BAYESIAN AND DOMINANT STRATEGY IMPLEMENTATION IN THE INDEPENDENT PRIVATE VALUES MODEL

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Abstract. We prove—in the standard independent private-values model—that the outcome, in terms of expected probabilities of trade and expected transfers, of *any* Bayesian mechanism, can also be obtained with a dominant-strategy mechanism.

Key words: Independent private values, incentive compatibility, Bayesian implementations, dominant-strategy implementation, adverse selection, bilateral trade, mechanism design.

1. Introduction

We prove that in the independent private-values model with linear utility, the outcome—in terms of expected probabilities of trade and expected transfers—of any Bayesian incentive-compatible mechanism, can also be obtained with a dominant-strategy mechanism. In other words, a mechanism is Bayesian incentive compatible if and only if there is a dominant-strategy incentive-compatible mechanism that generates the same expected probability of trade for every agent. This equivalence result is valuable. Dominant-strategy mechanisms have advantages over Bayesian mechanisms. For instance, one may be more confident that a rational agent will play a dominant strategy (if one is available) than that the same agent will play a Nash equilibrium strategy.[1]

The model has a single indivisible object and finitely many agents. Every agent has private information, customarily interpreted as the agent's valuation for the object. Payoffs are linear in valuation and transfer. From each agent's viewpoint, other agents' valuations are random variables independently distributed according to known distribution functions. The setup is sufficiently flexible to include a privately informed seller and heterogeneous buyers.

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[1]See Mas-Colell, Whinston, and Green (1995), page 870, for a brief discussion of this point.

A direct mechanism consists of two maps per agent, a probability-of-trade function and a transfer function. Every agent, after observing her valuation, sends a report to the mechanism designer. Given the profile of reported valuations, the probability-of-trade function specifies the probability that the agent receives the object, and the transfer function specifies the amounts that the agent must pay. Thus, a direct mechanism defines a game where an agent's strategy is her report given her private information, and an agent's payoff is determined by the two functions.

A direct mechanism is *Bayesian incentive compatible* if reporting truthfully is a Bayesian Nash equilibrium, and is *dominant-strategy incentive compatible* if reporting truthfully is an equilibrium in weakly-dominant strategies. In a Bayesian-Nash equilibrium an agent reports truthfully when doing so maximizes the agent's interim utility, i.e. the agent's expected payoff given the agent's true valuation (and assuming by way of equilibrium analysis that opponents report their valuations truthfully). The interim utility is determined by the agent's *expected* probability of trade and *expected* transfer. The actual probability of trade and transfer depend on the realization of opponents' reports.

Our result is of the form "for every Bayesian incentive-compatible mechanism there is an equivalent dominant-strategy mechanism." We consider that two mechanisms are equivalent if both yield the same interim utility to each agent. With independent private values and linear utilities, this is so if and only if every agent is assigned the same expected probability of trade and expected transfer by both mechanisms. Since expected transfers are determined up to a constant by the expected probability-of-trade functions (Myerson 1981), it suffices to consider the latter. Hence, we use the term outcome to refer to the expected probability-of-trade functions.

Our equivalence applies to *every* Bayesian incentive-compatible outcome. Hence, solely moving from dominant-strategy to Bayesian incentive compatibility imports no gain; any such gain must come from variations in other constraints such as ex ante or ex post budget balance. As an illustration, consider a well-known example: d'Aspremont and Gerard-Varet (1979a) and Arrow (1979) showed that a particular Bayesian mechanism, the expected externality mechanism, achieves ex post efficency, ex post budget balance, and ex ante individual rationality. Green and Laffont (1977) proved that no dominant-strategy mechanism can match those achievements. Our equivalence implies that there is a dominant-strategy mechanism that achieves the same interim (and therefore ex ante) allocation, transfer, and payoffs as the Bayesian mechanism; it does not, however, satisfy ex post budget balance.

Our result is specific to the independent private values model with linear utilities. Without linearity, for instance under risk aversion, the expected probability of trade does not suffice to determine interim utility. Thus, even if there is a dominant-strategy mechanism that generates the same expected probability-of-trade as the target Bayesian mechanism, agents need not be indifferent between both mechanisms. With nonindependent valuations, Crémer and McLean (1988, Appendix A) provide an example where the seller obtains the full surplus with a Bayesian mechanism, but cannot do so with a dominant-strategy mechanism. Once again there is no equivalent dominant-strategy mechanism.

We depart from the existing literature in the notion of equivalence that we employed, and consequently, in the technique of proof and the scope of the result. We comment first on the technique of proof.

Our proofs have a geometric quality. First, we define the set of expected probability-of-trade functions derived from Bayesian incentive-compatible mechanisms. Second, we identify the step functions that are extreme points of that set. Third, we demonstrate, by construction, that the identified extreme points can be obtained with dominant-strategy mechanisms. Fourth, we show that other elements of the set—as convex combinations of extreme points, and limits of those convex combinations—inherit the desired property, i.e. they also have a dominant-strategy equivalent.

We borrow several ideas from Border (1991) and Matthews (1984). Border characterizes the functions that are the expected probability of trade for some mechanism; he proves, elegantly and in great generality, a conjecture in Matthews (1984), first established for the real line by Chen (1986). Maskin and Riley (1984) prove, constructively, a variation of the characterization for a particular case. Border (2007) extends his own result to nonsymmetric environments. The relationship to the cited works will be indicated throughout this essay. Formally we only use a simple observation from the mentioned literature, i.e. the trivial part of Border's characterization (Lemma 4.1). Our debt, however, is larger in that our technique of proof stems from Border's geometric approach.

We turn to our equivalence notion: Two mechanisms are equivalent if both give each agent the same interim utility (or expected probability of trade). For example, the first and second-price, seal-bid auctions are equivalent in a stronger sense than the one we use: Both auctions provide the same interim utility to all agents including the seller but they also implement the same allocation, the same probability-of-trade function. (See for instance, Myerson (1981) for details.)

Previous literature on the relationship between Bayesian and dominant strategy mechanisms has used the stronger equivalence notion. Mookherjee and Reichelstein (1992), in their words, identify "...mechanism design problems for which there is no loss in replacing Bayesian incentive compatibility by the stronger requirement of dominant strategy." For them, two "equivalent" mechanisms must not only provide the same interim utility to participants but also implement the same allocation, i.e. the same ex post probability-of-trade functions. They consider an independent private values model where utilities are quasilinear and identify a monotonicity condition that is sufficient for dominant-strategy implementation. In the case of linear utility, their monotonicity condition is, simply, that the ex post probability-of-trade function be monotone.[2] Their contribution is to find conditions on utilities such as a single crossing property and their one-dimensional condensation property, so that particular allocations can be implemented in dominant strategy.

An extensive literature, including d'Aspremont and Gérard-Varet (1979b), Laffont and Maskin (1979), Makowski and Mezzetti (1994), and Williams (1999), shows in various cases that if an ex post efficient allocation is implemented by a Bayesian incentive-compatible mechanism, then it can also be implemented by a dominant-strategy mechanism. Williams (1999) obtains an equivalence for quasilinear utilities and provides interesting applications, a lucid discussion, and a summary of the literature.

We prove that the interim utilities of *any* Bayesian incentive-compatible mechanism can be obtained with a dominant-strategy mechanism. Our result is not specific to particular allocations and holds with heterogeneous agents and nonsymmetric mechanisms.

The formal results are presented in three sections. Section 4 deals with ex ante identical bidders and symmetric mechanisms. This case allows us to present the main ideas in the the proofs. Sections 5 and 6 use similar arguments to those introduced in Section 4. Section 5 treats heterogeneous agents and nonsymmetric mechanisms. Section 4 introduces a privately informed seller and ex ante identical buyers.

[2]This follows from the standard characterization of incentive compatibility applied to a single agent (Myerson (1981)).

2. Notation

Vectors are represented in bold face. If $\boldsymbol{b}$ is a vector in $\mathbb{R}^K$, b_k is its k^{th} coordinate, $\boldsymbol{b}_{-k} = (b_1, \ldots, b_{k-1}, b_{k+1}, \ldots, b_K) \in \mathbb{R}^{K-1}$, $(a, \boldsymbol{b}_{-k}) = (b_1, \ldots, b_{k-1}, a, b_{k+1}, \ldots, b_K) \in \mathbb{R}^K$, and

$$\boldsymbol{b}_k = (0, 0, \ldots, 0, b_k, b_{k+1}, \ldots, b_K) \in \mathbb{R}^K$$

The vector whose k^{th}'s coordinate is 1 and all others are 0 is denoted by $\boldsymbol{e}_k$. For $\boldsymbol{b}, \boldsymbol{b}' \in \mathbb{R}^K$, $\boldsymbol{b} \vee \boldsymbol{b}' = (\max\{b_1, b_1'\}, \ldots, \max\{b_K, b_K'\})$ and $\boldsymbol{b} \wedge \boldsymbol{b}' = (\min\{b_1, b_1'\}, \ldots, \min\{b_K, b_K'\})$.

A sum with no terms is defined to be zero; for instance, $\sum_{j=3}^{2} b_j = 0$.

Given a set B, B^c denotes its complement, χ_B its characteristic or indicator function, $int\, B$ its interior, and $|B|$ the number of elements it contains.

Let I be a positive integer and $\mathcal{I} = \{1, 2, \ldots, I\}$. For $i \in \mathcal{I}$, let $X_i \subseteq \mathbb{R}$, and λ_i be a probability distribution on X_i. Then, $\lambda = \prod_{i \in \mathcal{I}} \lambda_i$ is the product distribution and $\lambda_{-j} = \prod_{i \in \mathcal{I} \setminus \{j\}} \lambda_i$. All functions are assumed to be measurable with respect to the corresponding Borel σ-algebras; product spaces are endowed with the product σ-algebras. We use the following conventions for expectations. Given a function $q : \prod_{i=1}^{I} X_i \to \mathbb{R}$,

$$E_{\boldsymbol{x}_{-j}} q(x_j) = \int_{\prod_{i \in \mathcal{I} \setminus \{j\}} X_i} q(x_1, \ldots, x_I)\, d\lambda_1 \ldots\, d\lambda_{j-1}\, d\lambda_{j+1} \ldots\, d\lambda_I.$$

Thus the expectation is taken over $\boldsymbol{x}_{-j}$.

Analogous notation is applied to other objects.

Step functions are used prominently in our analysis. Any increasing step function can be represented by a collection of pairs $\{(b_k, \beta_k)\}_{k=1}^{K}$ where k represents the k^{th} step, β_k is the value of Q on that step, and b_k is the size of the step (in the domain) according to the underlying probability distribution. The following definition makes this observation precise.

Definition 2.1. *Let $X \subseteq \mathbb{R}$, let λ be a probability distribution on X, and let $Q : X \to [0, 1]$. We say that Q is a step function with K steps if $Q(X)$ has K elements and $[\beta \in Q(X) \implies \lambda(Q^{-1}(\beta)) > 0]$.*

Any nondecreasing step function Q with K steps can be represented by K pairs $\{(b_k, \beta_k)\}_{k=1}^{K}$ where

(a) $Q(X) = \{\beta_k\}_{k=1}^{K}$, $1 \geq \beta_K > \beta_{K-1} > \ldots > \beta_k > \beta_{k-1} > \ldots > \beta_1 \geq 0$ and
(b) for $k = 1, \ldots, K$, $b_k = \lambda\,(Q^{-1}(\beta_k))$.

We denote such Q by $\{(b_k, \beta_k)\}_{k=1}^{K}$ or by $(\boldsymbol{b}, \boldsymbol{\beta})$ where $\boldsymbol{b} = (b_1, \ldots, b_K)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_K)$. Abusing notation, we may write $Q = \{(b_k, \beta_k)\}_{k=1}^{K} = (\boldsymbol{b}, \boldsymbol{\beta})$.

Remark 2.1. *Nondecreasing step functions that differ only on λ-measure zero sets may have the same representation by K pairs.*[3]

If the function Q has K steps and is nondecreasing, then $\beta_k > \beta_{k-1}$ for all k with $1 < k \leq K$, otherwise it would have fewer than K steps.

3. Model

We use a standard independent private-values model. There is a single indivisible object and a finite set $\mathcal{I} = \{1, 2, \ldots, I\}$ of agents. Agent i's type is an element $x_i \in X_i = [\underline{x}_i, \overline{x}_i] \subseteq \mathbb{R}$, distributed according to a nonatomic probability distribution λ_i. Agents are risk neutral. Preferences are linear in type and money: If t_i is the amount paid by agent i, and q_i is the probability that i obtains the object, i's utility is $x_i q_i - t_i$; hence the interpretation of an agent's type as her valuation for the object.

A direct mechanism consists of two functions per agent, $q_i(\boldsymbol{x})$ and $t_i(\boldsymbol{x})$, where $q_i(\boldsymbol{x})$ is the probability that i is assigned the object and $t_i(\boldsymbol{x})$ is the amount i pays when the profile of reports is $\boldsymbol{x}$. The sum over i of the probabilities $q_i(\boldsymbol{x})$ must be less than or equal to one. (Strict inequality is allowed.) Henceforth, all mechanisms are direct unless specified otherwise.

Fix a mechanism $\{(q_i, t_i)\}_{i \in \mathcal{I}}$. If i reports her type truthfully (and other players report $\boldsymbol{x}_{-i}$), then i's payoff is $u_i(x_i, \boldsymbol{x}_{-i}) = q_i(x_i, \boldsymbol{x}_{-i})x_i - t_i(x_i, \boldsymbol{x}_{-i})$. Assuming other players also report truthfully, i's expected payoff is $Eu_i(x_i) = E_{x_{-j}} q_i(x_i)x_i - Et_i(x_i)$.

A mechanism is incentive compatible if truthful reporting is an equilibrium. For different equilibrium concepts, i.e. dominant-strategy or Bayesian-Nash equilibrium, there are different characterizations of incentive compatibility in terms of probabilities of trade. The following well-known results follow from Myerson (1981). A mechanism is

(a) dominant-strategy incentive compatible if and only if for all i and $\boldsymbol{x}_{-i}$, $q_i(x_i, \boldsymbol{x}_{-i})$ is nondecreasing on x_i and $t_i(x_i, \boldsymbol{x}_{-i}) = q_i(x_i, \boldsymbol{x}_{-i})x_i - \int_{\underline{x}_i}^{x_i} q(z, \boldsymbol{x}_{-i})dz - u(\underline{x}_i, \boldsymbol{x}_{-i})$.

[3] If $Q(X)$ is finite but $\lambda(Q^{-1}(\beta)) = 0$ for some $\beta \in Q(X)$, then Q belongs to the equivalence class of a step function in L_p space.

