# Duality in Procurement Design

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**Abstract**

Finding an optimal mechanism in a standard adverse selection model is equivalent to solving an infinite dimensional linear program. We begin with certain feasible mechanisms—those implemented by auctions, take-it-or-leave-it offers, and combinations of these polar mechanisms—and search for the environments that make them optimal. We prove the optimality of each mechanism using the dual program.

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# 1 Introduction

The adoption by the FCC of auction mechanisms to sell licenses for the radio-frequency spectrum and the emergence of the internet as a commercial medium has drawn attention to both the theory and practice of auctions over the past ten years. The growth in the popularity of auctions has obscured an otherwise fairly obvious fact—auctions represent only a very small part of the feasible set of trading institutions. Many alternative mechanisms such as bilateral negotiations, posted price institutions and randomized trading mechanisms are also feasible and potentially optimal.[1] This observation is of particular interest in procurement. When the private knowledge of a supplier may affect not only her marginal cost but also the valuation of the buyer, auctions are optimal mechanisms in only very narrow environments.[2]

In an earlier paper, we analyzed a procurement model of adverse selection and we characterized the environments in which two mechanisms, in a sense opposites of each other, maximize either social or buyer's surplus. We studied reserve price auctions and direct bargaining institutions—in the latter mechanism, sellers are placed in an order that is determined by prior beliefs; then take-it-or-leave-it offers are made sequentially.

In this paper, we study hybrid mechanisms, mechanisms that contain features of both auctions and successive take-it-or-leave-it-offers. We illustrate the environments where these more sophisticated institutions are optimal.

We use the standard, independent private-values model with quasi-linear utilities. The first hybrid mechanism we examine is one in which sellers are ordered and a fixed price is tendered to each seller. If a seller rejects the offer, the following seller has a chance to respond to the offer. If all sellers reject their offers, then sellers are invited to submit bids in a second price auction.

The second hybrid mechanism we examine reverses the use of the two main components—

[1] Manelli and Vincent (2003a) and (2003b) illustrate that in multi-object environments, optimal mechanisms may require randomization.

[2] The dependency between the valuation of one agent and the private information of other agents appears in other models; for instance Samuelson (1984) analyzes a model, in the context of bilateral trading, where the valuation of the uninformed party is a function of the private information of the other party. Recently Morand and Thomas (2003) analyze the separation properties of equilibria in adverse selection models with dependency in valuations.

an auction is followed by a sequence of take-it-or-leave-it offers. We illustrate the type of environments where these mechanisms are optimal (and dominate both, auctions and take-it-or-leave-it-offers).

Finally, we analyze a sequential offer mechanism where one seller is made an offer, upon rejection, an offer is made to another seller, and if that offer is rejected, a more generous offer is made to the first seller.

We use canonical examples to demonstrate what factors are relevant in determining the optimality of such mixtures of auctions and bargaining mechanisms. A rich class of trading institutions can thus be constructed by combining the two polar institutions. These institutions are simple to implement—participants can learn equilibrium strategies relatively easily—and are straightforward to analyze.

The mathematical structure underlying many essential questions in mechanism design is that of a linear program. It is a frequent practice (see for instance Myerson's 1981 seminal piece) to treat the prior distributions of bidders' types, and the relationship between bidders' private information and buyer's valuation as parameters of the linear program which we call the primal. Seeking a solution then amounts to identify the direct revelation mechanism that solves the linear program. Using the characterization of incentive compatibility (roughly that probabilities of trade are monotone), the linear program can be written solely in terms of probabilities of trade. In many environments, direct inspection of the objective function is often sufficient to guess the "solution" that maximizes the unconstrained problem—ignoring, for instance, incentive compatibility constraints. This guessed "solution" is sometimes feasible in the constrained problem and, therefore, solves the implementation problem. For general environments, though, once the linear program has been set, i.e, after using the characterization of incentive compatibility to simplify the objective function, there is not much guidance in the literature as to how to "solve" the linear program. In many cases, the dual program offers such guidance.

We begin our analysis with a precise description of the indirect mechanism that we desire to analyze, a hybrid mechanism including an auction and direct negotiations. We are concerned with identifying environments in which the proposed mechanism is socially optimal. We find that the dual program provides useful additional information in the form of additional inequalities that the direct mechanism and the environment must satisfy. It is with the aid of these inequalities that we come up with our results.[3]

Once we identify the environment, we state our results in the form "mechanism A is optimal in environment B." Once both the the mechanism and the environment are stated, it is possible to prove the theorem solving either the primal or dual problems.

## 2 Model

The set of players consists of a single buyer and $S$ potential suppliers or sellers indexed by $s = 1, \ldots, S$. The set of all sellers is denoted, in an abuse of notation, by $S$. Each supplier $s$ observes some private information $q_s \in I \equiv [0, 1]$. This private information determines the supplier's reservation value and may be directly relevant for the ex post valuation of the buyer. We will often refer to $q_s$ as the quality of the product offered as well as the reservation value of supplier $s$.

Each individual parameter $q_s$ is independently distributed according to a *continuous and strictly positive* density function $f_s(q_s)$. The corresponding cumulative distribution is denoted $F_s(q_s)$. The vector $q = (q_1, \ldots, q_s) \in I^S$ is a profile of types. We define $f(q) = f_1(q_1) \times f_2(q_2) \times \ldots \times f_S(q_S)$ and for any $i \in S$, $f_{-i} = \prod_{s \neq i} f_s$.

Agents maximize expected utility. Sellers' preferences are linear over money and the use of the good: if a seller with quality $q_s$ engages in trade and receives a money transfer of $m$, her net payoff is $m - q_s$. If no trade occurs, then the net payoff is zero.

The buyer's use value for a good of quality $q_s$ is $v_s(q_s)$. In this paper, we restrict attention to piecewise linear functions $v_s(q_s)$. If the buyer transfers $m$ dollars and receives an object of quality $q_s$, the buyer's net payoff is $v_s(q_s) - m$. No trade yields a payoff of zero for the buyer.

By consuming the good, the buyer perceives the quality of the object $q_s$ and realizes his benefit $v_s(q_s)$. Neither the quality of the object nor the actual benefit to the buyer are verifiable by a third party; it is not possible to contract contingent on an object's true quality or on the true benefit that may result from its consumption. To justify this assumption we observe that even when contracts may include numerous contingencies, there often are residual elements of private information on which it is not possible to contract. Those are the elements

[3]To our knowledge, the duality approach has not been widely used in mechanism design; a notable exception is the work of Gale, I., and T. Holmes (1993) who use a dual program to solve a mechanism design problem.

that we model; they include features of quality that are not easily observable.

