Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

NEW EXAMPLES OF S-UNIMODAL MAPS WITH A SIGMA-FINITE ABSOLUTELY CONTINUOUS INVARIANT MEASURE

Jawad Al-Khal

Department of Mathematics
Howard University, DC, USA

Henk Bruin

Department of Mathematics
University of Surrey, England

Michael Jakobson

Department of Mathematics
University of MD, College Park, MD, USA

(Communicated by the associate editor)

Dedicated to Yakov B. Pesin on the occasion of his 60-th birthday

Abstract. We combine the technique of inducing with a method of *Johnson boxes* and construct new examples of S-unimodal maps φ which do not have a finite absolutely continuous invariant measure, but do have a σ-finite one which is infinite on every non-trivial interval.
We prove the following dichotomy. Every absolutely continuous invariant measure is either σ-finite, or else it is infinite on every set of positive Lebesgue measure.

1. Introduction.

1.1. Overview.

We consider non-renormalizable $\mathcal{S}$-unimodal maps $\varphi : [0,1] \to [0,1]$, with $\varphi(0) = \varphi(1) = 0$ and having no attracting periodic orbits. We refer the reader to [22] for detailed properties of $\mathcal{S}$-unimodal maps. The topological behavior of such maps is easily described. The iterates of every point, except 0 and 1, eventually fall inside an interval I' bounded by the critical value $\varphi(c)$ and its image $\varphi^2(c)$, and φ restricted to this interval is topologically mixing. In addition, the ω-limit set $\omega_\varphi(x)$ coincides with I' for x belonging to a residual subset B of I'.
It was S. D. Johnson who first showed the existence of non-renormalizable S-unimodal maps with *no finite* acim, [15]. In [10, 18, 8] the question about whether such maps have a σ-finite acim was raised, and in [11] infinite σ-finite measures were shown to exist if omega limit set of the critical point is a Cantor set. For S-unimodal maps φ the omega limit set $\omega_\varphi(x)$ is the same for Lebesgue almost every

2000 *Mathematics Subject Classification.* Primary: 37E05; Secondary: 28D05, 37C40.
Key words and phrases. sigma-finite invariant measure, interval maps, infinite measure.
HB was supported by EPSRC grant GR/S91147/01.

point x. We refer to this set $\mathcal{A}_\varphi$ as *attractor*.
First examples of maps φ such that $\mathcal{A}_\varphi = I'$, φ has no finite a.c.i.m and φ has a σ-finite a.c.i.m were constructed in [3]. For these maps the graphs of certain iterates φ_n are almost tangent to the diagonal line $y = x$, exhibiting almost saddle-node bifurcations.

Another method of constructing maps with σ-finite a.c.i.m and no finite a.c.i.m was developed in [2]. Here iterates φ^n exhibit *Johnson boxes*, [15]. We use Johnson boxes to prove our main result:

Theorem A. *There are uncountably many maps in the quadratic family that admit no finite a.c.i.m. but that have a σ-finite a.c.i.m. measure that is infinite on every interval.*

Most of the results of this paper were first proved in [2]. However, Theorems 2.1 and 2.2 were first proved in [3] and [4]. To our knowledge, the phenomenon that μ is infinite on every interval was previously encountered only in invertible dynamics (circle diffeomorphisms) by Katznelson [17, Part II, Section 2]. Our work uses the tower construction from [12], [14]. Generally tower constructions are going back to Kakutani [16].

1.2. **The Power Map T and Acim ν.** For ease of exposition we will construct our examples from the one-parameter family $\{\, \varphi_t : t \in [0, 4] \,\}$ of quadratic maps $x \mapsto tx(1 - x)$. Our procedures generalize to any *full* family of $\mathcal{S}$-unimodal maps $\varphi_t(x)$ which depend continuously on the parameter t in the C^1 topology, and have topological entropy varying between 0 and $\log 2$.

Let $G\colon I \to I$ be the *first return map* on the interval $I := [q^{-1}, q]$ bounded by the fixed point $q \in [1/2, 1]$ of φ and its second preimage $q^{-1} \in [0, 1/2]$. When $t \approx 4$, G has many monotone branches G_i and a central parabolic branch h, which we also call critical. The domains of these branches form a partition $\tilde{\xi}_0$ of I.
Our construction starts by refining $\tilde{\xi}_0$ to a partition ξ_0 with sufficiently small elements. Starting from ξ_0 we construct inductively an increasing sequence of partitions ξ_n converging to a limit partition ξ_∞ of I into a countable union of non-overlapping intervals Δ_i and a complementary Cantor set of Lebesgue measure zero such that every Δ_i is mapped diffeomorphically onto I by some iterate G^{N_i}. The *power* map T defined by $T|\Delta_i = G^{N_i}$ satisfies the conditions of the *Folklore Theorem* [1] and therefore has a unique ergodic invariant probability measure ν, which is absolutely continuous with respect to Lebesgue measure $|\cdot|$, and has a density bounded away from zero and infinity.
Since ν is ergodic, Lebesgue almost every point in I satisfies $\omega_T(x) = I$ and $\omega_\varphi(x) = [\varphi^2(c), \varphi(c)]$. Next a φ-invariant measure μ is obtained from ν by using a tower construction.

2. Tower Construction and σ-finite Measures.

2.1. The Tower Construction.

2.1.1. Given the measure ν of the power map T, one can obtain an absolutely continuous invariant measure for the map G by defining

$$\mu(\,\cdot\,) = \sum_i \sum_{j=0}^{N_i - 1} \nu(G^{-j}(\,\cdot\,) \cap \Delta_i),$$

see e.g. [12] or [22, Chapter V, Lemma 3.1]. However μ is not a probability measure, and can only be normalized if $\sum_i N_i \nu(\Delta_i) < \infty$. Our set-up will be the following. Let

$$A_{ij} = G^j(\Delta_i) \qquad (i = 0, 1, \dots ;\ j = 0, 1, \dots, N_i - 1)$$

and let $\mathcal{H}$ be the disjoint union

$$\mathcal{H} = \bigsqcup_{i=0}^{\infty} \bigsqcup_{j=0}^{N_i - 1} A_{ij}.$$

We call the sets A_{ij} (i fixed; j varies) the tower over Δ_i. As $\mathcal{H}$ is a disjoint union of subintervals of I, and each $u \in \mathcal{H}$ belongs to some A_{ij}, we can define the map $\pi\colon \mathcal{H} \to I$ by letting $\pi(u)$ be the natural inclusion of $u \in A_{ij}$ into I. Let

$$\mathcal{G}(u) = \begin{cases} \pi^{-1} \circ G \circ \pi\,(u) \cap A_{i,j+1} & \text{if } u \in A_{ij}\ (j = 0, 1, \dots, N_i - 2); \\ \pi^{-1} \circ G \circ \pi\,(u) \cap (\cup_i A_{i,0}) & \text{if } u \in A_{i,N_i - 1}. \end{cases}$$

By construction, $G \circ \pi = \pi \circ \mathcal{G}$. Define a measure ρ on $\mathcal{H}$ by

$$\rho(A) = \nu(\mathcal{G}^{-j}(A)) \quad \text{when} \quad A \subset A_{ij}.$$

Since ν is T-invariant, ρ is $\mathcal{G}$-invariant. Notice that if we view I as the base of the tower $\mathcal{H}$ then T is the *first return map* and $\nu = \rho|I$.

Put $\mu = \pi_* \rho$. By construction μ is G-invariant. As T-invariant measure ν is equivalent to the Lebesgue measure and the piecewise smooth map G maps sets of Lebesgue measure zero into sets of Lebesgue measure zero, we get from the definition that μ and Lebesgue measure have the same sets of zero measure. So μ is equivalent to the Lebesgue measure.

2.1.2. An interesting fact is that if μ is not finite then *no finite* acim exists.

Theorem 2.1. *The map φ has a finite acim if and only if*

$$\sum_i N_i |\Delta_i| < \infty. \tag{1}$$

Proof. As G is a first return map with a bounded return time, the map φ has a finite acim if and only if G has. So we prove that G has a finite acim if and only if (1) holds.

(i) The convergence of the sum in (1) above is sufficient.

Suppose that $\mu = \pi_* \rho$ is given as above, then

$$\mu(I) = \rho(\mathcal{H}) = \sum_i \sum_{j=0}^{N_i - 1} \rho(A_{ij}) = \sum_i N_i \cdot \nu(\Delta_i) < \infty,$$

where the last inequality follows from (1) because ν has a bounded density. Thus G admits a finite acim.

(ii) The convergence of the sum in (1) above is necessary.

Assume there exits a G-invariant absolutely continuous probability measure (acip), μ on I. Then by a theorem of G. Keller [19] μ lifts to an acip $\hat{\mu}$ on the *canonical Markov extension* $(\hat{I}, \hat{G})$. As was shown in [4], the power map $(T, \cup_i \Delta_i)$ with $T|_{\Delta_i} = G^{N_i}|_{\Delta_i}$ corresponds to a first return map in the Hofbauer tower. More precisely, there is a subset $\hat{\Delta}$ of $\hat{I}$ consisting of (possibly countably many) disjoint copies of $\Delta := \cup_i \Delta_i$, such that if $\hat{x} \in \hat{\Delta}$ belongs to a copy of Δ_i, then $\hat{G}^{N_i}(\hat{x})$ is the first return of $\hat{x}$ to $\hat{\Delta}$.

Since $\hat{\mu}$ is $\hat{G}$-invariant, the (non-normalized) restriction $\hat{\mu}|_{\hat{\Delta}}$ of $\hat{\mu}$ to $\hat{\Delta}$ is invariant for the first return map to $\hat{\Delta}$. Let $\pi : \hat{\Delta} \to \Delta$ be the natural projection, and $\nu = \frac{1}{\hat{\mu}(\hat{\Delta})}\pi_*\hat{\mu}_{\hat{\Delta}}$. Since $T : \Delta \to \Delta$ corresponds to the first return map to $\hat{\Delta}$, ν is a T-invariant absolutely continuous probability measure. By the Folklore theorem such measure is unique and has continuous density bounded away from zero.
Let $\hat{\Delta}_i \subset \hat{\Delta}$ be the union of intervals, which are projected onto Δ_i. For such intervals the return time equals N_i, and we get

$$\frac{1}{\hat{\mu}(\hat{\Delta})} = \frac{\hat{\mu}(\hat{I})}{\hat{\mu}(\hat{\Delta})} = \sum_i N_i \hat{\mu}(\hat{\Delta}_i) = \frac{1}{\hat{\mu}(\hat{\Delta})} \sum_i N_i \nu(\Delta_i).$$

Because the density of ν w.r.t. Lebesgue measure is bounded away from 0, it follows that $\sum_i N_i|\Delta_i| < \infty$. □

2.2. **A Property of σ-finite Acims.**

2.2.1. Consider the T-invariant measure ν (equivalent to Lebesgue measure) and the measure ρ, which is defined on the tower as indicated above, with $\mu = \pi_*\rho$. Let m denote the normalized Lebesgue measure on I.

Theorem 2.2. *Either μ is σ-finite or else $\mu(B) = \infty$ for all B with $m(B) > 0$.*

Proof. Assume μ is not σ-finite and let $\mu(B) > 0$. Then $m(B) > 0$. As μ is G-invariant

$$\mu(B) = \mu(G^{-1}B) = \mu(G^{-2}B) = \cdots$$

Let $B_0 = B$ and

$$B_n = G^{-n}(B) \setminus \left(\bigcup_{i=0}^{n-1} B_i\right) \qquad (n = 1, 2, \dots)$$

Now, consider the set

$$\mathcal{A} = \bigcup_{n=0}^{\infty} G^{-n}(B) = \bigcup_{n=0}^{\infty} B_n. \tag{2}$$

Clearly $m(\mathcal{A}) > 0$. Now, if $m(\mathcal{A}) = 1$ and $\mu(B) < \infty$, then equality (2) gives us a decomposition of I into a countable union of disjoint sets B_n of finite μ measure, contradicting that μ is not σ-finite.