(b) Bayesian incentive compatible if and only if for all i, $E_{\boldsymbol{x}_{-i}} q_i(x_i)$ is nondecreasing on x_i and $E_{\boldsymbol{x}_{-i}} t_i(x_i) = E_{\boldsymbol{x}_{-i}} q_i(x_i) x_i - \int_{\underline{x}_i}^{x_i} E_{\boldsymbol{x}_{-i}} q_i(z) dz - E_{\boldsymbol{x}_{-i}} u_i(\underline{x}_i)$.[4]

This characterization justifies the usage summarized in the following definition.

Definition 3.1. *Let $\{q_i\}_{i \in \mathcal{I}}$ be a collection of I functions $q_i : \prod_{i=1}^{I} X_i \to [0,1]$ such that for every $x \in \prod_{i=1}^{I} X_i$, $\sum_{i \in \mathcal{I}} q_i(\boldsymbol{x}) \leq 1$.*

If for every i and $\boldsymbol{x}_{-i}$, $q_i(x_i, \boldsymbol{x}_{-i})$ is nondecreasing in x_i, then $\{q_i\}_{i \in \mathcal{I}}$ is a dominant-strategy incentive compatible mechanism.

If for every i, $E_{\boldsymbol{x}_{-i}} q_i(x_i)$ is nondecreasing in x_i, then $\{q_i\}_{i \in \mathcal{I}}$ is a Bayesian incentive-compatible mechanism.

The omitted transfer functions are recovered, up to a constant, using the corresponding incentive compatibility characterizations.

The framework presented is sufficiently flexible to include, among other things, a seller with private information. These and other features are discussed farther in the following sections.

4. Ex ante identical bidders

In this section we assume that the I agents or bidders are ex ante identical: Types are identically and independently distributed according to the nonatomic probability distribution λ_b in $X = [\underline{x}, \overline{x}]$. We denote by λ the product distribution λ_b^I.

We require that mechanisms be symmetric, i.e. that ex ante identical bidders be treated ex ante identically. (We introduce a privately informed seller, heterogeneous bidders, and nonsymmetric mechanisms in the following sections.)

Symmetric mechanisms are interesting on their own right. An alleged advantage of competitive bidding is that competitive bidding tends to reduce agency problems. Favoring a particular bidder when they are all ex ante identical may diminish this advantage. For instance, in several countries, government agencies must use competitive bidding for their purchases and are often not permitted to favor a particular bidder when all bidders are ex ante identical.

[4]Myerson (1981) assumes that distributions have densities, and that the densities are strictly positive on their supports. Monteiro and Svaiter (2007) extend the characterization to arbitrary measures.

Every symmetric mechanism can be represented by a single probability-of-trade function satisfying a permutation inequality. This representation is introduced in the definition below; its relationship to Definition 3.1 is made clear afterwards.

Definition 4.1. *Let $q : X^I \to [0,1]$ be such that for every $\boldsymbol{x} \in X^I$, $\sum_{i=1}^I q(\sigma_i(\boldsymbol{x})) \leq 1$, where $\sigma_i(x_1, \ldots, x_I) = (x_i, x_2, \ldots, x_{i-1}, x_1, x_{i+1}, \ldots, x_I)$, i.e $\sigma_i(\boldsymbol{x})$ interchanges the first and i^{th} coordinate of the vector $\boldsymbol{x}$.*

(a) q is a symmetric, dominant-strategy incentive compatible, mechanism with I bidders if $q(x_1, \boldsymbol{x}_{-1})$ is nondecreasing in x_1.

(b) q is a symmetric, Bayesian incentive compatible, mechanism with I bidders if $E_{\boldsymbol{x}_{-1}} q(x_1)$ is nondecreasing in x_1.

Remark 4.1. *Each bidder's probability-of-trade function q_i is derived from the single function q by setting $q_i(\boldsymbol{x}) = q(\sigma_i(\boldsymbol{x}))$. The omitted transfer functions are recovered, up to a constant, using the corresponding incentive compatibility characterizations.*

Theorem 1 is this section's main result: Every Bayesian incentive compatible mechanism has a dominant-strategy equivalent. Both mechanisms yield the same interim utilities for all agents.

Theorem 1. *If q' is a symmetric, Bayesian incentive-compatible mechanism with I bidders, then there is a symmetric, dominant-strategy incentive-compatible mechanism q with I bidders that generates the same expected probability of trade, i.e. $E_{\boldsymbol{x}_{-1}} q = E_{\boldsymbol{x}_{-1}} q'$ a.e.*

Remark 4.2. *We prove a stronger result: Even if the Bayesian incentive-compatible mechanism q' is not symmetric, provided $E_{\boldsymbol{x}_{-1}} q'$ is the same for all bidders, there is a* symmetric *dominant-strategy incentive-compatible mechanism q so that $E_{\boldsymbol{x}_{-1}} q = E_{\boldsymbol{x}_{-1}} q'$.*

Theorem 1 demonstrates that, ceteris paribus, going from dominant strategy to Bayesian implementation does not increase the set of implementable outcomes, when outcomes are defined in terms of *expected* probabilities of trade and expected transfers. This is the appropriate notion of outcome for Bayesian mechanisms in this setting because *expected* probabilities of trade determine bidders' expected payoffs given their private information, and expected transfers up to a constant.

Figure 1 depicts two mechanisms in an environment with two bidders whose valuations are uniformly distributed in $[0,1]$. Types are divided in five intervals of equal probability and types in the same interval are treated equally. The left diagram in the figure represents the mechanism $q'(x_1,x_2)$. Every type profile (x_1,x_2) belongs to a cell and the number in that cell is the value of $q'(x_1,x_2)$. (Cells without values indicate $q'(x_1,x_2)=0$.) The numbers below the horizontal axis are the expected probability of trade $E_{\boldsymbol{x}_{-1}}q'(x_1)$—the integral of the function q' for fixed x_1 along the vertical axis. Since $E_{\boldsymbol{x}_{-1}}q'(x_1)$ is nondecreasing, q' (with its implicit expected transfer) satisfies Bayesian incentive compatibility. (This is incentive compatibility's classic characterization.) It is clear, however, that q' does not satisfy dominant-strategy incentive compatibility because $q'(x_1,x_2)$ is *not* nondecreasing on x_1 for some x_2, say $x_2\in[1/5,2/5]$. The right diagram in the figure represents the mechanism $q(x_1,x_2)$ that is equivalent to q' in that yields the same expected probability of trade, $E_{\boldsymbol{x}_{-1}}q=E_{\boldsymbol{x}_{-1}}q'$, but it is also dominant-strategy incentive compatible. We go from mechanism q' on the left to mechanism q on the right by "rearranging the cells" in the diagram so that $q(x_1,x_2)$ is nondecreasing on x_1 for fixed x_2.

Care must be exercised so that the "rearrangement of cells" satisfies the symmetry of the mechanism: Given a type profile (x_1,x_2), if $q'(x_1,x_2)$ is the probability that agent 1 gets the object, the probability that bidder 2 gets the object is $q_2'(x_1,x_2)=q'(x_2,x_1)$, and thus $q'(x_1,x_2)+q'(x_2,x_1)\leq 1$. Thus, in both diagrams, the numbers in cells that are symmetric with respect to the diagonal must sum up to no more than one. While focusing on symmetric mechanisms allows us to use a single function q', it also *requires* us to use a single function q. In the example, the required "rearrangement of cells" is straightforward. That the required rearrangement can be carried out for any arbitrary mechanism q is the content of the theorem.

The proof of Theorem 1 follows from three lemmas of independent interest plus a convergence argument. We will conclude the section with an example.

The inequality in Lemma 4.1 is a feasibility constraint that must be satisfied by any mechanism not only Bayesian incentive compatible ones: The probability that a buyer with type in B wins, $I\int_B E_{\boldsymbol{x}_{-1}}q(x_1)\,d\lambda_b$, cannot exceed the probability that there is a buyer with type in B, $1-\lambda_b(B^c)^I$.

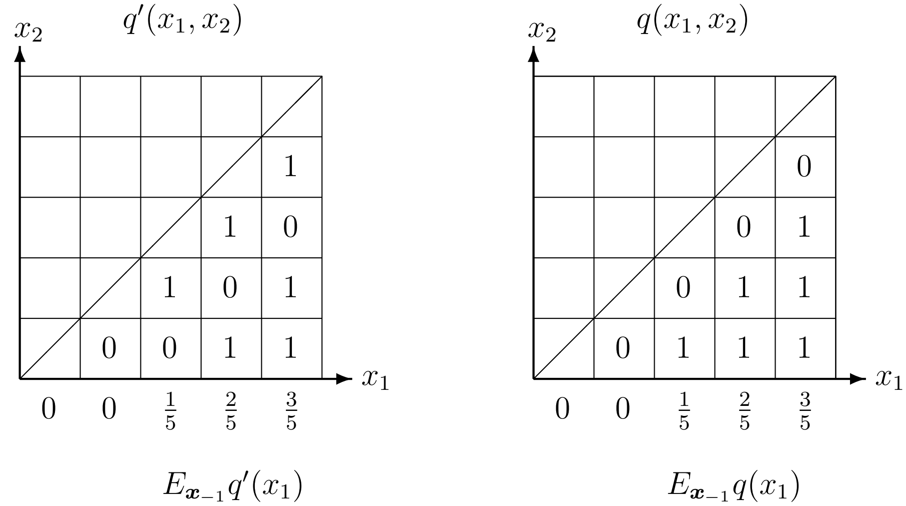

FIGURE 1. In cells with no values q' and q are 0

Lemma 4.1. *If q is a Bayesian incentive-compatible, symmetric, mechanism with I bidders, then $E_{\boldsymbol{x}_{-1}} q \in W$ where*

$$W = \Big\{ Q \mid Q : X \to [0,1] \text{ is nondecreasing and } B \subseteq X \implies I \int_B Q(x_1)\, d\lambda_b \leq 1 - [\lambda_b(B^c)]^I \Big\} \tag{1}$$

Proof. Bayesian incentive compatibility implies that $E_{\boldsymbol{x}_{-1}} q$ is nondecreasing. Lemma 5.1 in Border (1991) establishes the inequality. □

The feasibility constraint first appears in Matthews (1983) and in Maskin and Riley (1984). It plays a key role in our proof. Matthews (1984) conjectured that for any function $Q : X \to [0,1]$, not necessarily nondecreasing, that satisfies the feasibility constraint, there is a symmetric mechanism q with $E_{\boldsymbol{x}_{-1}} q = Q$, i.e. Q is the expected probability of trade of some symmetric mechanism q. Border (1991) proves Matthews' conjecture for general type spaces. He is not concerned with incentive compatibility; he is interested in determining when the expected probability of trade can be used as the primitive in the analysis. (Lemma 4.1 is a corollary to Lemma 5.1 in Border (1991).) Maskin and Riley (1984, Theorem 7 in their appendix) prove, constructively, a version of Matthews' conjecture for nondecreasing step functions Q. Matthews (1984) extends Maskin and Riley's result to arbitrary nondecreasing functions Q. All these authors restrict attention to symmetric mechanisms and ex ante identical bidders.

The rest of the proof proceeds as follows. Lemma 4.2 characterizes the step functions that are extreme points of W. Lemma 4.3 constructs a symmetric, dominant-strategy mechanism for each extreme point identified in Lemma 4.2. Finally, a convergence argument, provided after the proofs of the Lemmas, establishes the theorem: Every function in W is the limit of convex combinations of step functions.

Lemma 4.2 is based on the following observations. The domain of any step function can be partitioned into finitely many sets where the function is constant; the elements of the partition are the function's level sets. Lemma 4.2 arbitrarily fixes one such partition and identifies the step functions, relative to the fixed partition, that are extreme points of the feasible set W. Verifying that a step function is an extreme point is a finite dimensional matter: When applied to a step function, the inequality in (1) becomes a system of finitely many linear inequalities, determined by the fixed partition. To be an extreme point, the step function must make sufficiently many inequalities bind. (To visualize this point, imagine a set in $\mathbb{R}^2$ defined by finitely many linear inequalities, say a rectangle. The extreme points of the rectangle are its vertices. Vertices are defined by the intersection of sufficiently many lines, more precisely two lines per vertex because the rectangle is a subset of $\mathbb{R}^2$.)

The proof of Theorem 1 only uses one direction in Lemma 4.2: If a step function is an extreme point, it must be one of those identified by the Lemma.

Lemma 4.2. *Let $Q = (\boldsymbol{b}, \bar{\boldsymbol{\beta}}) = \{(b_k, \bar{\beta}_k)\}_{k=1}^K$ be a step function in W. Then $\{(b_k, \bar{\beta}_k)\}_{k=1}^K$ is an extreme point of W if and only if either*

(a) $\bar{\beta}_k = \frac{\left(\sum_{j=1}^k b_j\right)^I - \left(\sum_{j=1}^{k-1} b_j\right)^I}{I b_k}$ *for $k = 1, \ldots, K$, or*

(b) $\bar{\beta}_1 = 0$ *and $\bar{\beta}_k$ is as in* (a) *for $k = 2, \ldots, K$.*

Proof. Letting $B = \bigcup_{j=k}^K Q^{-1}(\bar{\beta}_k)$, the inequality in the definition of W becomes

$$\sum_{j=k}^K I b_j \beta_j \leq 1 - \left(\sum_{j=1}^{k-1} b_j\right)^I \tag{2}$$

or, in vector notation, $I\boldsymbol{b}_k \cdot \boldsymbol{\beta} \leq 1 - r_k$, where $\boldsymbol{b}_k$ is the vector $(0, \ldots, 0, 0, b_k, b_{k+1}, \ldots, b_K)$, and $r_k = \left(\sum_{j=1}^{k-1} b_j\right)^I$. Taking $k = 1, \ldots, K$, (2) becomes a system of K inequalities.

Recall $\boldsymbol{e}_k$ denotes the vector whose k^{th} coordinate is 1 and all others are zero. Define

$$P = \{\boldsymbol{\beta} \in \mathbb{R}^K : \text{ for } k = 1, \ldots, K,\ I\boldsymbol{b}_k \cdot \boldsymbol{\beta} \leq 1 - r_k \text{ and } \boldsymbol{e}_k \cdot \boldsymbol{\beta} \geq 0\} \tag{3}$$

The set $P \subseteq \mathbb{R}^K$ is defined by $2K$ inequalities; it is the set of all nonnegative vectors $\boldsymbol{\beta} \in \mathbb{R}^K$ (K inequalities), such that $(\boldsymbol{b}, \boldsymbol{\beta})$ satisfies the inequalities (2) (another K inequalities).

