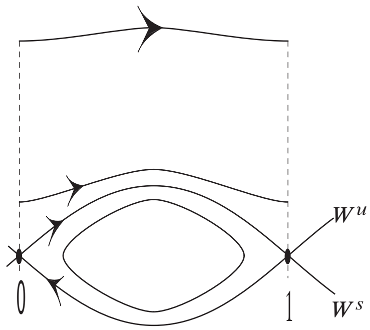

FIGURE 1.1. The "running" and the near-heteroclinic motion are the building blocks of the dynamics.

for all time. Nevertheless, we will show that for some motions the energy can slowly "seep" from one pendulum to another. We will in fact prove that for an arbitrarily small ε and for any sequence of integers $\sigma = (\dots, \sigma_{-1}, \sigma_0, \sigma_1, \dots)$ such that $\sigma_0 = 0$, $|\sigma_j - \sigma_{j+1}| = 1$, for all $j \in \mathbb{Z}$, there exists a sequence of times $(\dots, t_{-1}, t_0, t_1, \dots)$ (depending on ε) such that at time t_j the ${\sigma_j}^{\text{th}}$ pendulum has most of the system's energy. In particular, one can make the energy wander along the chain of the pendula in any prescribed fashion, advancing to the right by any number of steps, retreating to the left by any number of steps, and so on.

From now on we fix the energy of the system to be 1.[1] Below we shall concentrate on the case of a periodic collection of four pendula, i.e., of the index $i \pmod 4$. In our notation, the indices will be denoted $i = 1, 2, 3, 4$, and $5 \equiv 1 \pmod 4$. The proof in the general periodic case $i \in \mathbb{Z}/p\mathbb{Z}$ is quite similar, and necessary remarks are made along the proof.

We note as a side remark that the space discretization of the sine-Gordon equation $u_{tt} - u_{ss} = \sin u$ results in a system of pendula with elastic coupling [6,24,25]:

$$\beta(x_{j-1}, x_j, x_{j+1}) = a(x_{j-1} - 2x_j + x_{j+1}); \tag{1.3}$$

this corresponds to an elastic torsional coupling between the neighbors. In particular, as the angle $x_{j+1} - x_j \to \infty$ we have $a \to \infty$. By contrast, the coupling we consider in this paper can be interpreted as coming from a spring connecting points on a circle with angular coordinates x_j, as shown in Figure 1.2.

The coupling in our system is, however, localized, as described below. The class of coupling functions β for which our results hold is defined as follows: Let $\eta : \mathbb{R}_+ \to \mathbb{R}$ be a C^∞ bump function: $\eta(x) > 0$ for $|x| < 1$ and $\eta(x) = 0$ for $|x| \geq 1$.

[1] Energy 0 corresponds to all pendula "hanging upside down" and at rest. Indeed, the maximum of the potential energy $V(x) = -\cos x - 1$ of an individual pendulum is 0 and is achieved at $x = \pi$, an upside-down position.

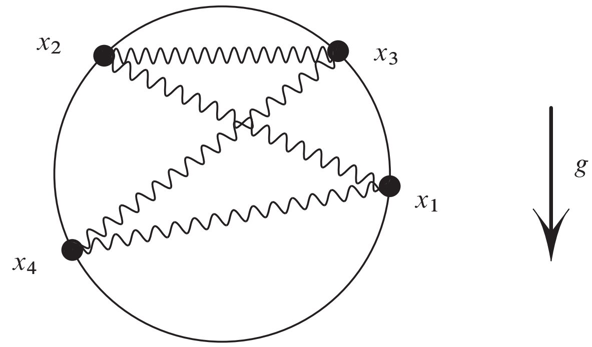

FIGURE 1.2. A mechanical interpretation of (1.3) with the coupling $\beta(x_{i-1}, x_i, x_{i+1}) = \sin(x_{j-1} - 2x_j + x_{j+1})$.

The exact form of η is not important, and in particular, no monotonicity properties are assumed. We can allow η to have many local maxima and minima, as long as the above conditions hold. From now on, fix $r \geq 3$. We define

$$\beta(t, \varepsilon) = \varepsilon^r \sum_{n \in \mathbb{Z}^3} \eta\left(\frac{|t - 2\pi n|}{\varepsilon}\right), \quad t = (t_1, t_2, t_3) \in \mathbb{R}^3. \tag{1.4}$$

This is a C^∞-smooth 2π-periodic function in each t_j, $j = 1, 2, 3$, or, equivalently, a function on $2\pi(\mathbb{R}^3/\mathbb{Z}^3)$. Note that the C^r-norm of $\varepsilon\beta(\,\cdot\,, \varepsilon)$ tends to 0 as $\varepsilon \to 0$, while the norms of order $r + 2$ and higher are unbounded for $\varepsilon \to 0$.

Please note that β is a function of three variables, though the configuration space in the subsequent proofs will be $\mathbb{R}^4$. The independent variables will always be explicitly written, e.g., $\beta(x_{j-1}, x_j, x_{j+1})$.

We will sometimes speak about the "connected components of the support of $\beta(x_{j-1}, x_j, x_{j+1})$ in the configuration space $\mathbb{R}^4$," with a little abuse of notation. These connected components will be referred to as *lenses*. They have a form of cylinders given by the product of a three-dimensional ball and a line. The name reflects the fact that, dynamically, the supports of β act by defocusing geodesics in the Jacobi metric, as explained later (see also [19]).

According to the main theorem, stated next, the energy

$$E_j := \frac{\dot{x}_j^2}{2} + V(x_j) \tag{1.5}$$

at the j^{th} site can pass from one site to another in an arbitrarily prescribed sequence of steps, as illustrated in Figure 1.3.

A path in the graph $\mathbb{Z}$ means that we have a solution which along a strictly monotone subsequence of times T_j has most energy concentrated in a single pendulum whose index σ_j is the corresponding vertex of the graph. In between consecutive times $[T_j, T_{j+1}]$ most of the energy gradually passes from σ_j to σ_{j+1}. Here is a precise statement.

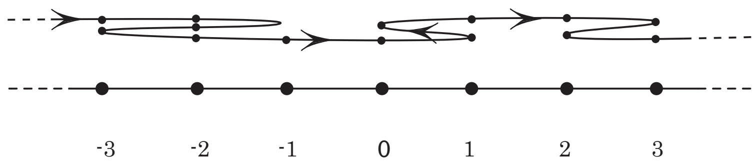

FIGURE 1.3. Any path in the graph $\mathbb{Z}$ can be shadowed by a solution of (1.1).

THEOREM 1.1. *Let us fix the total energy*[2] $E = 1$ *of system* (1.1) *with* β *satisfying* (1.4). *There exists* $\varepsilon_0 > 0$ *such that for any* $0 < \varepsilon < \varepsilon_0$ *and for any path* $\dots\sigma_{-1}\sigma_0\sigma_1\dots$ *in the graph* $\mathbb{Z}$ *there exists a solution of* (1.1) *and a sequence of times* $\dots t_{-1}t_0t_1\dots$ *such that the energies* (1.5) *of individual pendula satisfy*

$$|E_{\sigma_j}(t_j) - 1| < C\sqrt{\varepsilon} \quad \textit{and} \quad |E_\sigma(t_j)| < C\sqrt{\varepsilon} \quad \textit{for } \sigma \neq \sigma_j,$$

where C *is independent of* ε. *The times* t_j *can be chosen so that*

$$0 < t_{j+1} - t_j \leq C\varepsilon^{-4r-8}. \tag{1.6}$$

This theorem shows that, although system (1.1) is near-integrable, so that for most (in the sense of Liouville measure) solutions the action stays close to its initial value for all time, there exist solutions for which the action changes by $O(1)$ no matter how small ε is. In other words, the system exhibits Arnol′d diffusion. According to (1.6), the rate of this diffusion is polynomial. The bound in (1.6) is not sharp, but it can be improved by a more careful tracing of the estimates in our example. In the general case, polynomial upper bounds for the speed of diffusion for finitely differentiable systems have been obtained in [8].

The first example of Arnol′d diffusion was outlined in the well-known paper of Arnol′d [1]. Bessi [4] (see also [3]) proved diffusion in Arnol′d's example by a variational method, by considering the gradient flow of the Lagrangian action functional. John Mather [23] used a somewhat similar approach to construct accelerating orbits for time-periodic mechanical systems on a 2-torus (see also [7, 12, 15, 17]). References concerning the progress on Arnol′d diffusion go beyond the scope of this paper and can be found, e.g., in [18]. The most recent progress can be found in [10, 20], where Arnol′d diffusion for convex Hamiltonians of three degrees of freedom is discussed. In the present paper we use a slightly different version of this approach, based on using the Maupertuis principle. We construct the "diffusing" solutions as geodesics in a Jacobi metric so that all these solutions have a fixed prescribed energy. These geodesics are constructed by concatenating geodesic segments that follow a prescribed itinerary. The construction is fairly similar to [18, 19].

[2] In fact, any value strictly larger than the potential energy of an upside-down equilibrium works.

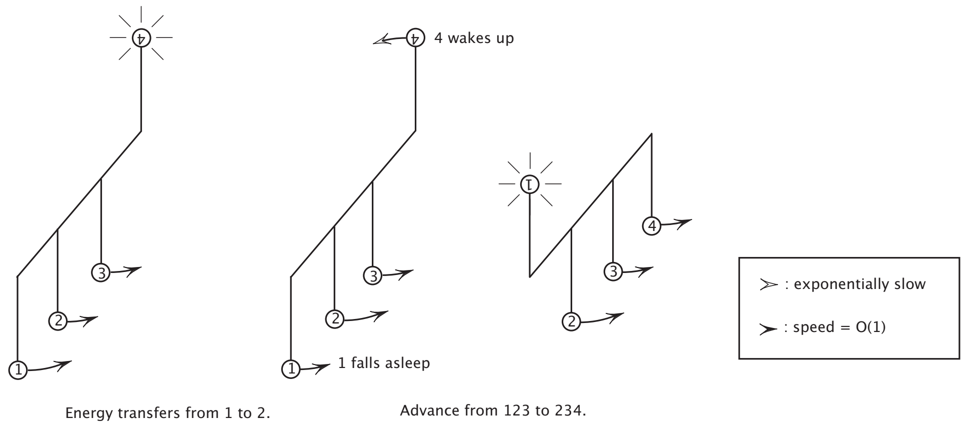

FIGURE 1.4. One full step in the propagation of the “kink”.