On the other hand, assume $0 < m(\mathcal{A}) < 1$. Since $G^{-1}(\mathcal{A}) \subset (\mathcal{A})$ we have $T^{-1}(\mathcal{A}) \subset (\mathcal{A})$, contradicting that T is ergodic with respect to the invariant measure ν equivalent to m. □

Notice that the power map T from [14] exists if and only if the measure of the set $C = I \setminus \bigcup \Delta_i$ equals zero (see also related results in [20]). If $|C| > 0$ then the map φ has a *wild attractor*, see [5]. In that case there exists a dissipative absolutely continuous invariant measure, see [21, 6]; the latter also establishes the existence of a σ-finite acim of φ is infinitely renormalizable. Combining this with Theorem 2.2 we get for any map, whether dissipative or conservative, the following:

Theorem 2.3. *Any $\mathcal{S}$-unimodal map has either a σ-finite acim, or it has an invariant measure μ such that $\mu(B) = \infty$ for all sets B with $m(B) > 0$.*

Remark 1. If φ has a quadratic critical point, but exhibits neither almost saddle-node bifurcations nor Johnson boxes, then φ has a finite acim, see [7]. If the critical orbit is nowhere dense, then φ has a σ-finite acim μ such that $\mu(J) < \infty$ for every interval J away from the critical orbit, see e.g. [3].

3. Preliminary Construction.

3.1. The Koebe Distortion Property.

Diffeomorphisms with negative Schwarzian derivative have bounded distortion in the following sense: Let $J, I, \hat{I}$ be intervals, with $\hat{I} = L \cup I \cup R$ where L is the interval adjacent to the left of I and R to the right. Note that L and R form a *collar* around I. Suppose

$$\min\left\{\frac{|L|}{|I|}, \frac{|R|}{|I|}\right\} > \tau.$$

Then there is $c = c(\tau)$ such that every diffeomorphism $F \colon \hat{J} \to \hat{I}$ with negative Schwarzian derivative satisfies

$$1/c < \left(\frac{|F'(x)|}{|F'(y)|}\right) < c$$

for all $x, y \in F^{-1}(I)$. We refer to c as the *Koebe distortion constant*, and say that a map has *small distortion*, whenever $c = 1 + \varepsilon$ for a small ε.

3.2. The First Return Map.

3.2.1. For any $t > 3$, the quadratic map φ_t has two repelling fixed points 0 and $q_t^+ = 1 - 1/t$. Let $q_t^- = 1/t$ denote the second preimage of q_t^+ and consider the first return map G_t induced by φ_t on the interval $I := [q_t^-, q_t^+]$, then G_t has $2K$ monotone branches (diffeomorphisms) and one central parabolic branch. When $t \to 4$, $K \to \infty$.

In our construction distortion and other properties of maps G_t hold for all t within certain parameter intervals. Therefore we often suppress dependence on the parameter in the notation. Let us denote the monotone branches by $f_i \colon \Delta_i^{\pm} \to I$, where Δ_i^- denotes the domain to the left of the critical point $1/2$ and Δ_i^+ denotes the symmetrical one to the right of $1/2$ that has the same return time $i = 2, 3, \ldots, K+1$. The central parabolic branch $h_0 \colon \delta_0 \to I$ has return time $K+2$. We denote the two boundary intervals of I with return time equal to 2 by Δ_l (l for left) and Δ_r (r for right). If we let $\underline{\varphi} = \varphi_t|[0, q]$, $\varphi_0 = \varphi_t|I$, and $\overline{\varphi} = \varphi_t|[q, 1]$, then $G \colon I \to I$ is given by:

$$\begin{aligned}
f_l &= \overline{\varphi} \circ \varphi_0 | \Delta_l, \\
f_r &= \overline{\varphi} \circ \varphi_0 | \Delta_r, \\
f_i^{\pm} &= \underline{\varphi}^{i-2} \circ \overline{\varphi} \circ \varphi_0 | \Delta_i^{\pm} \qquad (i = 3, 4, \ldots, K+1), \\
h_0 &= \underline{\varphi}^{K} \circ \overline{\varphi} \circ \varphi_0 | \delta_0.
\end{aligned} \tag{3}$$

Denote the resulting partition of I by $\tilde{\xi}_0$.

3.3. Uniform Extendibility.

3.3.1. In our construction $I = [q^{-1}, q]$ is extended to some interval $\hat{I} := [a^-, a^+]$ where $a^- \in (0, q^{-1})$, and $a^+ \in (q, 1)$, are specified below. We use the notation $\hat{f} \colon \hat{\Delta} \to \hat{I}$, where $\hat{\Delta} = \Delta_L \cup \Delta \cup \Delta_R$ and $\hat{f} \colon \Delta_L \to [a^-, q^{-1}]$, $\hat{f} \colon \Delta_R \to [q, a^+]$. When the collar $\hat{I} \setminus I$ remains the same for all branches, then we refer to these extensions as *uniform* and the collar is said to be a *Uniform Extendibility Collar*.

We define an extendibility collar by choosing a^- close to q^{-1} and a^+ close to q. Then for all monotone branches except for the boundary ones which domains Δ_l and Δ_r are the boundary domains of I, respective extensions are contained inside the adjacent domains. Since the fixed point q is repelling, sufficiently small intervals adjacent to q are contracted by G_t^{-1}. Then the extensions of Δ_l and Δ_r are both contained in $\hat{I}$. That implies that compositions $f_{j_1 j_2 \cdots j_k}$ of f_i are extendible.

3.3.2. In our construction critical branches have the form $h = F \circ Q$, where F is a diffeomorphism and Q is the restriction of the initial quadratic map to a small interval δ around the critical point $1/2$. Critical branches are also called *central* and their domains are called central domains. A central branch h is said to be *extendible* if F is extendible. In which case the extension $\hat{h} = \hat{f} \circ Q$ is a critical branch defined on $\hat{\delta} \supset \delta$ whose image contains either $[a^-, q^{-1}]$ or $[q, a^+]$. In particular, the initial critical branch $h_0 \colon \delta_0 \to I$ of the first return map is extendible and its extension $\hat{h}_0$ is given by equation (3). The image of $\hat{h}_0 \colon \hat{\delta} \to \hat{I}$ contains $[q, a^+]$.

Let

$$\chi \colon \delta^{-k} \to \delta$$

be a diffeomorphism from a preimage of a central domain δ onto δ. We call χ extendible whenever it extends up to a diffeomorphism $\hat{\chi}$ onto $\hat{\delta}$.

3.4. **The Initial Partition.**

3.4.1. For the purposes of our construction it is convenient to refine $\tilde{\xi}_0$ into a partition ξ_0 with sufficiently small elements, see [13]. It is done by using consecutive pull backs of $\tilde{\xi}_0$ by monotone branches of the first return map, and by their compositions. Then we get a partition ξ_0 called *initial partition*, from which we can start our inductive construction:

$$\xi_0 \colon I = (\cup_i \Delta_i) \cup (\cup_k \delta_0^{-k}) \cup \delta_0, \tag{4}$$

where Δ_i denotes domains of uniformly extendible monotone branches, δ_0^{-k} denotes preimages of δ_0 by extendible diffeomorphisms $\chi = G^k|\delta_0^{-k}$. and δ_0 is the domain of an extendible parabolic branch h_0.

The lemma below follows from straightforward estimates of derivatives of the first return map, see [13].

Lemma 3.1. *For every $\varepsilon > 0$ we can construct the partition ξ_0 to have the following properties:*
(i) Each monotone domain has length less than ε.
(ii) The aggregate sum of lengths of the "holes" δ_0^{-k} is less than ε.
(iii) The Extendibility Collar does not depend on ε.

Let us describe one property of ξ_0 which is used later. When $t = 4$, the first return map G_4 has an infinite number of monotone branches that converge toward the middle point $1/2$ and has no central parabolic branch. There exists a constant c_0, such that, $|\Delta_j| < \frac{c_0}{2^j}$ for every j, see [13].

Let us now suppose that G_t has $2K$ monotone branches and one central branch where K is extremely large. Then choose a large index $j_0 \ll K$ such that

$$\frac{c_0}{2^{j_0}} < \varepsilon \tag{5}$$

and consider the initial partition $\tilde{\xi}_0$ described in Section 2. When constructing ξ_0 out of $\tilde{\xi}_0$ we do not change the branches with indices $j \geq j_0$.
Expansions of all monotone branches f_j besides possibly the two branches $f_K^{\pm}$ next to the middle central branch satisfy

$$\left|\frac{df_j}{dx}\right| > c_1 2^j \tag{6}$$

and we assume that $c_1 2^j$ is large for $j \geq j_0$. If the height of the parabolic branch is small, then derivatives of $f_K^{\pm}$ can be small. However in our construction, we choose the position of the critical value $h_0(1/2)$ above $1/2$. Then the distance between Δ_K and the critical point is comparable to the size of Δ_K and the derivatives of $f_K^{\pm}$ will also satisfy (6).

4. **Construction of Partitions.**

4.1. **The Basic Step.**

4.1.1. Starting from ξ_0 we construct inductively an increasing sequence of partitions $\xi_0 \prec \xi_1 \prec \ldots \prec \xi_n \prec \ldots$ We assume by induction that after step $n-1$ we have constructed the following partitions $\xi_m,\ 0 \leq m \leq n-1$ of I:

$$\xi_m : I = (\cup \Delta) \cup (\cup_i \cup_k \delta_i^{-k}) \cup \delta_m \cup C_m.$$

Here $0 \leq i \leq m$, the collection $\{\Delta\}$ are monotone domains mapped onto I by uniformly extendible diffeomorphisms, δ_m is the domain of the extendible central parabolic branch and each δ_i^{-k} is a preimage of some δ_i by an extendible diffeomorphism χ. Sets C_m are Cantor sets with zero Lebesgue measure and $C_0 = \emptyset$. Partitions ξ_m and associated maps are defined for t which belong to parameter intervals $\Lambda_m \subset \Lambda_{m-1},\ 0 \leq m \leq n-1$. Notice that elements Δ of ξ_m are not changed at subsequent steps of induction, but δ_i^{-k} and δ_m are substituted by the new Δ, δ_j^{-k} and δ_{m+1}. Sometimes we call Δ *good intervals*, and we call δ_i^{-k} *holes*.

Depending on the step of induction we use one of the several operations described below. In particular the following operation is used throughout our construction.

(1) *Monotone Pullback:* Suppose

$$f_0 \colon \Delta_0 \to I$$

is a monotone branch and let ξ denote a partition of I. Then we refer to $f_0^{-1}(\xi)$ as the monotone pullback of the partition ξ onto Δ_0. This creates a partition of Δ_0 into domains of various types. For every domain J of the partition ξ we have the corresponding domain $f_0^{-1}(J) \subset \Delta_0$.

4.1.2. Let ξ_m, $0 \le m \le n-1$, be a partition constructed at the previous steps of induction. Assume the critical value $h_{n-1}(1/2)$ belongs to a certain element $\Delta_m^* \in \xi_m$. We refer to this as a *Basic step* and we proceed with the construction of the partition ξ_n using the following procedures.