A step function $(\boldsymbol{b}, \boldsymbol{\beta}) \in W$ is an extreme point of W if and only if $\boldsymbol{\beta}$ is an extreme point of P (Lemma A.2). A vector $\boldsymbol{\beta} \in P$ is an extreme point of P if and only if the set

$$R(\boldsymbol{\beta}) = \{\boldsymbol{b}_k : k \in \{1, \ldots, K\}, I\boldsymbol{b}_k \cdot \boldsymbol{\beta} = 1 - r_k\} \cup \{\boldsymbol{e}_k : k \in \{1, \ldots, K\}, \boldsymbol{e}_k \cdot \boldsymbol{\beta} = 0\} \tag{4}$$

has K linearly independent elements (Lemma A.1), i.e. if the inequalities defining P are evaluated at $\boldsymbol{\beta}$, they must include K linearly independent equations. We conclude that $(\boldsymbol{b}, \bar{\boldsymbol{\beta}})$ is an extreme point of W if and only if $R(\bar{\boldsymbol{\beta}})$ has K linearly independent vectors.

The set $R(\bar{\boldsymbol{\beta}})$ has K linearly independent vectors if and only if either $\{\boldsymbol{b}_k\}_{k=1}^K \subseteq R(\bar{\boldsymbol{\beta}})$ or $\{e_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_K\} \subseteq R(\bar{\boldsymbol{\beta}})$: Simple inspection reveals that both alternatives have K linearly independent vectors. To see that no other alternative is possible suppose that $e_k \in R(\bar{\boldsymbol{\beta}})$. Then then $\bar{\beta}_k = 0$. This implies $k = 1$, otherwise $\bar{\boldsymbol{\beta}}$ has less than K steps.

Finally, $\{\boldsymbol{b}_k\}_{k=1}^K \subseteq R(\tilde{\boldsymbol{\beta}})$ if and only if $\tilde{\boldsymbol{\beta}}$ is as defined in Lemma 4.2 (a): The system of equations $I\boldsymbol{b}_k \cdot \tilde{\boldsymbol{\beta}} = 1 - r_k$, $k = 1, \ldots, K$ has a unique solution (because the vectors $\{\boldsymbol{b}_k\}_{k=1}^K$ are linearly independent.) The solution to $I\boldsymbol{b}_K \cdot \tilde{\boldsymbol{\beta}} = 1 - r_K$ is $\tilde{\beta}_K = \bar{\beta}_K$. Pick any $k < K$. Subtracting $I\boldsymbol{b}_{k+1} \cdot \tilde{\boldsymbol{\beta}} = 1 - r_{k+1}$ from $I\boldsymbol{b}_k \cdot \tilde{\boldsymbol{\beta}} = 1 - r_k$ yields $\tilde{\beta}_k = \bar{\beta}_k$.

Similarly, $\{e_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_K\} \subseteq R(\bar{\boldsymbol{\beta}})$ if and only if $\bar{\boldsymbol{\beta}}$ is as in Lemma 4.2 (b). □

Lemma 4.3 constructs a dominant-strategy incentive compatible mechanism that implements the extreme points identified in Lemma 4.2. Let the step function Q be an extreme point of W. The mechanism is constructed as follows. Given a type profile $(x_1, \ldots, x_I)$, bidders are ranked using $Q(x_i)$. Those bidders with maximum rank, i.e. $\max_i Q(x_i)$, share the object with equal probability in the new mechanism; those bidders with less than maximum rank are assigned the object with probability zero. Thus, the new mechanism q takes values in $\{\frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{I}, 0\}$. It depends only on the partition $\{Q^{-1}(b_k)\}_{k=1}^K$ defined by Q and not on the actual values taken by Q.

Lemma 4.3. *Let the step function $Q = \{(b_k, \bar{\beta}_k)\}_{k=1}^K$ be an extreme point of W. Then, the symmetric mechanism*

$$q(x_1, x_2, \ldots, x_I) = \begin{cases} \frac{1}{|\{i: Q(x_1) = Q(x_i)\}|} & \text{if } Q(x_1) > 0, Q(x_1) \geq Q(x_i) \ \forall i \\ 0 & \text{otherwise} \end{cases}$$

is dominant-strategy incentive compatible and $E_{\boldsymbol{x}_{-1}} q = Q$.

Proof. Since q is nondecreasing in x_1 for any given $\boldsymbol{x}_{-1}$, q satisfies dominant-strategy incentive compatibility. We must prove that $E_{\boldsymbol{x}_{-1}} q = Q$.

Pick an arbitrary x_1. If $Q(x_1) = 0$, then $E_{\boldsymbol{x}_{-1}} q(x_1) = 0$.

Suppose then that $Q(x_1) = \bar{\beta}_k > 0$. By direct calculation,

$$E_{\boldsymbol{x}_{-1}} q(x_1) = \sum_{n=1}^{I} \frac{1}{n} \left(\begin{array}{c} I-1 \\ n-1 \end{array} \right) \left(\sum_{j=1}^{k-1} b_j \right)^{I-1-(n-1)} b_k^{n-1}$$

To see this note that $E_{\boldsymbol{x}_{-1}} q(x_1)$ is the integral of $q(x_1, \ldots, x_I)$ over all x_i with $i \neq 1$. Since q takes finitely many values, its integral is a summation. Each term in the expression above corresponds to a value of $q(x_1, x_{-1})$ as x_{-1} varies. The first factor in a typical term, $\frac{1}{n}$, is the value of q. The second factor is the number of ways in which q may take the value $\frac{1}{n}$: there are $I-1$ variables x_i and exactly $n-1$ of them must be in $Q^{-1}(\beta_k)$. The last two factors represent the probabilities: $\sum_{j=1}^{k-1} b_j$ is the probability that a given x_i is in $\bigcup_{j=1}^{k-1} Q^{-1}(\beta_j)$ and therefore $\left(\sum_{j=1}^{k-1} b_j\right)^{I-1-(n-1)}$ is the probability that $I-1-(n-1)$ of them will be in $\bigcup_{j=1}^{k-1} Q^{-1}(\beta_j)$. Similarly b_k^{n-1} is the probability that $n-1$ variables x_i will be in $Q^{-1}(\beta_k)$.

To show that $x_1 \in Q(\bar{\beta}_k) \implies E_{\boldsymbol{x}_{-1}} q(x_1) = \bar{\beta}_k$, we must prove that

$$\sum_{n=1}^{I} \frac{1}{n} \binom{I-1}{n-1} \left(\sum_{j=1}^{k-1} b_j \right)^{I-n} b_k^{n-1} = \frac{\left(\sum_{j=1}^{k-1} b_j + b_k\right)^I - \left(\sum_{j=1}^{k-1} b_j\right)^I}{I b_k}$$

Multiply both sides by $I b_k$, note that $\frac{I}{n}\binom{I-1}{n-1} = \binom{I}{n}$, and add $\left(\sum_{j=1}^{k-1} b_j\right)^I$ to both sides, to obtain

$$\sum_{n=1}^{I} \binom{I}{n} \left(\sum_{j=1}^{k-1} b_j \right)^{I-n} b_k^n + \left(\sum_{j=1}^{k-1} b_j \right)^I = \left(\sum_{j=1}^{k-1} b_j + b_k \right)^I$$

This is the binomial formula since $\left(\sum_{j=1}^{k-1} b_j\right)^I$ corresponds to the term, missing in the summation, for $n = 0$. □

That the type of mechanism employed in Lemma 4.3 can achieve the bounds in Lemma 4.1 was recognized by Border (1991), Lemma 5.2, page 1180.

The proof of Theorem 1 now follows from Lemmas 4.1, 4.2 and 4.3 plus a convergence argument. (See Border (1991), Lemma 5.4, for related material.) To develop the convergence argument we define the class of monotone dominant-strategy mechanisms (Definition 4.2) and prove Lemmas 4.4 and 4.5. Lemmas 4.4 identifies the set of monotone mechanisms as a subset of L_∞. Lemma 4.5 demonstrates that the set of monotone mechanisms is compact, thus contributing a convergent subsequence of dominant-strategy mechanisms to the proof of Theorem 1. After proving both lemmas, the convergence argument is presented in the proof of Theorem 1.

Definition 4.2. *A symmetric mechanism $q : X^I \to [0, 1]$ is monotone if $q(\boldsymbol{x}') \le q(\boldsymbol{x})$ for every $\boldsymbol{x}', \boldsymbol{x} \in X$ with $x_1' \le x_1$ and $x_i' \ge x_i$ for every $i > 1$.*

Remark 4.3. *The mechanism q in Lemma 4.3 is monotone. Monotone mechanisms are nondecreasing in x_1—hence dominant-strategy incentive compatible—and nonincreasing in x_i for $i > 1$.*

Definition 4.3. *For $\boldsymbol{x}', \boldsymbol{x} \in \mathbb{R}^I$, let $\boldsymbol{x}' \preceq \boldsymbol{x}$ if $x_1' \le x_1$ and $x_i' \ge x_i$ for $i > 1$; and let $\boldsymbol{x}' \curlywedge \boldsymbol{x} = (x_1' \wedge x_1, x_2' \vee x_2, x_3' \vee x_3, \ldots, x_I' \vee x_I)$.*

The lemma below defines a subset $\mathcal{D}$ of L_∞ and demonstrates that it is the set of monotone mechanisms: The elements of $\mathcal{D}$ are equivalence classes. Two functions that differ only on a set of λ_b^I-measure zero belong to the same equivalence class. Monotone mechanisms belong to the set because they satisfy its defining conditions everywhere. Lemma 4.4 proves that every equivalence class in the set contains a monotone mechanism. This is necessary for the intended interpretation of $\mathcal{D}$. (For instance, the set of continuous functions as a subset of L_∞ does not contain all functions continuous a.e.)

Lemma 4.4. *Let*

$$\mathcal{D} = \Big\{ q \in L_\infty(\lambda_b^I) \ : \exists B \subseteq [\underline{x}, \overline{x}]^I \ \textit{with}\ \lambda_b^I(B) = 1, \ \textit{and}\ \forall \boldsymbol{x}', \boldsymbol{x} \in B$$

$$(\forall i \in \mathcal{I}, q(\sigma_i(\boldsymbol{x})) \in [0,1]),$$

$$\sum_{i \in \mathcal{I}} q(\sigma_i(\boldsymbol{x})) \leq 1, \ \textit{and}$$

$$[\sigma_i(\boldsymbol{x}') \preceq \sigma_i(\boldsymbol{x})] \implies q(\sigma_i(\boldsymbol{x}')) \leq q(\sigma_i(\boldsymbol{x})) \Big\}$$

Suppose $\operatorname{supp}[\lambda_b] = X$. *Then,* $\tilde{q} \in \mathcal{D}$ *if and only if there is a monotone symmetric mechanism* q *such that* $q = \tilde{q}$ *a.e.*

Proof. One direction is trivial: Monotone mechanisms satisfy the defining conditions of $\mathcal{D}$ everywhere. We prove the converse, i.e. that every function in $\mathcal{D}$ belongs to the equivalence class of a monotone mechanism.

Let $\tilde{q} \in \mathcal{D}$ and let B be its corresponding set of full-measure as specified in the definition of $\mathcal{D}$. The restriction of $\tilde{q}$ to B is monotone and therefore the set of its discontinuity points has Lebesgue measure zero (Lavrič (1993)). Let $B' \subseteq B$ be the set of its continuity points. Since λ_b is absolutely continuous with respect to the Lebesgue measure, $\lambda_b^I(B') = \lambda_b^I(B) = 1$.

Let $B'' = B' \subseteq \{\boldsymbol{x} : \exists i \in \mathcal{I}, \sigma_i(\boldsymbol{x}) \notin B'\}$. Then $\lambda_b^I(B'') = \lambda_b^I(B') = 1$. Since $\operatorname{supp}[\lambda_b^I] = X^I$, B'' is dense in X^I.

We will construct a monotone mechanism q such that $q = \tilde{q}$ a.e. Let

$$\varphi(\boldsymbol{x}) = \{r \in [0,1] : \exists \{\boldsymbol{x}^n\} \text{ in } B'', \{(\boldsymbol{x}^n, \tilde{q}(\boldsymbol{x}^n))\} \to (\boldsymbol{x}, r)\}$$

For every $\boldsymbol{x} \in X^I$, $\varphi(\boldsymbol{x})$ is nonempty (because B'' is dense in X^I) and closed (because it is the set of limit points). For $\boldsymbol{x} \in X^I$, define

$$q(\boldsymbol{x}) = \begin{cases} 0 & \text{if } \textit{int}\,(\{\boldsymbol{x}' : \boldsymbol{x}' \preceq \boldsymbol{x}\}) = \emptyset \\ \min\{r : r \in \varphi(\boldsymbol{x})\} & \text{otherwise} \end{cases} \tag{5}$$

where for any set A, $\textit{int}\,(A)$ is the interior of A.

By construction, q takes values in $[0,1]$ and $q = \tilde{q}$ a.e. (because every $\boldsymbol{x} \in B''$ is a continuity point of the restriction of $\tilde{q}$ to B and therefore, $\varphi(\boldsymbol{x})$ is a singleton and $\tilde{q}(\boldsymbol{x}) = q(\boldsymbol{x})$).

We prove that $\forall \boldsymbol{x} \in X^I$, $\sum_i q(\sigma_i(\boldsymbol{x})) \leq 1$. Pick any $\boldsymbol{x} \in X^I$. Let $\{\boldsymbol{x}^n\}$ be a sequence in B'' such that $\{\boldsymbol{x}^n\} \to \boldsymbol{x}$. (Such a sequence exists because B'' is dense.) The sequence $\{(\boldsymbol{x}^n, \tilde{q}(\sigma_1(\boldsymbol{x}^n)), \ldots, \tilde{q}(\sigma_I(\boldsymbol{x}^n)))\}$ takes values in $X^I \times [0,1]^I$ and thus has a convergent subsequence, indexed by n_k, whose limit point we denote by $(\boldsymbol{x}, \bar{\boldsymbol{q}}) = (\boldsymbol{x}, \bar{q}_1, \ldots, \bar{q}_I)$. Since $\sum_{i\in\mathcal{I}} \tilde{q}(\sigma_i(\boldsymbol{x}^{n_k})) \leq 1$ for all n_k, $\sum_{i\in\mathcal{I}} \bar{q}_i \leq 1$. By construction, $\bar{q}_i \in \varphi(\sigma_i(\bar{\boldsymbol{x}}))$. By (5), $q(\sigma_i(\bar{\boldsymbol{x}})) \leq \bar{q}_i$. Therefore, $\sum_{i\in\mathcal{I}} q(\sigma_i(\boldsymbol{x})) \leq \sum_{i\in\mathcal{I}} \bar{q}_i \leq 1$.