Thiel (1988), Sinclair-Desgagné (1990) and Branco (1992) consider models in which private information is verifiable. As a result, agents may submit bids that consist of more than just prices. In our model seller types are determined exogenously. Thus we have a pure adverse selection model.[4]

By the revelation principle, we concentrate on *direct revelation* mechanisms, a pair of functions $p_s : I^S \longrightarrow I$ and $t_s : I^S \longrightarrow I\!R$, for each possible seller $s$. If sellers report $q = (q_1, \ldots, q_s)$, the probability that seller $s$ will trade is $p_s(q)$ and the expected transfer payment to seller $s$, which can be negative, is $t_s(q)$. Since the buyer wishes to purchase at most *one good* (OG), the following constraint must hold,

$$(\mathbf{OG}) \quad \sum_{s=1}^{S} p_s(q) \leq 1, \, \forall q.$$

If seller $s$ reports her type to be $x_s$ and supposes that the rest of the sellers report truthfully, $s$ would expect to receive a monetary transfer of $\int_{I^{S\setminus\{s\}}} t_s(x_s, q_{-s}) \, f_{-s}(q_{-s}) \, dq_{-s}$. We denote this expected monetary transfer by $E_s t_s(x_s)$. Supplier $s$'s expected payoff of reporting $x_s$ when having true quality $q_s$, is

$$\pi_s(x_s \mid q_s) = E_s t_s \,(x_s) - q_s \, E_s p_s \,(x_s).$$

Note that $E_s p_s(x_s)$ is the expected probability of trade when seller $s$ reports type $x_s$. Agents will truthfully reveal their private information only if the mechanism is incentive compatible (IC): for every seller $s$,

$$(\mathbf{IC}) \quad \pi_s(q_s \mid q_s) \geq \pi_s(x_s \mid q_s), \; \forall q_s(ae), \; \forall x_s \in I.$$

Using the well-known characterization of incentive compatibility, it follows that in any IC mechanism

$$(\mathbf{IC}) \quad \frac{d\, E_s p_s(q_s)}{dq_s} \leq 0, \text{ almost everywhere.}$$

Similarly if the expected probability of trade $E_s p_s(q_s)$ is non-increasing in $q_s$, then there exist transfers $t_s$ that will provide the correct incentives. (See for instance, Myerson (1981), and Myerson and Satterthwaite (1983)).

[4] We abstract of moral hazard issues such as the provision of incentives for investment in research and development. See for instance, Rob (1986) or Lang and Rosenthal (1991).

The buyer's expected profits are

$$\pi_b = \sum_{s=1}^{S} \int_{I^S} \left[ v_s(q_s)\, p_s(q) - t_s(q) \right] f(q)\, dq. \tag{1}$$

Let zero be the payoff for agents that do not participate in the mechanism. Individual rationality (IR) then requires that

$$(\mathbf{IR}) \quad \pi_b \geq 0 \;\text{ and }\; \pi_s(q_s \mid q_s) \geq 0,\ \forall q_s,\ \forall s.$$

We are interested in mechanisms that maximize social surplus. Procuring the object from seller $s$ with quality $q_s$, generates a total surplus of $w_s(q_s) = v_s(q_s) - q_s$. (For ease of notation and when confusion is unlikely, we drop the subindex $s$ from variables and functions.) Hence, for a given mechanism, $\{p_s, t_s\}_{s\in S}$, expected social surplus is [5]

$$\sum_{s=1}^{S} \int w_s(q_s)\, p_s(q)\, f(q)\, dq. \tag{2}$$

It has long been recognized that finding an optimal mechanism is formally equivalent to solving a linear program: the maximization of (2) subject to OG, IC and IR.[6]

We will proceed as follows. First we propose a particular trading institution. An equilibrium of a trading mechanism implies a particular mechanism $\{p_s, t_s\}_{s\in S}$ (which, by definition of equilibrium, represents a feasible solution to the optimization problem). For each seller $s$, the equilibrium outcome of the proposed institution specifies an expected probability of trade $E_s p_s$. We seek environments for which those functions $p_s$, $s \in S$, maximize (2) subject to (OG) and (IC). (We will refer to this linear program in infinite dimensions as $\mathcal{P}$.) We characterize those environments by solving the dual program. Finally, we verify that the proposed institution implements the solution, and that $IR$ is satisfied.

We now set up the conditions we use from the dual program. To simplify notation, we assume here that there are only two sellers, $s = 1, 2$. We also assume that for all $s$, $q_s$ are uniformly and independently distributed on $[0, 1]$.

[5] Any mechanism that maximizes expected social surplus will also maximize ex ante aggregate seller surplus, provided it is modified by adding additional transfers chosen so that expected buyer surplus is zero.

[6] Anderson and Nash (1987) is a good reference on linear programming in infinite-dimensional spaces.

To construct the Lagrangian of $\mathcal{P}$, we associate to each constraint a multiplier, $\gamma(q_1, q_2)$, $\lambda_1(q_1)$, $\lambda_2(q_2)$. Rearranging terms, the Lagrangian can be represented in two useful ways:[7]

$$\begin{aligned}
\mathcal{L}\left(p_1, p_2, \gamma, \lambda_1, \lambda_2\right) &= \int_0^1 \int_0^1 \left(w_1(q_1)\, p_1(q_1, q_2) + w_2(q_2)\, p_2(q_1, q_2)\right)\, dq_1\, dq_2 \\
&\quad + \int_0^1 \int_0^1 \gamma(q_1, q_2)\, \left(1 - p_1(q_1, q_2) - p_2(q_1, q_2)\right)\, dq_1\, dq_2 \\
&\quad + \int_0^1 \int_0^1 \left(-\lambda_1(q_1)\, \frac{d\, E_1 p_1(q_1)}{dq_1} - \lambda_2(q_2)\, \frac{d\, E_2 p_2(q_2)}{dq_2}\right)\, dq_1\, dq_2 \\
&= \int_0^1 \int_0^1 \gamma(q_1, q_2)\, dq_1\, dq_2 \\
&\quad + \int_0^1 \int_0^1 \left(w_1(q_1)\, p_1 - \gamma(q_1, q_2)\, p_1(q_1, q_2) - \lambda_1(q_1)\, \frac{d\, E_1 p_1(q_1)}{dq_1}\right)\, dq_1\, dq_2 \\
&\quad + \int_0^1 \int_0^1 \left(w_2(q_2)\, p_2 - \gamma(q_1, q_2)\, p_2(q_1, q_2) - \lambda_2(q_2)\, \frac{d\, E_2 p_2(q_2)}{dq_2}\right)\, dq_1\, dq_2.
\end{aligned}$$