(2) *Critical Pullback*: We induce on δ_{n-1} the partition $h_{n-1}^{-1}(\xi_m)$ thus creating preimages of all the elements of ξ_m that are contained in the image of h_{n-1}. This gives us domains inside δ_{n-1} of branches of the following type:

- Two new monotone branches $f \circ h_{n-1}$ for each monotone domain $\Delta(f)$ which lies inside the image of h_{n-1}.
- A central parabolic branch $h_n := f_n^* \circ h_{n-1}$, where $f_n^* \colon \Delta_n^* \to I$ is the monotone branch containing the critical value $h_{n-1}(1/2)$.
- We also obtain the diffeomorphisms $\chi \circ h_{n-1}$ from the corresponding diffeomorphisms $\chi \colon \delta_i^{-k} \to \delta_i$ of ξ_{n-1}. When the range of h_{n-1} contains the central domain δ_m, we also get two preimages $h_{n-1}^{-1}(\delta_m)$.

(3) *Grow-up procedure*: It may be that the range of the central branch $h_{n-1}(\delta_{n-1})$ is contained in the rightmost boundary domain Δ_r of the initial partition ξ_0, or in the leftmost boundary domain Δ_l. Notice that Δ_l and Δ_r are good intervals which are not changed at subsequent steps of induction, so they are as well the boundary domains of all partitions ξ_n. Then we replace the central branch respectively, by

$$f_l^m \circ h_{n-1} \quad \text{or} \quad f_l^{m-1} \circ f_r \circ h_{n-1},$$

where m is the smallest number such that the image of the new central branch covers more than just a boundary interval. The domain of definition of the new central branch remains the same, and we keep the same notation h_{n-1}.

(4) Extra Pullback Procedure: In our estimates on the measure of holes in Chapter 5 we use that ratio $|\delta_n|/|\delta_{n-1}|$ is small. According to Lemma 3.1, all elements belonging to the preliminary partition are of length less than ε. If the image of the central branch h_{n-1} covers more than half the length of I, then

$$\frac{|\delta_n|}{|\delta_{n-1}|} \le c\sqrt{2\frac{|\Delta_{n-1}^*|}{|I|}}$$

is small. However, if the image of h_{n-1} does not cover that much, then the length of Δ_{n-1}^* may be comparable to the height of that image. Therefore we introduce the following rule of *Extra Pullback*.

If $|\mathrm{Im}(h_{n-1})| < \frac{1}{2}|I|$, then we do one extra monotone pullback of ξ_0 onto Δ_{n-1}^* which ensures that after critical pullback the ratio

$$\frac{|\delta_n|}{|\delta_{n-1}|} \le \varepsilon_1 \tag{7}$$

is small. Here ε_1 depends on our choice of ε.

(5) *Boundary Refinement Procedure:* Suppose $F \colon \Delta \to I$ is an extendible monotone branch, where $\Delta \in \xi_m$, $\Delta \subset h_{n-1}(\delta_{n-1})$, and $h_{n-1}(1/2) \notin \Delta$. If Δ is too close to $h_{n-1}(1/2)$ then when we do critical pullback onto δ_{n-1}, the monotone domain $h_{n-1}^{-1}(\Delta)$ may be not extendible. In which case, we perform the *boundary refinement procedure* as follows:

The initial partition (4) contains the boundary branch $f_r \colon \Delta_r \to I$ which has a repelling fixed point q. We refine Δ_r by *monotone pullback*, thus creating the partition $f_r^{-1}(\xi_0)$ which has a boundary domain Δ_{rr} adjacent to q. Then we refine Δ_{rr} by *monotone pullback* of ξ_0 by f_r^{-2} and so on. The k^{th} step refinement creates a copy of ξ_0 on $\Delta_{\underbrace{rr\ldots r}_{k}}$ contracted approximately by $|f_r'(q)|^{-k}$. We call the resulting partition the k^{th} *right boundary refinement of* ξ_0; it is denoted by $\xi_{0,k}$. After constructing such a partition on Δ_r, we pull back $\xi_{0,k-1}$ by f_l onto the leftmost boundary interval Δ_l of ξ_0 to create the k^{th} *left boundary refinement of* ξ_0 denoted by $\xi_{k,0}$ with the most left interval $\Delta_{\underbrace{lrr\ldots r}_{k}}$. As the sizes of extensions of $\Delta_{\underbrace{rr\ldots r}_{p}}$ and $\Delta_{\underbrace{lrr\ldots r}_{p}}$ decrease exponentially there exists k such that all elements of $h_{n-1}\circ F^{-1}\xi_{0,k}$ or respectively $h_{n-1}\circ F^{-1}\xi_{k,0}$ are extendible.

Remark 2. . Notice that Δ remains unchanged. Its refinement is used to construct uniformly extendible monotone branches during the critical pullback. After doing boundary refinement for all elements Δ which need it, we get a partition of δ_{n-1}, which we denote by

$$\eta_{n-1} : \delta_{n-1} = \delta_n \cup (\cup\Delta) \cup (\cup_i \cup_p \delta_i^{-p}) \qquad (\text{mod } 0).$$

Notice that by construction at every step $i = 0, 1, \ldots, n-1$, similar partitions ξ_{i-1} and η_{i-1} are defined.

(6) *Filling-in:* We fill each preimage

$$\delta_j^{-k} = \chi^{-1}(\delta_j) \quad j = 0, 1, \ldots, n-1$$

with the pullback $\chi^{-1}(\eta_j)$. In this way we get a 'copy' of the elements of η_j inside each δ_j^{-k}.

After the above operations we get a new partition ξ_n which has the form

$$\xi_n = (\bigcup \Delta) \cup (\bigcup_{j\le n}\bigcup_{p>0} \delta_j^{-p}) \cup \delta_n. \tag{8}$$

Here all the monotone domains Δ are uniformly extendible due to the boundary refinement. Moreover, as explained in the next section, we choose the position of the critical value in such a way that all maps from δ_j^{-k} onto δ_j have small distortions.

4.2. Enlargements.

4.2.1. When constructing the partitions ξ_n we emphasized that the critical value $h_n(1/2)$ falls in a monotone domain. Clearly that excludes $h_n(1/2)$ from being inside a hole δ_i^{-k}. However, we will add the assumption that the critical value does not belong to an enlargement of δ_i^{-k} which we will define below. For δ_0 we define

$$\delta_0 = \bigcup_{m=2j_0}^{K} (\Delta_m^{\pm} \cup \delta_0)$$

where j_0 is defined by (5). Next we define *enlargements* as follows. If δ_i is a central domain of a basic step, then $\tilde{\delta}_i = \delta_{i-1}$. However if δ_i is a central domain of a Johnson step, then $\tilde{\delta}_i = H_i$, where H_i is a small subset of δ_{i-1} , see below.

When we apply the critical pullback procedure, we make sure that the critical value does not belong to the union of enlargements $\bigcup \tilde{\delta}_i$.

Then for any hole $\delta_i^{-k} = h_n^{-1}\delta_i^{-m}$, the restriction of h_n to δ_i^{-k} can be extended up to a diffeomorphism from $\tilde{\delta}_i^{-k}$ onto $\tilde{\delta}_i$ and respectively the enlargement $\tilde{\delta}_i^{-k}$ is well defined, and for any $\delta_i^{-k} \subset \delta_n$ its enlargement $\tilde{\delta}_i^{-k}$ also belongs to δ_n. As any hole is a diffeomorphic preimage of the respective central domain we get that if δ_i^{-k} is obtained by filling in of δ_j^{-p} then $\tilde{\delta}_i^{-k} \subset \delta_j^{-p}$.

Below we prove that the measure of the union $\cup_i \cup \delta_i^{-k}$ of holes at step n tends to zero when $n \to \infty$. The same holds for the measure of enlargements, because by construction the union of enlargements of step n is a subset of the union of the holes of step $n-1$. So the above choice of the position of the critical value outside of the enlargements is possible.

4.2.2. Recall that for all domains Δ, except Δ_r, Δ_l, extensions $\hat{\Delta}$ are contained in I and $\hat{\Delta}_r, \hat{\Delta}_l \subset \hat{I}$. Therefore extensions of $h_{n-1}^{-1}(\Delta)$ are contained in δ_{n-1}, and extensions of $h_{n-1}^{-1}(\Delta_r)$, $h_{n-1}^{-1}(\Delta_l)$ are contained in $\hat{\delta}_{n-1}$. At a basic step when we construct a new central domain δ_n, its extension is the critical pullback of the extension $\hat{\Delta}^*$ of the monotone domain Δ^* which contains the critical value. Therefore $h_n^{-1}(\Delta^*) \subset \delta_{n-1}$. As a result $\hat{\delta}_n \subset \delta_{n-1} = \tilde{\delta}_n$. The same holds at the Johnson step, see below. So all diffeomorphisms mapping δ_i^{-k} onto δ_i are extendible and extensions of their domains are subsets of respective enlargements.

4.2.3. For δ_n constructed at a basic step we have

$$\frac{|\delta_n|}{|\tilde{\delta}_n|} \leq \varepsilon_1 \tag{9}$$

for some small ε_1 determined by the sizes of elements in the preliminary partition ξ_0.

At a Johnson step described in Section 4.3, (9) holds as well. As all diffeomorphisms $\chi : \delta_i^{-k} \mapsto \delta_i$ are extendible up to $\tilde{\delta}_i^{-k} \mapsto \tilde{\delta}_i$ we obtain from the Koebe property that their distortions are small.

4.3. **The Delayed Basic or Johnson Step.**

4.3.1. At certain induction steps we use the method of S. Johnson [15] to get an infinite acim. We select parameter values such that $h_{n-1}(\frac{1}{2}) \in \delta_{n-1}$, the image of h_{n-1} contains $\frac{1}{2}$, $h_{n-1}(\frac{1}{2})$ is close to $\frac{1}{2}$, but the map remains non-renormalizable. According to terminology of [14] such steps are called *delayed basic*. We shall also call them *Johnson steps*.

After we construct the partition ξ_0 at the preliminary step, it is convenient to make a Johnson step.

4.3.2. *The Johnson Box:* Let h_0 be the parabolic branch of G. Note that the first return map reverses orientation and consequently h_0 has a minimum at the critical point. We choose an initial parameter interval Λ_0, so that for $t \in \Lambda_0$, $h_0(1/2) \in \delta_0$ with $h_0(1/2) < 1/2$. Then the image of h_0 contains all the domains of ξ_0 that are located to the right of δ_0. We define a *Johnson box* as the interval $B_0 = [q_0, q_0^{-1}]$ where q_0 is one of the two fixed point of h_0, the one which is farther away from $1/2$, and $q_0^{-1} = h_0^{-1} q_0$. Since we choose our maps non-renormalizable, we place the critical value outside of $[q_0^{-1}, q_0]$. We call the part of the graph outside this box the *hat* and denote its base by H_0.

4.3.3. *Constructing the First Step of the Staircase.* Let us denote $h_{0,\mathrm{right}}^{-1}$ the inverse branch of h_0^{-1} which image is on the right of $1/2$, and by $h_{0,\mathrm{left}}^{-1}$ the second branch. Then $\mathcal{S}_1 = h_{0,\mathrm{left}}^{-1}\xi_0 \cup h_{0,\mathrm{right}}^{-1}\xi_0 = \mathcal{S}_{1,\mathrm{left}} \cup \mathcal{S}_{1,\mathrm{right}}$ is the first step of the staircase.

