Almost dense orbit on energy surface

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We study a C^r nearly integrable Hamiltonian system $\mathcal{H}_\varepsilon(q,p) = \frac{1}{2}\langle p,p\rangle + \epsilon\mathcal{H}_1(q,p)$ defined on $\mathbb{T}^3\times\mathbb{R}^3$. Let $\Sigma = \{(q,p) : \mathcal{H}_\varepsilon(q,p) = \frac{1}{2}\}$ and μ_{Σ_1} be the restriction of Lebesgue measure on $\mathbb{T}^3\times\mathbb{R}^3$ to Σ. We prove there is a perturbation $\mathcal{H}_1(q,p)\in C^r, \|\mathcal{H}_1\|_{C^r}\le 1$ and an orbit $(q(t),p(t)) : \mathbb{R}\to\mathbb{T}^3\times\mathbb{R}^3$ of the Hamiltonian equation $\{\dot q = \partial_p\mathcal{H}_\varepsilon,\ \dot p = -\partial_q\mathcal{H}_\varepsilon\}$ such that $\mu_\Sigma(\overline{\bigcup_{t\in\mathbb{R}}(q(t),p(t))}) \ge \frac{1}{2}$.

1. Introduction

The famous question called the ergodic hypothesis suggested that for a typical Hamiltonian on a typical energy surface all, but a set of zero measure of initial conditions, have trajectories covering densely this energy surface itself. However, KAM theory showed that for nearly integrable systems there is a set of initial conditions of positive measure of quasi periodic trajectories. This disproved the ergodic hypothesis and forced to reconsider the problem.

A quasi ergodic hypothesis asks if a typical Hamiltonian on a typical energy surface has a dense orbit. A definite answer whether this statement is true or not is still far out of reach of modern dynamics. There was an attempt to prove this statement by E. Fermi,[5] which failed (see[6] for more detailed account).

To simplify the quasi ergodic hypothesis, M. Herman[7] formulated the following question: *Can one find an example of a C^∞ Hamiltonian $\mathcal{H}$ in a C^r small neighborhood of $\mathcal{H}_0(p) = \frac{\langle p,p\rangle}{2}$ such that on the unit energy surface $\{\mathcal{H}^{-1}(\frac{1}{2})\}$ there is a dense trajectory?* Many people believe that such examples do exist and are C^∞–generic (see,[4],[31]).

In this paper we make a step in the direction of answering Herman's question. For any r we construct a Hamiltonian, which is C^r close to $\mathcal{H}_0(p) = \frac{\langle p,p\rangle}{2}$ and has a trajectory dense in a set of Lebesgue measure $1/2$ on the energy surface. Here is the exact statement. Let $q\in\mathbb{T}^3$, $p\in\mathbb{R}^3$ and $\mathcal{H}_0(p) = \frac{\langle p,p\rangle}{2}$ be the unperturbed Hamiltonian, where $\langle p,p\rangle$ is the dot product in R^3.

Theorem 1.1. *For any $r\ge 2$ there is a C^r small perturbation $\mathcal{H}_\varepsilon(q,p) = \mathcal{H}_0(p) + \epsilon\mathcal{H}_1(q,p,\epsilon)$ and an orbit $(q(t),p(t)) : \mathbb{R}\to\mathbb{T}^3\times\mathbb{R}^3$ of*

$$\dot q = \partial_p\mathcal{H}_\varepsilon, \qquad \dot p = -\partial_q\mathcal{H}_\varepsilon \tag{1}$$

such that $\mu_\Sigma(\overline{\bigcup_{t\in\mathbb{R}}(q(t),p(t))}) \geq \frac{1}{2}$.[a]

Let $\Sigma = \{(q,p) : \mathcal{H}_\epsilon(q,p) = \frac{1}{2}\}$, we fix a subset $\mathcal{F} \subset \Sigma$ with $\mu_\Sigma(\mathcal{F}) \geq \frac{1}{2}$. It suffices to prove that for any $\delta > 0$ there exists T_δ such that the δ neighborhood of $\bigcup_{t\in[0,T_\delta]}(q(t),p(t))$ contains $\mathcal{F}$.