It remains to show that q is monotone. Let $\boldsymbol{x} \preceq \boldsymbol{y}$, $x \neq y$. If $int\,(\{\boldsymbol{x}' : \boldsymbol{x}' \preceq \boldsymbol{x}\}) = \emptyset$, then $q(\boldsymbol{x}) = 0$ and hence $q(\boldsymbol{x}) \preceq q(\boldsymbol{y})$.

If $int\,(\{\boldsymbol{x}' : \boldsymbol{x}' \preceq \boldsymbol{x}\}) \neq \emptyset$ then $int\,(\{\boldsymbol{y}' : \boldsymbol{y}' \preceq \boldsymbol{y}\}) \neq \emptyset$, and for sufficiently large n, $int\,(\{\boldsymbol{y}' : \boldsymbol{y}' \preceq \boldsymbol{y}^n\}) \neq \emptyset$. To prove that $q(\boldsymbol{x}) \leq q(\boldsymbol{y})$ it suffices to establish that for any $r \in \varphi(\boldsymbol{y}), \exists r' \in \varphi(\boldsymbol{x}), r' \leq r$. Since $r \in \varphi(\boldsymbol{y}), \exists\{\boldsymbol{y}^n\}$ in B'' such that $\{(\boldsymbol{y}^n, \tilde{q}(\boldsymbol{y}^n))\} \to (\boldsymbol{y}, r)$. For each $\boldsymbol{y}^n$, pick $\boldsymbol{x}^n \in \{\boldsymbol{x}' \in B'' : \boldsymbol{x}' \preceq (\boldsymbol{y}^n \curlywedge \boldsymbol{x})\}$ and $\|\boldsymbol{x}^n - (\boldsymbol{y}^n \curlywedge \boldsymbol{x})\| \leq 1/n$. Then $\{\boldsymbol{x}^n\}$ converges to $\boldsymbol{x}$ and $\boldsymbol{x}^n \preceq \boldsymbol{y}^n$ for every n. Let $(\boldsymbol{x}, r')$ be any accumulation point of $(\boldsymbol{x}^n, \tilde{q}(\boldsymbol{x}^n))$. Then $r' \leq r$. □

Lemma 4.5. *The set $\mathcal{D}$ defined in Lemma 4.4 is weak* compact.*

Proof. We now show that $\mathcal{D}$ is weak* compact. If $q \in \mathcal{D}$, q takes values in $[0,1]$ a.e. and thus $\mathcal{D}$ is L_∞-bounded. If in addition $\mathcal{D}$ is weak* closed, $\mathcal{D}$ is weak* compact (Banach-Alaoglu Theorem). Let $\{q^n\}$ be a sequence in $\mathcal{D}$ such that $\{q^n\} \xrightarrow{w^*} \bar{q}$. (Since in this case L_1 is separable, the weak* topology on the unit ball in L_∞ is metrizable and hence we may restrict attention to sequences.) We must prove that $\bar{q}$ belongs to (an equivalence class in) $\mathcal{D}$.

Let $A_1 = \{\boldsymbol{x} : \sum_{i\in\mathcal{I}} \bar{q}(\sigma_i(\boldsymbol{x})) \leq 1\}$ and suppose $\lambda(A_1) < 1$. For any $q^n \in \mathcal{D}$,

$$\sum_{i\in\mathcal{I}} \int \chi_{A_1^c} q^n(\sigma_i(\boldsymbol{x}))\, d\lambda_b^I \leq \int \chi_{A_1^c}\, d\lambda_b^I = \lambda_b^I(A_1^c)$$

Taking limits, $\sum_{i\in\mathcal{I}} \int \chi_{A_1^c} \bar{q}(\sigma_i(\boldsymbol{x}))\, d\lambda_b^I \leq \lambda_b^I(A_1^c)$, a contradiction. Hence, $\lambda_b^I(A_1^c) = 0$.

Let B', B be subsets of X^I, $\lambda_b(B') > 0$, $\lambda_b(B) > 0$, such that $[\boldsymbol{x}' \in B', \boldsymbol{x} \in B] \implies \boldsymbol{x}' \preceq \boldsymbol{x}$. By Lemma A.3,

$$q^n \in \mathcal{D} \implies \frac{1}{\lambda_b^I(B')} \int \chi_{B'} q^n\, d\lambda_b^I \leq \frac{1}{\lambda_b^I(B)} \int \chi_B q^n\, d\lambda_b^I$$

Taking limits: $\frac{1}{\lambda_b^I(B')}\int \chi_{B'}\bar{q}\,d\lambda_b^I \le \frac{1}{\lambda_b^I(B)}\int \chi_B \bar{q}\,d\lambda_b^I$. Lemma A.3 implies

$$\exists A_2,\ \lambda_b^I(A_2) = 1\ :\ [\forall \boldsymbol{x}', \boldsymbol{x} \in A_2, \boldsymbol{x}' \preceq \boldsymbol{x}] \implies \bar{q}(x') \le \bar{q}(x)$$

Let

$$A = A_1 \cap A_2$$

Since $\lambda_b^I(A_1) = \lambda_b^I(A_2) = 1$, $\lambda_b^I(A) = 1$. Similar arguments show that $\bar{q}$ takes values in $[0,1]$ a.e. This proves that $\mathcal{D}$ is weak* closed and hence weak* compact. $\square$

Combining Lemmas 4.4 and 4.5 we obtain

Corollary 4.5.1. *The set of monotone symmetric mechanisms, as a subset of $L_\infty(\lambda_b^I)$, is weak* compact.*

Proof of Theorem 1. We divide the proof in steps. Fix any Q in W.

a. There exists a sequence of step functions $\{Q^n\}$ in W such that $\{Q^n\} \xrightarrow{L_\infty} Q$.

Proof. For $n = 1, 2, \ldots$, define

$$Q^n(x_1) = \sup\left\{k2^{-n} : k \in \mathbb{N}, k2^{-n} \le Q(x_1), \lambda_b(Q^{-1}([k2^{-n}, (k+1)2^{-n}))) > 0\right\}$$

where the supremum over the empty set is defined to be zero. By construction,

$$\lambda_b\left(\left\{x_1\ :\ |Q^n(x_1) - Q(x_1)| \le 2^{-n}\right\}\right) = 1$$

and therefore $\{Q^n\} \xrightarrow{L_\infty} Q$. Also Q^n is a nondecreasing step function in the sense of Definition 2.1. Finally, since $Q^n \le Q$, it satisfies the inequality in (1). Thus, $Q^n \in W$.

b. For $n = 1, 2, \ldots$, the step function Q^n is a convex combination of step functions that are extreme points of W.

Proof. Suppose, without loss of generality, that Q^n has K steps. By Definition 2.1, $Q^n = (\boldsymbol{b}, \boldsymbol{\beta})$, $\boldsymbol{b}, \boldsymbol{\beta} \in \mathbb{R}^K$. Let $W(\boldsymbol{b}) = \left\{\boldsymbol{\beta}' \in [0,1]^K : (\boldsymbol{b}, \boldsymbol{\beta}') \in W\right\}$. The set $W(\boldsymbol{b})$ is a convex, compact, subset of $\mathbb{R}^K$. Then, $W(\boldsymbol{b})$ equals the convex hull of its extreme points (see for instance, Grünbaum (2003), page 18.) Every extreme point of $W(\boldsymbol{b})$ is an extreme point of W (Corollary A.2.1).

c. For $n = 1, 2, \ldots$, there exists a monotone mechanism q^n such that $E_{\boldsymbol{x}_{-1}} q^n = Q^n$.

Proof. Apply Lemmas 4.2 and 4.3, Remark 4.3, and the fact that the convex combination of monotone symmetric mechanisms is a monotone symmetric mechanism. Using (b) the claim is established.

d. The sequence $\{q^n\}$ has a subsequence $\{q^{n_k}\}$ such that $\{q^{n_k}\} \xrightarrow{w^*} q'$, and (a member of the equivalence class of) q' is a monotone mechanism with $E_{\boldsymbol{x}_{-1}} q' = Q$ a.e.

Proof. By Lemma 4.5, the sequence $\{q^n\}$ in $\mathcal{D}$ has a convergent subsequence, $\{q^{n_k}\} \xrightarrow{w^*} q'$, $q' \in \mathcal{D}$. (Since L_1 is separable, the weak* topology on the unit ball in L_∞ is metrizable and we may use sequences without loss of generality.) By Lemma 4.4, q' can be considered a monotone mechanism.

The map $q^{n_k} \mapsto E_{\boldsymbol{x}_{-1}} q^{n_k}$ is continuous when domain and range are endowed with their weak* topologies; therefore $\{E_{\boldsymbol{x}_{-1}} q^{n_k}\} \xrightarrow{w^*} E_{\boldsymbol{x}_{-1}} q'$. Since $\{Q^{n_k}\}$ is a subsequence of $\{Q^n\}$ in (a), $\{Q^{n_k}\} \xrightarrow{L_\infty} Q$. By (c), $E_{\boldsymbol{x}_{-1}} q^{n_k} = Q^{n_k}$ $\forall n_k$; hence $E_{\boldsymbol{x}_{-1}} q' = Q$ a.e.

□

It follows from Manelli and Vincent (2007), Theorem 21 and Corollary 21.1, that the set of Bayesian incentive-compatible mechanisms is the closed convex hull of the set of extreme points that are step functions (and that the set of extreme points that are step functions is norm dense in the set of extreme points). Thus (a) and (b) in the proof of Theorem 1 could be replaced by the cited results. We provide a direct proof for completeness, and because the results in Manelli and Vincent (2007) are expressed in terms of interim utilities in multiunit environments.

The example below illustrates Lemmas 4.2 and 4.3.

Example 1. *There are two bidders, $i = 1, 2$, $K_1 = K_2 = 4$, and for every i, x_i is uniformly distributed in $X = [0, 1]$.*

Lemma 4.2 states that there are at most two extreme points for any given partition. Fix a partition of $[0, 1]$, say $[0, 1/4]$, $(1/4, 2/4]$, $(2/4, 3/4]$, $(3/4, 1]$. The step function

$$Q = \{(1/4, 1/8), (1/4, 3/8), (1/4, 5/8), (1/4, 7/8)\}$$

is one extreme point of W for the proposed partition. The level sets of Q are the elements of the partition. The second extreme point of W (for the same partition), say Q', is obtained from Q by setting $Q'(x_1) = 0$ for all $x_1 \in [0, 1/4]$, and $Q'(x_1) = Q(x_1)$ elsewhere.

Figure 2 illustrates Lemma 4.3 as it applies to Q. The numbers in the cells of the figure indicate the values of $q(x_1, x_2)$; empty cells indicate $q(x_1, x_2) = 0$ for (x_1, x_2) in the cell. Note that $E_{\boldsymbol{x}_{-1}} q = Q$.

The mechanism that implements the second extreme point is $q'(x_1, x_2) = 0$ for $(x_1, x_2) \in [0, 1/4] \times [0, 1/4]$ and $q' = q$ elsewhere; then $E_{\boldsymbol{x}_{-1}} q' = Q'$.

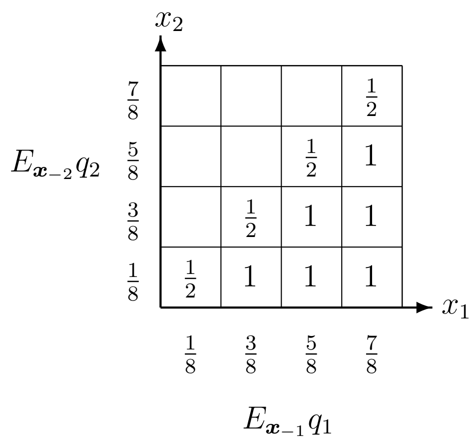

FIGURE 2. An extreme point and its dominant-strategy mechanism

5. HETEROGENEOUS BIDDERS

In this section, agents are potentially heterogeneous and therefore mechanisms are not required to be symmetric.

Theorem 2. *If $\{q'_i\}_{i \in \mathcal{I}}$ is a Bayesian incentive-compatible mechanism, then there exists a dominant-strategy, incentive-compatible mechanism $\{q_i\}_{i \in \mathcal{I}}$ that generates the same expected probability of trade, i.e. $E_{\boldsymbol{x}_{-i}} q'_i = E_{\boldsymbol{x}_{-i}} q_i$ a.e. for $i \in \mathcal{I}$.*

Remark 5.1. *Theorem 1 is not implied by Theorem 2. Theorem 2 establishes the existence of an equivalent dominant-strategy mechanism but does not ensure that it is symmetric, even if the original Bayesian mechanism is symmetric. Note that nonsymmetric mechanisms may yield symmetric expected probabilities of trade. See also Remark 4.2 following Theorem 1.*

The proof of Theorem 2 follows closely the proof of Theorem 1. It proceeds in three lemmas, the analogues of Lemmas 4.1, 4.2, and 4.3.

Lemma 5.1. *If $\{q_i\}_{i\in\mathcal{I}}$ is Bayesian incentive compatible, then $\{E_{\boldsymbol{x}_{-i}} q_i\}_{i\in\mathcal{I}}$ is in*

$$W'' = \Bigg\{ \{Q_i\}_{i\in\mathcal{I}} \big| \ \forall i,\ Q_i : X_i \to [0,1] \text{ is nondecreasing and } \\ \prod_{i\in\mathcal{I}} B_i \subseteq \prod_{i=1}^{I} X_i \implies \sum_{i\in\mathcal{I}} \int_{B_i} Q_i \, d\lambda_i \le 1 - \prod_{i\in\mathcal{I}} \lambda_i(B_i^c) \Bigg\} \tag{6}$$

Since the proof is analogous to that of Lemma 4.1, we provide only a sketch. Because of Bayesian incentive compatibility $E_{\boldsymbol{x}_{-i}} q_i$ must be nondecreasing. The probability that bidder i has type in B_i and wins the object is $\int_{B_i} E_{\boldsymbol{x}_{-i}} q_i \, d\lambda_i$. Hence the left-hand side of the inequality is the probability that one buyer's type is in her specified set and the buyer wins the object. The right-hand side is the probability that at least one buyer has type in her specified set. Therefore the inequality in (6) must hold.