Anderson localization is an important example of energy (non)transfer (see [21] for a survey), which is still not very well understood. The role of Arnol′d diffusion for destruction of Anderson localization is discussed in [2]. Probably the most popularized lattice model is the one introduced by Fermi, Pasta, and Ulam (FPU) in their seminal paper [13]. Although most small-amplitude solutions in the FPU model do not exhibit energy transfer (see, e.g., [16]), proving the existence of solutions with energy transfer is an interesting open problem. Other physically significant lattice models are discussed in [14].

Understanding the transfer of energy for Hamiltonian PDEs is one of emerging directions of research (see [9]; recent progress for the cubic defocusing nonlinear Schrödinger equation has been made in [11]).

1.1 Heuristic Description of Energy Propagation

In this section we give a purely heuristic picture of the physical motions exhibiting Arnol′d diffusion. As mentioned earlier, we consider the periodic case $x_{i+4} = x_i$ as a representative example.

STAGE 1: TRANSFER OF ENERGY. At this stage only three pendula, **1**, **2**, and **3**, governed by (1.1) with $k = 1, 2, 3$ are “active,” while **4** “sleeps” upside down (see Figure 1.4, left). By the reasons to be seen in a moment we refer to **1** as the “giver” and to **2** as the “taker.” Pendulum **3** is the “facilitator” of the transfer, while its own energy stays small during the whole stage.

This stage consists of many substages illustrated by Figure 1.5, left. At each of these substages, a small amount of energy is transferred from **1** to **2**. This transfer is somewhat similar to the one described in [19] for a metric on the 3-torus. At the last of these substages, **1** is left with just enough energy to climb upside down and to fall asleep there, while **2** rotates with speed $O(1)$, as shown in the middle

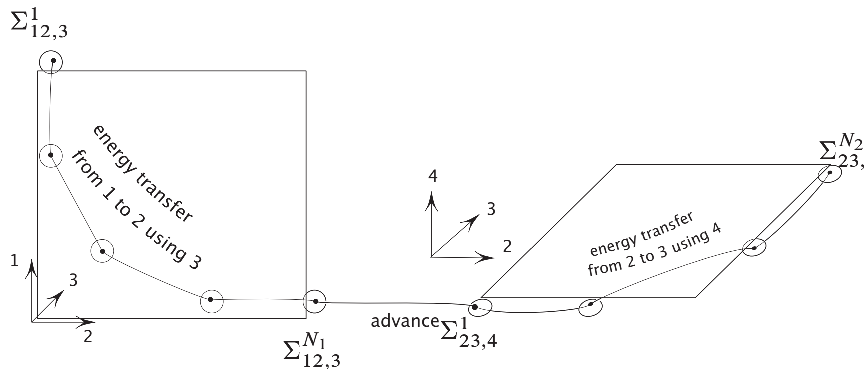

FIGURE 1.5. Energy transfer and sections in the configuration space $\mathbb{R}^4$.

of Figure 1.4. Figure 1.5 illustrates the same motion, viewed in the configuration space $\mathbb{R}^4$.

The motion just described is similar to the one studied in [18, 19] for a slightly simpler example.

STAGE 2: ADVANCE. This stage is sketched in Figure 1.4, from middle to the right, and Figure 1.5, middle. At $t = t_1$ three neighbors, say **1**, **2**, and **3**, are in the bottom position. The middle pendulum **2** is running: $\dot{x}_2 = O(1)$, while its two neighbors **1** and **3** have near-heteroclinic speeds close to the heteroclinic speeds $\sqrt{-2V(x_i)},\ i = 1, 3$, respectively, at $t = t_1$. The remaining pendulum **4** is upside-down (see Figure 1.4, middle). As the time goes on, while **2** is spinning with speed $O(1)$, **1** rises to the top equilibrium, where it will sleep until further notice, while the sleeper **4** "wakes up," i.e., falls from its perch, turning π at the exact moment when **2** finishes a large integer number of full spins. By that moment, x_3 makes a "gentle" turn by 2π, returning to the bottom position. In short, the accomplishment of this stage is the falling asleep of **1** and the awakening of **4**. The result is illustrated in Figure 1.4, right. We will call this stage the "advance" because of its similarity with the advancing caterpillar: a rear foot **1** is placed on the ground, while the front foot **4** is lifted, ready to move.

The ending moment of the second stage is the beginning moment of the first stage described above modulo the shift of the index by 1. We have, in other words, a "traveling wave"—a (very) discrete analogue of the kink in the sine-Gordon equation. However, in contrast to the standard traveling kink, ours can change the direction of its propagation arbitrarily, according to a prescribed itinerary.

2 Proof of Theorem 1.1

The full complexity of the problem is already seen in the case of four pendula, and we limit our consideration to this case. Now we restate the theorem in geometrical terms. The following is motivated by the heuristic outline of the energy transfer between the pendula: as mentioned above (see Figures 1.4 and 1.5), we want the energy to pass from one pendulum (e.g., **1**), which is called the "giver," to another (e.g., **2**), the "taker," in small increments over many steps. At this stage **3** is the "facilitator" of the transfer; its own energy stays small during this stage. Pendulum **4** at this stage is the "sleeper"—its trajectory stays close to the saddle fixed point (which corresponds to the pendulum hanging upside down). At the end of this stage almost the whole total energy is concentrated at **2**, and the energy of **1** is very small.

During the "advance" stage, the pendula change roles: **1** becomes the next "sleeper," **2** becomes the next "giver," **3** becomes the new "taker," and **4** becomes the new "facilitator."

This is reflected in the following geometrical construction. Consider the lift $\mathbb{R}^4$ of $\mathbb{T}^4$. In Section 2.1 we construct an *itinerary* for the desired orbit. This means that we choose a sequence of small three-dimensional sections (for the first stage, described above, they are called $\Sigma^j_{12,3}$), centered at the points of a certain lattice. The sections are defined in (2.2). The desired orbit of the system, providing the energy transfer, will be constructed to pass through these sections in the given order.

Moreover, the sections related to the deformation β are as follows: Recall that each connected component of the support of β is a (thin) cylinder in the configuration space. Each section intersects only one such cylinder. More about the form of the sections is in Section 2.1; see (2.2).

Recall that the system is integrable outside the support of β. Hence the velocity vector of a trajectory, connecting a pair of neighboring sections, is rather precisely defined by the angle between the centers of the sections. This velocity is related to the energy contained in different pendula.

The sections corresponding to the first stage, described above, will have centers with the same fourth component (this reflects the fact that pendulum **4** has velocity almost 0). The same holds for the component corresponding to the "sleeper" at each stage. Moreover, the angle between the centers of each pair of neighboring sections changes very slowly. This will be used in order to find a trajectory that passes through all the sections. These ideas are very close to those contained in [19].

The main contents of this paper deal with the "advance" part, when the pendula change their roles. This is analogous to passing a *strong double resonance*.

Later in this section we reformulate the problem of existence of an orbit passing through all the sections as a variational problem. Finally, we prove existence of a solution to this problem in Section 2.2. We use Lemmas 4.1 and 5.1, stated and proved in Sections 4 and 5, respectively.

2.1 Constructing an Itinerary

Without loss of generality, let us prove Theorem 1.1 in the case of *monotone energy transfer* (i.e., to the right neighbor): $\sigma_{j+1} = \sigma_j + 1 = j + 1$ for all $j \in \mathbb{Z}$.

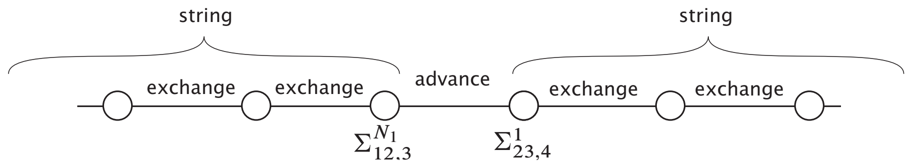

FIGURE 2.1. An itinerary.

Transferring the energy between the pendula in any other prescribed order poses no new difficulties. We thus consider an infinite sequence of codimension 1 sections in $\mathbb{R}^4$, grouped into finite strings, Figures 1.5 and 2.1:

$$\cdots \underbrace{(\Sigma^1_{12,3}, \Sigma^2_{12,3}, \dots, \Sigma^{N_1}_{12,3})}_{1\to 2} \underbrace{(\Sigma^1_{23,4}, \Sigma^2_{23,4}, \dots, \Sigma^{N_2}_{23,4})}_{2\to 3} \cdots, \tag{2.1}$$

where the sections and their spacings are defined according to the following rules.

(1) Section $\Sigma^1_{12,3}$, for example, is seen in Figure 1.5, left. The subscripts 12 indicate that **1** and **2** exchange energy, and **3** is the "facilitator":

$$\Sigma^1_{12,3} := \{x_3 = 0, \quad x_1^2 + x_2^2 \le \varepsilon,\ |x_4 - \pi| \le \sqrt{\varepsilon}\}. \tag{2.2}$$

We refer to the point $(0, 0, 0, \pi)$ as the *Center* of this section. Notice that the only connected component of the support of β that intersects $\Sigma^1_{12,3}$ is contained in the cylinder $\{x_1^2 + x_2^2 + x_3^2 \le 1\}$.

(2) All sections within each string in (2.1) are translates of each other by integer multiples of 2π. For example, in the first string $1 \to 2$:

$$\Sigma^{k+1}_{12,3} = \Sigma^k_{12,3} + 2\pi(m^k_{12,3}, n^k_{12,3}, 1, 0) =: \Sigma^k_{12,3} + \vec{n}^k_{12,3}, \quad k = 1, \dots, N_1.$$

The centers of the sections satisfy

$$\mathit{Center}(\Sigma^{k+1}_{12,3}) = \mathit{Center}(\Sigma^k_{12,3}) + \vec{n}^k_{12,3}, \quad k = 1, \dots, N_1.$$

Note that the fourth coordinate of the centers of the sections is kept constant for all the sections in this string.