We will construct $\mathcal{H}_\varepsilon$ in two steps. In step one we build $\mathcal{H}'_\varepsilon = \mathcal{H}_0 + \varepsilon\mathcal{H}'_1$ so that has a variety of good local normal forms and nice invariant sets. Then $\mathcal{H}_\varepsilon = \mathcal{H}'_\varepsilon + \varepsilon\mathcal{H}''_1$ is designed to have diffusing orbits shadowing these invariant sets.

2. Choice of $\mathcal{F}$

We will describe the choice of the positive measure set $\mathcal{F}$, as well as an approximate path of diffusion. We begin with a informal discussion of the diffusion path and what kind of perturbation we need. Usually diffusing orbits travel along resonant segments. To be able to saturate a set of positive measure one has to be able to move along infinitely many resonant segments. If size of a perturbation is fixed, the analysis of motions near resonances of larger and larger orders in the original coordinate system becomes increasingly complicated as explained in section 4 . To be able to control dynamics along some arbitrary high order resonant we define a convenient symplectic coordinate system $\Phi : (\theta, I) \to (q,p)$ on a neighborhood of $\{\|p\| = 1, p_1 \geq \frac{1}{2}\}$, such that $\mathcal{H}'_\varepsilon \circ \Phi(\theta, I) = H_0(I) + H_1(\theta, I)$, where $\|H_1(\theta, I)\|_{C^r}$ gets the smaller as the order of a corresponding resonance increases.

We consider the following set of Diophantine numbers:

$$\begin{aligned}\mathcal{D}_\gamma = \{\omega = (\omega_1,\omega_2,\omega_3); \|\omega\| = 1,\ &|k\cdot\omega| \geq \gamma|\omega||k|^{-2-\tau},\ \forall k \in \mathbb{Z}^3;\\ &|k_1\omega_1 + k_2\omega_2| \geq \gamma^{\delta(1+\delta)}|(k_1,k_2)|^{-1-\delta}, \forall (k_1,k_2) \in \mathbb{Z}^2;\\ &|k_1\omega_1 + k_3\omega_3| \geq \gamma^{\delta(1+\delta)}|(k_1,k_3)|^{-1-\delta},\ \forall (k_1,k_3) \in \mathbb{Z}^2\},\end{aligned} \tag{2}$$

where $\delta > 0$ is a small number. The set $\mathcal{D}_\gamma$ has positive measure on the surface $\{\|\omega\| = 1\}$. Let $B = \{\|\omega\| = 1; \omega_1 \geq \frac{1}{2}\}$ and we will choose a subset $\mathcal{D}_\gamma^\infty \subset \mathcal{D}_\gamma \cap B$ with positive measure. The family of Diophantine number corresponds to a family of KAM tori which has measure on the energy surface $\{\mathcal{H}_\epsilon = \frac{1}{2}\}$. Denote it $\mathcal{F}$.

The construction will be done in infinitely many stages, each stage we will define a set of paths in the set B, such that if the Hamiltonian H satisfies a list of properties, there exists an orbit such that $\dot\varphi$ shadows the chosen path. The path gets denser in each stage and in the limit $\dot\varphi$ accumulates to a set of positive measure.

For any integer vector $k \in \mathbb{Z}^3 \setminus \{0\}$, we can relate to it a resonant plane $\{\omega \in \mathbb{R}^3 : k\cdot\omega = 0\}$. If the plane intersects B, the intersection is a curve on the unit sphere, which we will refer to as Γ_k.