A pair $(\tilde{p}_1, \tilde{p}_2)$ solves $\mathcal{P}$ if and only if there are (non-negative) multipliers $\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\gamma}$, forming a saddle-point of $\mathcal{L}:$ for all $p_1, p_2, \gamma,\ \lambda_1, \lambda_2$,

$$\mathcal{L}\left(\tilde{p}_1, \tilde{p}_2, \gamma, \lambda_1, \lambda_2\right) \geq \mathcal{L}\left(\tilde{p}_1, \tilde{p}_2, \tilde{\gamma}, \tilde{\lambda}_1, \tilde{\lambda}_2\right) \geq \mathcal{L}\left(p_1, p_2, \tilde{\gamma}, \tilde{\lambda}_1, \tilde{\lambda}_2\right).$$

Manipulating these inequalities in both representations of the Lagrangian, and using the fact that $(\tilde{p}_1, \tilde{p}_2)$ must be feasible yields

$$\begin{aligned}
\mathcal{L}\left(\tilde{p}_1, \tilde{p}_2, \tilde{\gamma}, \tilde{\lambda}_1, \tilde{\lambda}_2\right) &= \int_0^1 \int_0^1 \left(w_1(q_1)\tilde{p}_1(q_1, q_2) + w_2(q_2)\tilde{p}_2(q_1, q_2)\right) dq_1\, dq_2 \\
&= \int_0^1 \int_0^1 \tilde{\gamma}(q_1, q_2)\, dq_1\, dq_2. \qquad (3)
\end{aligned}$$

This simply states that at a solution, the value of the dual program must equal the value of the primal.

Further manipulation of the saddle-point inequalities yields the following well-known result (Duality Theorem): *A feasible pair* $(\tilde{p}_1, \tilde{p}_2)$ *solves* $\mathcal{P}$ *if and only if there are (non-negative)*

[7] In all the cases studied in this paper, the IR constraint on the buyer is non-binding. Thus, although, formally, the IR constraint should be in the Lagrangian, for conciseness, we leave it out.

*multipliers $\tilde{\gamma}$, $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$, such that*[8]

$$\int_0^1 \int_0^1 \left(w_1(q_1)\tilde{p}_1(q_1,q_2) + w_2(q_2)\tilde{p}_2(q_1,q_2)\right)\, dq_1\, dq_2 = \int_0^1 \int_0^1 \tilde{\gamma}(q_1,q_2)\, dq_1\, dq_2 \tag{4}$$

$$\int_0^1 \int_0^1 \left[w_1(q_1)p_1(q_1,q_2) - \tilde{\gamma}(q_1,q_2)p_1(q_1,q_2) - \tilde{\lambda}_1(q_1)\frac{d\, E_1 p_1(q_1)}{dq_1}\right] dq_1\, dq_2 \le 0, \;\; \forall p_1 \tag{5}$$

$$\int_0^1 \int_0^1 \left[w_2(q_2)p_2(q_1,q_2) - \tilde{\gamma}(q_1,q_2)p_2(q_1,q_2) - \tilde{\lambda}_2(q_2)\frac{d\, E_2 p_2(q_2)}{dq_2}\right] dq_1\, dq_2 \le 0, \;\; \forall p_2 \tag{6}$$

In addition, complementary slackness implies that (5) and (6) hold with equality when evaluated at $\tilde{p}_1$ and $\tilde{p}_2$ respectively.

## 3 A Characterization

In this section, we characterize the environments (within our model) in which the basic mechanisms, a take-it-or-leave-it offer and an auction, are optimal.

Theorem 1 shows when each of these mechanisms, a take-it-or-leave-it offer or an auction, is optimal. (It is a direct consequence of Corollaries 2 and 3, Manelli and Vincent (1995).) We provide a proof here to illustrate the use of the dual program.

**Theorem 1** *Let $v(x) = a + bx$, where $a \ge 1$ and $b \in I\!R$, and suppose that the quality variables $\{q_s\}_{s\in S}$ are uniformly and independently distributed. Then*

1. *A take-it-or-leave-it offer of $k = 1$ to an arbitrary supplier $s$ maximizes social surplus if and only if $b \ge 1$;*
2. *A second price auction maximizes social surplus if and only if $b \le 1$.*

Figure 1 illustrates the hypotheses of Theorem 1.

**Proof** (Part 1, necessity) Suppose for simplicity of notation, that there are only two suppliers. Without loss of generality, let supplier $s = 1$ be the chosen one, and suppose $\tilde{p}_1(q_1,q_2) = 1\ \forall(q_1,q_2)$ is optimal. (Note that because of symmetry, $w_1 = w_2$.)

[8] Although in infinite dimensions a duality gap, a gap between the value of the dual and primal programs, is possible, it is not an issue in our applications. In previous work, we provide a more detailed discussion of the gap, and of the duality setting.

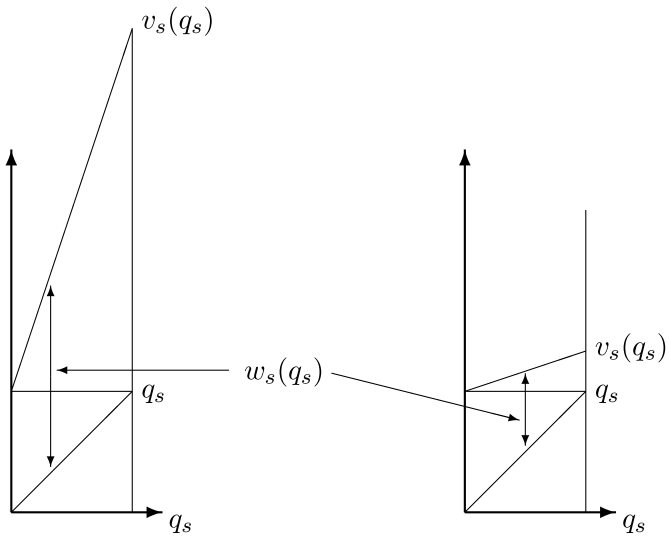

Figure 1: Take-it-or-leave-it Offer vs Auction

Note first that $\tilde{p}_1(q_1, q_2) = \mathbf{1}_{q_2 \leq r} + (1 - \mathbf{1}_{q_2 \leq r}) = 1$ where both terms are continuously differentiable in $q_1$ for all $r$. Complementary slackness (applied to (5)) implies

$$\int_0^1 \int_0^1 \left[ w_1(q_1) - \tilde{\gamma}(q_1, q_2) - \tilde{\lambda}_1(q_1) \frac{dE_1 \left[ \mathbf{1}_{q_2 \leq r} + (1 - \mathbf{1}_{q_2 \leq r}) \right]}{dq_1} \right] dq_1 \, dq_2 = 0. \tag{7}$$