To define the second string, we change the subindices in formula (2.2) in the following way: replace each subindex i by $i + 1$ modulo 4 (e.g., $\Sigma^1_{12,3}$ becomes $\Sigma^1_{23,4}$). This gives

$$\Sigma^1_{23,4} := \{x_4 = 0, \quad x_2^2 + x_3^2 \le \varepsilon,\ |x_1 - \pi| \le \sqrt{\varepsilon}\}. \tag{2.3}$$

We then define $\Sigma^k_{23,4}$ as a translate of $\Sigma^1_{23,4}$ by an integer multiple of 2π in coordinates (x_2, x_3, x_4). Namely, $\Sigma^{k+1}_{23,4} = \Sigma^k_{23,4} + \vec{n}^k_{23,4}$, where

$$\vec{n}^k_{23,4} := 2\pi(0, m^k_{23,4}, n^k_{23,4}, 1).$$

This corresponds to the fact that **1** is the "sleeper" at this stage, its velocity staying close to 0, so the first component of all the sections of this string is the same. Pendulum **4** is the "facilitator" at this stage, and pendula **2** and **3** exchange energy.

(3) The neighboring strings $(1 \to 2)$ and $(2 \to 3)$ (see Figure 2.1) are related via

$$\begin{aligned} Center(\Sigma^1_{23,4}) &= Center(\Sigma^{N_1}_{12,3}) + 2\pi\left(\frac{1}{2}, m^0_{23,4}, 1, \frac{1}{2}\right) \\ &:= Center(\Sigma^{N_1}_{12,3}) + \vec{n}^0_{23,4} \end{aligned}$$

for a suitable integer $m^0_{23,4}$.

(4) For solving the variational problem below, it will be important that the angle between the intervals connecting each pair of neighboring sections changes very slowly (this corresponds to the slow transfer of energy between the pendula). Since these centers are integers times 2π, we need to choose the vectors $\vec{n}^k_{\cdot\cdot,\cdot}$ very long: each

$$|\vec{n}^k_{\cdot\cdot,\cdot}| \geq \varepsilon^{-2r-4}. \tag{2.4}$$

(5) The turns are gradual in the sense that the unit vectors $e^k_{\cdot\cdot,\cdot} = \vec{n}^k_{\cdot\cdot,\cdot}/|\vec{n}^k_{\cdot\cdot,\cdot}|$ satisfy

$$|e^{k+1}_{\cdot\cdot,\cdot} - e^k_{\cdot\cdot,\cdot}| \leq \varepsilon^{2r+4}, \quad |e^1_{\cdot\cdot,\cdot+1} - e^{N_j}_{\cdot\cdot,\cdot}| \leq \varepsilon^{2r+4}. \tag{2.5}$$

2.2 A Variational Problem and Its Solution

We note that energy one solutions of (1.1) are geodesics in the Jacobi metric[3]

$$d\rho(\mathbf{x}) = \sqrt{1 - \sum_{i=1}^{4}\big(V(x_i) + \varepsilon\beta(x_{i-1}, x_i, x_{i+1}, \varepsilon)\big)}\, ds, \tag{2.6}$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4)$, $V(x) = -\cos x - 1 \leq 0$, ds is the euclidean metric, and indices of the x_i are taken mod 4 (thus, in our notation $i = 1, 2, 3, 4$).

With the sections defined in items 1–5 above, we now sketch the main steps of the proof of Theorem 1.1 and fill in the details in the following sections.

Step 1. *Defining geodesic segments.* Let Σ_0, Σ_1 be two consecutive sections in the chain of sections (2.1) and let $p_i \in \Sigma_i$, $i = 0, 1$. Centers of these sections differ by $2\pi\vec{n}$ with $\vec{n}$ being either $(m, n, 1, 0)$ or $(\frac{1}{2}, s, 1, \frac{1}{2})$ with $m, n, s \in \mathbb{Z}$ and satisfying (2.4). According to Lemma 5.1 from Section 5, there exists a connecting geodesic $\gamma(p_0, p_1)$ of (2.6) that depends smoothly on its ends p_0, p_1. At this stage the integer parameters, either m and n or s, are still free.

[3] Up to the factor $\sqrt{2}$, the square root above is the speed of the energy one solution in the configuration space.

Step 2. *Constructing a long shadowing geodesic.* Consider a finite segment of $N+2$ sections from sequence (2.1). To simplify the notation, we denote these sections by Σ_i, $0 \le i \le N+1$ (N here is arbitrarily large). We also choose arbitrary points $p_i \in \Sigma_i$. Later we will treat p_0, p_{N+1} as fixed and $p_1, \dots, p_N$ as variable. According to the preceding item, there exists a broken geodesic $\gamma(p_0, \dots, p_{N+1})$, a concatenation of energy one orbits $\gamma(p_i, p_{i+1})$ of (1.1). The length (in the Jacobi metric) of this broken geodesic,

$$L(p_1, \dots, p_N) = L(p_0, p_1) + \cdots + L(p_N, p_{N+1}), \tag{2.7}$$

is a function of the "break points" p_j, $j = 1, \dots, N$; we omit p_0 and p_{N+1} from the left-hand side since they will be considered as fixed.

We will show that L has a *minimum in the interior of its domain* $\Sigma_1 \times \cdots \times \Sigma_N$. Such an interior minimum corresponds to a true geodesic. We will thus establish the existence of a geodesic with a prescribed itinerary.

Step 3. *Existence of an interior minimum for* (2.7). In this key step, let us fix a j, $1 \le j \le N$, and consider the two consecutive terms from the sum (2.7) that contain p_j:

$$S(p_j) = L(p_{j-1}, p_j) + L(p_j, p_{j+1}), \quad p_k \in \Sigma_k,\ k = j-1, j, j+1. \tag{2.8}$$

where $p_{j\pm1} \in \Sigma_{j\pm1}$ are fixed and $p \in \Sigma_i$ is variable. To prove the existence of an interior minimum of (2.7) it suffices to show that for each j, the minimum of $S(p_j)$ is achieved in the interior of Σ_j. Without loss of generality we take $\Sigma_j = \Sigma^1_{12,3}$, given by (2.2).[4]

To simplify the notation, in the argument below we denote p_j by p, $p_{j\pm1}$ by $p_\pm$, Σ_j by Σ, and $\Sigma_{j\pm1}$ by $\Sigma_\pm$. Hence, we shall minimize $S(p) = L(p_-, p) + L(p, p_+)$.

In the remaining part of Section 2 we prove existence of the interior minimum for this problem (modulo certain technical lemmas). To begin with, we alter the Jacobi metric (2.6) by setting $\beta = 0$ *only* in the cylinder $\{x_1^2 + x_2^2 + x_3^2 < \varepsilon^2\}$ that passes through the center of Σ (we do not alter β anywhere else). The new Jacobi metric is defined by formula (2.6) with $\beta = 0$. Denote the corresponding geodesic distance between points p_0 and p_1 by $L^0(p_0, p_1)$.

Before restoring β to its original form, we study the associated length

$$S^0(p) := L^0(p_-, p) + L^0(p, p_+), \quad p \in \Sigma. \tag{2.9}$$

Once the properties of $S^0(p)$ are established (see (2.10) and (2.11) below), we will show that restoring β to its original form creates a minimum for $S(p)$ in the interior of Σ.

We have assumed above that $\Sigma = \Sigma^1_{12,3}$ as in (2.2), so $S^0(p)$ does not depend on the third component of p, which is 0. With a slight abuse of notation, we write $S^0(p) = S^0(x_1, x_2, x_4)$. This choice of the section does not produce any loss of

[4] For future reference, we note that the triple may or may not be entirely in one string in the sequence (2.1) of sections.

generality, since all the other sections have the same structure, up to a permutation of indices.

An important step in the proof is to show that $S^0(x_1, x_2, x_4)$ is nearly constant as a function of the first two variables, and has a minimum near the "equator" $x_4 = \pi$:

$$|S^0(x_1, x_2, x_4) - S^0(0, 0, x_4)| \leq c\varepsilon^{r+2.5}, \tag{2.10}$$

$$S^0(x_1, x_2, \pi \pm \sqrt{\varepsilon}) > S^0(x_1, x_2, \pi) + \frac{\varepsilon}{2}, \tag{2.11}$$

for any $(x_1, x_2, \pi) \in \Sigma$.

PROOF OF (2.10) AND (2.11). By lemma 1 from [19], for $p = (x_1, x_2, x_3, x_4)$ we get

$$\frac{\partial L^0(p_-, p)}{\partial x_i} = \dot{x}_i^-, \quad \frac{\partial L^0(p, p_+)}{\partial x_i} = -\dot{x}_i^+,$$

where $\mathbf{x}^-(t) = (x_1^-, x_2^-, x_3^-, x_4^-)$ is the energy one solution with the modified β, connecting p_- to p, and where the differentiation is taken at the moment the solution passes through p. This solution exists by Lemma 5.1 below. We use a similar notation $\mathbf{x}^+$ for the energy one solution connecting p with p_+. We thus conclude that

$$\frac{\partial S^0}{\partial x_i}(p) = \dot{x}_i^- - \dot{x}_i^+, \quad i = 1, 2, 4; \tag{2.12}$$

this identity[5] will allow us to analyze S^0. Now, due to the fact that the perturbation β near p is removed, the pendula are decoupled and the velocity is explicitly given in terms of energy distribution (1.5)

$$|\dot{x}_i| = \sqrt{2(E_i - V(x_i))},$$

where E_i is the energy of the i^{th} pendulum near p. Estimation of (2.12) is now reduced to studying the difference of velocities. We have to consider two cases: in one, Σ_-, Σ, and Σ_+ belong to the same string in (2.1) (the case of "energy transfer"), and in the other they do not (the "advance").

Case 1. *Energy Transfer*. In this case all sections lie in the same string in (2.1)—say, in $(1 \to 2)$. The displacements $2\pi\vec{n}_-$ and $2\pi\vec{n}_+$ are then of the form $2\pi(m_\pm, n_\pm, 1, 0)$ with integer $m_\pm$ and $n_\pm$. Assuming the integers to be positive (we can always assume them to be of the same sign), we have

$$\dot{x}_i^- > 0, \quad \dot{x}_i^+ > 0, \quad i = 1, 2, 3, \tag{2.13}$$

at the moment when Σ is crossed.

[5] We do not differentiate by x_3 since we only need to define S and S^0 on $\Sigma \subset \{x_3 = 0\}$.

