PARAMETER CHOICE FOR FAMILIES OF MAPS WITH MANY CRITICAL POINTS

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ABSTRACT. We consider families of smooth one-dimensional maps with several critical points and outline the main ideas of the construction in the parameter space that allows to get infinite Markov Partitions and SRB measures for a positive set of parameter values.

The construction is based on the properties of uniformly scaled Markov partitions from [6]. The same approach works for families of Henon-like maps.

1. FORMULATION OF THE THEOREM

1.1. There are several approaches to the construction of absolutely continuous invariant measures in families of non-hyperbolic maps depending on the parameter. The original method of [4] provides infinite Markov partitions for the related family of piecewise continuous expanding iterates $F_t \mid \Delta_i = f_t^{n_i}$ of the initial smooth map f_t. The method of [1], [2], see also [10], [3] and subsequent papers, based on large deviations is non-Markov. It results in an a priori smaller set of parameter values satisfying Collet-Eckmann type conditions. The method of [13], [12] combines Markov property and Collet-Eckmann conditions. For further references see a recent survey [9] which in particular contains a detailed discussion of parameter exclusion.

In any method the choice of parameter is an important part of the construction. Here we outline several features of a method based on the uniform scaling of Markov partitions, see [6].

There are similarities and differences between constructions in dimensions one and two. For example, distortion estimates in dimension two are more complicated, see [7], [11]. In the method based on uniformly scaled Markov partitions the problem of parameter choice for two-dimensional families is similar to the problem for one-dimensional families with several critical points . We mostly discuss that one-dimensional case and at the end outline specifics of the 2-dimensional construction.

1.2. After some preliminary construction, which includes transition to a first return map and taking several iterates of that map, see [5], [6], we get a family of one-dimensional C^2 mappings F_t depending on the parameter $t \in \mathcal{T}_0 = [t_0, t_1]$ with the following properties.

For each t, F_t is piecewise continuous with a finite number of branches. The union of the domains of these branches is an interval I which can be considered independent of t , say $I = [0,1]$ for all t. The branches of F_t are of three types.

(1) There are m critical branches $h_l, l = 1, \dots m$, whose domains are called *central domains* . Each central domain δ_l contains a single critical point O_l of F_t. Without loss of generality one can assume that O_l do not depend on t and so for $l = 1, \dots m$ and for all t we have

$$h_{lx}(O_l) = 0$$

(2) Monotone expanding branches which we also call *good branches*

$$f_i \colon \Delta_i \to I \tag{1}$$

satisfying for all t

$$|f_{ix}| > R_0 \tag{2}$$

where $R_0 > 1$ is a large constant.

(3) Branches g_i which map preimages of central domains δ_l diffeomorphically onto δ_l

$$g_i :\ \delta_l^{-n_i} \to \delta_l \tag{3}$$

satisfying for all t

$$|g_{ix}| > a_0 > 1 \tag{4}$$

The above domains form a partition ξ_0 of I and we assume that the elements of that partition vary continuously with t . All new branches constructed in the inductive process are created inside δ_l and their preimages $\delta_l^{-n_i}$.

1.3. Let $W_l(t) = h_l(O_l)$ be the critical values of h_l. We assume their speeds are bounded away from zero by some $V_0 > 0$.

$$|W_{lt}| > V_0 \tag{5}$$

When t varies in $\mathcal{T}_0$, each critical value $W_l(t)$ moves through the elements of some partition η_l. Although geometrically different partitions may have common elements, or just be parts of the same partition of the phase space, we consider them separately.

It is convenient to imagine $W_l(t)$ moving along its own copy of the y axis partitioned by η_l .

In the case of the quadratic family

$$x \to ax(1-x)$$

when a is close to 4 we take for the interval I the domain of the first return map between the fixed point and its preimage. One can make a change of variables which makes I fixed. At the same time the critical value moves far away from I across a partition created by adjacent preimages of I which accumulate toward 1. Similar examples with two critical values each moving near a fixed point and other examples with several critical points are described in [5] .

We assume that partitions η_l consist of domains Δ_i and $\delta_l^{-n_i}$ of the same types (2) and (3) as elements of ξ_0 in the subsection 1.2 above.

1.4. We define the *distortion* $\Theta(f)$ of a diffeomorphism f defined on a domain Δf as the following supremum over $z \in \Delta f$

$$\Theta(f) = \sup \frac{|f_{xx}(z)|}{|f_x(z)|} |\Delta f| \tag{6}$$

We assume the maps defined above satisfy the following conditions.