Inequality (5) applied to $p_1(q_1, q_2) = \mathbf{1}_{q_2 \leq r}$ becomes

$$\int_0^1 \int_0^1 \left[ w_1(q_1) \mathbf{1}_{q_2 \leq r} - \tilde{\gamma}(q_1, q_2) \mathbf{1}_{q_2 \leq r} - \tilde{\lambda}_1(q_1) \frac{dE_1 \mathbf{1}_{q_2 \leq r}}{dq_1} \right] dq_1 \, dq_2 \leq 0.$$

We show now that the expression above must be actually zero. Applying (5) to $p_1(q_1, q_2) = (1 - \mathbf{1}_{q_2 \leq r})$ would also yield a non positive expression. But since both expressions must add up to zero because of (7), they must be both zero.

In addition, since $\frac{dE_1 \mathbf{1}_{q_2 \leq r}}{dq_1} = 0$,

$$\int_0^r \bar{w} \, dq_2 = \int_0^1 \int_0^1 \mathbf{1}_{q_2 \leq r} \, \tilde{\gamma}(q_1, q_2) \, dq_1 \, dq_2, \tag{8}$$

where

$$\bar{w} = \int_0^1 w_s(q_s) \, dq_s.$$

Inequality (6) applied to $p_2 = \mathbf{1}_{q_2 \le r}$ implies

$$\begin{aligned}\int_0^1 \int_0^1 \mathbf{1}_{q_2\le r}\tilde{\gamma}(q_1,q_2)\,dq_2\,dq_1 \quad &\ge \quad \int_0^1 \int_0^1 w_2(q_2)\mathbf{1}_{q_2\le r} - \tilde{\lambda}_2(q_2)\frac{dE_2\mathbf{1}_{q_2\le r}}{dq_2}\,dq_2\,dq_1 \\ &\ge \quad \int_0^r w_2(q_2)\,dq_2, \end{aligned} \tag{9}$$

where the last inequality follows because $\int_0^1\int_0^1 \tilde{\lambda}_2(q_2)\frac{dE_2\mathbf{1}_{q_2\le r}}{dq_2}\,dq_1\,dq_2 \le 0$ (since $\frac{dE_2\mathbf{1}_{q_2\le r}}{dq_2} \le 0$).

Combining (9) and (8), we obtain

$$\int_0^r \bar{w}\,dq_2 \ge \int_0^r w_2(q_2)\,dq_2, \quad \forall r, \tag{10}$$

or equivalently, that

$$\bar{w} \ge E[w_1(q_1)|q_1 \le r], \quad \forall r \in [0,1]. \tag{11}$$

Using the definition $w_1(q_1) = a + bq_1 - q_1$ and integrating, (11) becomes

$$\int_0^1 [a+(b-1)q_1]\,dq_1 = a + \frac{(b-1)}{2} \ge \int_0^r [a+(b-1)q_1]\,\frac{1}{r}dq_1 = a + \frac{(b-1)r}{2}, \quad \forall r \in [0,1].$$

This condition holds if and only if $b \ge 1$.

(Part 1, sufficiency) Note that the condition $a \ge 1$ implies that a buyer obtains non-negative expected utility in the take-it-or-leave-it mechanism. Thus, the IR constraint is satisfied. We must demonstrate that if $b \ge 1$, then (4), (5) and (6) hold. To define $\tilde{\gamma}$ and $\tilde{\lambda}_1$, let

$$\tilde{\gamma}(q_1,q_2) = \bar{w} \text{ and } \tilde{\lambda}_1(q_1) = \int_0^{q_1} (\bar{w} - w_1(t))dt.$$

It is immediate that (4) holds. The non-negativity of $\tilde{\lambda}_1(q_1)$ follows from the requirement $b > 1$. Condition (5) is

$$\int_0^1 \int_0^1 \left[ (w_1(q_1) - \bar{w})\,p_1 - \int_0^{q_1} (\bar{w} - w(t))\;dt\;\frac{d\,E_1 p_1(q_1)}{dq_1} \right] dq_1\,dq_2 \le 0.$$

Integrating by parts the last term on the left hand side, we have

$$\int_0^1 \left[ (w_1(q_1) - \bar{w}) \int_0^1 p_1(q_1,q_2)\,dq_2 - (w_1(q_1) - \bar{w}) \int_0^1 p_1(q_1,q_2)\,dq_2 \right] dq_1 = 0.$$

(The second inequality is handled in a similar fashion.)

This completes the proof of the first part of the Theorem: a take-it-or-leave-it offer of $k = 1$ to seller 1 is optimal if and only if $b \ge 1$.

A similar argument demonstrates the second part: an auction is optimal if and only if $b \le 1$. **QED**

**Remark 1** *Let* $\bar{w} = \int_0^1 w_s(q_s)\, dq_s$*. The proof of Theorem 1—see expression (11)—shows that a take-it-or-leave-it offer of* 1 *is socially optimal if and only if*

$$\bar{w} \geq 0 \ \ \textit{and} \ \ \bar{w} \geq E[w_1(q_1) \mid q_1 \leq r], \ \ \textit{for all possible offers } r.$$

Thus, to determine if a take-it-or-leave-it offer of 1 is the optimal mechanism, the designer only needs to compare that institution against other take-it-or-leave-it offers, ignoring all other possible mechanisms.

As long as the buyer values quality more than the sellers on the margin, a take-it-or-leave-it offer to an arbitrarily selected seller is socially optimal; competition among suppliers would not elicit a higher surplus. On the contrary, if sellers value quality more than the buyer on the margin, an auction is the socially optimal mechanism; competition among suppliers is useful.