4.3.4. *The Infinite Staircase Construction.* We proceed by constructing the *infinite staircase* $\mathcal{S} = \cup_{j\geq 1}\mathcal{S}_j$ where each $\mathcal{S}_j$ consists of two components $\mathcal{S}_{j,\mathrm{left}} = h_{0,\mathrm{left}}^{-1}\mathcal{S}_{j-1,\mathrm{right}}$ and $\mathcal{S}_{j,\mathrm{right}} = h_{0,\mathrm{right}}^{-1}\mathcal{S}_{j-1,\mathrm{right}}$ symmetric about $1/2$. These preimages are adjacent and form an infinite staircase

$$\mathcal{S} = \mathcal{S}_{\mathrm{left}} \cup \mathcal{S}_{\mathrm{right}}.$$

They are outside the Johnson box, in fact $\mathcal{S} = \delta_0 \setminus B_0$.

4.3.5. *Filling-in the Box.* Define

$$r_0 = \min\{\, r : h_0^r(1/2) \notin \delta_0 \,\}.$$

We choose parameter values such that $h_0(\frac{1}{2})$ belongs to some monotone domain $\Delta \subset \mathcal{S}_{r_0,\mathrm{left}}$. Then we fill the base of the hat H_0 by critical pullback $h_0^{-1}\left(\bigcup_{j=r_0}^{\infty} \mathcal{S}_{j,\mathrm{left}}\right)$ thus creating a new partition ζ_0 inside H_0, which in particular contains a new critical branch

$$h_1 := f_0^* \circ h_0^{r_0}.$$

Here f_0^* is the monotone branch whose domain $\Delta_0^* \in \mathcal{S}_1$ contains the iterate $h_0^{r_0}(1/2)$ of the critical point.

Restricting h_0 to the two symmetric intervals of $B_0 \setminus H_0$, we obtain two monotone maps g_1, g_2. Since $H_0 \neq \emptyset$, we get that g_1, g_2 and all their iterates are uniformly extendible branches of an $\mathcal{S}$-unimodal map. Thus they have uniformly bounded distortions. Then B_0 is a countable union of preimages $g_{i_k}^{-1} \circ \ldots \circ g_{i_1}^{-1} H_0$ and a Cantor set of zero Lebesgue measure. Therefore we get a partition (mod 0) of B_0 which is a union of ζ_0 and all pullbacks $g_{i_k}^{-1} \circ \ldots \circ g_{i_1}^{-1}\zeta_0$. We combine that partition with staircases and get the partition η_0 of δ_0

$$\eta_0 : \delta_0 = (\cup\Delta) \cup (\cup\delta_0^{-p}) \cup (\cup\delta_1^{-p}) \qquad (\mathrm{mod}\ 0). \tag{10}$$

4.3.6. *Staircases and Extendibility.* Consider the domains $\Delta_i, i > j_0$ of the partition ξ_0. These domains are not refined at the preliminary construction because their sizes are small enough. Then, as for all domains of the first return map, the extensions of Δ_i are contained inside the adjacent domains Δ_{i-1} and Δ_{i+1}. And the left and right extensions of the central domain $\delta_0 = \Delta_N$ are contained inside $\Delta_{N_\pm}$. Notice that the extensions of the boundary elements of the partition $h_0^{-1}\xi_0$ are inside extensions of δ_0. Thus they are contained respectively inside Δ_{N_+} and Δ_{N_-}.

Each step $\mathcal{S}_k$ has two boundary elements $\Delta_{k,int}$ located closer to the critical point and $\Delta_{k,ext}$. As $\mathcal{S}_k = h_0^{-1}\mathcal{S}_{k-1}$ we get that extension of $\Delta_{k,ext}$ is contained inside $\Delta_{k-1,int}$.

As δ_0 is small, $\mathcal{S}_1$ is the preimage of almost one half of the interval I. Thus $\mathcal{S}_1$ covers almost one half of δ_0 and all remaining steps cover a small fraction of δ_0. By choosing a small extendibility collar we ensure that interior extensions of $\Delta_{N_\pm}$ are contained respectively inside $\mathcal{S}_{1,left}$ and $\mathcal{S}_{1,right}$. Moreover these extensions do not intersect the domains $h_0^{-1}\Delta_i, \;\; i > j_0$ located in the "middle" of $\mathcal{S}_1$.

The extension of $\Delta_{k-1,int}$ is contained inside $\mathcal{S}_k$ and moreover does not intersect preimages $h_0^{-k}\Delta_i,\ \ i > j_0$ located in the "middle" of $\mathcal{S}_k$. This implies

Corollary 1. *For every k and for every domain $h_0^{-k}\Delta_i,\ \ i > j_0$ located inside $\mathcal{S}_k$ one can choose the position of the critical value of h_0 inside $h_0^{-k}\Delta_i$ so that all maps constructed at Johnson step are extendible.*

In this situation we do not need to do boundary refinement and η_0 constructed above is the partition of δ_0 at the first step of induction.

Finally we get the partition ξ_1 of I by filling-in every element δ_0^{-k} of ξ_0 by the pullback of the partition η_0.

4.3.7. Now let $n = n_k$ be a step of induction, when the k-th Johnson step occurs. Then $h_{n-1}(1/2) \in \delta_{n-1}$ and $1/2 \in \operatorname{Im}(h_{n-1})$. We define

$$r_k = \min\{\, r : h_{n-1}^r(1/2) \notin \delta_{n-1}\,\}.$$

We define the *Johnson box* B_k bounded by the points q_k, q_k^{-1} where q_k is one of the two fixed point of h_{n-1} — the one farther away from $1/2$ — and $q_k^{-1} = h_{n-1}^{-1} q_k$. The part of the graph, which contains the critical value and is located outside this box is called the *hat*. We denote its base by H_{n-1}. As in the first step we construct an *infinite staircase* $\mathcal{S} = \cup_{j\geq 1}\mathcal{S}_j$ where each $\mathcal{S}_j$ consists of two components $\mathcal{S}_{j,\mathrm{left}}$ and $\mathcal{S}_{j,\mathrm{right}}$, symmetric about $1/2$.

As at the first Johnson step we can choose parameter in such a way that the critical value $h_{n-1}(1/2)$ belongs to a preimage of one of the elements $\Delta_i, i > j_0$ of ξ_0, and boundary refinement is not needed at Johnson step. Then we fill the base of the hat H_k by using critical pullback. In particular, we get a new critical branch $h_n := f_n^* \circ h_{n-1}^{r_k}$. Here f_n^* is the monotone branch whose domain Δ_n^* contains $h_{n-1}^{r_k}(1/2)$. Restricting h_{n-1} to the two symmetric intervals of $B_k \setminus H_k$, we obtain two monotone maps g_1 and g_2. So, as before almost every point of $B_k \setminus H_k$ under the iterations of g_1 and g_2 eventually 'escapes' the box through H_k. The preimages of the partition of H_k under the two monotone branches g_1 and g_2 generates a partition of $B_k \setminus H_k$ (modulo a Cantor set of zero Lebesgue measure). This partition of B_k adjoined with that of the staircase $\mathcal{S}$ constitute the desired partition η_{n-1} of δ_{n-1}. Finally, the partition (8) is obtained by filling in each domain δ_j^{-k} of ξ_{n-1}.

4.4. **The Limit Partition.**

4.4.1. Let

$$\mathcal{H}_{n-1} = \bigcup_{j<n;p\geq 0} \delta_j^{-p}$$

denote the collection of holes. At each step of induction we construct domains of monotone branches which are not changed anymore, domains δ_i^{-k} which are filled-in at the next steps and Cantor sets of zero measure. As δ_j^{-k} are mapped onto δ_j with uniformly bounded distortions, the relative measure of new holes obtained after the filling-in of δ_j^{-k} is bounded away from one, if and only if the measure of the new holes obtained at step $j+1$ inside δ_j, is bounded away from one. This implies

Proposition 1. *Suppose that at each step n of our construction the relative measure of $\mathcal{H}_{n-1}$ within δ_n is less than a uniform constant $\theta < 1$. Then as $n \to \infty$*

we obtain a limiting partition $\xi = \xi_\infty$ of I consisting of an infinite number of uniformly extendible domains Δ_i of monotone branches $f_i\colon \Delta_i \to I$ and a Cantor set of Lebesgue measure zero.

As $|\Delta_i| < \varepsilon$, where ε can be made arbitrary small and distortions of f_i are bounded by a constant independent of ε, we get that for any $R > 1$ one can find $\varepsilon > 0$ such that expansions of all f_i in Proposition 1 are greater than R. Under the conditions of Proposition 1 above, we obtain that all monotone branches f_i are expanding and have uniformly bounded distortion.

5. **The Proof of Theorem A.**

5.1. **Preliminary Definitions.**

5.1.1. (i) In the course of our construction we need to keep track of certain quantities associated with the successive partitions ξ_n. Let

$$\mathcal{H}_n := \bigcup_{j\leq n} \bigcup_{p>0} \delta_j^{-p} \tag{11}$$

denote the union of all holes δ_j^{-p} at step n. These are preimages of central domains δ_j for $j = 0, 1, 2, \ldots, n$, which are elements of ξ_n. Let $\alpha_n = |\mathcal{H}_n|$ be the Lebesgue measure of $\mathcal{H}_n$.

(ii) If Δ needs a boundary refinement, then we define $R_n(\Delta)$ to be the minimal number of boundary refinements needed so that all new elements constructed inside $h_{n-1}^{-1}(\Delta)$ are extendible.

(iii) If $n = n_k$ is a delayed basic step and we have the box B_k and the base of the hat H_k we will have the ratio

$$|H_k|/|B_k| \leq \beta_k.$$

where β_k, to be specified later, is chosen in advance to be small enough in order that the acim μ is on the one hand

5.2. **Strategy of the Construction.**

5.2.1. The examples we give are constructed by a decreasing sequence of nested parameter intervals Λ_n such that for all $t \in \Lambda_n$ the map φ_t admits the partition ξ_n as described in Section 4 In addition, we will arrange that ξ_n satisfies certain conditions specified below, so that for $t = \cap_n \Lambda_n$, φ_t has a non-integrable invariant density. At each step either $h_n(1/2)$ falls in a monotone domain Δ_n^* created at one of the previous steps (*Basic Case*); Or $h_n(1/2)$ is "delayed" in δ_n and falls instead in a preimage of a monotone domain Δ_n^* belonging to ξ_n, so that $h_n^{r_n}(1/2) \in \Delta_n^*$ (*Delayed Basic Case*). Notice that in the latter case $h_n(1/2)$ still falls in a monotone domain, except that this monotone domain is created at the current step, that is, it belongs to the partition ξ_{n+1}.

Thus, in either situation, the critical value falls in a domain which is mapped *onto* I by a monotone branch. It follows from the monotonicity of the kneading invariant, (see [9]), that if the critical value enters a certain domain $\Delta = [a_1, a_2]$, say through a_1 when the parameter $t = t_1$, then it remains inside Δ until the parameter reaches

$t = t_2$ when it then leaves Δ through a_2. Therefore, by varying the parameter, we can arrange that the new critical value

$$h_{n+1}(1/2) = \begin{cases} f_n^* \circ h_n(1/2) & \text{at a basic step,} \\ f_n^* \circ h_n^{r_n}(1/2) & \text{at a delayed basic step,} \end{cases}$$

is mapped anywhere in I. In this way, we can ensure that the forward G-orbit of the critical point is dense, i.e., $\omega_G(1/2) = I$ and hence $\omega_{\varphi_t}(1/2) = [\varphi_t^2(1/2), \varphi_t(1/2)]$.

Moreover, every time the critical value $h_n(1/2)$ is delayed in the box, the level of the staircase, r_n, which contains $h_n(1/2)$, as well as the size of the hat can be chosen independent of the topological requirements on the critical orbit because each level of the infinite staircase consists of the preimage of the previous level.