At stage 1 the construction consists of the following components:

[a]In[8] there is a construction of $\mathcal{H}_\varepsilon$ and an orbit of $\mathcal{H}_\varepsilon$ whose closure has maximal Hausdorff dimension

(1) Let $\gamma_1 = \gamma^4$. We will choose a discrete set $\mathcal{DN}^1 \subset \mathcal{D}_\gamma \cap B$, and disjoint neighborhoods $\mathcal{U}(\omega_i)$ of $\omega_i \in \mathcal{DN}^1$, such that each $\mathcal{U}(\omega_i)$ contains a ball of radius γ_1, and is contained in a ball of radius $3\gamma_1$, both centered at ω_i.
(2) Let $\mathcal{D}^1_\gamma = \mathcal{D}_\gamma \cap \bigcup_{\omega_i \in \mathcal{DN}^1} \mathcal{U}(\omega_i)$, we have the sets $\mathcal{U}(\omega_i)$ is chosen in such that a way that the measure of $\mathcal{D}_\gamma \setminus \mathcal{D}^1_\gamma$ is small.
(3) There exists a collection $\mathbb{F}^1$ of integer vectors, such that for any $\omega_i \in \mathcal{DN}^1$ there exists some $k \in \mathbb{F}^1$ such that Γ_k enters $\gamma_1/2$ neighborhood of ω_i. Furthermore, the union $\mathcal{F}^1 := \bigcup_{k\in\mathbb{F}^1} \Gamma_k$ is connected.

In stage 2, let $\gamma_2 = \gamma_1^{1+\alpha}$ for some $\alpha > 0$. For each neighborhood $\mathcal{U}(\omega_i)$ of stage 1, we similarly define the following:

(1) A discrete set $\mathcal{DN}^2_i \subset \mathcal{D}^1_\gamma \cap \mathcal{U}(\omega_i)$, and for each $\omega_{ij} \in \mathcal{DN}^2_i$, we have neighborhoods $\mathcal{U}(\omega_{ij})$, whose radius is between γ_2 and $3\gamma_2$.
(2) $\mathcal{D}^2_i = \mathcal{D}^1_\gamma \cap \bigcup_{\omega_{ij}\in\mathcal{DN}^2_i} \mathcal{U}(\omega_{ij})$. The measure of $\mathcal{D}^1_\gamma \cap \mathcal{U}(\omega_i) \setminus \mathcal{D}^2_i$ is small.
(3) For the neighborhood $\mathcal{U}(\omega_i)$, there exists $k' \in \mathbb{F}^1$, such that the resonant line $\Gamma_{k'}$ enters the neighborhood. We further define a collection $\mathbb{F}^2_i$ of integer vectors, such that for any $\omega_{ij} \in \mathcal{DN}^2_i$, there exists some $k \in \mathbb{F}^2_i$ such that Γ_k enters $\gamma_2/2$ neighborhood of ω_{ij}. Write $\mathcal{F}^2_i = \bigcup_{k\in\mathbb{F}^1_i} \Gamma_k$, we assume that $\mathcal{F}^2_i \cup \Gamma_{k'}$ is connected. Denote also $\mathbf{F}^n = \cup_{i=1}^n \mathcal{F}^i$.

We do this for every neighborhood $\mathcal{U}(\omega_i)$ and let $\mathcal{DN}^2 = \bigcup \mathcal{DN}^1_i$, $\mathcal{D}^2_\gamma = \bigcup \mathcal{D}^2_i$, $\mathbb{F}^2 = \bigcup \mathbb{F}^2_i$, $\mathcal{F}^2 = \bigcup \mathcal{F}^2_i$. We then continue this construction inductively: for each multi-index $(i_1 \cdots i_n)$, assume that we have the neighborhood $\mathcal{U}(\omega_{i_1\cdots i_n})$, we can define $\mathcal{DN}^{n+1}_{i_1\cdots i_n}$, $\mathcal{D}^{n+1}_{i_1\cdots i_n}$ and $\mathbb{F}^{n+1}_{i_1\cdots i_n}$ in a similar fashion. Union over all multi-indices of same order is denoted by $\mathcal{DN}^{n+1}$, $\mathcal{D}^{n+1}_\gamma$ and $\mathbb{F}^{n+1}$. Then $\mathcal{D}^\infty_\gamma$ is the intersection of $\mathcal{D}^n_\gamma$ and has almost full measure in $\mathcal{D}_\gamma \cap B$. Finally, using ideas from,[15,16] we have the following