**Remark 2** *Although, for simplicity, we have used the uniform distribution, the characterization in Remark 1 holds for any distribution (with a continuous and strictly positive density function).*

Theorem 1 highlights a sometimes overlooked difference between situations where one object is sold to many bidders, and situations where one object is *purchased* from many bidders.

When selling an object to potentially many buyers, it is natural to assume that buyers are the privately informed parties—they typically know how much they are willing to pay for the object. It is perhaps also natural to assume that the buyers' valuations do not affect the valuation of the seller.[9]

In a procurement context, however, the private information is frequently possessed by the potential sellers. A given seller's private information might index *only* the private marginal cost of the seller, and all sellers may have identical objects to sell. If this is so, then it is natural to assume that the buyer's valuation does not depend on the sellers' private information; one may assume $v_s(q_s) = \bar{v}$, a constant.

A given seller's private information, however, may be an index of both marginal cost and quality or other such attribute. If the buyer's valuation depends on quality, one may obtain a

[9]In some contexts, for example an insurance market, this assumption is less plausible.

non-trivial functional form for $v_s(q_s)$. In this case, an auction because it biases trade to sellers with lower quality, may be very undesirable. In fact, as the number of potential sellers grows, an auction will lead to trade with the lowest quality supplier. In this case, Theorem 1 shows it is best to forgo competition entirely in favor of direct negotiation with an arbitrary seller.

Even if the buyer valuation is increasing in quality, a take-it-or-leave-it mechanism might not be optimal. It follows from Theorem 1 that the optimality of take-it-or-leave-it mechanisms requires that social surplus $(v_s(q_s) - q_s)$ be increasing in $q_s$ (i.e., $b > 1$). Similarly, an auction is optimal only if $b < 1$. Theorem 1 is particularly useful when social surplus is monotone on $q_s$, admittedly a restrictive environment. The proof of Theorem 1, however, by illustrating the role played by monotonicity in constructing the dual variables, helps us explore how and when hybrid mechanisms are optimal.

# 4 Hybrid Mechanisms

Consider an environment where increases in the quality of the object up to a certain level, significantly increase the buyer's well being. Beyond that threshold, however, increases in quality do not increase the buyer's payoff. Higher values of $q_s$ may cost more to produce but provide the buyer no additional utility. In this case, social surplus increases with quality for low levels of $q_s$ and decreases for high levels. Figure 2 illustrates this situation. The conditions for optimality of a take-it-or-leave-it offer hold for low $q_s$'s and those for the optimality of an auction hold for high $q_s$'s. Theorem 2 demonstrates in a canonical example that a hybrid mechanism with an auction following rejected take-it-or-leave-it offers is optimal.

**Theorem 2** *For $s = 1, 2$, let*

$$v_s(q_s) = \begin{cases} 1/2 + 3/2 q_s, & \text{if } q_s \in [0, 1/2] \\ 5/4 & \text{otherwise} \end{cases}$$

*and suppose that $q_s$ is uniformly and independently distributed in the unit interval. The following trading mechanism maximizes social surplus: a take-it-or-leave-it offer of $5/8$ is made to seller one; if rejected, a take-it-or-leave-it offer of $3/4$ is made to seller two; and if rejected, a second price auction is conducted.*

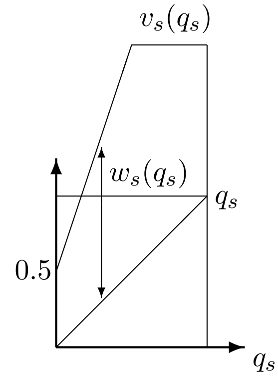


Figure 2: A Take-it-or-leave-it Offer and an Auction

The proof below proceeds by proposing values for the dual variables and checking that a saddle point has been obtained. Before providing the details of the proof, we briefly described how we arrived at the proposed variables. The value of the take-it-or-leave-it offers is determined by noting that the supplier with $q_s = 1/2$ must be indifferent between accepting the offer or waiting for the chance of an auction. The expected price in the auction is 3/4. The first supplier anticipates that by rejecting the offer the auction occurs with probability 1/2, the second supplier knows that by rejecting the offer the auction occurs with probability 1. Thus, the equilibrium probabilities of trade of this mechanism are as illustrated in Figure 3. Any seller with $q_s \leq 1/2$ accepts the initial offer. Otherwise, the seller rejects and submits a bid in the auction (if there is one) equal to her valuation. Expected buyer surplus gross of payment is 7/8 in the offer phase and 5/4 in the auction phase which exceeds the highest possible auction price so the buyer IR is satisfied in equilibrium.

**Proof** Let $k = 1/2$, $\bar{w}_k = E[w_s(q_s)|q_s \leq k] = 2\int_0^1 \mathbf{1}_{q_s \leq k}\, w_s(q_s)\, dq_s$,

$$\tilde{\gamma}(q_1, q_2) = \begin{cases} \bar{w}_k, & \text{if } q_1 \leq k \\ \bar{w}_k, & \text{if } q_1 > k,\ q_2 \leq k \\ w_1(q_1), & \text{if } k < q_1 \leq q_2 \\ w_2(q_2), & \text{if } k < q_2 < q_1 \end{cases}$$

and

$$\tilde{\lambda}_s(q_s) = \begin{cases} \int_0^{q_s} (\bar{w}_k - w_s(t))dt, & \text{if } q_s \leq k \\ 0, & \text{if } q_s > k. \end{cases}$$

It is straightforward to verify that the value of the dual equals the value of the primal, expression (3). Since $w_s(q_s)$ is monotone increasing on $q_s$ for all $q_s \in [0, 1/2]$, $\tilde{\lambda}_s$ is non-

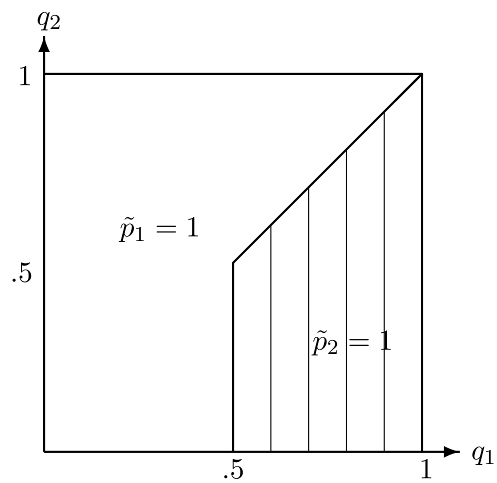


Figure 3: The Constrained Optimum–Theorem 2

negative.