The fact that the signs are the same for x^+ and x^- is of key importance because it provides a near cancellation in (2.12) for $i = 1, 2, 4$. Thus, we have

$$\frac{\partial S^0}{\partial x_i} = \sqrt{2(E_i^- - V(x_i))} - \sqrt{2(E_i^+ - V(x_i))}, \quad i = 1, 2. \tag{2.14}$$

Note that if $0 \le A \le B$ then $\sqrt{B} - \sqrt{A} \le \sqrt{B - A}$; this, used in (2.14), gives

$$\left|\frac{\partial S^0}{\partial x_i}\right| \le \sqrt{2|E_i^+ - E_i^-|}, \quad i = 1, 2. \tag{2.15}$$

Now, according to Lemma 4.1 and assumptions (2.4) and (2.5) we have

$$|E_i^+ - E_i^-| \le c\varepsilon^{2r+4}, \quad i = 1, 2, 3, 4, \tag{2.16}$$

and by (2.15), we get

$$\left|\frac{\partial S^0}{\partial x_i}\right| \le c\varepsilon^{r+2}, \quad i = 1, 2. \tag{2.17}$$

By integrating this with respect to x_i (recall that $|x_i| \le \sqrt{\varepsilon}$), we obtain (2.10), which shows that S^0 is "flat" in the first two variables.

To estimate $\partial S^0/\partial x_4$, note that $(x_4^-, \dot{x}_4^-)$ and $(x_4^+, \dot{x}_4^+)$ stay at the $\sqrt{\varepsilon}$-neighborhood of the saddle $(x_4, \dot{x}_4) = (\pi, 0)$. Since the distance between the sections is large (see (2.4)), and the velocity is bounded (by the choice of fixed energy), the duration of each stage is at least ε^{-2r-4}. This implies that, at the moment when the solution crosses Σ, the distance between $(x_4^-, \dot{x}_4^-)$ and the unstable manifold of the saddle is at most $O(\exp(-\varepsilon^{-1}))$ (if ε is sufficiently small). The unstable manifold has the form

$$y_4 = U(x_4) = (x_4 - \pi) + O((x_4 - \pi)^2).$$

At the same time, the distance between $(x_4^+, \dot{x}_4^+)$ and the stable manifold, $y_4 = -U(x_4)$, is at most $O(\exp(-\varepsilon^{-1}))$ for sufficiently small ε. That is,

$$\dot{x}_4^- = U(x_4) + O(\exp(\varepsilon^{-1})) \tag{2.18}$$

and

$$\dot{x}_4^+ = -U(x_4) + O(\exp(\varepsilon^{-1})), \tag{2.19}$$

so that

$$\frac{\partial S^0}{\partial x_4} = 2U(x_4) = 2(x_4 - \pi) + O((x_4 - \pi)^2) + O(\exp(\varepsilon^{-1})). \tag{2.20}$$

Integration with respect to x_4 gives (2.11).

Case 2. *Advance*. In this case not all sections lie in the same string in (2.1); without loss of generality, assume that Σ_- and Σ are the last two sections in the string $(1 \to 2)$, while Σ_+ is the first section in the following string, $(2 \to 3)$. The corresponding displacement vectors are of the form $\vec{n}_- = 2\pi(m, n, 1, 0)$, $\vec{n}_+ = 2\pi(\frac{1}{2}, s, 1, \frac{1}{2})$. In this case we still have (2.13), and following (2.14) and (2.15) we obtain (2.10). Since the sign of $\dot{x}_4$ is unknown, we treat it separately, observing, as before, that (2.18) and (2.19) hold. This implies (2.20) and thus (2.11).

This completes the proof of (2.10) and (2.11) in both cases. □

PROOF OF THE INTERIOR MINIMIUM FOR S. Using the properties of S^0 and the positivity of β we now show that S has a minimum inside Σ. We do so for Case 1; the remaining case is treated almost verbatim.

The boundary $\partial\Sigma = \partial_v\Sigma \cup \partial_h\Sigma$ consists of the "vertical" and the "horizontal" parts (after possible reindexing of the coordinates):

$$\begin{aligned} \partial_v\Sigma &= \{x_1^2 + x_2^2 = \varepsilon,\ |x_4 - \pi| \le \sqrt{\varepsilon}\}, \\ \partial_h\Sigma &= \{x_1^2 + x_2^2 \le \varepsilon,\ |x_4 - \pi| = \sqrt{\varepsilon}\}. \end{aligned} \tag{2.21}$$

A key observation we will use shortly is this:

$$S(p) = S^0(p) \quad \text{for all } p \in \partial_v\Sigma. \tag{2.22}$$

PROOF. We wish to show that the energy one solution $\gamma(p_{j-1}, p)$ with p lying on $\{x_1^2 + x_2^2 = \varepsilon,\ x_3 = 0\}$ does not intersect the cylinder $\{x_1^2 + x_2^2 + x_3^2 \le \varepsilon^2\}$ (and hence it does not intersect the connected component of the support of the deformation β, contained in this cylinder). To that end assume the contrary: the solution travels from one set to the other, taking some time $t = t^* > 0$. Since the distance between the above sets is $\ge \frac{1}{2}\sqrt{\varepsilon}$, while the speed is ≤ 2, the time of travel is $t^* > \frac{1}{4}\sqrt{\varepsilon}$. But $\dot{x}_3 \ge 1$, at least as long as $|x_3| \le \sqrt{\varepsilon}$. Thus, during time t^*, x_3 changes by an amount of $\Delta x_3 > \frac{1}{4}\sqrt{\varepsilon}$, which means that the solution lies outside of the cylinder $\{x_1^2 + x_2^2 + x_3^2 \le \varepsilon^2\}$, contradicting the definition of t^*. This proves (2.22). □

We now show that the restriction of S to each horizontal disk in Σ,

$$D_h := \{x_1^2 + x_2^2 \le \varepsilon,\ x_4 = \pi + h\}, \quad |h| \le \sqrt{\varepsilon},$$

has a minimum in the interior of D_h. To that end, we first note a crucial fact that β decreases S (as compared to S^0) near the center $C_h = (0, 0, \pi + h)$ of each D_h. Note that by the definition (1.4) the infimum of $\beta(\,\cdot\,, \varepsilon)$ taken over the set $x_1^2 + x_2^2 + x_3^2 \le \varepsilon^2/2$ is bounded from below by $b\varepsilon^r$ for some $b > 0$ independent of ε. Therefore, comparing the geodesic length in the original Jacobi metric with the truncated one, we obtain

$$S(C_h) \le S^0(C_h) - \varepsilon \inf \beta(\,\cdot\,, \varepsilon) \le S^0(C) - b\varepsilon^{r+1}. \tag{2.23}$$

See the proof of lemma 4 in [19] for more details on this argument.

On the other hand, by (2.10) we have, for any $p \in \partial D_h$, $|h| \le \sqrt{\varepsilon}$:

$$S^0(C_h) \le S^0(p) + 2c\varepsilon^{r+2.5}.$$

Combining this with (2.23) and (2.22) we obtain

$$S(C_h) \le S(p) + 2c\varepsilon^{r+2.5} - b\varepsilon^{r+1} < S(p) \quad \forall p \in \partial D_h,$$

provided that ε is sufficiently small (since b and c are independent of ε). We showed that the minimum of S cannot be achieved on $\partial_v \Sigma$, and it remains to show that it cannot be achieved on $\partial_h \Sigma$ either. Estimate (2.11) shows that S^0 has a pronounced minimum near the equator $x_4 = \pi$.

By the same estimate as we used for (2.23), we have a two-sided result: $|S^0(x) - S(x)| \leq b\varepsilon^{r+1}$, which together with (2.11) gives

$$S(x_1, x_2, \pi \pm \sqrt{\varepsilon}) > S(x_1, x_2, \pi).$$

This proves that the minimum is achieved *inside* Σ.

To complete the proof of the main theorem, it remains to observe that the existence of the internal minimum for S implies the existence of the internal minimum for $L(p_1, \ldots, p_N)$, as well as the existence of an internal minimum for an infinitely long sequence (2.1). The details can be found in [22]. $\square$

3 The Pendulum Lemma

In this section we state and prove an auxiliary lemma that is used in the proofs of the main two lemmas in the following two sections. This lemma asserts that there exists a unique trajectory of the pendulum along which the angle changes in a certain prescribed way between two prescribed values in a prescribed amount of time.

LEMMA 3.1. *For any $T > 0$ and for any α, β satisfying*

$$|\alpha| \leq 1, \quad |\beta - \pi| < 1, \tag{3.1}$$

there exists a unique solution $x(t) = x(t; T, \alpha, \beta)$ of $\ddot{x} + \sin x = 0$ satisfying $x(0) = \alpha$, $x(T) = \beta$, and

$$\alpha \leq x(t) \leq \max\{\beta, \pi\} \quad \text{for } 0 \leq t \leq T. \tag{3.2}$$

This solution depends smoothly on T, α, and β.

All these conclusions also hold if conditions (3.1) *are replaced by one of the following conditions*:

$$|\alpha| \leq 1, \quad |\beta + \pi| \leq 1, \quad \alpha \leq x(t) \leq \max(\beta, \pi), \ \ t \in [0, T], \tag{3.3}$$

or

$$|\alpha + \pi| \leq 1, \quad |\beta| \leq 1, \quad \min(\alpha, \pi) \leq x(t) \leq \beta, \ \ t \in [0, T], \tag{3.4}$$

or

$$|\alpha| \leq 1, \quad |\beta - 2\pi| \leq 1, \quad \alpha \leq x(t) \leq \beta, \ \ t \in [0, T], \tag{3.5}$$

or finally

$$\beta - \alpha \geq 2\pi, \quad \alpha \leq x(t) \leq \beta, \ \ t \in [0, T]. \tag{3.6}$$

The solution's energy $E(T) = \dot{x}^2/2 - (1 + \cos x)$ (with α and β fixed) is continuous in T, with $\lim_{T \to 0} E(T) = \infty$ and $\lim_{T \to \infty} E(T) = E_{\text{saddle}} = 0$. If $|\beta - \alpha| > 2\pi$,

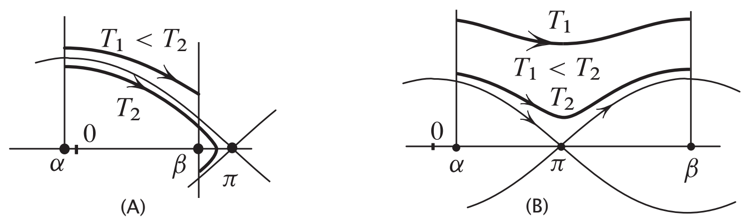

FIGURE 3.1. Potential energy for one degree of freedom systems. If $\beta < \pi$, the solution "turns around" for T large, as illustrated in (A).

then $E(T)$ is monotone decreasing. If $|\beta - \alpha| \le 2\pi$ then E depends exponentially weakly on T for large T:

$$\left|\frac{\partial E}{\partial T}\right| < c_1 e^{-c_2 T}, \tag{3.7}$$

where c_1, c_2 are positive constants independent of T. Finally, for any constant $c_3 > 0$ there exists $c_4 > 0$ independent of T such that if $c_4 T \ge |\beta - \alpha| \ge c_3 T$, then

$$\frac{\partial E}{\partial T} \le -\frac{c_5}{T}. \tag{3.8}$$

Remark 3.2. If α and β are not separated by a saddle, as in Figure 3.1(A), then $E(T)$ approaches $E_{\text{saddle}} = 0$ (as T increases) as follows. Define T_β by the relation $\dot{x}(T_\beta; \alpha, \beta, T_\beta) = 0$; that is, we consider the solution that reaches the line $x = \beta$ at the point $(\beta, 0)$, Figure 3.1. Then $E(T)$ decreases on the interval $(0, T_\beta)$ and increases on the interval (T_β, ∞), approaching $E_{\text{saddle}} = 0$. The minimal value $E(T_\beta) = -1 - \cos\beta$.