5.2.2. Every monotone branch $f_i : \Delta_i \to I$ is by construction a composition of iterates of the first return map G. Accordingly $f_i = G^{N_i}|\Delta_i$ and we call N_i the *power* of f_i. Every critical branch h_n can be factored into $h_n = F_n \circ h_0$, where F_n is a composition of monotone branches and h_0 is the central parabolic branch of the first return map G restricted to a small neighborhood of the critical point. In this case, we define the power of h_n as 1 plus the sum of powers of each of the monotone branches in the composition F_n. Notice that in this sense, the power of all branches of the first return map G is 1, and all monotone branches can be factored into compositions of branches of G. In terms of the *Tower Construction* of Chapter 2, N_i corresponds to the number of domains in the tower over Δ_i and thus may be referred to as the *height* of Δ_i.

We define a map $T \colon I \to I$ piecewise by

$$T|\Delta_i = f_i := G^{N_i}|\Delta_i \colon \Delta_i \to I,$$

and T is expanding with uniformly bounded distortion for all branches f_i. It satisfies the hypothesis of the so-called Folklore Theorem, see [1]. Therefore T has an ergodic acim ν with a density function that is continuous and bounded away from zero.

5.2.3. The G-invariant measure μ on I given by the formula

$$\mu(E) = \sum_i \sum_{j=0}^{N_i-1} \nu(\Delta_i \cap G^{-j}E) \tag{12}$$

for every measurable set $E \subset I$. Since G is a smooth map, formula (12) implies that μ is an absolutely continuous invariant measure. Since ν has a bounded density, $\mu(E) < \infty$ if and only if

$$\Sigma := \sum_i \sum_{j=0}^{N_i-1} \left|(\Delta_i \cap G^{-j}E)\right| \tag{13}$$

converges, and μ is finite if and only if

$$\sum_i N_i|\Delta_i| < \infty. \tag{14}$$

Our aim is to construct the map T in such a way that:

(A) The convergence of the sum in (14) does not hold.
(B) There exists a set E with positive Lebesgue measure for which the sum Σ in (13) converges.
(C) The μ-measure of every interval is infinite.

Theorem 2.1 implies by property (B) that the measure μ is σ-finite.

5.3. The Parameter Choice Lemma.

5.3.1. The first return map $G\colon I \to I$ induced by φ_t has $2K$ monotone branches for all parameter values t inside an interval of parameter denoted by (t_K, t_{K+1}). When $t = t_{K+1}$, the critical branch splits into two new monotone branches and a new critical branch is born in between.

So, our first parameter interval is given by $\Lambda_0 = [t_K, t_{K+1}]$ and for all parameter values in the interior of Λ_0, a partition ξ_0 is defined and its elements vary continuously with t. In the course of our construction we determine a nested sequence of closed parameter intervals $\Lambda_n \subset \Lambda_{n-1}$ such that for all parameter values $t \in \Lambda_n$, φ_t induces the partition ξ_n with desired properties. Then for

$$\tau = \cap_i^\infty \Lambda_i \tag{15}$$

we obtain the limit maps φ_τ, G_τ and the limit partition ξ_∞ corresponding to the power map T_τ. In order to do this, we will use the following lemma.

Lemma 5.1. (Parameter Choice Lemma) *At each step n, there exists a parameter interval $\Lambda_n \subset \Lambda_{n-1}$, such that as t varies in the interior of Λ_n, the following two properties hold:*

- $(i)_n$ *All intervals of the partition ξ_n vary continuously, in particular none of them disappear and no new ones appear.*
- $(ii)_n$ *The critical value $h_n(1/2)$ moves continuously across the whole interval I.*

Proof. Assume by induction that the two properties $(i)_j$ and $(ii)_j$ hold for all $j \leq n$. Then using $(i)_n$, continuity of $h_n(1/2)$ and monotonicity of the kneading invariant ([9]), we get that given a prescribed element $\Delta \in \xi_n$, which is a domain of a monotone branch, there exists a parameter subinterval $\Lambda_{n+1} \subset \Lambda_n$ such that when $t \in \Lambda_{n+1}$, $h_n(1/2)$ moves all the way through Δ without leaving Δ. According to our inductive construction of Chapter 4, the next central branch is $h_{n+1} = F_n \circ h_n$, where $F_n = f_n^*$ at a basic step, and $F_n = f_n^* \circ h_n^{r_n}$ at a delayed basic step. Since, in both cases, F_n maps Δ *onto* the whole interval I, it follows that $h_{n+1}(1/2)$ satisfies $(ii)_{n+1}$ as $h_n(1/2)$ moves across the interval Δ. Next, since $h_n(1/2)$ depends continuously on the parameter t and stays inside the domain Δ when $t \in \Lambda_{n+1}$, the new partition of δ_n which we had denoted by η_n will satisfy $(i)_{n+1}$. Moreover, the new branches of the partition ξ_{n+1} constructed outside δ_n are compositions of branches of ξ_n with those branches inside δ_n. As both vary continuously, all new branches satisfy $(\mathrm{i})_{n+1}$. □

5.4. Generating Partitions.

5.4.1. In this section, we define an additional sequence of partitions which allows us to ensure that the forward orbit of the critical point is dense in I. Using the sequence of partitions ξ_n constructed in Chapter 4, we define a sequence of partitions $\mathcal{P}_n \succeq \xi_n$ as follows:

Let $\mathcal{P}_0 = \xi_0$ be the preliminary partition constructed in Section 3.

Let $\mathcal{P}_{n-1} \succeq \xi_{n-1}$ be the partition of step $n-1$. By construction elements of $\mathcal{P}_{n-1}$ are of the same types as elements of ξ_{n-1} : domains Δ of monotone branches and δ_i^{-k}, $0 \leq i \leq n-1$, $k \geq 0$.

We construct $\mathcal{P}_n$ by refining elements of $\mathcal{P}_{n-1}$ as follows.

1. We are doing filling-in for each element δ_i^{-k}.
2. For each Δ which size exceeds $\frac{1}{3^n}$ we pull-back on Δ the partition ξ_0.

Remark 3. Sizes of elements depend on the parameter. So we are doing the above partition if the size of Δ is too large at least for one parameter value under consideration.

5.4.2. When constructing ξ_0 we made expansions of all monotone branches of ξ_0 greater than some $R \gg 1$, and at the same time kept distortions bounded by $c(tau)$ independently of R. If R is big enough, then the above construction provides elements Δ in $\mathcal{P}_n$ with sizes less than $\frac{c}{3^n}$. At the same time sizes of holes in ξ_n and respectively in $\mathcal{P}_n$ satisfy $\delta_i^{-k} < \varepsilon^n$, where ε is a small constant. Therefore sizes of elements in the increasing sequence of partitions $\mathcal{P}_n$ decrease uniformly. So if the critical orbit eventually visits every element of every $\mathcal{P}_n$, then the T-orbit and hence the G-orbit of the critical point is dense in I.

5.5. **Positioning the Critical Value at a Johnson Step.**

5.5.1. In this section we describe how to achieve at step n the following two properties.

(i) The trajectory of the critical point visits certain good intervals between two consecutive delayed basic steps.
(ii) Given a sequence of numbers γ_k, at each delayed basic step $n = n_k$, the hat is so small that the ratio $|H_k|/|B_k| < \gamma_k$.

We may start the construction of Chapter 4 with a delayed basic step, that is

$$h_0(1/2) \in \delta_0, \dots, h_0^{r_0-1}(1/2) \in \delta_0 \quad \text{and} \quad h_0^{r_0}(1/2) \in I \setminus \delta_0.$$

Let $\Delta_0^* \in \xi_0$ denote the monotone domain that contains $h_0^{r_0}(1/2)$.

The idea is to look ahead. Since $h_0^{r_0}(1/2)$ falls in a monotone domain Δ_0^* that is mapped *onto* the whole interval I, the location of $h_0(1/2)$ may be chosen so as for some finite collection of good intervals Δ of $\mathcal{P}_0$ there correspond basic steps such that $h_j(1/2) \in \Delta$. This determines a sequence of basic steps $j = 1, 2, \dots, n_1 - 1$. Then the following step is delayed basic: $h_{n_1}(1/2) \in \delta_{n_1}$. For each of these basic steps we let f_j^* denote the monotone branch whose domain Δ_j^* contains the critical value $h_j(1/2)$. Then $h_{j+1} = f_j^* \circ h_j$, and

$$h_{n_1} = f_{n_1-1}^* \circ f_{n_1-2}^* \circ \cdots \circ f_0^* \circ h_0^{r_0}.$$

Therefore the above requirement on the critical value for steps $n = 1, 2, \dots, n_1$ is that the collection of domains

$$\Delta_1^*, \Delta_2^*, \dots, \Delta_{n_1-1}^*$$

includes a given collection of good intervals of $\mathcal{P}_0$. Notice that this requirement is independent of the value of r_0 which is chosen so large that $|H_0|/|B_0| < \gamma_0$ for any prescribed γ_0.

Using the Parameter Choice Lemma for each of the steps $n = 1, 2, \dots, n_1$, we obtain a sequence of parameter intervals

$$\Lambda_0 \supset \Lambda_1 \supset \cdots \supset \Lambda_{n_1}$$

such that for $t \in \Lambda_{n_1}$, the trajectory of the critical point has the properties described above.

Observe that Λ_{n_1} contains a subinterval such that when the parameter runs through this subinterval, $h_{n_1}(1/2)$ moves across the staircase $\mathcal{S} = \cup_j \mathcal{S}_j$ belonging to δ_{n_1}. Now, $h_{n_1}^{r_1}(1/2)$ falls in a monotone domain $\Delta_{n_1}^* \in \xi_{n_1}$, and so the location of the critical value $h_{n_1}(1/2)$ may be chosen so that for the next series of basic steps, the critical value $h_j(1/2)$, $(j = n_1+1, n_1+2, \ldots, n_2-1)$, falls in a prescribed collection of good intervals in $\mathcal{P}_1$. At the same time we can take r_1 arbitrary large, which will make $\frac{|H_1|}{|B_1|} < \gamma_1$ for any given γ_1.

Then follows a delayed basic step n_2. At that step $h_{n_2}(1/2) \in \delta_{n_2}$ and $h_{n_2}^{r_2}(1/2)$ is the smallest iterate of $h_{n_2}(1/2)$ outside the central domain δ_{n_2}. At that step we choose $|H_2|/|B_2| < \gamma_2$. At the next basic steps $j = n_2+1, n_2+2, \ldots, n_3-1$, the critical value $h_j(1/2)$ visits a prescribed collection of good intervals of $\mathcal{P}_2$. Next we consider a delayed basic with $|H_3|/|B_3| < \gamma_3$, and so on.

In this way, we may select a sequence of nested parameter intervals Λ_{n_k} such that for $t = \cap \Lambda_{n_k}$, the orbit of $1/2$ under G_t is ε_k-dense, where $\varepsilon_k \downarrow 0$. It follows that for $t \in \cap_k \Lambda_{n_k} (= \cap_n \Lambda_n)$, φ_t has $\omega_G(1/2) = I$.

At the same time the above property (ii) is satisfied.

5.6. Making the Acim μ Infinite.

5.6.1. We assume that at step $n = n_k$, the central branch $h_n \colon \delta_n \to I$ falls in the delayed basic situation and we construct the box B_k with hat H_k. Set

$$H_k^{-j} = h_n^{-j}(H_k),$$

i.e., if g_1 and g_2 denote the two monotone branches of $h_n|(B_k \backslash H_k)$ then H_k^{-j} consists of the collection of 2^j intervals that are mapped onto H_k by the compositions $g_{i_1 \cdots i_j}$ of g_1 and g_2 for all possible $i_1 \cdots i_j$. These intervals are called preimages of the hat of *order* j.