Theorem 2.1. *The Hamiltonian $H(\theta, I) = H_0(I) + H_1(\theta, I)$ has the following property: Consider the resonant lines $\mathcal{F}^n$ of stage n, there exists an open cover U_j of $\mathcal{F}^n$, such that for each U_j, there exists a neighborhood $\mathbf{U}_j \times \mathbb{T}^3 \supset (\partial_I H)^{-1}(U_j)$, on which H is in one of the two normal forms:*

(1) Single and ghost[b] *resonances: There exist local coordinates $\Psi : (\hat\theta, \hat I) \to (\theta, I)$ such that*

$$H \circ \Psi(\hat\theta, \hat I) = \hat H_0(\hat I) + a_k \cos(\pi k \cdot \hat\theta) + R, \tag{3}$$

where $k \in \mathbb{F}^n$ and $\|R\| \ll |a_k|$.

(2) Double resonance:

$$H(\theta, I)|\mathbf{U}_j \times \mathbb{T}^3 = H_0(I) + a_k \cos(\pi k \cdot \theta) + a_{k'} \cos(\pi k' \cdot \theta) + R, \tag{4}$$

where $k \in \mathbb{F}^n$, k' is in $\mathbb{F}^{n-1}$, $\mathbb{F}^n$ or $\mathbb{F}^{n+1}$, $\|R\|_{C^3} \ll \max\{|a_k|, |a_{k'}|\}$.

[b]there are certain $k'' \notin \mathbb{F}^{n-1} \cup \mathbb{F}^n \cup \mathbb{F}^{n+1}$ such that Γ_k intersects $\Gamma_{k''}$ inside U_j. We call such an intersection *a ghost double resonance*

3. A proof of existence of a $\delta-$dense orbit using a variational problem with constrains

In this section we reformulate a problem of existence of an orbit following a Cantor set of lines as a variational problem with constrains (following Mather). Recall that under the convenient coordinate system we have the Hamiltonian $H(\theta, I) = H_0(I) + H_1(\theta, I)$.

Due to the convexity with respect to I, the Hamiltonian system (1) is equivalent to the dynamics of the E-L equation with Lagrangian L as $L(\theta, \dot{\theta}) = l_0(\dot{\theta}) + L_1(\theta, \dot{\theta})$, which is positive definite with respect to $\dot{\theta}$ for any $\theta \in \mathbb{T}^3$.

Select $\{\omega_n^k\}_{k=1}^{N_n}$ be a set of points in $\mathbf{F}^n$ such that $|\omega_n^k - \omega_n^{k+1}|$ is sufficiently small.

Denote by $\mathcal{A}^\omega$ a special invariant set of orbits (to be defined later) with rotation vector ω. In our case velocity of these orbits will stay close to ω. Our goal is to construct a transition chain from these sets $\{\mathcal{A}^{\omega_n^k}\}_{k=1}^{N_n}$ and an orbit shadowing these sets. Such an orbit will stay close to the union of the stable set $W^s(\mathcal{A}^{\omega_n^k})$ and the unstable set $W^s(\mathcal{A}^{\omega_n^k})$ for all time. We find these orbits by constructing a variational problem with constrains. This construction is fairly involved and relies heavily on Mather's ideas. We describe its construction into several steps.

Let $\theta \in \mathbb{T}^3$, denote $\hat{\theta} \in \mathbb{R}^3$ a lift to $\mathbb{R}^3$. Let η be a closed one form, denote $\hat{\eta}$ a lift of it to a periodic close one form on $\mathbb{R}^3$. Fix a lift. One can proof existence of the following set of objects:

collections of numbers α_i, periodic functions $B_i^\pm$ and closed one forms η_i on the 3-torus $\mathbb{T}^3$, errors (negligibly small numbers) δ_i, smooth manifolds S_i with a boundary diffeomorphic to a 2–disk inside the 3-torus $\mathbb{T}^3$ such that the following variational problem with constrains has an interior solution: Given $T^* \gg 1$ and $\mathbf{T} \gg NT^*$, consider