COROLLARY 3.3. *Consider the uncoupled system*[6]

$$\ddot{x}_i + \sin x_i = 0, \quad i = 1, \dots, 4. \tag{3.9}$$

For any pair of points $q, p \in \mathbb{R}^4$ with $|p - q| \ge c$ with $\alpha = q_i$, $\beta = p_i$ satisfying conditions of Lemma 3.1 *for each $i = 1, \dots, 4$, and for any $T > 0$ there exists a unique solution $X(t, q, p, T)$ of the system* (3.9) *that satisfies $X(0) = q$ and $X(T) = p$, and each of whose coordinates $x(t) = X_i(t)$ satisfies one of the bounds of Lemma* 3.1. *Moreover, there exists a constant $D > 0$ such that if, in addition to the above, $|p - q| \ge D$, then there is a unique $T > 0$ such that energy of this solution is* 1.

PROOF. Since $q_i = \alpha$, $p_i = \beta$ satisfy the conditions of Lemma 3.1 by assumption, the lemma applies to each equation in (3.9). Consequently, for any $T > 0$ there exists a unique solution $x_i(t) = x_i(t; T, q_i, p_i)$ of $\ddot{x}_i + \sin x_i = 0$ with the

[6] Since we chose to concentrate on $n = 4$ pendula, we formulate the lemma for this case, although the proof carries over verbatim for an arbitrary n.

desired boundary conditions and lying in one of the ranges stated in the lemma, (3.2)–(3.6). This proves the existence of the solution with the prescribed boundary conditions.

It remains to prove the uniqueness of the solution with energy one for $|p-q| \geq D$ with D sufficiently large. To that end, note that the velocity of the energy one solution $X(t; q, p, T)$ has an upper bound

$$|\dot{X}| = \sqrt{2(1 - V(X))} \leq 3\sqrt{2},$$

since $V(X) = -\sum_i (1 + \cos x_i) \geq 8$. With the upper bound on the speed, we get the lower bound on the time:

$$T \geq \frac{|p-q|}{3\sqrt{2}} \geq \frac{D}{3\sqrt{2}}.$$

Let us now choose D, and hence T, so large that the right-hand side of (3.8) dominates the right-hand side of (3.7):

$$c_1 e^{-c_2 T} < \frac{c_5}{T}.$$

Consider now the energy $E(T) = \sum E_i(T)$ of the solution in question. Each $E_i(T)$ satisfies either (3.7) or (3.8). Since $|p-q| > D$, at least one E_i satisfies (3.8). This shows that $E'(T) < 0$ whenever $E(T) = 1$, and thus such T is unique. This completes the proof of the corollary. □

PROOF OF LEMMA 3.1. We concentrate on the first case; the proofs of the remaining ones are the same.

We are seeking a solution of

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -\sin x, \end{cases} \tag{3.10}$$

which starts on the line $x = \alpha$ and ends at time $t = T$ on the line $x = \beta$. The desired solution is obtained by finding the point of intersection of the image of this line at time T and the line $x = \beta$, Figure 3.2. We have to show that the solution with the properties stated in the lemma exists and is unique.

Let $O \in \{x = \alpha\}$ be the point whose orbit crosses the x-axis at $x = \pi - 1$. None of the solutions starting below O on the line $x = \alpha$ can satisfy $x(T) = \beta$ for $T > 0$, so that we can concentrate on seeking the initial condition on the ray OM.

Let φ^t be the flow of (3.10), let S be the strip $S = \{(x, y) : x \in [\pi - 1, \pi + 1]\}$, and let π_x denote the projection onto the x-axis. Referring to Figure 3.2, we fix any $T > 0$ and define the set

$$I_T = \{z : z \in OM,\ \varphi^T z \in S,\ \pi_x(\varphi^t z) \in [\alpha, \pi + 1]\ \forall t \in [0, T]\}. \tag{3.11}$$

We claim the following: *for any $T > 0$, the set I_T is an interval, and $\varphi^T(I_T)$ is a curve with a positive slope connecting the two boundaries of the strip S*, Figure 3.2. This claim amounts to the existence and uniqueness of the desired solution.

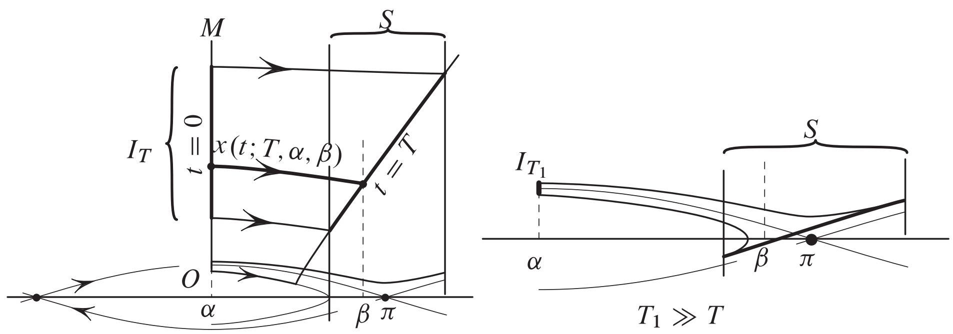

FIGURE 3.2. The slope of the image of the vertical interval is positive.

If T is small, the solutions are fast and thus lie above the separatrix, and the result is obvious. However, for larger T, some solutions starting on I_T "turn around," as in Figure 3.2 (right), and the proof requires a little care.[7]

Our goal is thus to prove that I_T is an interval, and that its image is a curve with positive slope. To that end it suffices to show that for any $z_0 = (\alpha, y_0) \in I_T$ we have

$$\frac{\partial}{\partial y_0}(\pi_x \varphi^t(\alpha, y_0)) > 0 \quad \text{for any } 0 < t \leq T. \tag{3.12}$$

To that end, consider the linearization of (3.10):

$$\begin{cases} \dot{\xi} = \eta, \\ \dot{\eta} = -(\cos x)\xi, \end{cases} \tag{3.13}$$

where x is the solution of (3.10) with the chosen initial condition. Note that the solution $\zeta = (\xi, \eta)$ of (3.13) with $\zeta(0) = (0, \eta_0)$, $\eta_0 > 0$, is a tangent vector to the image curve $\varphi^T(OM)$ at the point $\varphi^T z_0$. To prove (3.12) it suffices, therefore, to prove that $\zeta(T)$ *lies in the first quadrant.* (More formally, we note that the left-hand side in (3.12) is simply $\xi(t)$, so that the goal is to show that $\xi(t) > 0$). The idea of proof is to compare $\zeta(t)$ with $\varphi^t z_0$.

To that end, let $\tau \in (0, T]$ be the time of entrance into S:

$$x(t) \in [\alpha, \pi - 1] \quad \text{for } t \in [0, \tau]$$

and

$$x(t) \in [\pi - 1, \pi + 1] \quad \text{for } t \in [\tau, T].$$

We will first show that $\frac{\eta}{\xi} > 0$ at $t = \tau$.

[7] In fact, the last condition in (3.11) is crucial for uniqueness: without this condition, I_T would be a union of several intervals, giving rise to solutions that make extra "turns" around the focus. These solutions, however, violate the condition $x(t) \geq \alpha$ in (3.2), so that their existence does not contradict our claim of uniqueness.

The key idea is to observe that the vector $z = (x, \dot{x}) = \varphi^t z_0$ rotates clockwise faster than the vector $\zeta = (\xi, \eta)$; this will be shown shortly. Since the slope of z is positive at $t = \tau$, the same will then be true of $\zeta(\tau)$, which will prove (3.12). We claim the following:

$$\frac{d}{dt}\left(\frac{y}{x}\right) < \frac{d}{dt}\left(\frac{\eta}{\xi}\right) < 0 \quad \text{whenever} \quad \frac{y}{x} = \frac{\eta}{\xi}. \tag{3.14}$$

To prove (3.14) we carry out the differentiations and use the equality of the slopes to reduce the inequality to an equivalent one:

$$\frac{\dot{y}x - y\dot{x}}{x^2} < \frac{\dot{\eta}\xi - \eta\dot{\xi}}{\xi^2} < 0,$$

or, using (3.10) and (3.13),

$$\frac{-x\sin x - y^2}{x^2} < \frac{-\cos x\ \xi^2 - \eta^2}{\xi^2} < 0.$$

Using the equality of slopes, this reduces to

$$\frac{\sin x}{x} > \cos x,$$

which holds true due to $x \in (0, \pi)$. We conclude: since $y(\tau)/x(\tau) \geq 0$, then $\eta(\tau)/\xi(\tau) > 0$.

We now show that the slope of ζ remains positive for the remaining time $[\tau, T]$. During this time we have $|x - \pi| \leq 1$ and thus $\cos x < 0$. Hence the linearized vector field (3.13) crosses *into* the first quadrant, and since $\zeta(\tau)$ lies in that quadrant, it is still there at $t = T$. This completes the proof of (3.12), and thus of the existence and uniqueness of the desired solution with the boundary conditions (3.1). The treatment of the remaining cases involves no new ideas.