Let t_k be the parameter value at which H_k disappears. Then for $t \to t_k$ the ratio $|H_k|/|B_k| \to 0$ and at the same time $|B_k(t)|/|B_k(t_k)| \to 1$.

Let N_k be the power of the central branch:

$$h_{n_k} = G^{N_k}.$$

When h_{n_k} exhibits a box B_k, one of the boundary points of B_k is a fixed point for h_{n_k}, and hence a periodic point of G with the period N_k. We call N_k the *period* of the box B_k. The G orbit of any point in H^{-j} includes jN_k iterates such that it returns to B_k at multiples of N_k and finally escapes through the hat.

Let $s = \lceil 2/|B(t_k)| \rceil$. Then $s > 1/|B(t)|$ for all t close enough to t_k.

Lemma 5.2. *There exists $w_k \in (0,1)$ such that if $|H_k|/|B_k| < w_k$ then*

$$|H_k| + \cdots + |H_k^{-s}| < \frac{1}{2}|B_k|$$

Proof. Obvious by continuity, cf. [15]. ☐

This leads to

Proposition 2. *Assume that in the construction of ξ there are infinitely many delayed basic steps $n = n_k$ such that $|H_k|/|B_k| < w_k$, where the w_k are given by Lemma 5.2. Then the measure μ is infinite, and moreover the measure of every box B_k is infinite.*

Proof. As each interval in H_k^{-s} visits B_k s times before exiting through the hat, the tower construction implies that $\mu(H_k) \geq \sum_j j|H_k^{-j}|$. From the previous lemma we get

$$\sum_j j|H_k^{-j}| > s\sum_{j>s}|H_k^{-j}| \geq \frac{1}{2}|B_k|\frac{1}{|B_k|} = \frac{1}{2}. \tag{16}$$

Let $G_\tau = \lim_{k\to\infty} G_{t_k}$ be the limit map, where τ is from 15. The above argument proves that the part of the sum 14 for G_τ contributed by the intervals $\Delta_i \subset B_k \setminus H_k$ satisfies

$$\sum_{\Delta_i\subset B_k\setminus H_k} N_i|\Delta_i| > \frac{1}{2} \tag{17}$$

Let $d > 0$ be a lower bound for the density of the T_τ invariant measure ν_τ. Then for the G_τ invariant measure $\mu = \mu_\tau$ we get from 17

$$\mu(B_k \setminus H_k) \geq \frac{d}{2} \tag{18}$$

As the next box is contained inside H_k, we get infinitely many disjoint annuli $B_k\setminus H_k$ satisfying (18). This proves $\mu(B_k) = \infty$ for every k. In particular,the sum Σ of (13) diverges and μ is infinite. □

This satisfies condition (A) given in Section 5.2.3.

5.7. **Construction of the Set E.**

5.7.1. Recall from Section 5.2.3 that we wish to construct a set E with non-zero Lebesgue measure for which the sum in (13) converges. From the previous section we see that we need to exclude the intervals that go back and forth within a Johnson box. With this in mind, we construct the set E by defining a sequence of open sets U_k which contain many iterates of B_k and such that their union $U = \bigcup_k U_k$ does not have full measure in I. Then $E := I \setminus U$ has positive Lebesgue measure and we prove that E has the desired properties. Take $U_0 = \delta_0$ and define U_k inductively by using the partition ξ_{n_k} as follows. At each delayed basic step $n = n_k$ we have $h_n(1/2) \in \delta_n$. Let N be the power of h_n with respect to G, and let $R = h_n^{-1}(\delta_n)$. We define U_k as the union

$$G(R) \cup G^2(R) \cup \ldots \cup G^N(R), \qquad \text{where} \qquad G^N(R) = \delta_n \cap \mathrm{Im}(h_n).$$

Let $B = B_k$ denote the associated Johnson box with hat $H = H_k$.

Proposition 3. *There exists a sequence b_k such that if at each delayed basic case $|H_k| < b_k$ then $|E| > 0$.*

Proof. Let $n = n_k$ and $m = n_{k-1}$ be two consecutive delayed basic steps. In our construction we will have many basic steps in between. Therefore

$$h_n = f_{n-1}^* \circ f_{n-2}^* \circ \cdots \circ f_m^* \circ h_m^{r_m},$$

where the branches f_i^* for $i = m, m+1, \ldots, n-1$ are chosen in order to ensure that the orbit of the critical point is everywhere dense. By construction $h_m^{r_m}(1/2)$ is the first iterate of $h_m(1/2)$ that falls outside δ_m, i.e., $h_m(1/2) \in \mathcal{S}_{r_m}(\delta_m)$ — the r_m^{th} level of the staircase construction belonging to δ_m. Take $R = h_n^{-1}(\delta_n)$ and let N_n be the power of h_n. Set $S = h_m^{r_m-1}(R) \in \mathcal{S}_1(\delta_m)$ — the first level of the staircase belonging to δ_m. Then we decompose the orbit

$$U_k = R \cup G(R) \cup \cdots \cup (G^{N_n}(R) = \delta_n \cap \mathrm{Im}\, h_n)$$

into two blocks

$$\begin{aligned} \mathcal{B}_1 &= R \cup G(R) \cup \cdots \cup S, \\ \mathcal{B}_2 &= G(S) \cup G^2(S) \cup \cdots \cup (\delta_n \cap \operatorname{Im} h_n). \end{aligned}$$

Clearly $R \subset h_m^{-1}\delta_m$ since $\delta_n \subset \delta_m$ because $n > m$. Consequently, $\mathcal{B}_1 \subset U_{k-1}$ and

$$|U_k \setminus U_{k-1}| \leq |\mathcal{B}_2|.$$

The key point is now, that the number of iterates of S which make up the union in the second block $\mathcal{B}_2$ is **independent** of r_m, (remember that by construction $h_m^{r_m}(S) \subset \Delta_m^*$, irrespective of r_m). So if M denotes the power of h_m, then $\mathcal{B}_2$ consists of a union of $M + N(\Delta_m^*) + N(\Delta_{m+1}^*) + \cdots + N(\Delta_{n-1}^*)$ G-iterates of S. It follows by continuity, that $\mathcal{B}_2$ can be made arbitrarily small provided $\delta_n \subset H_{k-1}$ is small enough, which in turn can be arranged by choosing r_m sufficiently large. Therefore at each delayed basic step, we can determine in advance the level of the staircase because the series of basic steps and the following Johnson box depends only on the location of $h_m^{r_m}(1/2)$ within Δ_m^* and not on r_m. Consequently, there exists a sequence b_k such that if $|H_k| < b_k$ then $|U| < |I|$ and $|E| > 0$. □

Let $\gamma_k = \min\{a_k, b_k\}$. If

$$|H_k| < |H_k|/|B_k| < \gamma_k, \tag{19}$$

then the hypotheses of Propositions 2 and 3 are both satisfied.

5.7.2. Having established that μ is infinite, because $\sum_i N_i|\Delta_i| = \infty$ in (14), we continue to show that Σ in (13) is finite. Then by Theorem 2.2, property (B) implies the measure μ is σ-finite.

If we only count the intervals $G^k(\Delta_i)$ that intersect E and denote their number by $N_E(\Delta_i)$ we get that the sum Σ given by formula (14) is majorized by

$$\sum_n \sum_{\Delta_i \in \xi_n} N_E(\Delta_i)|\Delta_i|. \tag{20}$$

Terminology: We call $N_E(\Delta_i)$ *the height through* E of the monotone branch f_i with the domain Δ_i.

Let us consider the preliminary partition ξ_0. Since this partition consists of a finite number of intervals we can set

$$N_0 = \max\{\, N(J) : J \in \xi_0 \,\}$$

We define the *power through* E *of* h_n as

$$N_E(h_n) := 1 + N_0^* + N_1^* + \cdots + N_{n-1}^* \qquad (N(h_0) = 1) \tag{21}$$

where $N_i^* = N_E(\Delta_i^*)$ is the height through E of the domain Δ_i that contains the critical value $h_i(1/2)$.

5.8. Properties of Boundary Refinement.

5.8.1. When estimating N_E we must take into account boundary refinement. In this section we show how to control it by choosing the appropriate position of the critical value.

Recall that by the choice of parameter, we can ensure that no boundary refinement is needed at Johnson steps. However we will usually have many basic steps $j = n+1, n+2, \dots, m-1$ between two Johnson steps in order to make the orbit of the critical point dense. In particular for any element Δ there is a step of induction when we put the critical value inside Δ and arbitrary close to its boundary. To make adjacent branches extendible we need many steps of boundary refinement. We will use the following notation:

(i) Suppose Δ, $\Delta_0 \in \xi_n$ are monotone domains and Δ_0 contains the critical value $x_0 = h_n(1/2)$. Then we let $R_n(\Delta, \Delta_0, x_0)$ denote the minimum number of boundary refinements needed for Δ in order to make monotone domains $h_n^{-1}(\Delta) \in \xi_{n+1}$ extendible.

(ii) Let
$$R_n(\Delta_0, x_0) = \max_{\Delta \in \xi_n} \{ R_n(\Delta, \Delta_0, x_0) \}.$$

Remark 4. (i) If the map F is diffeomorphic then $R_n(F^{-1}\Delta, F^{-1}\Delta_0, F^{-1}x_0) = R_n(\Delta, \Delta_0, x_0)$, since extensions of preimages are preimages of extensions.

(ii) By the construction of enlargements in Section 5, each hole δ^{-k} belonging to a given partition ξ has an enlargement $\tilde{\delta}^{-k}$ such that for all elements constructed as a result of filling in δ^{-k}, their extensions are inside $\tilde{\delta}^{-k}$. As a consequence we obtain that no additional boundary refinement is needed after a filling-in operation.

The following lemma is a straightforward consequence from the definition of extendibility. To simplify notation, we may assume $I = [0, 1]$.

Lemma 5.3. (The Boundary Refinement Lemma) *Suppose $f\colon [a, b] \to [0, 1]$ is an extendible monotone branch with $f(b) = 1$ and let $J = [b, d]$ be an interval that is adjacent to $[a, b]$. Let us consider the refinements of $[a, b]$ and let ζ_k be the boundary interval of the k^{th} refinement which is adjacent to b. Then there exists $k_0 = k_0(|J|)$ such that the extension of the boundary interval ζ_{k_0} is contained in J.*

Suppose ξ is a partition with the critical value $x_0 = h(1/2)$ contained in $\Delta_0 \in \xi$. Also assume $\Delta_a \subset \operatorname{Im}(h)$ is the monotone domain adjacent to Δ_0. Then, using the boundary refinement lemma we get the following corollary:

Corollary 2. *If $\Delta \neq \Delta_a$ belongs to ξ and requires boundary refinement, then we will need no more than $k_0(\Delta_a)$ steps of boundary refinements.*

Recall that if $\xi_0 \prec \xi_1 \prec \cdots \prec \xi_n \prec \cdots$ are partitions constructed in the course of our induction. Let $\xi_\infty = \lim_{n\to\infty} \xi_n$ denote the limit partition.