$$M(\theta_0, \dots, \theta_N) = \min_{\substack{\theta_i \in S_i,\ T_{i+1}-T_i \geq T^* \\ T_0=0, T_N=\mathbf{T}}} \sum_{i=0}^{N} h_i(\theta_i, \theta_{i+1}, T_{i+1} - T_i), \tag{5}$$

where

$$h_i(\theta_i, \theta_{i+1}, T) = \min \int_0^T (L - \eta_c)(\gamma(s), \dot{\gamma}(s))\, dt, \tag{6}$$

and the minimum is taken over all absolutely continuous curves $\gamma : [0, T] \to \mathbb{T}^3$ such that $\gamma(0) = \hat{\theta}_i = \theta_i(\text{mod}1)$, and $\gamma(T) = \hat{\theta}_{i+1} = \theta_{i+1}(\text{mod}1)$. In addition, we need some constraints on the homology class of the minimizing orbit, which can be achieved by going to a proper covering of $\mathbb{T}^3$. We clarify this later in the section. It turns out that for each $i = 0, \dots, N$ we have

$$|h_i(\theta_i, \theta_{i+1}, \Delta T_i) - \alpha_i(T_{i+1} - T_i) + B_i^-(\theta_i) + B_i^+(\theta_{i+1})| \leq \delta_i.$$

Thus, to have an interior minimum it suffices to have a sufficiently deep interior minimum of $B_i^-(\theta_i) + B_i^+(\theta_{i+1})$. It also turns out that η_i and η_{i+1} can be chosen so that they coincide near the disk S_{i+1}.

Having this as the motivating goal, we shall define related objects from the Mather theory([13,14]). Here is the correspondence:

$B_i^{\pm}(\theta)$ are one-sided barrier functions, defined by Mather.[9] These functions form a 3-parameter family, naturally parametrized by $c \in H^1(\mathbb{T}^3, \mathbb{R})$. It turns out that cohomology class of the one form η_i is given by $[\eta_i]_{H^1(\mathbb{T}^3,\mathbb{R})} = c_i$. To determine position of $S_i \subset \mathbb{T}^3$ we need to determine a location of certain invariant sets, usually called Aubry sets $\mathcal{A}_{c_i}$ also naturally parametrized by c.

Let $I = [a, b]$ be an interval of time and $c \in H^1(\mathbb{T}^3, \mathbb{R}) = \mathbb{R}^3$. A curve $\gamma \in C^1(I, \mathbb{T}^3)$ is called $c-$minimizer if

$$A_c(\gamma) := \int_a^b (L - \eta_c)(\gamma(s), \dot\gamma(s))dt = \min_{\substack{\xi(a)=\gamma(a),\xi(b)=\gamma(b) \\ \xi \in C^1(I,\mathbb{T}^3)}} \int_a^b (L - \eta_c)(\gamma(s), \dot\gamma(s))dt,$$

where η_c is a closed $1-$form on $\mathbb{T}^3$ such that $[\eta_c] = c$. Let $\mathcal{M}_L$ be the set of Borel probability measures on $\mathbb{T}^3 \times \mathbb{R}^3$, invariant for the E-L flow φ_l^t. For any $\nu \in \mathcal{M}_L$, the action $A_c(\nu)$ is defined as $A_c(\nu) = \int (L - \eta_c)d\nu$. A probability measure μ is called $c-$minimal invariant measure if $A_c(\mu) = \min_{\nu \in \mathcal{M}_L} A_c(\nu)$. Denote $\mathcal{M}(c)$ the supports of $c-$minimal invariant measures and call it *Mather set*. A function $\alpha(c) := -A_c(\mu) : H^1(\mathbb{T}^3, \mathbb{R}) \to \mathbb{R}$ is called $\alpha-$*function*, where μ is a $c-$minimal invariant measure. Define

$$h_c(\theta, \theta'; t) = \min_{\substack{\gamma \in C^1([0,t],\mathbb{T}^3) \\ \gamma(0)=\theta, \gamma(t)=\theta'}} \int_0^t (L - \eta_c + \alpha(c))(\gamma(s), \dot\gamma(s))ds,$$

$$F_c(\theta, \theta') = \inf_{t \geq 0} h_c(\theta, \theta'; t), \qquad h_c^{\infty}(\theta, \theta') = \lim_{t \geq 0 \ t \to +\infty} h_c(\theta, \theta'; t).$$