(1) There exists $D_0 > 0$ such that all good maps $f : \Delta f \to I$ satisfy

$$\Theta(f) < D_0 \tag{7}$$

and there exists a small $\varepsilon_0 > 0$ such that all maps $g :\ \delta_l^{-n_i} \to \delta_l$ satisfy

$$\Theta(g) < \varepsilon_0 \tag{8}$$

(2) For all t the relative measure of good branches in ξ_0 is close to one

$$meas \bigcup \Delta f > 1 - \varepsilon_0 \tag{9}$$

(3) Let $f\colon \Delta f \to I$ or $g\colon \delta_l^{-n_i} \to \delta_l$ be maps defined on elements of partitions η_k. Then they are moving much slower than the critical values:

$$\frac{|f_t|}{|f_x|}, \frac{|g_t|}{|g_x|} < \varepsilon_0 \ll V_0. \tag{10}$$

(4) Let $I_k = [W_k(t_1) - W_k(t_0)] = \bigcup_{t\in\mathcal{T}_0} W_k(t)$ be the interval of variation of the k-th critical value. Then for all $t \in \mathcal{T}_0$ elements Δf, $\delta_l^{-n_i}$ of the partitions η_k are small compared to $|I_k|$. For that purpose it is enough to assume

$$|\Delta|, |\delta_l^{-n_i}| < \varepsilon_0 V_0 |\mathcal{T}_0| \tag{11}$$

(5) The variation of lengths $|\Delta(t)|$ is small

$$1 - \varepsilon_0 < \frac{|\Delta(t_1)|}{|\Delta(t_2)|} < 1 + \varepsilon_0 \tag{12}$$

for all $t_1, t_2 \in \mathcal{T}_0$.

(6) Let $M_l(t) = \bigcup_i \Delta_i(t)$ be the union of good branches which are elements of η_l such that the critical value $W_k(t_i)$ belongs to $\Delta_i(t_i)$ for some $t_i \in \mathcal{T}_0$, and let $|M_l(t)|$ be the measure of $M_l(t)$. Then for all l and all t

$$\frac{|M_l(t)|}{|I_l|} > 1 - \varepsilon_0. \tag{13}$$

Remark 1.1. (1) *The above conditions are satisfied for C^2-perturbations of quadratic family and for multi-modal families considered in* [5]. *In these examples the interval of parameter $\mathcal{T}_0$ is small and the domains δ_l are small. At the same time derivatives of all branches of the first return map , except for the critical branches are greater than some $c > 1$. As arbitrary compositions of a finite number of uniformly expanding C^2 maps have uniformly bounded distortions, one can satisfy simultaneously 2 and 7.*

(2) *As δ_l are small and distortions are uniformly bounded one gets 8 and 9.*

(3) *The condition 10 is satisfied because the elements of the partitions η_l are mapped onto I after many iterates, so the respective $|f_x|$, $|g_x|$ are large.*

(4) *The condition 12 is satisfied because the interval of parameter $\mathcal{T}_0$ is small.*

(5) *The condition 13 follows from 9 when the partitions η_l are obtained using various pull-backs of the partition ξ_0.*

(6) *In order to satisfy 11 it is enough to have elements of ξ_0 small and do one extra pullback of ξ_0 onto the elements Δf of η_l.*

1.5. The relations between the parameters R_0, V_0, ε_0, etc., that appear above determine the measure of parameters with SRB measures, see [6]. For families considered in [5] one can use a preliminary construction as described in [6] and get a family of maps F_t satisfying the above conditions, where ε_0 can be made arbitrary small, R_0 arbitrary large and other parameters uniformly bounded. That motivates the theorem below. In the general case one can vary the preliminary construction and use computer assisted estimates.

Theorem 1.2. *There exist $\overline{R}_0, \overline{\varepsilon}_0$ such that if the above conditions are satisfied with $R_0 > \overline{R}_0$ and $\varepsilon_0 < \overline{\varepsilon}_0$ and the other parameters D_0, V_0 fixed, then there is a set of parameters of positive measure such that the respective maps F_t have SRB measures and the relative measure of such parameters tends to one when $\overline{R}_0 \to \infty$ and $\varepsilon_0 \to 0$.*

2. Specifics of the Proof in the Parameter Space in the Presence of Several Critical Points

2.1. The inductive construction in the phase space that proves the above theorem is similar to the one for *Unimodal Maps*, see [4], [6]. However in the parameter space there are some specifics due to the presence of several critical points. Below we outline that part of the proof.