Lemma 5.2 below serves the same purpose as Lemma 4.2 did in the symmetric environment of Section 4. Given a partition of the type space, the lemma identifies all the step functions (relative to that partition) that are extreme points of W''. Similar arguments are used in the proofs of both lemmas: Identifying the extreme points of W'' is equivalent to finding the solution to a system of equations obtained from the feasibility condition in (6). This is what the proof of Lemma 4.2 accomplished.

The differences in details between Lemmas 5.2 and 4.2 arise from the selection of the system of equations and the number of unknowns to be determined. Generally there are more inequalities than necessary to determine an extreme point. To see this, imagine that W'' is a rectangle in $\mathbb{R}^2$. Four inequalities suffice to define the rectangle but each of its extreme points, i.e. each vertex, is determined by only two inequalities; each vertex is a point were two inequalities become binding. To identify a vertex, the inequalities must be chosen judiciously: if two inequalities represented by parallel lines are chosen, an extreme point will not be identified.

In Lemma 4.2, because the I ex ante identical bidders must be treated symmetrically, the selection of equations to determine the unknowns is trivial: For a fixed partition of the type space, Lemma 4.2 identifies a single "family" of extreme points containing one main extreme point and another one obtained through a small variation (i.e. $\bar{\beta}_1 = 0$).

In Lemma 5.2, the situation is more involved. Since bidders need not be treated symmetrically, even for a fixed partition of the type spaces, the feasible set has many extreme points. Each one of them is identified by a different system of equations. Modulus the selection of the system of equations, however, the arguments used to prove Lemma 5.2 are the same as those used to prove Lemma 4.2. To characterize the different extreme points without listing them individually, we use a labeling system. The labeling system identifies the equations that determine the extreme points of W''. This allows us to prove Lemma 5.2 for a canonical extreme point.

Definition 5.1. *Let $\{K_i\}_{i=1}^I$ be a collection of I nonnegative integers. A labeling relative to $\{K_i\}_{i\in\mathcal{I}}$ is a function $g : \{0, 1, \ldots, \sum_{i\in\mathcal{I}} K_i\} \to \prod_{i\in\mathcal{I}}\{1, \ldots, K_i + 1\}$ such that*

(a) $g(0) = (K_1 + 1, \ldots, K_I + 1)$,

(b) for $n \geq 1$, $g(n) - g(n-1) = -\boldsymbol{e}_i$ for some $i \in \{1, \ldots I\}$.

For $k \in \{0, \ldots, \sum_{i\in\mathcal{I}} K_i\}$, define $g_i^{-1}(k) = \min\{n : g_i(n) = k\}$.[5]

Let $I = 2$. Intuitively, a labeling is a collection of multi-indices ordered from highest $(K_1 + 1, K_2 + 1)$ to lowest $(1, 1)$ with the restriction that each element in the collection is obtained from the previous one by decreasing a single index by one unit. (We use the term multi-index because, as we will see after Example 2, each $g(n)$ is used in Lemma 5.2 to index a feasibility constraint.) For instance, if $K_1 = K_2 = 3$, one possible labeling is $g(0) = (4, 4), g(1) = (3, 4), g(2) = (3, 3), g(3) = (2, 3), g(4) = (2, 2), g(5) = (1, 2), g(6) = (1, 1)$. The collection $g(0) = (4, 4), g(1) = (3, 4), g(2) = (3, 2), g(3) = (2, 2), g(4) = (1, 2), g(5) = (1, 1), g(6) = (1, 1)$ is not a labeling.

More generally, consider the set $\prod_{i\in\mathcal{I}}\{1, \ldots, K_i+1\}$ with the partial order defined by the standard vector inequality, i.e. $\mathbf{k}' < \mathbf{k}$ if $k_i' \leq k_i$ for every i with strict inequality for some i. A labeling g selects $\sum_{i\in\mathcal{I}} K_i$ vectors in decreasing order (i.e. $g(n) < g(n-1)$) and at each step n only one component decreases by the minimum possible (i.e.$g(n) - g(n-1) = -\boldsymbol{e}_i$ for some i). Thus, a labeling determines a finite, ordered sequence of multi-indices.

Example 2 contains three labelings that we also use in Example 3.

Example 2. *There are two bidders, $i = 1, 2$, and $K_1 = K_2 = 3$. Three labeling systems are described in the table below:*

[5] Note that $g_i^{-1}(k)$ is well defined: $g_i(0) = K_i + 1$ and $g(\sum_{i\in\mathcal{I}} K_i) = \mathbf{1}$; therefore $g_i(\sum_{i\in\mathcal{I}} K_i) = 1$ and there must be an n' such that $g_i(n') = k$.

	Labeling (a)	*Labeling (b)*	*Labeling (c)*
$g(0)$	$(4,4)$	$(4,4)$	$(4,4)$
$g(1)$	$(3,4)$	$(4,3)$	$(4,3)$
$g(2)$	$(3,3)$	$(3,3)$	$(4,2)$
$g(3)$	$(2,3)$	$(3,2)$	$(4,1)$
$g(4)$	$(2,2)$	$(2,2)$	$(3,1)$
$g(5)$	$(1,2)$	$(2,1)$	$(2,1)$
$g(6)$	$(1,1)$	$(1,1)$	$(1,1)$

Figures 3 and 4 may be used to visualize the labelings in Example 2. Let the lattice formed by the intersections of the solid lines represent the set $\prod_{i\in\mathcal{I}}\{1,\ldots,K_i+1\} = \{1,2,3,4\} \times \{1,2,3,4\}$: Then $g(0) = (4,4)$ is the north-east corner of the diagram; if $g(1) = (3,4)$ then $g(1) - g(0) = (-1,0)$ is the horizontal arrow with origin at $g(0)$. Thus, the arrows in the diagram indicate the finite sequence of multi-indices and their order.

The set of all step functions relative to a fixed partition in W'' is determined by a system of finitely many inequalities derived from the feasibility inequality in (6). The role of the labeling is to select sufficiently many, linear-independent inequalities so that when binding, they determine an extreme point.

Before stating Lemma 5.2, we offer an example. Example 3 shows that, given a partition of the type space, labelings (a) and (b) in Example 2 correspond to extreme points.

Example 3. *There are two bidders, $i = 1,2$, $K_1 = K_2 = 3$, and for every i, x_i is uniformly distributed in $X_i = [0,1]$. Although bidders are ex ante identical, mechanisms are not required to be symmetric.*

Fix a partition of $[0,1]$, say $[0,1/3]$, $(1/3,2/3]$, $(2/3,1]$ and index its elements by $k_i = 1,2,3$ respectively when referring to X_i.

Each diagram in Figure 3 corresponds to a pair of step functions (Q_1,Q_2) in W''. Each pair is an extreme point of W''.

The left-hand-side diagram is associated with

$$\begin{aligned} Q_1 &= \{(b_k^1,\beta_k^1)\}_{k=1}^3 = \{(1/3,1/3),(1/3,2/3),(1/3,3/3)\} \\ Q_2 &= \{(b_k^2,\beta_k^2)\}_{k=1}^3 = \{(1/3,0),(1/3,1/3),(1/3,2/3)\}. \end{aligned}$$

The right-hand-side diagram illustrates the reciprocal extreme point. (See Lemma 5.2.)

Numbers in the cells of the diagrams indicate the values of the mechanism $q_1(x_1, x_2)$*; empty cells signify* $q_1 = 0$*. (See Lemma 5.3.) Note that in both diagrams* $E_{\boldsymbol{x}_{-i}} q_i = Q_i$

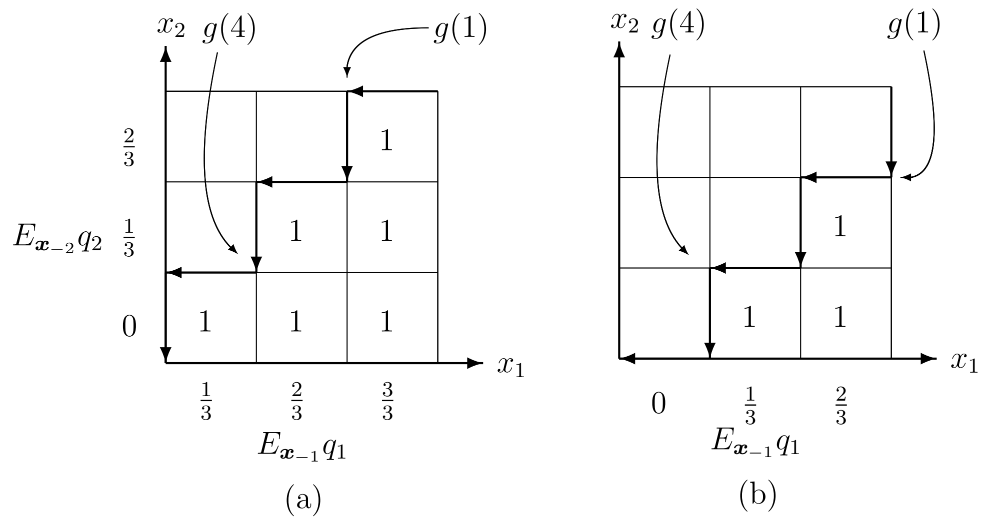

FIGURE 3. Labelings (a) and (b), two extreme points

Labelings (a) and (b) in Example 2 correspond to the extreme points in Example 3. Figure 3 is helpful to see this. We associate each index k_i with a set as follows: If $k_i = 3$, then let $B_i = (2/3, 3/3]$; if $k_i = 2$, then let $B_i = (1/3, 2/3] \cup (2/3, 3/3]$; if $k_i = 1$, then let $B_i = [0, 1/3) \cup (1/3, 2/3] \cup (2/3, 3/3]$. Therefore to any multi-index (k_1, k_2) corresponds a set $B_1 \times B_2$. Applying the feasibility inequality in (6) to this set yields a linear constraint

$$\sum_{i=1}^{2} \sum_{k=k_i}^{K_i} b_k^i \beta_k^i \leq 1 - \prod_{i=1}^{2} \sum_{k=1}^{k_i - 1} b_k^i$$

Every labeling thus determines $\sum_{i \in \mathcal{I}} K_i$ equations (i.e. binding inequalities), six in the example. If the equations are linearly independent, the solution to the equations is an extreme point. Similarly, for each extreme point there is an implicit labeling. Lemma 5.2 makes this precise.

Consider, as an illustration, labeling (a). Since $g(1) = (3, 4)$, the inequality in (6) becomes $(1/3)\beta_3^1 \leq 1 - [(1/3 + 1/3) \times 1]$. (When binding, the solution is $\beta_3^1 = 1$.) From $g(2) = (3, 3)$, the inequality in (6) becomes $(1/3)\beta_3^1 + (1/3)\beta_3^2 \leq 1 - [(1/3 + 1/3) \times (1/3 + 1/3)]$. When binding and using the found $\beta^1_3 = 1$, the solution is $\beta^2_3 = 2/3$. The process continues until all $\beta^i_{k_i}$ have been identified. It is not difficult to verify that the six equations selected by labeling (a) are linearly independent. Thus labeling (a) corresponds to an extreme point.

A similar argument will show that labeling (b) also corresponds to an extreme point. Applying the same argument to labeling (c), however, yields as solution, a function with fewer than three steps. Thus, labeling (c) does not correspond to an extreme point with three steps (Figure 4).

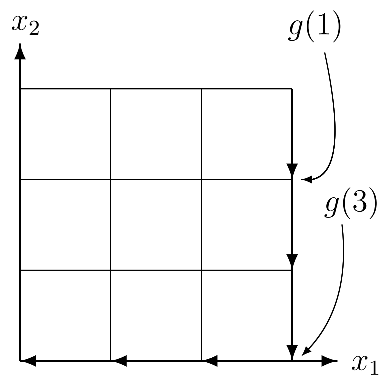

FIGURE 4. Labeling (c)

In a first reading of Lemma 5.2, it may be useful to assume that $K_1 = K_2 = \ldots = K_I$, and that for $n \geq 1$ the difference $g(n) - g(n-1) = -\boldsymbol{e}_{(n \mod I)}$. Under this assumption, the labeling is the natural one, i.e. $g(1) = (K_1, K_2 + 1, \ldots, K_I + 1)$, $g(2) = (K_1, K_2, K_3 + 1, \ldots, K_I + 1)$, etc.

The proof of Theorem 2 only uses one direction in Lemma 5.2: If a step function is an extreme point, it must be one of those identified by the Lemma.

Lemma 5.2. *Let* $\{\{(b^i_k, \bar{\beta}^i_k)\}_{k=1}^{K_i}\}_{i\in\mathcal{I}}$ *be a collection of step functions in* W''*. The collection* $\{\{(b^i_k, \bar{\beta}^i_k)\}_{k=1}^{K_i}\}_{i\in\mathcal{I}}$ *is an extreme point of* W'' *if and only if there exists a labeling* g *relative to* $\{K_i\}_{i\in\mathcal{I}}$ *such that either*

(a) $\forall i \in \mathcal{I}$ *and* $k \in \{1, \ldots, K_i\}$, $\bar{\beta}^i_k = \prod_{j\in\mathcal{I}\setminus\{i\}} \sum_{\ell=1}^{g_j(g_i^{-1}(k))-1} b^j_\ell$ *or*

(b) $\bar{\beta}^i_k$ *is defined as above for all* i *and* k *with the following exceptions: there is* i' *such that* $\bar{\beta}^{i'}_1 = 0$*, and for every* i *and* k *with* $k > g_i(g_{i'}^{-1}(1))$*,* $\bar{\beta}^i_k = 0$.

Proof ($\Rightarrow$). We prove that if a step function is an extreme point of W'', then it is one of those identified by the Lemma. Its converse, not used in the proof of Theorem 2, is proven in Appendix A.

Fix $\{\{b^i_k\}_{k=1}^{K_i}\}_{i\in\mathcal{I}}$ as in the Lemma's statement, define

$$\begin{aligned} K &= \prod_{i\in\mathcal{I}}\{1,\ldots,K_i+1\} \\ \mathcal{K} &= \left\{1,2,\ldots,\sum_{i\in\mathcal{I}}K_i\right\} \end{aligned}$$

and note that g maps $\mathcal{K}\cup\{0\}$ into K.

Any family of step functions $\{\{(b^i_k,\beta^i_k)\}_{k=1}^{K_i}\}_{i\in\mathcal{I}}\in W''$ must satisfy the inequality in (6), and therefore

$$\forall\mathbf{k}=(k_1,\ldots,k_I)\in K,\ \sum_{i\in\mathcal{I}}\sum_{k=k_i}^{K_i}b^i_k\beta^i_k\le 1-\prod_{i\in\mathcal{I}}\sum_{k=1}^{k_i-1}b^i_k \tag{7}$$

where sums with no terms are defined to be zero.