Let $\gamma : \mathbb{R} \to \mathbb{T}^3$ be a C^1 curve

- It is called *c-semi-static* if $A_c(\gamma|_{[a,b]}) + \alpha(c)(b - a) = F_c(\gamma(a), \gamma(b), b - a)$ for any $a < b$.
- It is called $c-$*static* if it is c-semi-static and $A_c(\gamma|_{[a,b]}) + \alpha(c)(b - a) = -F_c(\gamma(b), \gamma(a), b - a)$.

Denote the set of c–semi-static and c–static orbits as $\mathcal{N}(c)$ and $\mathcal{A}(c)$ respectively. Usually $\mathcal{N}(c)$ is called *a Mañé set* and $\mathcal{A}(c)$ is called *an Aubry set*.

At the n-th stage of the induction we construct a collection $\{c_n^k\}_{k=1}^{N_n}$ such that orbits in the corresponding Aubry sets $\mathcal{A}_{c_n^k}$ have velocity $\dot\theta$ close to ω_n^k. Then we find orbits following the stable set $W^s(\mathcal{A}_{c_n^k})$ and the unstable set $W^u(\mathcal{A}_{c_n^k})$ as local minimizer of Euler-Lagrange equation. There are two drastically different cases in our problem: *single resonance and double resonance.*

3.1. *Single resonance case*

In Theorem 2.1 near a single resonance $k \in \mathbb{Z}^3 \setminus \{0\}$ we obtain a normal form (3). In order to construct a variational problem whose solutions diffuses along this resonance. Associate to $k \in \mathbb{Z}^3 \setminus \{0\}$ an integer linear transformation $A \in SL_3(\mathbb{Z})$ such that A induces a new coordinate system on $\mathbb{T}^3$, denote $\mathbb{T}^3 = \mathbb{T}^2_f \times \mathbb{T}_s \ni \theta = (\theta_1, \theta_2, \theta_s)$ so that θ_s is parallel to k. After an associated linear transformation we can consider the following Lagrangian system

$$L(\theta, \dot\theta) = l_0(\dot\theta) + a\cos^2\frac{\pi\theta_s}{2} + \delta L_1(\theta, \dot\theta), \tag{7}$$

where $l_0(\dot\theta)$ is close to $\langle\dot\theta, \dot\theta\rangle/2$, $\theta \in \mathbb{T}^3$ and δ is sufficiently small compare to a. The form of the Lagrangian implies that there is a co-dimensional 2 normally hyperbolic cylinder $\Lambda_k = \{\dot\theta_s = \theta_s = 0\}$. Rotation vectors associated to the single resonant are of form $\omega = (\omega_1, \omega_2, 0)$ in this new coordinates system. Restricted to an energy surface the normally hyperbolic is a 3 dimensional invariant manifold which is diffeomorphic to $T\mathbb{T} \times \mathbb{T}$. View the second $\mathbb{T}$ component as time, the dynamics of the Poincare return map on the invariant cylinder is an exact area-preserving twist map. For exact area-preserving twist maps structure of Mather and Aubry sets is well understood (see e.g.[10,14]) For example, minimal invariant measures have rotation number ω_1/ω_2. If ω_1/ω_2 is irrational, then there is unique $c' = (c_1, c_2)$ corresponds to (ω_1, ω_2). After we add the hyperbolic part into the dynamics, then there is an open interval $I_\omega \subset \mathbb{R}$ such that $\mathcal{A}(c) = \mathcal{N}(c)$ for any $c = \{(c_1, c_2, c_3) : c_3 \in I_\omega\}$. Moreover, $\mathcal{A}(c)$ is on the invariant cylinder. If ω_1/ω_2 is rational, the situation is a little bit complicated, because there is an open set of $c = (c_1, c_2, c_3)$'s with the same $\mathcal{A}(c)$. It is still true that $\mathcal{A}(c)$ belongs to the invariant cylinder Λ_k.