Lemma 5.4. *For $\Delta_0 \in \xi_n$ and $x_0 \in \Delta_0$, we have*
$$R_\infty(\Delta_0, x_0) := \sup_{m \geq n} R_m(\Delta, \Delta_0, x_0) = R_{n+1}(\Delta_0, x_0).$$

Proof. Notice that all monotone domains created after step n are inside the holes of ξ_n. After the filling-in of any hole δ we get two monotone domains adjacent to the boundary points of δ. Therefore we get a domain Δ adjacent to Δ_0 no later than at step $n+1$. $\square$

For $x_0 \in \Delta_0$, where Δ_0 is a monotone domain of some ξ_m, we define

$$R(x_0) = R_\infty(\Delta_0, x_0). \tag{22}$$

Using that $|\bigcup_{\Delta \in \xi_\infty} \Delta| = 1$ we can now prove

Proposition 4. *The limit* $\lim_{n\to\infty} |\{ x_0 : R(x_0) < n \}| = 1$.

Proof. For a given $\Delta_0 \in \xi_m$ we have $R_m(\Delta, \Delta_0, x_0) < k_0(\Delta_a)$ for all Δ non-adjacent to Δ_0. As for the adjacent interval Δ_a the number of boundary refinements is finite for any fixed x_0 inside the interior of Δ_0 and tends to ∞ as x_0 approaches the common boundary between Δ_0 and Δ_a. However,

$$\lim_{n\to\infty} \frac{|\{ x_0 : R_m(\Delta_a, \Delta_0, x_0) > n \}|}{|\Delta_0|} = 0$$

Hence, for every finite union U of intervals Δ_0 and every union V of open subintervals of Δ_0 that is separated from the boundary points of Δ_0 and has relative measure (in U) close to 1, there exists an n such that for any m

$$\max_{\Delta_0, x_0 \in V} R_m(\Delta_0, x_0) < n,$$

proving the proposition. □

Proposition 4 implies that we can carry out the construction of partitions ξ_n, and make the trajectory of the critical point everywhere dense, under an additional assumption that the maximum number of boundary refinements needed to make all elements of ξ_n extendible does not exceed, say, 2^n.

Assume at step n according to our itinerary we must visit certain domains, but it involves more than $M > 2^n$ refinements. Then we interrupt our itinerary and just pullback ξ_0 consecutively. We use that for ξ_0 there are many positions of the critical value such that no boundary refinement is needed. After that we return to our original predetermined itinerary.

5.9. Growth of the $\mathbf{N_E}$.

5.9.1. We want to show that the sum $\sum_n \sum_{\Delta \in \xi_n} N_E(\Delta)|\Delta|$ in (20) converges. Since the partitions ξ_n have the property that once a uniformly extendible monotone domain is created it is never changed, it follows that all new monotone domains come from the critical pullback into the central domain and then from the subsequent filling in procedure. So to calculate the sum in (20) we will estimate the contribution at each step n due to the these procedures.

5.9.2. Let

$$N_E(\xi_n) = \max_{J \in \xi_n} N_E(J).$$

By definition the maximum is taken over all elements J of ξ_n including δ_n. We define $N_E(\delta_n) := N_E(h_n)$, where $N_E(h_n)$ was defined in (21).

Then

$$N_E(\delta_n) \le N_E(\xi_n)$$

Since the preliminary partition ξ_0 is a finite partition, we have $N_0 := N_E(\xi_0) < \infty$. Next we prove

Proposition 5. *For all $n \ge 0$, we have*

$$(a)_n \qquad N_E(\xi_n) < N_0 5^n.$$

Proof. Clearly $(a)_0$ holds. Now, assume by induction $(a)_n$ and let us consider the partition ξ_{n+1}. All new elements inside δ_n are obtained by using critical pullback and boundary refinement. At a basic step we are doing one critical pullback. Then for the new critical branch $h_{n+1} = f_n^* \circ h_n$ we have

$$N_E(h_{n+1}) \le N_E(h_n) + N_E(\xi_n) < N_0 5^n + N_0 5^n \tag{23}$$

The same estimate holds for other elements inside δ_n obtained by critical pullback.

By construction at step n we need no more than 2^n boundary refinements. If we are doing a grow-up operation, we choose the position of the critical value, so that we need no more than 2^n additional compositions. Then we may need one extra pull-back operation. Taking into account all these possibilities we get

$$\max_{J \in \delta_n} N_E(J) < 2N_0 5^n + N_0(4^n + 1). \tag{24}$$

Recall that at a Johnson step we delete trajectories of all intervals until they enter the first step of the staircase. Deleted intervals cannot contribute to N_E. The points from the first step of the staircase are mapped by h_n onto the elements of ξ_n just as at a basic step. So at a Johnson step we get the same estimate (24). Finally, when we do the filling-in procedure, we add one more term not exceeding $N_E(\xi_n)$. So we obtain

$$N_E(\xi_{n+1}) \le 3N_0 5^n + (4^n + 1)N_0 < N_0 5^{n+1}$$

which proves $(a)_{n+1}$ as required. □

5.10. **Estimates at Step $\mathbf{n+1}$.**

5.10.1. Recall that elements Δ constructed at step n are not changed anymore. New elements at step $n+1$ are constructed inside δ_n and inside holes δ_i^{-k}, $i = 0, 1, \dots,\ k \ge 0$.

Let us now estimate the contribution to (20) from the elements constructed inside the preimages δ_i^{-p} created by the filling in procedure. Suppose $\Delta, \delta_j^{-k} \subset \delta_i^{-p}$ are elements obtained by filling in δ_i^{-p}. Then we can subdivide the orbit of these elements into two segments. The first segment consists of the trajectory of δ_i^{-p} until they reach δ_i, the second segment then follows the orbit of the elements inside δ_i that are constructed at step $i+1$. In our estimates we accounted for the first segment at step n. At that step we counted the contribution of the hole δ_i^{-p} without any partition. In order to estimate the new contribution at step $n+1$ we need to account for the second segment of that orbit. For a given i that contribution does not exceed

$$N_E(\xi_{i+1}) \Big(\sum_{\delta_i^{-p} \in \xi_n} |\delta_i^{-p}| \Big) \tag{25}$$

Since $i \le n$, Proposition 5 implies that $N_E(\xi_{i+1}) \le N_0 5^{n+1}$ and consequently estimate (25) is at most

$$N_0 5^{n+1} \sum_{\delta_i^{-p} \in \xi_n} |\delta_i^{-p}|.$$

Therefore, the total contribution to the sum in (20) at step $n+1$ due the preimages δ_i^{-k} for $i = 0, 1, 2, \dots, n$ and $p \ge 0$ does not exceed

$$N_0 5^{n+1} \Big(\sum_n \sum_{\delta_i^{-p} \in \xi_n} |\delta_i^{-p}| \Big) \tag{26}$$

In the next section we will prove that

$$\sum_{\delta_i^{-p} \in \xi_n} |\delta_i^{-p}| < a^i b^n s_0, \tag{27}$$

where $a = a(\delta_0)$, $b = b(\delta_0)$ and $s_0 = s_0(\delta_0)$ all tend to zero with δ_0. We can ensure that he sum

$$\sum_{i=0}^{\infty} a^i b^n s_0 = \frac{1}{1-a} s_0 b^n < 6^{-n} \tag{28}$$

provided a, b and s_0 are sufficiently small. Combining this with (26) proves the convergence of the sum of the new contributions, and respectively the sum in formula (20).

5.11. Estimating the Measure of Holes $\cup\delta_{\mathbf{i}}^{-\mathbf{k}}$ inside $\xi_{\mathbf{n}}$.

5.11.1. In our construction every central branch is a composition $h(x) = F \circ Q(x)$, where $Q(x)$ is the standard quadratic map and F is a composition of monotone domains with uniformly bounded distortion. For the quadratic map $Q(x)$ we know that, if $J \subset \delta$ are both symmetric intervals containing the critical point, then

$$\frac{|J|}{|\delta|} = \sqrt{\frac{|Q(J)|}{|Q(\delta)|}}. \tag{29}$$

Since F has bounded distortion we obtain for similar intervals J and δ

$$|J| < c|\delta| \sqrt{\frac{|h(J)|}{|h(\delta)|}}. \tag{30}$$

Let δ be the domain of a central branch h. By the grow-up procedure the image of the central branch covers at least a fixed length I_0. So we may write

$$|J| < c|\delta|\sqrt{|h(J)|} \tag{31}$$

where c is another uniform constant.

5.11.2. For a given i let $\alpha_i^{(n)} = \left|\bigcup_{\xi_n} \delta_i^{-k}\right|$ be the total measure of the holes $\cup\delta_i^{-k}$ that belong to ξ_n. In order to estimate from above the relative measure of the holes created inside δ_n as a result of the critical pullback procedure, we assume the worst position of these holes. By this, we mean that we assume that all the holes are contiguous with one end being bounded by the critical value $w = h_n(1/2)$. Let $M_i^{(n+1)}$ denote the measure of the union of all preimages of δ_i created inside δ_n at step $n+1$. For $i < n+1$, inequality (31) implies

$$\frac{M_i^{(n+1)}}{|\delta_n|} < c\sqrt{\alpha_i^{(n)}}. \tag{32}$$

This gives us the worst estimate on the relative measure of $M_i^{(n+1)}$ inside δ_n.

For $i = n+1$ we get in the basic case

$$M_{n+1}^{(n+1)} = \delta_{n+1} < \beta\delta_n, \tag{33}$$

where β is a small constant depending on the maximal size of elements in ξ_0.

We get estimates (32), (33) at basic steps. At a Johnson step, the estimate (32) still holds for preimages which belong to the first step of the staircase. For subsequent preimages we prove

Lemma 5.5. *The union of the box and all stairs except $\mathcal{S}_1$ satisfies:*
$$|\delta_n \setminus \mathcal{S}_1| \le c_1 |\delta_n|^{3/2}. \tag{34}$$

Proof. Let $h_n = F \circ Q$, where Q is the initial quadratic map. Let $J = \delta_n \setminus \mathcal{S}_1$. By definition $h_n(J) = h_n(\delta_n) \cap \delta_n$ and $|\delta_n| > |h_n(J)| > \frac{1}{2}|\delta_n|$. Since Q is quadratic, (29) gives $\frac{|J|}{|\delta_n|} = \sqrt{\frac{|Q(J)|}{|Q(\delta_n)|}}$. Using (30), $|h_n(\delta_n)| > 1/2|I|$ and $h_n(J) \subset \delta_n$ we obtain $|J| < c|\delta_n|^{3/2}$. □

So at a Johnson step we get for $i < n+1$
$$M^i_{n+1} < c|\delta_n|(\sqrt{\alpha_i^{(n)}} + |\delta_n|^{1/2}). \tag{35}$$

Remark 5. At a Johnson step $n+1$ we get infinitely many preimages of δ_{n+1} all created inside the box B_n. Since the tip of the hat is small comparatively to the box, we have $|h_n(B)| = |B|(1+\varepsilon)$. Taking $J = B$ in (31) we obtain $|B| < c|\delta_n|\sqrt{|B|(1+\varepsilon)}$ which shows that the box is of order $|\delta_n|^2$. Therefore, when $i = n+1$ we get a stronger estimate
$$M^{(n+1)}_{n+1} < c_1|\delta_n|^2. \tag{36}$$

As (35) in the Johnson case majorizes (32), we use the estimate (35) in all cases.