We now verify that inequality (5) holds. An integration by parts yields that

$$\int_0^1 \int_0^1 \tilde{\lambda}_1(q_1) \frac{d\, E_1 p_1(q_1)}{dq_1}\, dq_1\, dq_2 = \int_0^1 \int_0^1 \mathbf{1}_{q_1 \le k}\, p(q_1, q_2)(w_1(q_1) - \bar{w}) dq_1 dq_2$$

for all $p(\cdot, \cdot)$ with $E_1 p \in C^1$. Therefore,

$$\begin{aligned} &\int_0^1 \int_0^1 \left[ (w_1(q_1) - \tilde{\gamma}(q_1, q_2)) p_1(q_1, q_2) - \tilde{\lambda}_1(q_1) \frac{d\, E_1 p_1(q_1)}{dq_1} \right] dq_1\, dq_2 \\ &\quad = \int_0^1 \int_0^1 \left( \mathbf{1}_{q_1 \le k} \left[ w_1 - \tilde{\gamma} p_1 - (w_1 - \bar{w}) p_1 \right] + \mathbf{1}_{q_1 \ge k} \left[ w_1 - \tilde{\gamma} \right] p_1 \right) dq_1\, dq_2 \\ &\quad = \int_0^1 \int_0^1 \mathbf{1}_{q_1 \ge k} \left[ w_1(q_1) - \tilde{\gamma}(q_1, q_2) \right] p_1(q_1, q_2)\, dq_1\, dq_2 \le 0, \quad \forall p_1, \end{aligned}$$

where the last inequality uses the definitions of $\tilde{\lambda}_s$ and $\tilde{\gamma}$, and establishes (5): it is non-positive by definition of $\tilde{\gamma}$ and because $w_2(q_2)$ is decreasing in $q_2$ for $q_2 \ge 1/2$.

A similar argument establishes condition (6).

We conclude that $(\tilde{p}_1, \tilde{p}_2, \tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\gamma})$ is a saddle point of $\mathcal{P}$.

**QED**

It is instructive to compare the optimal hybrid mechanism found in Theorem 2 with the mechanism that maximizes *ex post* social surplus. The latter mechanism, illustrated in

Figure 4, is an unconstrained optimum, violates incentive compatibility, and is therefore not feasible. The figure represents the space of qualities $(q_1, q_2)$. Vertical lines shade the area

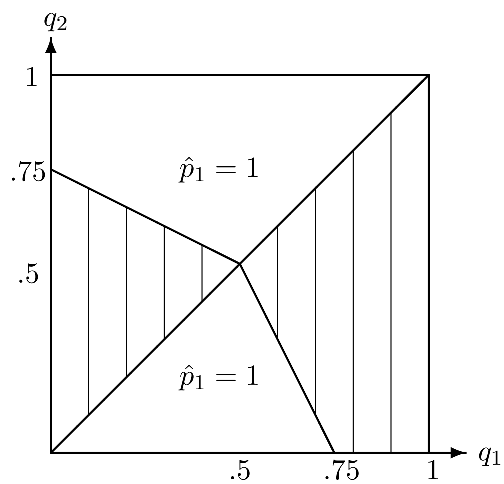


Figure 4: The Unconstrained Optimum–Theorem 2

where seller 2 is allocated the trade, i.e., $\hat{p}_2(q_1, q_2) = 1$. The complement represents the region where seller 1 is allocated the trade. Pick any $q_2$ in the vertical axis and consider a horizontal line through $q_2$. Since bidder types are uniformly and independently distributed, the length of the horizontal line *on the shaded region* represents the expected probability of trade for seller 2 with quality $q_2$, $E_2\hat{p}_2(q_2)$. Direct observation reveals that the expected probability of trade is not compatible with IC; it is increasing over the region $q_2 \in [0, .5]$ and therefore the mechanism is not implementable. Theorem 2 identifies the optimal mechanism.

Consider now situations in which the buyer's payoff only increases with increases in quality when quality is high.

**Theorem 3** *For* $s = 1, 2$, *let*

$$v_s(q_s) = \begin{cases} 3/4, & \text{if } q_s \in [0, 1/2] \\ 3/2q_s, & \text{otherwise,} \end{cases}$$

*and suppose also that* $q_s$ *is uniformly and independently distributed in the unit interval. Let* $k < 1/2$ *satisfy*

$$w_s(k) = E[w_s(q_s) | q_s \in (k, 1]].$$

*The following trading mechanism maximizes social surplus: a second price auction with reserve price $(1+k)/2$ is conducted; if no bids meet the reserve, a take-it-or-leave-it offer of* 1 *is made with equal probability to sellers one or two.*[10]

**Proof** We show first that the subgame perfect equilibrium in this mechanism consists of the following strategies: in the auction phase, bidders with valuations $q_s \in [0, k]$ submit bids equal to their valuations. In the second phase, all bidders accept the price offer of 1. To see this, suppose a rival bidder is expected to follow this strategy. Consider the expected return of bidding $\tilde{q} \leq k$ when the true quality is $q_s$,

$$\int_{\tilde{q}}^{k} (b - q_s)db + (1-k)((1+k)/2 - q_s).$$

If $q_s \leq k$, the first term is positive and achieves its maximum at $\tilde{q} = q_s$. A bid $\tilde{q} \in [k, (1+k)/2]$ wins if and only if the rival has valuation greater than $k$ and the reserve price $(1+k)/2$. This yields an expected return of $(1-k)\left[\frac{1+k}{2} - q_s\right]$.

A bidder with $q_s > (1+k)/2$ receives a strictly negative payoff from such a bid and so will prefer to wait for the take-it-or-leave-it offer. For a bidder with $q_s \in [k, (1+k)/2]$, if no bid is submitted, then the expected payoff from relying on the take-it-or-leave-it game is

$$\frac{1}{2}(1-k)(1-q_s) = (1-k)\left[\frac{1+k}{2} - q_s + \frac{q_s - k}{2}\right]$$

which is greater than the highest achievable payoff in the auction game for any $q_s > k$. Note that, in the auction phase, the actual price is less than or equal to $k \leq 3/4 = v_s(q_s), q_s \in [0, k]$ and in the offer phase, the price is 1 which is less than the expected buyer valuation given $q_s \in (k, 1]$ so the buyer's IR constraint is satisfied.