According to Bernard's theorem[2] Aubry and Mather sets are invariant under symplectic transformation. Once we establish a structure of Aubry-Mather sets in the normal form (3) we can construct a variational problem in the original coordinate system.

To describe the variational problem for the original coordinate system, we consider the covering space $N_k = \mathbb{T}^2_f \times \mathbb{R}$ of $\mathbb{T}^2_f \times \mathbb{T}_s$ by unfolding the θ_s direction. Denote by $\pi_k : N_k \to \mathbb{T}^2_f \times \mathbb{T}_s$ the natural projection.

Now we construct a variational problem to diffuse along Γ_k. Consider a sufficiently dense set of c's whose corresponding Aubry sets are on Λ_k, denoted $\{c_j\}_{0\le j\le N}$. We will construct relative open sets $S_j \subset \{\theta_{ss} = j + \frac{1}{2}\}$, a collection of closed one forms η_i on $\mathbb{T}^3$ such that $[\eta_i]_{H^1(\mathbb{T}^3,\mathbb{R})} = c_i$ and η_i coincides with η_{i+1} near $\pi_k(S_i)$. We would like to show that for this choice of S_j and η_j there is $T^* \gg 1$ and $\mathbf{T} \gg NT$ the variational problem as in the notations (5) attains an interior minimum. This can be done, if we add an additional perturbation to $L(\theta, \dot\theta)$. This additional perturbation will give some necessary information on the minimizer of variational problem which is related to the regularity of barrier function with respect to c and its proof heavily depends on.[11,12]

3.2. *Double resonance*

In Theorem 2.1 near double resonances we obtain a normal form (4). We would like to diffuse first along Γ_k, come to the intersection with $\Gamma_{k'}$, and then diffuse along

$\Gamma_{k'}$. In order to construct a variational problem, whose solutions diffuse along these resonances we distinguish three regimes: diffusing along Γ_k, switching from Γ_k to $\Gamma_{k'}$, and diffusing along $\Gamma_{k'}$.

Associate to $k,\ k' \in \mathbb{Z}^3 \setminus \{0\}$ an integer linear transformation $A \in SL_3(\mathbb{Z})$ such that A induces a new coordinate system on $\mathbb{T}^3$, denote $\mathbb{T}^3_A = \mathbb{T}_f \times \mathbb{T}_s \times \mathbb{T}_{ss} \ni \theta = (\theta_f, \theta_s, \theta_{ss})$ so that θ_s is parallel to k' and θ_{ss} is parallel to k. After such a transformation we have the following Lagrangian system

$$L(\theta, \dot\theta) = l_0(\dot\theta) + a_1 \cos^2(\frac{\pi}{2}\theta_s) + a_2 \cos^2(\frac{\pi}{2}\theta_{ss}) + \delta L_1(\theta, \dot\theta), \tag{8}$$

$a_1, a_2 > 0$ and sufficiently small and $\delta = \min\{a_1^{100}, a_2^{100}\}$. Denote by $L^0(\theta, \dot\theta) = L - \delta L_1$.