5.11.3. When doing filling-in of a hole δ_j^{-p} we pullback the structure of δ_j that was created by critical pullback at step $j+1$. So we handle this at step $j+1$ as we did above at step $n+1$ and get $M_i^{(j+1)} < c|\delta_j|\left(\sqrt{\alpha_i^{(j)}} + |\delta_j|^{1/2}\right)$. Then we pullback with small distortion onto the preimage δ_j^{-p} and obtain new preimages δ_i^{-k} with measure less than $c_1|\delta_j^{-p}|\left(\sqrt{\alpha_i^{(j)}} + |\delta_j|^{1/2}\right)$ inside each preimage δ_j^{-p}. Notice that the central domain δ_i is constructed at step i. Respectively, preimages of δ_i can only appear at steps $i, i+1, \dots$ Taking the union over all preimages δ_j^{-p} for $j = i, \dots, n$, we get that at step $n+1$ the total measure of all preimages δ_i^{-r} appearing after filling in all preimages δ_m^{-k}, $(m = i, i+1, \dots, n)$, is at most
$$c\sum_{m=i}^{n} \alpha_m^{(n)} \left(\sqrt{\alpha_i^{(m)}} + |\delta_m|^{1/2}\right).$$

Recall that at a basic step $\frac{|\delta_i|}{|\delta_{i-1}|} \le \beta$, where β can be made arbitrary small by choosing elements of the initial partition ξ_0 small.

We choose $\delta_0 \ll \beta$. Then at a Johnson step we get $|\delta_i| < c|\delta_{i-1}|^2 \ll \beta|\delta_{i-1}|$ and moreover from (36)
$$M_i^i \ll \beta|\delta_{i-1}|. \tag{37}$$

The filling-in operation produces at the middle of any domain δ_{i-1}^{-k} a new central preimage δ_i^{-k} or a union of such preimages $\bigcup \delta_i^{-m}$, if i was a Johnson step.
Since the diffeomorphisms $\chi : \delta_{i-1}^{-k} \to \delta_{i-1}$ have small distortions, we get for preimages
$$\frac{|\delta_i^{-k}|}{|\delta_{i-1}^{-k}|} \le (1+\varepsilon)\beta.$$

We change notation and use the same constant β as an estimate for the ratio of these preimages.

Combining the previous estimates we get at step $n+1$

$$\alpha_i^{(n+1)} < \beta\alpha_{i-1}^{(n)} + c_1 \sum_{j=i}^{n} \alpha_j^{(n)} \left(\sqrt{\alpha_i^{(j)}} + |\delta_j|^{1/2}\right) \tag{38}$$

where β and c_1 do not depend on i and on n.

5.11.4. Now, we prove

Proposition 6. *There exists small positive constants s_0, a and b, such that for all $n \geq 0$ and all $i \leq n$ we have*

$$(\Gamma_{(i,n)}) \qquad \alpha_i^{(n)} < a^i b^n s_0. \tag{39}$$

Moreover, one can choose s_0, a and b that tend to zero as $|\delta_0| \to 0$.

Proof. 1. We may assume that δ_0 is small enough — to be specified below. Recall that $\frac{|\delta_{i+1}|}{|\delta_i|} < \beta$, where β is small. Consequently, in our estimates below, we use that

$$\begin{cases} |\delta_i| < \beta^i |\delta_0|, \\ \alpha_i^{(i)} < c_0 \beta^i |\delta_0|. \end{cases} \tag{40}$$

Here a constant c_0 appears because ξ_0 contains not only δ_0, but also its preimages.

2. By construction of our initial partition ξ_0 we decrease an element until its size becomes smaller than ε_0. We recall that when the image of the critical branch covers less than one half of I , we are using an extra pull-back operation. Hence $\beta < c_2\sqrt{\varepsilon_0}$, where c_2 is a uniform constant.
The key observation is that the size of the central domain δ_0 does not depend on ε_0, and we can choose $|\delta_0| \ll \varepsilon_0$.
3. Let us choose a constant s_0 such that $|\delta_0| \ll s_0$, say $|\delta_0| < s_0^2$. In addition , we choose small constants $a = \beta^x$ and $b = \beta^y$ where $0 < x, y < 1/2$ and $ab > 3\beta$. Combining all the above, we will use in our estimates below the following inequalities

$$\begin{cases} |\delta_0| < s_0^2, \\ \beta < \frac{1}{3}ab, \quad \beta < b^2, \quad \beta < a^2. \end{cases} \tag{41}$$

4. Let us first check the case $i = 0$. In this case (38) becomes

$$\begin{aligned} \alpha_0^{(n+1)} &< c_1 \sum_{m=0}^{n} s_0 a^m b^n \left(\sqrt{s_0} b^{m/2} + |\delta_0| \beta^{m/2}\right) \\ &< c_1\sqrt{s_0} \left[s_0 b^n \Big(\sum_{m=0}^{\infty} a^m b^{m/2} + |\delta_0| \sum_{m=0}^{\infty} \beta^{m/2}\Big) \right]. \end{aligned}$$

If a, b and β are small enough, then the sums of geometric progressions are close to 1, and we get

$$\alpha_0^{(n+1)} \leq c_1 s_0^{3/2} b^n (1+\varepsilon). \tag{42}$$

Also, we can arrange that the elements of the initial partition are small enough to ensure that

$$c_1\sqrt{s_0} < b/10 \tag{43}$$

Then (42) implies formula $(\Gamma_{(0,n+1)})$.

5. Now we assume by induction that $\Gamma_{(i,n)}$ holds for all $i \leq n$. Then for all $i = 1, 2, \ldots, n$ we get using (38), (40) and (41)

$$\begin{aligned} \alpha_i^{(n+1)} &< \beta a^{i-1} b^n s_0 + c_1 \sum_{j=i}^{n} s_0 a^j b^n \Big[\sqrt{s_0} a^{i/2} b^{j/2} + \sqrt{|\delta_0|} \beta^{j/2}\Big] \\ &< \frac{1}{3} s_0 a^i b^{n+1} \\ &\quad + s_0 a^{\frac{i}{2}} b^n \Big[c_1 \sqrt{s_0} \Big(\sum_{j=i}^{n} a^j b^{j/2}\Big) + c_1 \sqrt{|\delta_0|} \sum_{j=i}^{n} \beta^{j/2}\Big]. \end{aligned} \tag{44}$$

The first term in the square brackets is $\leq c_1 a^i b^{\frac{i}{2}} \sqrt{s_0} \left(\frac{1}{1-ab^{\frac{1}{2}}}\right)$. From $c_1\sqrt{s_0} < b/10$ given in (43) and since $\frac{1}{10}\left(\frac{a^{\frac{i}{2}} b^{\frac{i}{2}}}{1-ab^{\frac{1}{2}}}\right) \leq \frac{1}{3}$ which holds for small a, b, we obtain that

$$c_1 a^i b^{\frac{i}{2}} \sqrt{s_0} \left(\frac{1}{1-ab^{\frac{1}{2}}}\right) \leq a^{\frac{i}{2}} b \left[\frac{1}{10}\left(\frac{a^{\frac{i}{2}} b^{\frac{i}{2}}}{1-ab^{\frac{1}{2}}}\right)\right] \leq \frac{1}{3} a^{\frac{i}{2}} b. \tag{45}$$

As $|\delta_0| < s_0^2$ and $\beta < a^2$ we get as above that the second term in the square brackets in (44) satisfies

$$c_1 \sqrt{|\delta_0|} \beta^{\frac{i}{2}} \left(\frac{1}{1-\beta^{\frac{1}{2}}}\right) < \frac{1}{3} a^{\frac{i}{2}} b. \tag{46}$$

Combining (45), (44) and (46) we get $\alpha_i^{(n+1)} < s_0 a^i b^{n+1}$ proving formula $(\Gamma_{(i,n+1)})$ for all $i \leq n+1$.

Thus Proposition 6 follows by induction. $\square$

As discussed in Section 5.10, Proposition 6 implies that μ is a σ-finite acim. Finally we get from Proposition 2 that every interval $J \subset I$ has infinite μ-measure. Let J be any interval in I . By construction there is an n such that $h_n(c)$ passes through J and hence there is a Johnson step k_0 such that the forward orbit of the box B_{k_0} passes through J. We proved that B_{k_0} has infinite μ-measure. As μ is G-invariant, every iterate of B_{k_0} has infinite μ-measure. Respectively J which contains an iterate of B_{k_0} also has infinite μ-measure.
This finishes the proof of the main Theorem A.

Acknowledgements. The authors are grateful to G. Świątek for useful discussions.

REFERENCES

[1] R. L. Adler, *Afterword to R. Bowen, Invariant measures for Markov maps of the interval,* Commun. Math. Phys. **69** (1979), 1–17.

[2] J. Al-Khal, “New Examples of S-unimodal Maps with a Sigma-finite Absolutely Continuous Invariant Measure,” Ph.D. Thesis, University of Maryland, College Park, 2004.

[3] H. Bruin, “Invariant measures of interval maps,” Ph.D. Thesis, University of Delft, 1994.

[4] H. Bruin, *Induced maps, Markov extensions and invariant measures in one-dimensional dynamics,* Commun. Math. Phys. **168** (1995), 571–580.

[5] H. Bruin, G. Keller, T. Nowicki and S. van Strien, *Wild Cantor sets exist,* Annals of Mathematics, **143** (1996), 97–130.

[6] H. Bruin, J. Hawkins, *Exactness and maximal automorphic factors of unimodal interval maps,* Ergod. Th. & Dyn. Sys. **21** (2001), 1009–1034.

[7] H. Bruin, W. Shen, S. van Strien, *Invariant measures exist without a growth condition,* Commun. Math. Phys. **241** (2003), 287–306.

[8] J. Guckenheimer and S. D. Johnson, *Distortion of $\mathcal{S}$-unimodal maps,* Annals of Mathematics, **132** (1990), 71–130.

[9] J. Graczyk and G. Świątek, "The real Fatou conjecture," Annals of Mathematics Studies, Princeton University Press, **144**, 1998.

[10] F. Hofbauer and G. Keller, *Quadratic maps without asymptotic measures,* Commun. Math. Phys., **127** (1990), 319–337.

[11] F. Hofbauer and G. Keller, *Some remarks on recent results about $\mathcal{S}$-unimodal maps,* Ann. Institute Henri Poincaré, **53** (1990), 413–425.

[12] M. V. Jakobson, *Absolutely continuous invariant measures for one-parameter families of one-dimensional maps,* Commun. Math. Phys., **81** (1981), 39–88.

[13] M. V. Jakobson, *Piecewise smooth maps with absolutely continuous invariant measures and uniformly scaled Markov partitions,* Proceedings of Symposia in Pure Mathematics, **69** (2001), 825–881.

[14] M. V. Jakobson and G. Świątek, *Metric properties of non-renormalizable $\mathcal{S}$-unimodal maps. Part I: Induced expansion and invariant measures,* Ergodic Th. & Dynam. Sys., **14** (1994), 721–755.

[15] S. D. Johnson, *Singular measures without restrictive intervals,* Commun. Math. Phys., **110** (1987), 185–190.

[16] S. Kakutani, *Induced measure preserving transformations,* Proc. Japan. Acad., **19** (1943), 635–641.

[17] Y. Katznelson, *σ-finite invariant measures for smooth mappings of the circle,* Journ. d'Anal Math. **31** (1977), 1–18.

[18] G. Keller, *Exponents, attractors and Hopf decompositions for interval maps,* Ergodic Th. & Dynam. Sys., **10** (1990), 717–744.

[19] G. Keller, *Lifting measures to Markov extensions,* Monatsh. Math. **110** (1989), 183–200.

[20] M. Martens, "Interval Dynamics," Ph.D. Thesis, Delft 1990.

[21] M. Martens, *The existence of σ-finite measures, applications to real one-dimensional dynamics,* Stony Brook Preprint **1992-1**

[22] W. de Melo and S. van Strien, "One-dimensional dynamics," Springer, Berlin, 1993.

E-mail address: `jawad7@comcast.net`

E-mail address: `h.bruin@surrey.ac.uk`

E-mail address: `mvy@math.umd.edu`