For $s = 1$, the equilibrium probability of trade in this mechanism is therefore (see Figure 6)

$$\tilde{p}_1(q_1, q_2) = \begin{cases} 1, & \text{if } q_1 \leq k, q_1 \leq q_2 \\ .5, & \text{if } q_1 > k,\ q_2 > k \\ 0, & \text{otherwise.} \end{cases}$$

Define $K_1 = \{(q_1, q_2)|q_1 \leq q_2 \leq k\}$, $K_2 = \{(q_1, q_2)|q_2 \leq q_1 \leq k\}$, $K_3 = \{(q_1, q_2)|q_1 \leq k, q_2 > k\}$, $K_4 = \{(q_1, q_2)|q_2 \leq k, q_1 > k\}$, $K_5 = \{(q_1, q_2)|q_1 > k, q_2 > k\}$.

For any integrable $f$, $\int_0^1 \int_0^1 f(q_1, q_2)\, dq_1\, dq_2 = \sum_{i=1}^{5} \int_{K_i} f(q_1, q_2)\, dq_1\, dq_2$.

[10] Computation yields $k = .38$ in the case presented.

Let

$$\bar{w}_B = E[w_s(q_s)|q_s \in (k,1]].$$

Define

$$\tilde{\gamma}(q_1,q_2) = \begin{cases} w_1(q_1), & \text{if } q \in K_1 \cup K_3 \\ w_2(q_2), & \text{if } q \in K_2 \cup K_4 \\ \bar{w}_B, & \text{if } q \in K_5. \end{cases}$$

It is straightforward to verify that the value of the dual equals the value of the primal. Define

$$\tilde{\lambda}_s(q_s) = \begin{cases} 0, & \text{if } q_s \leq k \\ \int_k^{q_s} (\bar{w}_B - w_s(t))dt, & \text{if } q_s > k. \end{cases}$$

To see that $\tilde{\lambda}_s$ is non-negative, note that it can be written as

$$\tilde{\lambda}_s(q_s) = (q_s - k)\left\{\bar{w}_B - E[w_s(x)|x \in (k,q_s]]\right\}.$$

We show the term in braces is always non-negative. This is immediate for $q_s \in (k, 1/2]$, since $w_s(\cdot)$ is strictly decreasing and $w_s(k) = \bar{w}_B$ by definition of $k$. Note that

$$\frac{\partial}{\partial q_s} E[w_s(x)|x \in (k,q_s]] = \frac{w_s(q_s) - E[w_s(x)|x \in (k,q_s]]}{(q_s - k)}.$$

Thus, $E[w_s(x)|x \in (k,q_s]]$ is decreasing in $q_s$ if and only if it exceeds $w_s(q_s)$. Let $\hat{q}$ be such that

$$\hat{q} = min_{q_s \geq 1/2}\{q_s | E[w_s(x)|x \in (k,q_s]] = \bar{w}_B\}.$$

Since $E[w_s(x)|x \in (k,q_s]]$ is continuous and equals $\bar{w}_B$ at $q_s = 1$, $\hat{q}$ exists. Furthermore, $\hat{q} > 1/2$. Suppose $\hat{q} < 1$. Since $E[w_s(x)|x \in (k,q_s]]$ is continuous and is less than $\bar{w}_B$ for $q_s = 1/2$, then $E[w_s(x)|x \in (k,q_s]]$ must be increasing at $\hat{q}$ which implies $w_s(\hat{q}) \geq E[w_s(x)|x \in (k,\hat{q}]] = \bar{w}_B$. Since $w_s(\cdot)$ is strictly increasing in $(1/2, 1] \supset (\hat{q}, 1]$, we must have $E[w_s(x)|x \in (k,q_s]] > \bar{w}_B$ for all $q_s > \hat{q}$. This violates the definition, $E[w_s(x)|x \in (k,1]] = \bar{w}_B$. Therefore, $\hat{q} = 1$ and

$$\tilde{\lambda}_s(q_s) = (q_s - k)\left\{\bar{w}_B - E[w_s(x)|x \in (k,q_s]]\right\} \geq 0, \forall q_s.$$

Integrating by parts, we obtain

$$\begin{aligned} \int_0^1 \int_0^1 \tilde{\lambda}_1(q_1) \frac{d\, E_1 p(q_1)}{dq_1}\, dq_1\, dq_2 &= \int_0^1 \int_0^1 \mathbf{1}_{q_1 > k}\, p(q_1,q_2)(w_1(q_1) - \bar{w}_B) dq_1 dq_2 \\ &= \sum_{i \in \{4,5\}} \int_{K_i} p(q_1,q_2)(w_1(q_1) - \bar{w}_B) dq_1 dq_2 \end{aligned}$$

for all $p(\cdot,\cdot)$ with $E_1 p \in C^1$.

Therefore,

$$\begin{aligned}
&\int_0^1 \int_0^1 \left[(w_1(q_1) - \tilde{\gamma}(q_1,q_2))p_1(q)dq - \tilde{\lambda}_1(q_1)\frac{d\,E_1 p_1(q_1)}{dq_1}\right] dq_1\,dq_2 \\
&\quad = \int_{K_1} [w_1 - \tilde{\gamma}]p_1\,dq_1\,dq_2 + \int_{K_3} [w_1 - \tilde{\gamma}]p_1\,dq_1\,dq_2 + \int_{K_2} [w_1 - \tilde{\gamma}]p_1\,dq_1\,dq_2 \\
&\qquad + \int_{K_4} [w_1 - \tilde{\gamma} - (w_1 - \bar{w}_B)]\,p_1\,dq_1\,dq_2 + \int_{K_5} [w_1 - \tilde{\gamma} - (w_1 - \bar{w}_B)]\,p_1\,dq_1\,dq_2 \\
&\quad = \int_{K_2} [w_1 - w_2]\,p_1(q_1,q_2)\,dq_1\,dq_2 + \int_{K_4} [\bar{w}_B - w_2]\,p_1(q_1,q_2)\,dq_1\,dq_2 \le 0,\ \forall p_1.
\end{aligned}$$

The second equality follows by applying the definition of $\tilde{\gamma}$. The inequality follows because, for $q \in K_2$, $w_1(q_1) \le w_2(q_2)$ and for $q \in K_4$, $w_2(q_2) \ge w_s(k) = \bar{w}_B$. (A similar argument may be applied to $\tilde{\lambda}_2$.)

We conclude that $(\tilde{p}_1, \tilde{p}_2, \tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\gamma})$ is a saddle point of $\mathcal{P}$.

**QED**

Figure 5 illustrates the allocation that maximizes *ex post* social surplus. Again, this allocation is infeasible. The constrained optimal mechanism is illustrated in Figure 6.