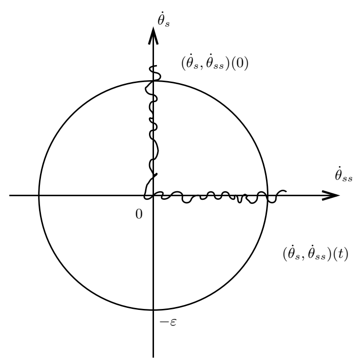

Fig. 1. Velocity diffusing across a double resonance

Let $K = a_1^{2\tau}$. Suppose $\sqrt{a_1}\, l$ is $(K, 1)$-Diophantine, i.e. $|p - q\sqrt{a_1}\, l\,| > K/|q|^2$ for any $q, p \in \mathbb{Z},\ q \neq 0$. Consider c such that the corresponding Aubry set $\mathcal{A}_c$ for L^0 has rotation vector $\omega = (1, \sqrt{a_1}\, l, 0)$ satisfying the above Diophantine condition. Consider a sufficiently dense set of c's with this property, denoted $\mathbb{R}_k = \{c_j\}_{0 \geq j \geq N}$.

To diffuse along Γ_k, similar to the single resonance case, we can define the manifold N_k, S_j and η_j, and our goal is to prove existence of the interior minimum for for the sum as in (5).

One can show that for each c_i's above the corresponding $\mathcal{A}_{c_i}$ for the Lagrangian system on $\mathbb{T}^3$ can be lifted to a countable collection $\{\mathcal{A}^j_{c_j}\}_{j \in \mathbb{Z}}$ so that projection on to $ss-$component belongs to $[j - \frac{1}{2}, j + \frac{1}{2}]$. Define $B^+_{j,c}(\theta) = \inf_{\theta' \in \mathcal{A}^j_c} h^\infty_c(\theta, \theta')$ and $B^-_{j,c}(\theta) = \inf_{\theta' \in \mathcal{A}^j_c} h^\infty_c(\theta, \theta')$. Notice that θ and θ' belong to the lift N_k. We show that for $T > T_*$ there is α_j such that we have

$$h_j(\theta, \theta'; T) = \alpha_j T + B^+_{j,c_j}(\theta) + B_{j+1,c_j}(\theta') + \delta_j,$$

where δ_j is sufficiently small. For c satisfying this condition we prove that

Theorem 3.1.

$$\max_{\theta_{ss}=1} B^\pm_c(\theta) - \min_{\theta_{ss}=1} B^\pm_c(\theta) = O(\delta^a \epsilon^{-b}).$$

We add a localized potential perturbation close to $(\theta_f, \theta_s, \theta_{ss}) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ such that the Barrier function $B^{\pm}_{j,c}$ has an isolated local minimum. The diffusion for $N_{k'}$ is similar and we can glue the diffusion orbits together by using a common covering of N_k and $N_{k'}$.

4. Competition between order of resonance and distance to a KAM torus

In this section we show why we need a careful selection of symplectic coordinates near resonant segments. To illustrate the problem consider dynamics of $\mathcal{H}_\varepsilon(q,p) = \mathcal{H}_0(p) + \varepsilon\mathcal{H}_1(q,p)$ near a double resonance given by two resonant segments $\Gamma_k \cap \Gamma_{k'}$. If $k'' := k \times k'$ is sufficiently large, then typically orbits of the unperturbed system at the double resonance are periodic of length $\sim |k''|$. If $|k''| \cdot \varepsilon$ is not small, then standard averaging does not apply.

On the other side, consider a Diophantine number $\omega \in \mathcal{D}_\gamma$. Then for small ε the Hamiltonian $\mathcal{H}_\varepsilon$ has a KAM torus $\mathbb{T}_\omega$. In a certain neighborhood of $\mathbb{T}_\omega$ one can choose a Birkhoff normal form of some order m: $\mathcal{H}_\varepsilon \circ \Phi_\omega(\theta, I) = H^\omega(I) + H^\omega_1(\theta, I)$. Notice that in a ρ-neighborhood of $\mathbb{T}_\omega$ with small ρ perturbation $\|H^\omega_1\|_{C^r}$ is bounded by ρ^m. Notice now that if a double resonance $\Gamma_k \cap \Gamma_{k'}$ belongs to this neighborhood and $|k''|^3 \times \rho^m$ is small then averaging does apply and there is a hope to control dynamics. Selection of resonant segments in (2) is so that on one side resonant segments stay close enough to Diophantine numbers and on the other they fill a set of almost maximal measure.

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