We find it convenient to use vector notation. To that end, define for $i=1,\ldots,I$,

$$\begin{array}{rclcrcl} \boldsymbol{b}^i &=& (b^i_1,\ldots,b^i_{K^i}) & \textit{and} & \boldsymbol{b} &=& (\boldsymbol{b}^1,\ldots,\boldsymbol{b}^I) \\ \boldsymbol{b}^i_{k_i} &=& (0,\ldots,0,b^i_{k_i},b^i_{k_i+1},\ldots,b^i_{K^i}) & \textit{and} & \boldsymbol{b}_{\mathbf{k}} &=& (\boldsymbol{b}^1_{k_1},\ldots,\boldsymbol{b}^I_{k_I}) \\ \boldsymbol{\beta}^i &=& (\beta^i_1,\ldots,\beta^i_{K^i}) & \textit{and} & \boldsymbol{\beta} &=& (\boldsymbol{\beta}^1,\ldots,\boldsymbol{\beta}^I) \end{array}$$

and let $\boldsymbol{b}^i_{K_i+1}=\mathbf{0}$, the null vector in $\mathbb{R}^{K_i}$. Also for every $\mathbf{k}\in K$, define

$$r(\mathbf{k})=\prod_{i\in\mathcal{I}}\sum_{k=1}^{k_i-1}b^i_k$$

In vector notation, inequality (7) becomes

$$\forall\mathbf{k}\in K,\boldsymbol{b}_{\mathbf{k}}\cdot\boldsymbol{\beta}\le 1-r(\mathbf{k}) \tag{8}$$

Note that $\boldsymbol{\beta}$ is a vector in $\mathbb{R}^{\sum_{i\in\mathcal{I}}K_i}$. To express nonnegativity constraints in vector form, for any $i\in\mathcal{I}$ and $k\in\{1,\ldots,K_i\}$, let $\boldsymbol{e}^i_k\in\mathbb{R}^{\sum_{i\in\mathcal{I}}K_i}$ such that

$$\boldsymbol{e}^i_k=(0,\ldots,0,1,0,\ldots,0)$$

where the 1 corresponds to the element k of bidder i. Thus writing $\boldsymbol{e}_k^i \cdot \boldsymbol{\beta} \geq 0$ is equivalent to writing $\beta_k^i \geq 0$.

Define the set containing all nonnegative vectors $\boldsymbol{\beta}$ such that $(\boldsymbol{b}, \boldsymbol{\beta})$ satisfies (8).

$$P'' = \left\{\boldsymbol{\beta} : [\mathbf{k} \in K \implies \boldsymbol{b}_{\mathbf{k}} \cdot \boldsymbol{\beta} \leq 1 - r(\mathbf{k})] \text{ and } [i \in \mathcal{I}, k \in \{1, \ldots, K_i\} \implies \boldsymbol{e}_k^i \cdot \boldsymbol{\beta} \geq 0]\right\}$$

An element $(\boldsymbol{b}, \boldsymbol{\beta})$ in W'' is an extreme point of W'' if and only if $\boldsymbol{\beta}$ is an extreme point of P'' (Lemma A.2). In turn $\boldsymbol{\beta}$ is an extreme point of P'' if and only if

$$R(\boldsymbol{\beta}) = \{\boldsymbol{b}_{\mathbf{k}} : \boldsymbol{b}_{\mathbf{k}} \cdot \boldsymbol{\beta} = 1 - r(\mathbf{k}),\ \mathbf{k} \in K\} \cup \left\{\boldsymbol{e}_k^i : \boldsymbol{e}_k^i \cdot \boldsymbol{\beta} = 0,\ i \in \mathcal{I}, k \in \{1, \ldots, K_i\}\right\} \tag{9}$$

contains $\sum_{i=1}^{I} K^i$ linearly independent vectors (Lemma A.1).

In summary, fix $\boldsymbol{b}$ and let $(\boldsymbol{b}, \bar{\boldsymbol{\beta}})$ be an extreme point of W''. Then $\bar{\boldsymbol{\beta}}$ is an extreme point of P''. Therefore the set $R(\bar{\boldsymbol{\beta}})$ must have $\sum_{i \in \mathcal{I}} K_i$ linearly independent vectors.

First, suppose that $\bar{\boldsymbol{\beta}}$ is strictly positive, i.e. $\bar{\beta}_k^i > 0$ for every i and k. Let $\{\boldsymbol{b}_{\mathbf{k}}\}$ be the collection, with $\sum_{i \in \mathcal{I}} K_i$ elements, of linearly independent vectors in $R(\bar{\boldsymbol{\beta}})$. Each vector $\boldsymbol{b}_{\mathbf{k}}$ in the collection satisfies

$$\boldsymbol{b}_{\mathbf{k}} \cdot \bar{\boldsymbol{\beta}} = 1 - r(\mathbf{k}) \tag{10}$$

Order the $\sum_{i \in \mathcal{I}} K_i$ indices of these vectors from largest to smallest so that $\mathbf{k} > \mathbf{k}' > \ldots$. We will show below that it is always possible to order indices strictly as described. Then define the labeling g as follows: $g(0) = (K_1 + 1, \ldots, K_I + 1)$ and $g(n)$ is the n^{th} element in the ordered sequence.

We now prove that the strict ordering of indices is possible. Arguing by contradiction, suppose it is not. Then $\exists \mathbf{k}, \mathbf{k}' \in K$, $\mathbf{k} \neq (\mathbf{k} \wedge \mathbf{k}') \neq \mathbf{k}'$ such that $\boldsymbol{b}_{\mathbf{k}} \cdot \bar{\boldsymbol{\beta}} = 1 - r(\mathbf{k})$, $\boldsymbol{b}_{\mathbf{k}'} \cdot \bar{\boldsymbol{\beta}} = 1 - r(\mathbf{k}')$, and $\boldsymbol{b}_{\mathbf{k} \wedge \mathbf{k}'} \cdot \bar{\boldsymbol{\beta}} = 1 - r(\mathbf{k} \wedge \mathbf{k}')$. Note that $\mathbf{k} \vee \mathbf{k}' = \mathbf{k} + \mathbf{k}' - \mathbf{k} \wedge \mathbf{k}'$ and thus $\boldsymbol{b}_{\mathbf{k}} + \boldsymbol{b}_{\mathbf{k}'} - \boldsymbol{b}_{\mathbf{k} \wedge \mathbf{k}'} = \boldsymbol{b}_{\mathbf{k} \vee \mathbf{k}'}$. Therefore

$$1 - r(\mathbf{k}) + 1 - r(\mathbf{k}') - 1 + r(\mathbf{k} \wedge \mathbf{k}') = \boldsymbol{b}_{\mathbf{k} \vee \mathbf{k}'} \cdot \bar{\boldsymbol{\beta}} \leq 1 - r(\mathbf{k} \vee \mathbf{k}')$$

This implies that $r(\mathbf{k} \vee \mathbf{k}') + r(\mathbf{k} \wedge \mathbf{k}') \leq r(\mathbf{k}) + r(\mathbf{k}')$. This is a contradiction because $r(\mathbf{k} \vee \mathbf{k}') + r(\mathbf{k} \wedge \mathbf{k}') \geq r(\mathbf{k}) + r(\mathbf{k}')$, and the inequality is strict except when $\mathbf{k}' \wedge \mathbf{k} \in \{\mathbf{k}', \mathbf{k}\}$. This establishes that ordering the indices strictly is possible.

We now demonstrate that $\bar{\boldsymbol{\beta}}$ is item (a) in the Lemma's statement relative to the labeling g defined above.

Using the labeling, (10) can be rewritten as $\boldsymbol{b}_{g(n)} \cdot \bar{\boldsymbol{\beta}} = 1 - r(g(n))\ \forall n \in \mathcal{K}$. Pick any $i \in \mathcal{I}$ and $k \in \{1, \ldots, K_i\}$, and let $n' = g_i^{-1}(k)$. Subtracting $\boldsymbol{b}_{g(n'-1)} \cdot \bar{\boldsymbol{\beta}} = 1 - r(g(n'-1))$ from $\boldsymbol{b}_{g(n')} \cdot \bar{\boldsymbol{\beta}} = 1 - r(g(n'))$ yields

$$[\boldsymbol{b}_{g(n')} - \boldsymbol{b}_{g(n'-1)}] \cdot \bar{\boldsymbol{\beta}} = r(g(n'-1)) - r(g(n'))$$

By definition of n', $g_i(n') = k$, $g_i(n'-1) = k+1$ and for all $j \neq i$, $g_j(n') = g_j(n'-1)$. Therefore $[\boldsymbol{b}_{g(n')} - \boldsymbol{b}_{g(n'-1)}] \cdot \bar{\boldsymbol{\beta}} = b_k^i \bar{\beta}_k^i$ and the expression above becomes

$$\begin{aligned}
b_k^i \bar{\beta}_k^i &= r(g(n'-1)) - r(g(n')) \\
&= \textstyle\prod_{j\in\mathcal{I}} \sum_{\ell=1}^{g_j(n'-1)-1} b_\ell^j - \prod_{j\in\mathcal{I}} \sum_{\ell=1}^{g_j(n')-1} b_\ell^j \\
&= \left(\textstyle\prod_{j\in\mathcal{I}\setminus\{i\}} \sum_{\ell=1}^{g_j(n')-1} b_\ell^j\right) \left[\textstyle\sum_{\ell=1}^{g_i(n'-1)-1} b_\ell^i - \sum_{\ell=1}^{g_i(n')-1} b_\ell^i\right] \\
&= \left(\textstyle\prod_{j\in\mathcal{I}\setminus\{i\}} \sum_{\ell=1}^{g_j(n')-1} b_\ell^j\right) \left[\textstyle\sum_{\ell=1}^{k+1-1} b_\ell^i - \sum_{\ell=1}^{k-1} b_\ell^i\right] \\
&= \left(\textstyle\prod_{j\in\mathcal{I}\setminus\{i\}} \sum_{\ell=1}^{g_j(n')-1} b_\ell^j\right) b_k^i
\end{aligned}$$

Therefore, $\bar{\beta}_k^i = \prod_{j\in\mathcal{I}\setminus\{i\}} \sum_{\ell=1}^{g_j(n')-1} b_\ell^j$. This establishes that $\bar{\boldsymbol{\beta}}$ is item (a) in the Lemma's statement.

Second, suppose that $\bar{\beta}_k^i = 0$ for some i and k, i.e. $\boldsymbol{e}_k^i \cdot \tilde{\boldsymbol{\beta}} = 0$ and thus $\boldsymbol{e}_k^i \in R(\bar{\boldsymbol{\beta}})$ is one of the linearly independent vectors. Note that $\bar{\beta}_k^i = 0$ implies $\bar{\beta}_{k-1}^i = 0$. Therefore, unless $k = 1$ the function will not have the required number of steps. A similar argument to the one we used above (when assuming $\bar{\beta}_k^i > 0$) yields that $\bar{\boldsymbol{\beta}}$ is item (b) in the Lemma's statement. □

Lemma 5.3. *Let $\{\{(b_k^i, \bar{\beta}_k^i)\}_{k=1}^{K_i}\}_{i\in\mathcal{I}}$ be an extreme point of W'' and let g be its labeling. The two mechanisms $\{q_i\}_{i=1}^I$ defined below satisfy dominant-strategy incentive compatibility and for every i $E_{\boldsymbol{x}_{-i}} q_i = \{(b_k^i, \bar{\beta}_k^i)\}_{k=1}^{K_i}$.*

For $i = 1, \ldots, I$, *let* $\iota_i(x_i) = k : x_i \in Q_i^{-1}(\bar{\beta}_k^i)$. *For alternative* (a) *in Lemma 5.2, the implementing mechanism is*

$$q_i(x_1, \ldots, x_I) = \begin{cases} 1 & \textit{if } \iota_j(x_j) \leq g_j(g_i^{-1}(\iota_i(x_i))) - 1 \; \forall j \neq i \\ 0 & \textit{otherwise} \end{cases}$$

For alternative (b) *in Lemma 5.2, the implementing mechanism is*

$$q_{i'}(x_1, \ldots, x_I) = \begin{cases} 1 & \textit{if } \iota_j(x_j) \leq g_j(g_{i'}^{-1}(\iota_{i'}(x_{i'}))) - 1 \; \forall j \neq i' \textit{ and } \iota_{i'}(x_{i'}) \neq 1 \\ 0 & \textit{otherwise} \end{cases}$$

$$q_i(x_1, \ldots, x_I) = \begin{cases} 1 & \textit{if } \iota_j(x_j) \leq g_j(g_i^{-1}(\iota_i(x_i))) - 1 \; \forall j \neq i \textit{ and} \\ & \quad \iota_i(x_i) = g_i(n), \textit{ for } n \leq \min\{n' : n' \in g_i^{-1}(1)\} \\ 0 & \textit{otherwise} \end{cases}$$

The proof is by direct calculation.

Proof. For alternative (a) in Lemma 5.2, pick i and x_i. Let $\iota_i(\boldsymbol{x}_i) = k$ and $g_i^{-1}(\iota(x_i)) = n'$. We must show that $E_{\boldsymbol{x}_{-i}} q_i(x_i) = \bar{\beta}_k^i$, i.e, that $E_{\boldsymbol{x}_{-i}} q_i(x_i) = \prod_{j \in \mathcal{I} \setminus \{i\}} \sum_{k'=1}^{g_j(n')-1} b_{k'}^j$. Using definitions,

$$E_{\boldsymbol{x}_{-i}} q_i(x_i) = \int_{X_{-i}} q_i(x_i, x_{-i}) \, d\lambda_{-i} = \prod_{j \in \mathcal{I} \setminus \{i\}} \int_{\underline{x}_j}^{g_j(n')-1} d\lambda_j = \prod_{j \in \mathcal{I} \setminus \{i\}} \sum_{k'=1}^{g_j(n')-1} b_{k'}^j.$$

For alternative (b), apply the same argument. □

6. Bilateral differential information

The environment in this section has I ex ante identical buyers plus a distinct agent that we call the seller. (We discuss the interpretation of the distinct agent as a seller after Definition 6.1.)

All agents, including the seller, have private information. To emphasize the difference between the seller and the bidders, the seller's private information is denoted by $y \in Y$, distributed according to a probability distribution λ_s. (Every buyer's private information $x \in X$ is independently distributed according to the same distribution λ_b.)

We require that mechanisms be symmetric with respect to the ex ante identical buyers as in Section 4. After stating the definition, we explain its content. To avoid cumbersome notation, we will use Eq or $Eq(x_1)$—i.e the expectation of $q(\boldsymbol{x}, y)$ taken over $\boldsymbol{x}_{-1}$ and y—instead of $E_{y,\boldsymbol{x}_{-1}}q$.