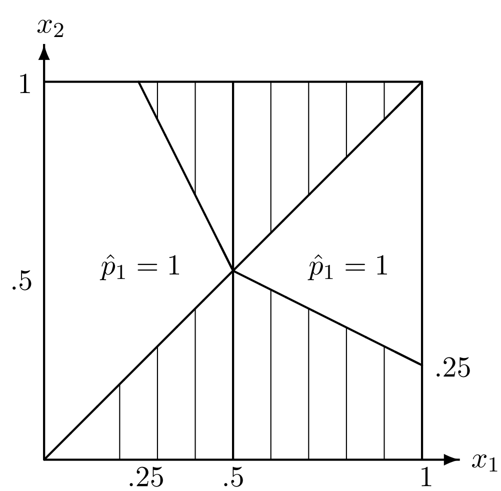


Figure 5: The Unconstrained Optimum–Theorem 3

Sequential offer mechanisms could also be of the following form: the first seller is made a take-it-or-leave-it offer, upon rejection, the second seller is made an offer and if that is rejected,

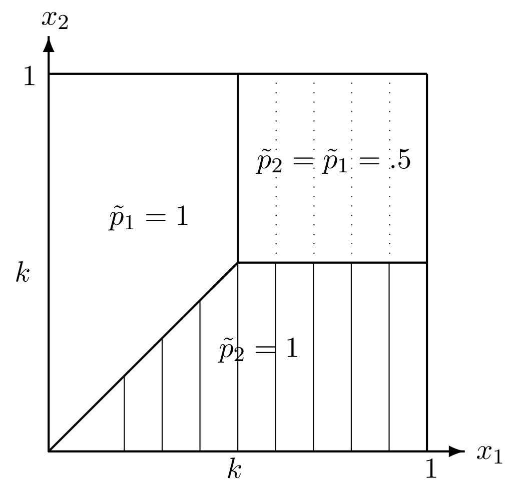


Figure 6: The Constrained Optimum–Theorem 3

the first seller is made a second, higher take-it-or-leave it offer. Our final result, Theorem 4, illustrates environments where the mechanism just described is an optimal mechanism. Figure 7 depicts the relevant social welfare functions.

**Theorem 4** *Let*

$$w_2(q_2) = v_2(q_2) - q_2 = \begin{cases} 3q_2, & \textit{if } q_2 \in [0, 1/2] \\ 0 & \textit{otherwise} \end{cases}$$

*and*

$$w_1(q_1) = v_1(q_1) - q_1 = \begin{cases} 4q_1, & \textit{if } q_1 \in [0, 1/2] \\ 1/4 & \textit{otherwise,} \end{cases}$$

*and suppose that for $s = 1, 2$, $q_s$ are independently and uniformly distributed on the unit interval. The following trading mechanism maximizes social surplus: a take-it-or-leave-it offer of $3/4$ is made to seller one; if rejected, a take-it-or-leave-it offer of $1/2$ is made to seller two; if rejected, a take-it-or-leave-it offer of $1$ is made to seller one.*

**Proof** The proof is similar to that of Theorem 2. Let

$$\gamma(q_1, q_2) = \begin{cases} 1, & \text{if } q_1 \leq 1/2 \\ 3/4, & \text{if } q_1 > 1/2,\ q_2 \leq 1/2 \\ 1/4, & \text{otherwise.} \end{cases}$$

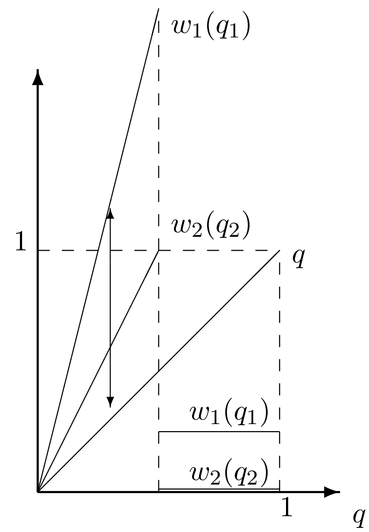


Figure 7: Sequential take-it-or-leave-it Offers

Observe that

$$\tilde{\lambda}_1(q_1) = \begin{cases} \int_0^{q_1}(1 - w_1(t))dt, & \text{if } q_1 \leq 1/2 \\ 0, & \text{if } q_1 > 1/2. \end{cases}$$

and

$$\tilde{\lambda}_2(q_2) = \begin{cases} \int_0^{q_2}(3/4 - w_2(t))dt, & \text{if } q_2 \leq 1/2 \\ 0, & \text{if } q_2 > 1/2. \end{cases}$$

Note that $\tilde{\lambda}_s$ is non-negative. Integrating by parts, we have

$$\int_0^1 \int_0^1 \tilde{\lambda}_1(q_1) \frac{d\, E_1 p_1(q_1)}{dq_1}\, dq_1\, dq_2 = \int_0^1 \int_0^1 \mathbf{1}_{q_1 \leq \frac{1}{2}}\, p_1(q_1, q_2)(w_1(q_1) - 1)dq_1 dq_2$$

and

$$\int_0^1 \int_0^1 \tilde{\lambda}_2(q_2) \frac{d\, E_2 p_2(q_2)}{dq_2}\, dq_1\, dq_2 = \int_0^1 \int_0^1 \mathbf{1}_{q_2 \leq \frac{1}{2}}\, p_2(q_1, q_2)(w_2(q_2) - 3/4)dq_1 dq_2.$$

**QED**

The mechanism described in Theorem 4 belongs to a larger class of mechanisms in which seller types are partitioned into intervals, low seller types are made (appropriately chosen) offers in turn, then higher seller types are made offers in turn and so on until potentially all seller types are made offers. Some reflection will reveal that as the partition is made finer, such mechanisms actually approximate low-price mechanisms.

## 5 Conclusion

Our results illustrate that mechanisms constructed by combining features of both auctions and direct negotiations, can easily dominate the component mechanisms in terms of social surplus. We illustrate the environments in which each richer hybrid mechanism is optimal. We exploit the duality of the implicit linear program. The responsiveness of the social surplus function to changes in $q_s$ is fundamental in the construction of the dual variables that demonstrate the optimality of the hybrid institutions. These techniques could be used with any trading institution that can be described as a direct revelation mechanism, and with any additional constraints on outcomes that can be incorporated as linear constraints on the-probability-of-trade functions.

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