It remains to prove the monotonicity of $E(T)$ for $|\beta - \alpha| > 2\pi$, and the estimates (3.7) and (3.8) on $E'(T)$.

Since $\dot{x} > 0$ for all $t \in [0, T]$, due to the assumption $\beta - \alpha > 2\pi$,[8] we have

$$T = \frac{1}{\sqrt{2}} \int_\alpha^\beta \frac{dx}{\sqrt{E(T) - 1 - \cos x}}.$$

Differentiation by T shows at once that $E'(T) < 0$:

$$1 = -\frac{1}{2\sqrt{2}} \int_\alpha^\beta \frac{dx}{(E(T) - 1 - \cos x)^{3/2}}\, E'(T).$$

Now if $c_3 > 0$ is fixed and $\beta - \alpha > c_3 T$, then there exists a constant k independent of T such that $\dot{x} = \sqrt{2}(E(T) - 1 - \cos x) \geq k > 0$ for all t. The last integral is then bounded from above by a constant multiple of $\beta - \alpha$, and hence by a constant multiple of T, since $\beta - \alpha \leq c_4 T$. This amounts to the estimate (3.8).

[8] The case $\beta - \alpha < -2\pi$ is treated identically.

It remains to prove (3.7). The above integral formula for $E(T)$ does not apply if and only if $\dot{x}$ changes sign, which may happen in the case $|\beta-\alpha| \le 2\pi$ under consideration, as in Figure 3.1(A). In this case, a separate argument is needed. One can use the modification of the above formula, but it is more illuminating to use the hyperbolicity property rather than the integrable nature of the equation. Let $x(t, y_0) = \pi_x(\varphi^t(\alpha, y_0))$. We showed that for any T there exists $y_0 = y_0(T)$ such that

$$x(T, y_0(T)) = \beta.$$

Differentiating by T, we obtain

$$\underbrace{\frac{\partial x(T, y_0)}{\partial T}}_{\dot{x}(T)} + \underbrace{\frac{\partial x(T, y_0)}{\partial y_0}}_{\xi(T)} y_0'(T) = 0,$$

where $y_0 = y_0(T)$, so that

$$y_0'(T) = -\frac{\dot{x}(T)}{\xi(T)}, \tag{3.15}$$

where ξ satisfies (3.13). For $\beta - \alpha < 2\pi$ the solution spends time $O(T)$ in a finite neighborhood of the saddle; from our analysis of (3.13) it follows that $\xi(T) \ge c' e^{c'' T}$ for some positive constants c', c'' independent of T. From (3.15) we conclude that $|y_0'(T)| < c''' e^{-c'' T}$. Differentiating $E(T) = y_0(T)^2/2 - 1 - \cos\alpha$, we conclude that (3.7) holds.

The proof of Lemma 3.1 is complete. □

4 The Hyperbolic Lemma

By Corollary 3.3 of Lemma 3.1, given any points $q, q' \in \mathbb{R}^4$ with their coordinates x_i and x_i' lying in $[-1, 1] \bmod 2\pi$ or in $[\pi - 1, \pi + 1] \bmod 2\pi$, there exists a solution $X(t; q, q', T)$ of (3.9) with $\varepsilon = 0$ that travels from q to q' in time T. By the same corollary, there exists a unique $T(q, q')$ for which the total energy of the solution $X(t; q, q', T(q, q'))$ is 1:

$$\sum_{i=1}^{4} E_i = 1 \quad \text{where } E_i = \frac{\dot{X}_i^2}{2} + V(X_i). \tag{4.1}$$

We thus associate with the energy one solution (of (1.1) with $\varepsilon = 0$) connecting q and q', the energy vector

$$\mathbf{E}(q, q') \stackrel{\text{def}}{=} (E_1, E_2, E_3, E_4);$$

according to Lemma 3.1, this vector is uniquely determined by the endpoints q, q'.

LEMMA 4.1. *If two pairs of points q_1, q_1' and q_2, q_2' in $\mathbb{R}^4$ satisfy conditions*[9]

$$|q_k' - q_k| \geq \varepsilon^{-2r-4}, \quad k = 1, 2, \tag{4.2}$$

and

$$|e_2 - e_1| < \varepsilon^{2r+4} \quad \textit{where } e_k = \frac{q_k' - q_k}{|q_k' - q_k|}, \;\; k = 1, 2, \tag{4.3}$$

then the energy vector $\mathbf{E}$ *of the connecting solution of* (1.1) *with* $\beta = 0$ *satisfies*

$$|\mathbf{E}(q_2, q_2') - \mathbf{E}(q_1, q_1')| < \varepsilon^{2r+4}. \tag{4.4}$$

Moreover, there exists a constant C *such that for all* q, q' *with* $|q' - q| \geq 1$ *we have*

$$\left| \frac{d}{dq} \dot{X}(0; q, q', T(q, q')) \right| < C; \tag{4.5}$$

here the notation $\frac{d}{dq}$ *is used to emphasize that the* q*-dependence enters* $\dot{X}$ *in two places—one through the boundary condition, and the other through* $T(q, q')$.

PROOF. Statement (4.4) follows from the proof of Lemma 3.1; the main difficulty is in proving (4.5).[10] To prove (4.5), we expand its left-hand side:

$$\frac{d}{dq} \dot{X}(0; q, q', T(q, q')) = \partial_q \dot{X}(0; q, q', T) + \partial_T \dot{X}(0; q, q', T) \cdot \partial_q T(q, q'), \tag{4.6}$$

where $T = T(q, q')$ is to be substituted after the differentiations on the right-hand side. We will now estimate each of the summands on the right-hand side separately.

ESTIMATE OF $\partial_q \dot{X}(0; q, q', T)$. The proof of Lemma 3.1 shows that each component X_i of X depends on the boundary conditions x_i, x_i' and T only,[11] but not on x_j, x_j' with $j \neq i$. This implies that the matrix $\partial_q \dot{X}(0; q, q', T)$ is diagonal, with the diagonal entries $\partial \dot{X}_i(x_i, x_i', T)/\partial x_i$. But this derivative is simply the slope of the image of the line $x = x_i'$ under the map φ^{-T}, where φ^t is the phase flow of the pendulum equation. The argument of Lemma 3.1 shows that, because of the shear in the phase velocity field, this slope is always bounded once T exceeds a fixed constant. It remains to prove the upper bound for the last summand in (4.6).

ESTIMATE OF $\partial_T \dot{X}(0; q, q', T) \cdot \partial_q T(q, q')$. We will first show that this term is expressible via the first factor alone, thus reducing the number of estimates needed. Note that $\partial_T \dot{X} \cdot \partial_q T$ is a square matrix with entries $\partial_T \dot{X}_i(0; x_i, x_i', T) \cdot \partial_{x_j} T(q, q')$. To prove the lemma, it remains to show that each entry is bounded:

$$|\partial_T \dot{X}_i(0; x_i, x_i', T) \cdot \partial_{x_j} T(q, q')| < C \quad \text{for } |q' - q| \geq 1. \tag{4.7}$$

[9] Since we chose to concentrate on $n = 4$ pendula, we formulate the lemma for this case, although the proof carries over verbatim for an arbitrary n.

[10] This estimate can be strengthened: C can be replaced by $C/|q' - q|$, but we do not need this in our proof.

[11] We recall the notation $q = (x_1, x_2, x_3, x_4)$.

SOME IDENTITIES. Let

$$K_i = \left(\int_{x_i}^{x_i'} \frac{dx}{(2(E_i - V(x)))^{3/2}}\right)^{-1} \quad \text{and} \quad K = \sum_{s=1}^{4} K_s. \tag{4.8}$$

We will show that

$$\partial_{x_i} T(q, q') = K^{-1} \partial_T \dot{X}_i(0; x_i, x_i', T), \tag{4.9}$$

thus reducing (4.7) to an equivalent inequality

$$|K^{-1}\, \partial_T \dot{X}_i(0; x_i, x_i', T)\, \partial_T \dot{X}_j(0; x_j, x_j', T)| < C. \tag{4.10}$$

Heuristically, one expects that $|\partial_T \dot{X}_i(0; x_i, x_i', T)| \le cT^{-1}$. Indeed, the y-coordinate of the intersection in the $(X, \dot{X})$-plane of the line $\{X = x_i\}$ and the curve $\varphi^{-T}\{X = x_i'\}$ is $\dot{X}_i(0; x_i, x_i', T)$, where φ^t is the phase flow of the pendulum equation. Now because of the shear in the phase flow, one expects the line $\ell_T = \varphi^{-T}\{X = x_i'\}$ to form an angle at most cT^{-1} with the trajectories. Thus the point $\ell_T \cap \{X = x_i\}$ is expected to move with speed $\le cT^{-1}$, suggesting that indeed $|\partial_T \dot{X}_i(0; x_i, x_i', T)| \le cT^{-1}$.

We carry out a precise proof by an alternative, purely analytical method (which ultimately reduces to the same estimates). Namely, we will use the following identity:

$$\partial_T \dot{X}_i(0; x_i, x_i', T) = -\frac{K_i}{\sqrt{2(E_i - V(x_i))}}, \tag{4.11}$$

which, together with (4.9), is proven in a separate section below.

ESTIMATE OF K^{-1}. Since $\Sigma_{i=1}^4 E_i = 1$, we have $\frac{1}{4} \le E_i \le 1$ for some i, and thus for some C we have

$$K_i^{-1} = \int_{x_i}^{x_i'} \frac{dx}{(2(E_i - V(x)))^{3/2}} \le C \int_{x_i}^{x_i'} \frac{dx}{\sqrt{2(E_i - V(x))}} = CT,$$

so that

$$K^{-1} = \Big(\sum_{j=1}^{4} K_j\Big)^{-1} < K_i^{-1} \le CT. \tag{4.12}$$

ESTIMATE OF $\partial_T \dot{X}_i(0; x_i, x_i', T)$. We consider two separate cases: (1) $|x_i' - x_i| < 2\pi$ and (2) $|x_i' - x_i| \ge 2\pi$.