Definition 6.1. *Let $q : Y \times X^I \to [0,1]$, $q_s : Y \times X^I \to [0,1]$ be such that for every $(y, \boldsymbol{x}) \in Y \times X^I$, $\sum_{i=1}^{I} q(y, \sigma_i(\boldsymbol{x})) + q_s(y, \boldsymbol{x}) \leq 1$.*

If $q(y, \boldsymbol{x})$ and $q_s(y, \boldsymbol{x})$ are nondecreasing in x_1 and y respectively, then (q, q_s) is a symmetric, dominant-strategy incentive-compatible mechanism with I bidders and a seller.

If $Eq(x_1)$ and $E_{\boldsymbol{x}}q_s(y)$ are nondecreasing, then (q, q_s) is a symmetric, Bayesian incentive compatible mechanism with I bidders and a seller.

The omitted transfer functions are recovered, up to a constant, using the corresponding incentive compatibility characterizations. (See Section 3 and the paragraph following this definition.)

To interpret the distinct agent as a seller, it suffices to set $Y = [\underline{y}, \overline{y}] \subseteq \mathbb{R}_-$. A nondecreasing $E_{\boldsymbol{x}}q_s(y)$ becomes nonincreasing as a function of $|y|$.[6]

Fix a profile $(y, \boldsymbol{x})$. While $q_s(y, \boldsymbol{x})$ is the probability that the seller ends up with the object, it is *not* the probability that the object is not given to some buyer. If the probability sum (for the given type profile) is strictly less than one, then the object might not be assigned to either buyers or the seller. This flexibility in the definition of a mechanism increases the set of mechanisms for which the equivalence (between dominant-strategy and Bayesian implementation) is obtained. Since we show the equivalence of any mechanism, not just a revenue maximizing one, the additional generality is valuable.

Theorem 3. *If (q', q'_s) is a symmetric, Bayesian incentive-compatible mechanism with I bidders and a seller, then there is a symmetric dominant-strategy, incentive-compatible mechanism (q, q_s) with I bidders and a seller that generates the same expected probability of trade, i.e. $Eq'(x_1) = Eq(x_1)$ a.e. and $E_{\boldsymbol{x}}q'_s = E_{\boldsymbol{x}}q_s$ a.e.*

If there is a single buyer, Theorem 3 is a particular case of Theorem 2 (with two agents, a buyer and a seller). If there are at least two ex ante identical buyers that must be treated symmetrically, Theorem 3 does not follow from Theorem 2. The proof, however, is similar

[6] The seller's preferences, defined as $u_s(y, \boldsymbol{x}) = q_s(y, \boldsymbol{x})y - t_s(y, \boldsymbol{x})$ (where $t_s(y, \boldsymbol{x}) \geq 0$ represents transfers from the agent to the mechanism designer) can be written as $u_s(y, \boldsymbol{x}) = -t_s(y, \boldsymbol{x}) - q_s(y, \boldsymbol{x})|y|$.

to the proofs of Theorems 1 and 2. The nontrivial direction also proceeds in three lemmas. Given the similarities, we state them without proof in Appendix B.

7. Concluding comments

a. An outcome in a game is customarily defined as a distribution on the terminal nodes that results from a strategy profile and nature's moves. For conciseness, suppose there are only two agents in our model and consider the implicit game in a direct revelation mechanism. An outcome is a distribution μ on

$$X_1 \times X_2 \times X_1 \times X_2 \times [0,1] \times [0,1] \times \mathbb{R} \times \mathbb{R}$$

where from left to right we have the type spaces, the action spaces, the probabilities of trade, and the transfers. The linearity of preferences imply that the marginal distribution of μ on the first, fifth and seventh space ($\mu_{X_1\times[0,1]\times\mathbb{R}}$) suffices to determine player 1's payoff. (This is the marginal distribution on player 1's own type, own probability of trade and own transfer.) If two outcomes μ and ν generate the same relevant marginal distributions (i.e. $\mu_{X_i\times[0,1]\times\mathbb{R}} = \nu_{X_i\times[0,1]\times\mathbb{R}}$) for all players), they are equivalent in the sense that players are indifferent between them. We have proved that for any Bayesian Nash equilibrium outcome μ, there is a dominant-strategy equilibrium outcome ν such that the relevant marginal distributions of μ and ν are the same. The actual outcomes μ and ν will generally be different.

A mechanism design problem is often cast in terms of the maximization of an objective function subject to constraints. If the objective function and constraints depend only on the expected probabilities of trade as is often the case, then, per our equivalence theorems, there is no loss in requiring dominant-strategy over Bayesian incentive compatibility.

b. Our work is related to Border (1991) and (2007). Border (1991)'s objective is to identify the functions $Q : X \to [0,1]$ for which there is a symmetric mechanism $q : X^I \to [0,1]$ (with I identical bidders) and $E_{\boldsymbol{x}_{-1}} q = Q$. He demonstrates that a necessary and sufficient condition for this is that Q satisfy the feasibility inequality in Lemma 4.1). (From that characterization, we only use the simple part (Lemma 4.1.) Border (1991) assumes ex ante identical bidders and considers only symmetric mechanisms. He does not require incentive compatibility and therefore his expected probabilities of trade $E_{\boldsymbol{x}_{-1}} q$ need not be nondecreasing. Border (2007) extends his own characterization to general nonsymmetric environments but assumes finite types.

As a byproduct, Theorem 2 extends Border's result in that it applies to heterogeneous agents (and thus, to bilateral trade) and to nonsymmetric mechanisms with a continuum of types (but assuming nondecreasing Q).

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Appendix A

We complete the proof of Lemma 5.2.

Proof of Lemma 5.2 ($\Leftarrow$). We now prove that for $\bar{\boldsymbol{\beta}}$ defined in Lemma 5.2-(a), $(\boldsymbol{b}, \bar{\boldsymbol{\beta}})$ is an extreme point of W''. By hypothesis, $(\boldsymbol{b}, \bar{\boldsymbol{\beta}})$ belongs to W''. Therefore, it suffices to demonstrate that $R(\bar{\boldsymbol{\beta}})$ has $\sum_{i=1}^{I} K_i$ linearly independent vectors. We do so in two steps.

First, simple inspection shows that the $\sum_{i\in\mathcal{I}} K_i$ vectors $\{\boldsymbol{b}_{g(n)}\}_{n\in\mathcal{K}}$ are linearly independent where g is the labeling used to defined $\bar{\boldsymbol{\beta}}$.

Second, we demonstrate that $\{\boldsymbol{b}_{g(n)}\}_{n\in\mathcal{K}} \subseteq R(\bar{\boldsymbol{\beta}})$. We must show that

$$n \in \mathcal{K} \implies \boldsymbol{b}_{g(n)} \cdot \bar{\boldsymbol{\beta}} = 1 - r(g(n)) \tag{11}$$

Let $\tilde{\boldsymbol{\beta}}$ be a solution to the system of equations $\boldsymbol{b}_{g(n)} \cdot \tilde{\boldsymbol{\beta}} = 1 - r(g(n))\ \forall n \in \mathcal{K}$. (Such a solution always exists because the vectors $\{\boldsymbol{b}_{g(n)}\}_{n\in\mathcal{K}}$ are linearly independent.) We will show that $\tilde{\boldsymbol{\beta}} = \bar{\boldsymbol{\beta}}$. Pick any $i \in \mathcal{I}$ and $k \in \{1,\ldots,K_i\}$, and let $n' = g_i^{-1}(k)$.

Subtracting $\boldsymbol{b}_{g(n'-1)} \cdot \tilde{\boldsymbol{\beta}} = 1 - r(g(n'-1))$ from $\boldsymbol{b}_{g(n')} \cdot \tilde{\boldsymbol{\beta}} = 1 - r(g(n'))$ yields

$$[\boldsymbol{b}_{g(n')} - \boldsymbol{b}_{g(n'-1)}] \cdot \tilde{\boldsymbol{\beta}} = r(g(n'-1)) - r(g(n'))$$

By definition of n', $g_i(n') = k$, $g_i(n'-1) = k+1$ and for all $j \neq i$, $g_j(n') = g_j(n'-1)$. Therefore $[\boldsymbol{b}_{g(n')} - \boldsymbol{b}_{g(n'-1)}] \cdot \tilde{\boldsymbol{\beta}} = b^i_k \tilde{\beta}^i_k$ and the expression above becomes

$$\begin{aligned}
b^i_k \tilde{\beta}^i_k &= r(g(n'-1)) - r(g(n')) \\
&= \textstyle\prod_{j\in\mathcal{I}} \sum_{\ell=1}^{g_j(n'-1)-1} b^j_\ell - \prod_{j\in\mathcal{I}} \sum_{\ell=1}^{g_j(n')-1} b^j_\ell \\
&= \left(\textstyle\prod_{j\in\mathcal{I}\setminus\{i\}} \sum_{\ell=1}^{g_j(n')-1} b^j_\ell\right) \left[\textstyle\sum_{\ell=1}^{g_i(n'-1)-1} b^i_\ell - \sum_{\ell=1}^{g_i(n')-1} b^i_\ell\right] \\
&= \left(\textstyle\prod_{j\in\mathcal{I}\setminus\{i\}} \sum_{\ell=1}^{g_j(n')-1} b^j_\ell\right) \left[\textstyle\sum_{\ell=1}^{k+1-1} b^i_\ell - \sum_{\ell=1}^{k-1} b^i_\ell\right] \\
&= \left(\textstyle\prod_{j\in\mathcal{I}\setminus\{i\}} \sum_{\ell=1}^{g_j(n')-1} b^j_\ell\right) b^i_k
\end{aligned}$$

Therefore, $\tilde{\beta}^i_k = \prod_{j\in\mathcal{I}\setminus\{i\}} \sum_{\ell=1}^{g_j(n')-1} b^j_\ell = \bar{\beta}^i_k$. This establishes (11).

We have proved that $R(\bar{\boldsymbol{\beta}})$ has $\sum_{i\in\mathcal{I}} K_i$ linearly independent vectors and therefore $(\boldsymbol{b}, \bar{\boldsymbol{\beta}})$ is an extreme point of W''.

We now prove that for $\bar{\boldsymbol{\beta}}$ defined in Lemma 5.2 (b), $(\boldsymbol{b}, \bar{\boldsymbol{\beta}})$ is an extreme point of W''. Once again, we need to show that $R(\bar{\boldsymbol{\beta}})$ has $\sum_{i\in\mathcal{I}} K_i$ linearly independent vectors.

Let $\bar{\beta}^{i'}_1 = 0 < \prod_{j\in\mathcal{I}\setminus\{i'\}} \sum_{\ell=1}^{g_j(g_{i'}^{-1}(1))-1} b^j_\ell$. (If there is no i' for which this holds, then $\bar{\boldsymbol{\beta}}$ is as in Lemma 5.2 (a) and we are done.)

Define $n' = g_{i'}^{-1}(1)$.

For $n < n'$, $\bar{\beta}^i_{g_i(n)}$ is as defined in Lemma 5.2 (a) and therefore, by (11),

$$\boldsymbol{b}_{g(n)} \cdot \bar{\boldsymbol{\beta}} = \sum_{i \in \mathcal{I}} \sum_{k=g_i(n)}^{K_i} b^i_k \bar{\beta}^i_k = 1 - r(g(n))$$

Therefore for every $n < n'$, $\boldsymbol{b}_{g(n)} \in R(\bar{\boldsymbol{\beta}})$.

For $n \geq n'$,

$$\boldsymbol{b}_{g(n)} \cdot \bar{\boldsymbol{\beta}} = \sum_{i \in \mathcal{I}} \sum_{k=g_i(n)}^{K_i} b^i_k \bar{\beta}^i_k < 1 - r(g(n))$$

This is so because $\bar{\boldsymbol{\beta}}^{i'}_{g_{i'}(n')} = \bar{\boldsymbol{\beta}}^{i'}_1 = 0$ and this variable was strictly positive when (11) applied. Therefore, $\boldsymbol{b}_{g(n')}$ does not belong to $R(\bar{\beta})$ but $\boldsymbol{e}^{i'}_i$ does. For every $n > n'$ the same argument applies: $\bar{\boldsymbol{\beta}}^i_k = 0$ if $k > g_i(n')$, $\boldsymbol{b}_{g(n)} \notin R(\bar{\beta})$, and $\boldsymbol{e}^i_k \in R(\bar{\beta})$.

It is immediate that all vectors in $R(\bar{\beta})$ are linearly independent. Hence $\bar{\boldsymbol{\beta}}$ is an extreme point of P''. Since $(\boldsymbol{b}, \tilde{\boldsymbol{\beta}}) \in W''$, $(\boldsymbol{b}, \tilde{\boldsymbol{\beta}})$ is an extreme point of W''. □

The following well-known property is included here for the reader's convenience.

Lemma A.1. *For $j = 1, \ldots, J$, let $\mathbf{a_j} \in \mathbb{R}^K$ and let $r_j \in \mathbb{R}$. Let $P = \{\boldsymbol{\beta} \in \mathbb{R}^K : \mathbf{a_j} \cdot \boldsymbol{\beta} \leq r_j, j = 1, \ldots, J\}$. Then a vector $\beta \in P$ is an extreme point of P if and only if the set $A_{\boldsymbol{\beta}} = \{\mathbf{a_j} : \mathbf{a_j} \cdot \boldsymbol{\beta} = r_j, j \in \{1, \ldots, J\}\}$ contains K linearly independent vectors.*

Proof. See for instance Bertsekas (2003), Proposition 3.3.3, page 184. □

The following lemma adds detail to the proof of Lemmas 4.2 and 5.2. We state it and prove it using W and P used in Lemma 4.2. The result and its proof remain valid for W'' and P'' as used in Lemma 5.2.[7]

Lemma A.2. *Let W be defined as in Lemma 4.1. Let $(\boldsymbol{b}, \boldsymbol{\beta}) \in W$ be a step function with K steps and let P be as defined in (3). Then, $(\boldsymbol{b}, \boldsymbol{\beta})$ is an extreme point of W if and only if $\boldsymbol{\beta}$ is an extreme point of P.*

Proof. First, if $(\boldsymbol{b}, \boldsymbol{\beta})$ is not an extreme point of W, $\boldsymbol{\beta}$ is not an extreme point of P. If $Q = (\boldsymbol{b}, \boldsymbol{\beta})$ is not an extreme point of W, then there are $Q^1, Q^2 \in W$ such that $Q = \frac{1}{2}Q^1 + \frac{1}{2}Q^2$. Let β_k be the k^{th} component of $\boldsymbol{\beta}$ and pick any $x', x \in Q^{-1}(\beta_k)$ with $x' > x$. For $i = 1, 2$, Q^i

[7] Manelli and Vincent (2007), Theorems 17 and 19, observe that the domain partition defining a step function determines a face of W. Lemma A.2 and its corollary are based on this observation.

is nondecreasing (because $Q^i \in W$) and therefore $Q^i(x') \geq Q^i(x)$. Suppose this inequality is strict for some i. Then, $\beta_k = \frac{1}{2}Q^1(x') + \frac{1}{2}Q^2(x') > \frac{1}{2}Q^1(x) + \frac{1}{2}Q^2(x) = \beta_k$, a contradiction. We conclude that for $i = 1, 2$, Q^i is constant in $Q^{-1}(\beta_k)$. Since k was chosen arbitrarily, Q^i is constant in every interval in which Q is constant. Therefore, we may write $Q^i = (\boldsymbol{b}, \boldsymbol{\beta}^i)$. Then, $\boldsymbol{\beta} = \frac{1}{2}\boldsymbol{\beta}^1 + \frac{1}{2}\boldsymbol{\beta}^2$ and $\boldsymbol{\beta}$ is not an extreme point of P.