As the endpoints of the elements of η_l move slower than $W_l(t)$ it follows that to each element $\Delta \in \eta_l(t)$ there corresponds a parameter interval $\mathcal{T}_\Delta$ such that $W_l(t) \in \Delta$ when $t \in \mathcal{T}_\Delta$. The elements that contain $W_l(t_0)$ and $W_l(t_1)$ are exceptional, meaning that $W_l(t)$ moves across a part of such elements. We delete the respective parameter values. Let $\mathcal{S}_l$ be the remaining interval of parameters. It follows from 11 that

$$\frac{|\mathcal{S}_l|}{|\mathcal{T}_0|} > 1 - \varepsilon_1 \tag{14}$$

where ε_1 is small.

We restrict t to $\mathcal{S}_l$ and additionally delete parameter intervals $\mathcal{T}_\Delta$ corresponding to the movement of $W_l(t)$ through the good branches that are too close to the domains $\delta_k^{-n_i}$. So we consider the location of $W_l(t)$ inside some enlargements of $\delta_k^{-n_i}$ as inadmissible . By definition *admissible parameter values* belong to the remaining $\mathcal{T}_\Delta$. Let

$$\mathcal{T}_l = \bigcup \mathcal{T}_\Delta \tag{15}$$

be the union of l- *admissible* parameter intervals. Assuming that ε_0 from 13 is small enough, one can choose enlargements that are sufficiently big and imply 8 and at the same time satisfy

$$\frac{|\mathcal{T}_l|}{|\mathcal{T}_0|} > 1 - \varepsilon_2, \tag{16}$$

where ε_2 is small.

Let us define

$$\mathcal{A}_0 = \bigcap_{l=1}^{m} \mathcal{T}_l. \tag{17}$$

Then $\mathcal{A}_0$ is the initial set of parameters that are admissible for all critical values simultaneously.

The relative measure of $\mathcal{A}_0$

$$\frac{|\mathcal{A}_0|}{|\mathcal{T}_0|} > 1 - l\varepsilon_2 \tag{18}$$

is arbitrary close to one if ε_0 is sufficiently small.

2.2. For any t that belongs to an l-admissible parameter interval we get a partition ξ_1^l of the central domain δ_l by considering the pullback

$$\xi_1^l = h_l^{-1}\eta_l. \tag{19}$$

When $W^l(t)$ belongs to an admissible domain Δ_1^l, t belongs to the respective interval of parameters $\mathcal{T}_{\Delta_1^l}$, and the new central domain, which contains the critical point O_l, is $\delta_1^l = h_l^{-1}\Delta_1^l$.

Notice that differently from the partitions ξ_0 and η_l, which vary continuously for all t, the partitions ξ_1^l are defined and vary continuously only for $t \in \mathcal{T}_{\Delta_1^l}$. When $t \in \mathcal{T}_{\Delta_1'^l}$ we get a different partition of δ_l. In particular, there is no well-defined partition of δ_l when $t \in \mathcal{T}_{\Delta_1^k}$ where $k \neq l$.

2.3. In our construction in order to get consecutive refinements of the central domains δ_n^l at steps $n = 1, 2, \dots$ we first pull back some partition onto a domain Δ_{n-1}^l, which contains the critical value $W^l(t)$, and after that we pull back that new partition from Δ_{n-1}^l onto δ_{n-1}^l by h_l^{-1}.

At the first several steps of the induction we pull back onto Δ_{n-1}^l the initial partition ξ_0 which is defined for all t.

However, in order to keep the measure of parameters positive we have to consider at some step n_1 new admissible intervals in the phase space lying inside δ_l.

Let us denote by

$$I_1 = \bigcap_{l=1}^{m} \mathcal{T}_{\Delta_1}^{l} \tag{20}$$

one of the nonempty intersections of l-admissible parameter intervals at the first step of induction. We call it the *intersection of rank one*. By construction at each step of induction the parameter intervals $\mathcal{T}_{\Delta_{i_1 i_2 \dots i_k}}^{l}$ of rank k are partitioned into intervals $\mathcal{T}_{\Delta_{i_1 i_2 \dots i_k i_{k+1}}}^{l}$ of the next rank $k+1$ and the respective intersections I_k and I_{k+1} of ranks k and $k+1$ are considered. By construction each I_{k+1} belongs to only one I_k.