Case 1. In this case, an estimate of (4.11) is easier done geometrically, as follows: Consider the graph $y = U_T(x)$ of the time T-preimage of the line $x = x'$ in the phase plane of the pendulum. Let $y = U(x)$ be the graph of the stable manifold of the saddle $(\pi, 0)$; by a standard hyperbolic argument, the flow in the reverse direction takes the line exponentially close to the stable manifold:

$|U_T(x) - U(x)| < e^{-cT}$ for $|x| \le \pi$, and, moreover, the motion of the line becomes exponentially slow:

$$\left| \frac{d}{dT} U_T(x) \right| < e^{-cT} \quad \text{for } |x| \le \pi; \tag{4.13}$$

here T is greater than a fixed positive constant because of the assumption $|q'-q| \ge 1$. But $U_T(x_i) = \dot{X}_i(0; q, q', T)$ and (4.13) gives

$$|\partial_T \dot{X}_i(0; q, q', T)| \le e^{-cT} \quad \text{for } |x_i' - x_i| < 2\pi. \tag{4.14}$$

This completes the proof of (4.10), and thus of the lemma in case Case 1.

Case 2. In this case we have $x_i' - x_i = 2\pi n_i + r$, $0 \le r < 2\pi$, with integer $n \ne 0$. We will use (4.11) to prove (4.10), to which end we need an upper bound on K_i. Recall that $V(x) = -(\cos x + 1) = -2\cos^2 \frac{x}{2}$. From (4.8) we have

$$\begin{aligned} K_i^{-1} &\ge n_i \int_{-\pi}^{\pi} \frac{dx}{(E_i - V(x))^{3/2}} \\ &= 2n_i \int_0^{\pi} \frac{dx}{(E_i + 2\cos^2 x/2)^{3/2}} \ge \frac{n_i}{cE_i}, \end{aligned} \tag{4.15}$$

where c is a constant. To see this, note that $\cos^2 x = \sin^2(\frac{\pi}{2} - x) \le (\frac{\pi}{2} - x)^2$ and use the table integral

$$\int \frac{dx}{(a + x^2)^{2/3}} = \frac{x}{a^2\sqrt{x^2 + a^2}}.$$

Now the number of revolutions n_i is the integer part of $T/\mathcal{T}_{E_i}$, where $\mathcal{T}(E_i)$ is the time of one full revolution:

$$\mathcal{T}_{E_i} = \int_{-\pi}^{\pi} \frac{dx}{\sqrt{2(E_i + V(x))}} = 2\int_0^{\pi} \frac{dx}{\sqrt{2(E_i + 2\cos^2 x/2)}} \le c(1 - \ln E_i)$$

for some constant c. To see this, note that for small E_i the main contribution in this integral comes from a neighborhood of $x = \pi$. Here we can estimate the denominator to be larger than $\sqrt{2(E_i + (\pi - x)^2/4)}$. Then one uses the table integral

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln(x + \sqrt{x^2 + a^2}).$$

Substituting this into (4.15) we get

$$K_i \le c\,\frac{E_i}{n_i} \le c_1 \frac{E_i \mathcal{T}_{E_i}}{T} \le c_2 \frac{E_i(1 - \ln E_i)}{T}.$$

Finally, we substitute this estimate into (4.11):

$$|\partial_T \dot{X}_i(0; x_i, x_i', T)| \le c_2 \frac{E_i(1 - \ln E_i)}{T\sqrt{E_i}} \le \frac{c_3}{T}.$$

Together with (4.12) this proves (4.10). The proof of the lemma is thus complete. □

PROOF OF IDENTITIES (4.9) AND (4.11). Let us denote the energy of the solution $X(t; x_i, x_i', T)$ by $E(x_i, x_i', T)$. Then E is a smooth function of (x_i, x_i', T), and we can express

$$\dot{X}(0; x_i, x_i', T) = \sqrt{2(E(x_i, x_i', T) - V(x_i))}.$$

Differentiating with respect to T we get

$$\frac{\partial}{\partial T}\dot{X}(0; x_i, x_i', T) = \frac{\partial E(x_i, x_i', T)/\partial T}{\sqrt{2(E(x_i, x_i', T) - V(x_i))}}. \tag{4.16}$$

To estimate the numerator, we differentiate the identity

$$T = \int_{x_i}^{x_i'} \frac{dx}{\sqrt{2(E(x_i, x_i', T) - V(x))}} \tag{4.17}$$

with respect to T and solve for $\partial E/\partial T$, obtaining

$$\frac{\partial E(x_i, x_i', T)}{\partial T} = -\left(\int_{x_i}^{x_i'} \frac{dx}{(2(E(x_i, x_i', T) - V(x)))^{3/2}}\right)^{-1} = -K_i; \tag{4.18}$$

see (4.8). Substituting this into (4.16) proves (4.11). To prove the remaining identity (4.9), we recall that $T(q, q')$ is the time that gives energy one to the solution

$$\sum_{k=1}^{4} E(x_k, x_k', T(q, q')) = 1. \tag{4.19}$$

Differentiating this with respect to x_i gives

$$\left.\frac{\partial E(x_i, x_i', T)}{\partial x_i}\right|_{T=T(q,q')} + \frac{\partial T(q, q')}{\partial x_i}\sum_{k=1}^{4}\frac{\partial E(x_k, x_k', T)}{\partial T} = 0. \tag{4.20}$$

The above sum, according to (4.18), can be replaced by $-\sum_{k=1}^{4} K_k \overset{\text{def}}{=} -K$; solving for $\partial T/\partial x_i$ gives

$$\frac{\partial T(q, q')}{\partial x_i} = K^{-1}\left.\frac{\partial E(x_i, x_i', T)}{\partial x_i}\right|_{T=T(q,q')}. \tag{4.21}$$

Here and below, we write E_i instead of $E(x_i, x_i', T)$. To estimate the last derivative, we differentiate the identity (4.17) by x_i:

$$0 = -\frac{1}{\sqrt{2(E_i - V(x_i))}} - \underbrace{\int_{x_i}^{x_i'} \frac{dx}{(2(E_i - V(x)))^{3/2}}}_{K_i^{-1}} \frac{\partial E(x_i, x_i', T)}{\partial x_i}$$

or

$$\frac{\partial E(x_i, x_i', T)}{\partial x_i} = -\frac{K_i}{\sqrt{2(E_i - V(x_i))}}.$$

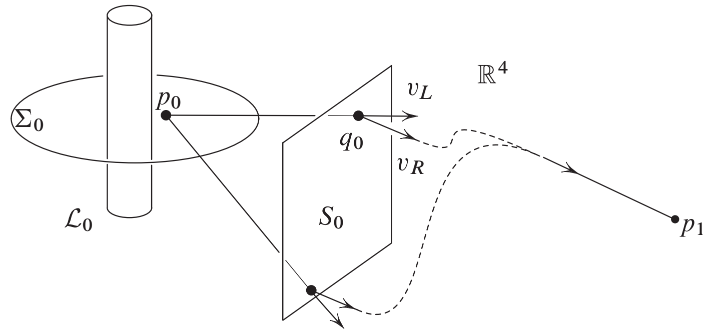

FIGURE 5.1. Towards proof of Lemma 5.1.

Substituting this into (4.21) results in the proof of (4.9):

$$\frac{\partial T(q,q')}{\partial x_i} = -K^{-1}\frac{K_i}{\sqrt{2(E_i - V(x_i))}} \overset{(4.11)}{=} K^{-1}\partial_T \dot{X}_i(0; x_i, x_i', T).$$

The proof of the two identities is now complete. □

5 The Connection Lemma

The following lemma is a building block in the construction of shadowing geodesics. Since we chose to concentrate on $n = 4$ pendula, we formulate the lemma for this case, although the proof carries over verbatim for an arbitrary n.

LEMMA 5.1 (Existence of Geodesic Segments). *There exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the following holds: Consider any two sections, which we denote by Σ_0 and Σ_1, in the itinerary* (2.1) *such that the vector $2\pi\vec{n}$ connecting the centers of Σ_0, Σ_1 satisfies $|\vec{n}| > \frac{1}{\varepsilon}$. This vector is of the form $2\pi(m, n, 1, 0)$ or $2\pi(\frac{1}{2}, s, 1, \frac{1}{2})$ (with integers m, n, and s). Then for all $p_0 \in \Sigma_0$, $p_1 \in \Sigma_1$ there exists a geodesic $\gamma(p_0, p_1)$ in the Jacobi metric* (2.6) *connecting p_0 with p_1 and depending smoothly on p_0 and p_1.*

PROOF.

Step 1. First we define a section S_0, which is shown in Figure 5.1, as follows: By Lemma 3.3 there exists a (unique) solution $X_0(t)$ of the unperturbed system (3.9) having energy one and connecting $c_0 = \mathit{Center}(\Sigma_0)$ with $c_1 = \mathit{Center}(\Sigma_1)$. Let $e_0 = \dot{X}_0(0)/|\dot{X}_0(0)|$ be the "initial direction" of $X_0(t)$. Section S_0 is defined to be a codimension-1 disk in $\mathbb{R}^4$ of radius $\varepsilon^{1/3}$ centered at the point $c_0 + e_0\varepsilon^{1/3}$ and perpendicular to e_0:

$$S_0 = \{q : (q - (c_0 + \varepsilon^{\frac{1}{3}} e_0)) \cdot e_0 = 0,\ |q - (c_0 + \varepsilon^{\frac{1}{3}} e_0)| < \varepsilon^{\frac{1}{3}}\},$$

where $\cdot$ denotes the usual dot product. Analogously, we define the section S_1 near c_1.

Step 2. For any pair of points $q_0 \in S_0$, $q_1 \in S_1$ we consider three geodesic segments (in the metric (2.6)): $\gamma(p_0, q_0)$, $\gamma(q_0, q_1)$, and $\gamma(q_1, p_1)$, along with the velocities of the associated solutions of (1.1) v_L, v_R, w_L, and w_R, as shown in Figure 5.1. It should be noted that v_L, v_R, w_L, and w_R all depend on q_0, q_1. The lemma will be proved once we show that there exists a pair q_0, q_1, smoothly depending on p_0, p_1, for which

$$v_L = v_R \quad \text{and} \quad w_L = w_R. \tag{5.1}$$

To that end we first list the properties of each of the three geodesic segments.