Second, if $\boldsymbol{\beta}$ is not an extreme point of P, then $\boldsymbol{\beta} = \boldsymbol{\beta}'/2 + \boldsymbol{\beta}''/2$ for some $\boldsymbol{\beta}', \boldsymbol{\beta}'' \in P$. If $\boldsymbol{\beta}', \boldsymbol{\beta}''$ are both nondecreasing, then $(\boldsymbol{b}, \boldsymbol{\beta}'), (\boldsymbol{b}, \boldsymbol{\beta}'') \in W$ and the proof is complete.

For every $\alpha \in (0, 1)$, $\boldsymbol{\beta} = [\alpha\boldsymbol{\beta}' + (1-\alpha)\boldsymbol{\beta}]/2 + [\alpha\boldsymbol{\beta}'' + (1-\alpha)\boldsymbol{\beta}]/2$. For every α sufficiently small, $\alpha\boldsymbol{\beta}' + (1-\alpha)\boldsymbol{\beta}$ and $\alpha\boldsymbol{\beta}'' + (1-\alpha)\boldsymbol{\beta}$ are nondecreasing members of P. Hence, $\boldsymbol{\beta}$ is not an extreme point of W. □

Corollary A.2.1. *Let $(\boldsymbol{b}, \boldsymbol{\beta}) \in W$ be a step function with K steps and let $W(\boldsymbol{b}) = \left\{\boldsymbol{\beta}' \in [0,1]^K : (\boldsymbol{b}, \boldsymbol{\beta}') \in W\right\}$. Then $(\boldsymbol{b}, \boldsymbol{\beta})$ is an extreme point of W if and only if $\boldsymbol{\beta}$ is an extreme point of $W(\boldsymbol{b})$.*

Proof. Since $\boldsymbol{\beta}^1, \boldsymbol{\beta}^2$ in the proof of the lemma are nondecreasing, if $(\boldsymbol{b}, \boldsymbol{\beta})$ is not an extreme point of W, $\boldsymbol{\beta}$ is not an extreme point of $W(\boldsymbol{b})$. The converse—if $(\boldsymbol{b}, \boldsymbol{\beta})$ is an extreme point of W, then it is an extreme point of $W(\boldsymbol{b})$—is trivial. □

The following lemma, used in the proof of Theorem 1, provides a convenient characterization of functions that are monotone almost everywhere.

Lemma A.3. *Let $X = [\underline{x}, \overline{x}] \subseteq \mathbb{R}$, λ_b be a probability measure on the Borel σ-algebra of X. Define an order $\preceq$ on X^I by $\boldsymbol{x}' \preceq \boldsymbol{x}$ if $x'_1 \leq x_1$ and $x'_i \geq x_i$ for $i > 1$. Let $f : X^I \to [0,1]$ be a measurable function. The following statements are equivalent.*

(i) $\exists A \subseteq X,\ \lambda_b^I(A) = 1\ :\ [x', x \in A, x' \preceq x] \implies f(x') \leq f(x)$.

(ii) $\forall B := \prod_{i=1}^I B_i$ *and* $B' := \prod_{i=1}^I B'_i, [B'_i, B_i \subseteq X]$, $\lambda_b(B_i) > 0$, $\lambda_b(B'_i) > 0$, *such that* $[x' \in B', x \in B \implies x' \preceq x]$,

$$\frac{1}{\lambda_b^I(B')} \int_{B'} f \, d\lambda_b^I \leq \frac{1}{\lambda_b^I(B)} \int_B f \, d\lambda_b^I$$

Proof. $(i) \implies (ii)$.

$$[x' \in (B' \cap A), x \in B \cap A] \implies f(x') \leq f(x)$$

Therefore $E[f|B' \cap A] \leq E[f|B \cap A]$. Since $\lambda_b^I(B \cap A) = \lambda_b^I(B)$ and similarly for B', $E[f|B'] \leq E[f|B]$.

$(ii) \implies (i)$. Without loss of generality, let $X = [0,1]$. For $n = 1, 2, \ldots$, let

$$G_n = \{[0, 1/2^n), [1/2^n, 2/2^n), \ldots, [k/2^n, (k+1)/2^n), \ldots, [(2^n-1)/2^n, 2^n/2^n]\}$$

Then $\{G_n\}$ is a sequence of increasingly finer, nested partitions of $[0,1]$; each partition is a collection of disjoint intervals of $[0,1]$. Define

$$f_n(x) = \begin{cases} \frac{1}{\lambda_b^I(B)} \int_B f \, d\lambda_b^I & \text{if } x \in B \in G_n^I, \text{ and } \lambda_b^I(B) > 0 \\ 0 & \text{otherwise} \end{cases}$$

By construction f_n satisfies (ii), and it is simple to verify that there is $A_n \subseteq X^I, \lambda_b^I(A_n) = 1$ such that f_n satisfies (i) in A_n. Also by construction, $f_n = E[f|\sigma(G_n^I)]$. Since $\sigma(\bigcup_{n=1}^{\infty} \sigma(G_n^I))$ is the Borel σ-field, $\exists A' \subseteq X^I, \lambda_b^I(A') = 1$ such that $\forall x \in A', \{f_n(x)\} \to f(x)$ (see for instance, Shiryaev (1991), Theorem 3, page 510).

Finally, let $A = \bigcap_{n=1}^{\infty} A_n \cap A'$. Since A is the countable intersection of sets of measure one, $\lambda_b^I(A) = 1$. Suppose $\exists x', x \in A, x' \preceq x$ and $f(x') > f(x)$. Then, for sufficiently large n, $f_n(x') > f_n(x)$, a contradiction since f_n satisfies (i) with respect to A_n and $A \subseteq A_n$. Therefore, f satisfies (i). □

Appendix B

We discuss here the three Lemmas leading to the proof of Theorem 3. Lemma B.1 follows as a corollary to Lemma 5.1.

Lemma B.1. *If (q, q_s) is a Bayesian incentive-compatible, symmetric mechanism with I bidders and a seller, then $(Eq, E_{\boldsymbol{x}} q_s)$ is in W', where*

$$W' = \Big\{(Q_b, Q_s) |\, Q_b : X \to [0,1],\ Q_s : Y \to [0,1] \text{ are nondecreasing and} \\ [B \subseteq X, S \subseteq Y] \implies I \int_B Q_b \, d\lambda_b + \int_S Q_s \, d\lambda_s \leq 1 - \lambda_b(B^c)^I \lambda_s(S^c)\Big\} \tag{12}$$

Ex ante identical buyers must be treated symmetrically by the mechanism.

We proceed to identify the extreme points of W'. Lemma B.2 is the analogue of Lemmas 4.2 and 5.2.[8] Once the domain's partition (for the step function) is determined, these

[8] In reading the Lemma's statement, recall that summations with no terms are assumed to be zero.

lemmas identify all the step functions (defined by the partition) that are extreme points of the feasible set.

Lemma B.2. *Let* $(\{b_k, \bar{\beta}_k\}_{k=1}^{K_b}, \{b_k^s, \bar{\beta}_k^s\}_{k=1}^{K_s})$ *be a pair of step functions in* W'. *The pair* $(\{b_k, \bar{\beta}_k\}_{k=1}^{K_b}, \{b_k^s, \bar{\beta}_k^s\}_{k=1}^{K_s})$ *is an extreme point of* W' *if there exists a labeling* g *relative to* (K_b, K_s), *such that for every* $(k_b, k_s) \in \{1, \ldots, K_b\} \times \{1, \ldots, K_s\}$ *either*

(1)
$$\begin{cases} \bar{\beta}_{k_b}^b = & \frac{1}{Ib_{k_b}} \left[\left(\sum_{k=1}^{k_b} b_k \right)^I - \left(\sum_{k=1}^{k_b - 1} b_k \right)^I \right] \sum_{k=1}^{g_s(g_b^{-1}(k_b))-1} b_k^s \\ \bar{\beta}_{k_s}^s = & \left(\sum_{k=1}^{g_b(g_s^{-1}(k_s))-1} b_k \right)^I \end{cases}$$

or

(2) one or both of $\bar{\beta}_1$ *and* $\bar{\beta}_1^s$ *are zero, and all other* $(\bar{\beta}_{k_b}, \bar{\beta}_{k_s}^s)$ *are as in* (1).

A direct proof of Lemma B.2 can be obtained following the same steps employed in Lemmas 4.2 and 5.2. Instead of repeating those steps, we make a few heuristic observations that lead to the result.

Suppose that there are only two heterogeneous agents, a buyer and a seller, $i \in \{b, s\}$. From Lemma 5.2 (a), the extreme point mechanism for the buyer is

$$\bar{\beta}_{k_b}^b = \sum_{k=1}^{g_s(g_b^{-1}(k_b))-1} b_k^s \tag{13}$$

for some labeling g.

Suppose instead that there is no seller but that there are I ex ante identical bidders. From Lemma 4.2 (a), the symmetric extreme point mechanism is

$$\bar{\beta}_{k_b} = \frac{\left(\sum_{k=1}^{k_b} b_j \right)^I - \left(\sum_{k=1}^{k_b - 1} b_j \right)^I}{Ib_{k_b}} \tag{14}$$

This mechanism assigns the object to one of the I bidders.

Finally, suppose that there are I ex ante identical bidders plus a seller, a heterogeneous agent. The probability that one of the I bidders gets the object is given by (13), $\sum_{k=1}^{g_s(g_b^{-1}(k_b))-1} b_k^s$. This probability must be distributed among the I symmetric bidders; this is done according to (14). The result is precisely Lemma B.2 (1). Similar arguments lead to Lemma B.2 (2).

Lemma B.3. *Let the pair of step functions* $(Q_b, Q_s) = (\{b_k, \bar{\beta}_k\}_{k=1}^{K_b}, \{b_k^s, \bar{\beta}_k^s\}_{k=1}^{K_s})$ *be an extreme point of* W' *and let* g *be its labeling. The symmetric, mechanism* (q, q_s) *defined* *below satisfies dominant-strategy incentive compatibility and* $(Eq, E_{\boldsymbol{x}} q_s) = (Q_b, Q_s)$.

$$q(y,\boldsymbol{x}) = \begin{cases} \frac{1}{|\{i:\iota_b(x_i)=\iota_b(x_1)\}|} & \text{if } Q_b(x_1) = \max\{Q_b(x_i)\}_{i=1}^I > 0 \text{ and} \\ & \quad \iota_s(y) \leq g_s(g_b^{-1}(\iota_b(x_1))) - 1 \\ 0 & \text{otherwise} \end{cases}$$

$$q_s(y,\boldsymbol{x}) = \begin{cases} 1 - \sum_{i \in \mathcal{I}} q(y, \sigma_i(\boldsymbol{x})) & \text{if } Q_s(y) > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\iota_b(x_i) = k : x_i \in Q_b^{-1}(\bar{\beta}_k)$ *and* $\iota_s(y) = k : y \in Q_s^{-1}(\bar{\beta}_k^s)$.

Proof. We first show that $Eq = Q_b$. If $\iota_b(x_1) = 1$ and $\bar{\beta}_1 = 0$, then $Eq(x_1) = 0$ trivially and we are done. Suppose then $\iota_b(x_1) = k_b$ and $\bar{\beta}_{k_b} \neq 0$. By direct calculation, the expected probability of trade is

$$Eq(x_1) = \left[\sum_{i=1}^{I} \frac{1}{i} \binom{I-1}{i-1} \left(\sum_{k=1}^{k_b - 1} b_k \right)^{I-1-(i-1)} (b_{K-\ell})^{i-1} \right] \sum_{k=1}^{g_s(g_b^{-1}(k_b))-1} b_k^s \tag{15}$$

The argument in Lemma 4.3 yields the desired result.

We now show that $E_{\boldsymbol{x}} q_s = Q_s$. If $\iota_s(y) = 1$ and $\bar{\beta}_1^s = 0$, then $E_{\boldsymbol{x}} q_s(y) = 0$ trivially. Suppose then $\iota_s(y) = k_s$ and $\bar{\beta}_{k_s}^s \neq 0$.

Note then that $q_s(y,\boldsymbol{x}) \neq 0 \iff \sum_{i \in \mathcal{I}} q(y, \sigma_i(\boldsymbol{x})) \neq 1 \iff \forall i, q(y, \sigma_i(\boldsymbol{x})) = 0$. It follows from the definition of $q(y,\boldsymbol{x})$ in the Lemma's statement, that $q(y, \sigma_i(\boldsymbol{x})) = 0, \forall i \iff k_s > g_s(g_b^{-1}(\iota_b(x_i))) - 1$, or equivalently, $k_s \geq g_s(g_b^{-1}(\iota_b(x_i)))$. In turn, this is so if and only if $g_s^{-1}(k_s) < g_b^{-1}(\iota_b(x_i)) \iff g_b(g_s^{-1}(k_s)) > \iota_b(x_i) \iff g_b(g_s^{-1}(k_s)) - 1 \geq \iota_b(x_i)$. The probability that x_i is such that $\iota_b(x_i) \leq g_b(g_s^{-1}(k_s)) - 1$ is $\sum_{k=1}^{g_b(g_s^{-1}(k_s))-1} b_k$. This occurs for the I bidders with probability $(\sum_{k=1}^{g_b(g_s^{-1}(k_s))-1} b_k)^I$. □

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