Let us consider an intersection of rank n_1

$$I_{n_1} = \bigcap_{l=1}^{m} \mathcal{T}_{\Delta_{i_1 \dots i_{n_1}}}^{l} \tag{21}$$

and a respective Intersection of rank one

$$I_{n_1} \subset I_1 = \bigcap_{l=1}^{m} \mathcal{T}_{\Delta_1}^{l}, \tag{22}$$

where

$$\mathcal{T}_{\Delta_{i_1 \dots i_{n_1}}}^{l} \subset \mathcal{T}_{\Delta_1}^{l} \tag{23}$$

Let us define a *union of rank* n_1 corresponding to the intersection 21 of rank n_1 by

$$\mathcal{U}_{n_1} = \bigcup_{l=1}^{m} \mathcal{T}_{\Delta_{i_1 \dots i_{n_1}}}^{l}. \tag{24}$$

As at step n_1 we pull back partitions of rank 1, we get that pull-backs are well defined if the *union of rank* n_1 lies inside the respective *intersection of rank* 1

$$\mathcal{U}_{n_1} \subset I_1 \tag{25}$$

We delete Intersections I_{n_1} that do not satisfy 25.

Let us estimate the measure of the deleted parameter intervals. By construction 25 is not satisfied if and only if one of the intervals $\mathcal{T}_{\Delta_{i_1 \dots i_{n_1}}}^{l}$ contains a boundary point of some $\mathcal{T}_{\Delta_1}^{k}$ with $k \neq l$.

Let N_1 be the number of intervals $\mathcal{T}_{\Delta}$ of rank 1, and let s_{n_1} be the maximum of the lengths of the intervals $\mathcal{T}_{\Delta_{i_1 \dots i_{n_1}}}^{l}$ of rank n_1. Then the total measure of the deleted intervals I_{n_1} does not exceed

$$2N_1 s_{n_1} \tag{26}$$

Notice that s_n decreases exponentially:

$$s_n < CR_0^{-n}. \tag{27}$$

Therefore the measure of the deleted intervals also decreases exponentially.

2.4. At the general step of induction we use *uniform scaling* in the phase space, see [6]. By construction the domains of good branches at step n of the induction satisfy

$$c_1 b^n < |\Delta f_n| < c_2 a^n, \tag{28}$$

where

$$0 < b < a < 1.$$

Estimates of speeds imply that parameter intervals corresponding to the movement of $W_l(t)$ through Δf_n satisfy similar inequalities with another choice of constants:

$$c_1' b^n < |\mathcal{T}_{\Delta n}| < c_2' a^n. \tag{29}$$

By construction at step n we are using pull-backs of partitions $\xi_{[nx_0]}$ of step $[nx_0]$ where $x_0 > 0$ is a small constant.

In order to get well-defined partitions we need each n-union to be a subset of the respective $[nx_0]$-intersection

$$\mathcal{U}_n \subset \mathcal{I}_{[nx_0]} \tag{30}$$

We delete $\mathcal{I}_n$ which do not satisfy 30. The measure of the deleted intervals is less than

$$Clb^{-[nx_0]}a^n, \tag{31}$$

which is exponentially small for large n if

$$\frac{a}{b^{x_0}} < 1. \tag{32}$$

If several first steps of induction are adjusted, then one can make the constants a and b arbitrary close, see [6]. At the same time one can choose x_0 arbitrary small. So it is easy to satisfy 32, although one should notice that at these special first steps of induction, when we pull the same partition ξ_0 back several times, we can lose a lot of measure in the parameter space.

2.5. Finally we discuss the problem of parameter choice in the Markov construction for Henon-like maps, see [8].

In the two-dimensional case the role of critical branches is played by thin horseshoes. At step n of inductive construction their number is less than $2^{y_0 n}$, where $y_0 > 0$ is a small constant.

So 31 is replaced by

$$C2^{y_0 n} b^{-[nx_0]} a^n. \tag{33}$$

Using a preliminary construction one can make the expansion R_0 sufficiently large. That makes the constant a small and compensates the the factor $2^{y_0 n}$. Then as above, if x_0 is sufficiently small, the estimates 33 decrease exponentially for large n .

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