Step 3. Since the radius of Σ_0 is $\varepsilon^{1/2}$, we have $|q_0 - p_0| = O(\varepsilon^{1/3})$, which is small compared to the injectivity radius of the metric (2.6). By the standard arguments from differential geometry (using the smoothness of solutions of the ODEs and the implicit function theorem), we conclude that

$$v_L = v_0 \frac{q_0 - p_0}{|q_0 - p_0|} + r_{0L}(p_0, q_0, \varepsilon), \quad |r_{0L}|_{C^1} = O(\varepsilon^{\frac{1}{3}}); \tag{5.2}$$

here $v_0 > 0$ is the speed of the solution at c_0. A similar estimate holds for the right end:

$$w_R = v_1 \frac{p_1 - q_1}{|p_1 - q_1|} + r_{1R}(q_1, p_1, \varepsilon), \quad |r_{1R}|_{C^1} = O(\varepsilon^{\frac{1}{3}}). \tag{5.3}$$

Step 4. The intermediate segment $\gamma(q_0, q_1)$ avoids the lenses, and thus Lemma 4.1 applies; in particular, the C^1-bound (4.5) holds, implying that

$$v_R = \dot{X}_0(t_0) + r_{0R}(q_0, q_1, \varepsilon), \quad |r_{0R}|_{C^1} < C\varepsilon^{\frac{1}{3}}, \tag{5.4}$$

and

$$w_L = \dot{X}_0(t_1) + r_{1L}(q_1, p_1, \varepsilon), \quad |r_{1L}|_{C^1} < C\varepsilon^{\frac{1}{3}}. \tag{5.5}$$

Here t_i $(i = 0, 1)$ is the time when $X_0(t)$ intersects the section S_i.

Step 5. We will prove the existence of the pair q_0, q_1 satisfying (5.1) by applying the implicit function theorem. To that end, let $\hat{v}$ denote the orthogonal projection of $v \in \mathbb{R}^4$ onto $\mathbb{R}^3 \supset S_i$ (we shall use projections on either S_0 or S_1, the choice being clear from the context). We will also treat $q_i \in \mathbb{R}^4$ as an element of $S_i \subset \mathbb{R}^3$, denoting it by $\hat{q}_i \in \mathbb{R}^3$.

To prove (5.1) it suffices to prove that the projected equations

$$\hat{v}_L = \hat{v}_R \quad \text{and} \quad \hat{w}_L = \hat{w}_R \tag{5.6}$$

hold. Indeed, if equalities (5.6) hold, then the remaining components, orthogonal to S_i, must match as well by the conservation of energy. Substituting estimates (5.2), (5.3), (5.4), and (5.5) into (5.1) and projecting onto S_0 (respectively, S_1 for the second equality) we obtain the new matching conditions, which are equivalent to (5.1):

$$v_0 \frac{q_0 \,\widehat{-}\, p_0}{|q_0 - p_0|} = \hat{r}_0(p_0, \hat{q}_0, \hat{q}_1, \varepsilon), \quad v_1 \frac{p_1 \,\widehat{-}\, q_1}{|p_1 - q_1|} = \hat{r}_1(\hat{q}_0, \hat{q}_1, p_1, \varepsilon). \tag{5.7}$$

Here the remainders are C^1:

$$|\hat{r}_i|_{C^1} < C\varepsilon^{\frac{1}{3}}.$$

In arriving at (5.7), we made use of the fact that $\hat{\dot{X}}_0(t_0) = O(\varepsilon^{1/3})$, as follows from the choice of S_0 to be orthogonal to $\dot{X}_0(0)$ (so that $\hat{\dot{X}}_0(0) = 0$) and the fact that $t_0 = O(\varepsilon^{1/3})$.

Step 6. To apply the implicit function theorem, instead of the variables $\hat{q}_i$ we introduce

$$Q_0 = v_0 \frac{\widehat{q_0 - p_0}}{|q_0 - p_0|}, \quad Q_1 = v_1 \frac{\widehat{p_0 - q_1}}{|p_1 - q_1|},$$

$Q_i \in \mathbb{R}^3$. Expressing

$$\hat{q}_0 = \hat{p}_0 + \frac{1}{v_0}|q_0 - p_0|Q_0, \quad \hat{q}_1 = \hat{p}_1 + \frac{1}{v_1}|p_1 - q_1|Q_1,$$

and substituting into (5.6), we obtain

$$Q_0 = R_0(p_0, Q_0, Q_1, \varepsilon), \quad Q_1 = R_1(Q_0, Q_1, p_1, \varepsilon).$$

Introducing $Q = (Q_0, Q_1) \in \mathbb{R}^6$ and $R = (R_0, R_1)$ we rewrite the matching condition (5.1) in the final form

$$Q = R(Q, p_0, p_1, \varepsilon),$$

where $|R|_{C^1} < C\varepsilon^{1/3}$. It is important to observe that R is defined (at least) on the entire ball $|Q| \leq \frac{1}{2}$, independently of ε. Thus for all sufficiently small ε there exists a unique solution Q depending differentiably on the parameters p_0, p_1.

This completes the proof of Lemma 5.1. $\square$

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Bibliography

[1] Arnol′d, V. I. Instability of dynamical systems with many degrees of freedom. *Dokl. Akad. Nauk SSSR* **156** (1964), 9–12.

[2] Basko, D. M. Weak chaos in the disordered nonlinear Schrödinger chain: destruction of Anderson localization by Arnold diffusion. *Ann. Physics* **326** (2011), no. 7, 1577–1655. doi:10.1016/j.aop.2011.02.004

[3] Berti, M.; Bolle, P. A functional analysis approach to Arnold diffusion. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **19** (2002), no. 4, 395–450. doi:10.1016/S0294-1449(01)00084-1

[4] Bessi, U. An approach to Arnol′d's diffusion through the calculus of variations. *Nonlinear Anal.* **26** (1996), no. 6, 1115–1135. doi:10.1016/0362-546X(94)00270-R

[5] Bessi, U. Arnold's diffusion with two resonances. *J. Differential Equations* **137** (1997), no. 2, 211–239. doi:10.1006/jdeq.1996.3251

[6] Boesch, R.; Willis, C. R.; El-Batatouny, M. Spontaneous emission of radiation from a discrete sine-Gordon kink. *Phys. Rev. B* **40** (1989), no. 4, 2284–2296. doi:10.1103/PhysRevB.40.2284

[7] Bolotin, S.; Treschev, D. Unbounded growth of energy in nonautonomous Hamiltonian systems. *Nonlinearity* **12** (1999), no. 2, 365–388. doi:10.1088/0951-7715/12/2/013

[8] Bounemoura, A. Nekhoroshev estimates for finitely differentiable quasi-convex Hamiltonians. *J. Differential Equations* **249** (2010), no. 11, 2905–2920. doi:10.1016/j.jde.2010.06.004

[9] Bourgain, J. Problems in Hamiltonian PDE's. GAFA 2000 (Tel Aviv, 1999). *Geom. Funct. Anal.* **2000** (2000), Special Volume, Part I, 32–56. doi:10.1007/978-3-0346-0422-2_2

[10] Cheng, C.-Q. Arnold diffusion in nearly integrable Hamiltonian systems. Preprint, 2012. arXiv:1207.4016 [math.DS]

[11] Colliander, J.; Keel, M.; Staffilani, G.; Takaoka, H.; Tao, T. Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation. *Invent. Math.* **181** (2010), no. 1, 39–113. doi:10.1007/s00222-010-0242-2

[12] Delshams, A.; de la Llave, R.; Seara, T. M. A geometric approach to the existence of orbits with unbounded energy in generic periodic perturbations by a potential of generic geodesic flows of T^2. *Comm. Math. Phys.* **209** (2000), no. 2, 353–392.

[13] Fermi, E.; Pasta, J.; Ulam, S. Studies of nonlinear problems. I. Los Alamos Scientific Laboratory, LA-1940, 1955.

[14] Fröhlich, J.; Spencer, T.; Wayne, C. E. Localization in disordered, nonlinear dynamical systems. *J. Statist. Phys.* **42** (1986), no. 3–4, 247–274. doi:10.1007/BF01127712

[15] Gelfreich, V.; Turaev, D. Unbounded energy growth in Hamiltonian systems with a slowly varying parameter. *Comm. Math. Phys.* **283** (2008), no. 3, 769–794. doi:10.1007/s00220-008-0518-1

[16] Henrici, A.; Kappeler, T. Results on normal forms for FPU chains. *Comm. Math. Phys.* **278** (2008), no. 1, 145–177. doi:10.1007/s00220-007-0387-z

[17] Kaloshin, V. Geometric proofs of Mather's connecting and accelerating theorems. *Topics in dynamics and ergodic theory*, 81–106. London Mathematical Society Lecture Note Series, 310. Cambridge University Press, Cambridge, 2003. doi:10.1017/CBO9780511546716.007

[18] Kaloshin, V.; Levi, M. An example of Arnold diffusion for near-integrable Hamiltonians. *Bull. Amer. Math. Soc. (N.S.)* **45** (2008), no. 3, 409–427. doi:10.1090/S0273-0979-08-01211-1

[19] Kaloshin, V.; Levi, M. Geometry of Arnold diffusion. *SIAM Rev.* **50** (2008), no. 4, 702–720. doi:10.1137/070703235

[20] Kaloshin, V.; Zhang, K. A strong form of Arnold diffusion for two and a half degrees of freedom. Preprint, 2012. arXiv:1212.1150 [math.DS]

[21] Lagendijk, A.; van Tiggelen, B.; Wiersma, D. A. Fifty years of Anderson localization. *Phys. Today* **62** (2009), no. 8, 24–29. doi:10.1063/1.3206091

[22] Levi, M. Shadowing property of geodesics in Hedlund's metric. *Ergodic Theory Dynam. Systems* **17** (1997), no. 1, 187–203. doi:10.1017/S0143385797060999

[23] Mather, J. Variational construction of trajectories for time periodic Lagrangian systems on the two torus. Preprint, 1996.

[24] Peyrard, M.; Kruskal, M. D. Kink dynamics in the highly discrete sine-Gordon system. *Phys. D* **14** (1984), no. 1, 88–102. doi:10.1016/0167-2789(84)90006-X

[25] Willis, C.; El-Batanouny, M.; Stancioff, P. Sine-Gordon kinks on a discrete lattice. I. Hamiltonian formalism. *Phys. Rev. B* **33** (1986), no. 3, 1904–1911. doi:10.1103/PhysRevB.33.1904

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