Economic Inquiry

DESIGNED TO FAIL: THE MEDICARE AUCTION FOR DURABLE MEDICAL EQUIPMENT

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We examine the theoretical properties of the auction for Medicare Durable Medical Equipment. Two unique features of the Medicare auction are (1) winners are paid the median winning bid and (2) bids are nonbinding. We show that median pricing results in allocation inefficiencies as some high-cost firms potentially displace low-cost firms as winners. Further, the auction may leave demand unfulfilled as some winners refuse to supply because the price is set below their cost. We also introduce a model of nonbinding bids that establishes the rationality of a lowball bid strategy employed by many bidders in the actual Medicare auctions and recently replicated in Caltech experiments. We contrast the median-price auction with the standard clearing-price auction where each firm bids true costs as a dominant strategy, resulting in competitive equilibrium prices and full efficiency. (*JEL* D44, I11, H57)

I. INTRODUCTION

The Centers for Medicare and Medicaid Services (CMS) conducted auctions in nine major metropolitan areas in November 2009 to establish reimbursement prices and identify suppliers for durable medical equipment. The impetus for these auctions was the 1997 Balanced Budget Act, which specified that competitive bidding be used as a means of "harnessing market forces" to decrease Medicare costs. The prices that resulted from the 2009 auctions took effect on January 1, 2011 and the program is currently being expanded to 90 other cities.

Medicare's program is unique in that it uses a never before seen median-price auction and does not make winning bids binding.[1] This article examines the theoretical properties of the median-price auction and compares those properties to the well-known clearing-price auction under the independent private values (IPV) paradigm. Our focus is on two important efficiencies that should result from a well-designed auction. *Allocation efficiency* occurs if the auction always leads to outcomes where winners have lower costs than losers. *Quantity efficiency* occurs if the auction results in a quantity being supplied at the point where supply meets demand.

From a modeling perspective, allocation efficiency results if a unique, symmetric, increasing equilibrium bid function exists since firms with lower costs always submit lower bids in such an equilibrium. If no such equilibrium exists, an auction can generate an inefficient allocation as some high-cost firms may displace low-cost firms as auction winners. Quantity inefficiency can arise from two sources. First, if the auction rules discourage participation, then too few units might be supplied (this is common when a reserve price is

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1. Nonbinding bids have been used on rare occasions, just not in conjunction with median pricing which has never been used. A notorious example of non-binding bids was the April 1993 auction for Australian satellite television services. Because bids were not binding these auctions were marred by bidder default and political embarrassment. See McMillan (1994) for details.

ABBREVIATIONS

CMS: Centers for Medicare and Medicaid Services
IPV: Independent Private Values

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used). Second, if the auction sets the price below any winner's cost, then that winner will likely refuse to supply.[2]

When bids are binding, it is well known that the IPV clearing-price auction elicits the dominant strategy of bidding one's cost. With this strategy, full economic efficiency is achieved as the price is set at the point where supply meets demand and the lowest-cost firms provide the goods for a price that is greater than their costs. Alternatively, we show that the median-price auction suffers both allocation and quantity inefficiencies. Allocation inefficiency arises because symmetric equilibrium bid functions do not exist under realistic assumptions.[3] Quantity inefficiency occurs because the median price is set below some winning bidders' costs and thus the median-price auction is not ex post *Individually Rational,* leading some demand to go unfulfilled.

The median-price inefficiencies will likely result in supply shortages, diminished quality and service to Medicare beneficiaries, and an increase in long-term total cost as Medicare beneficiaries are forced into more expensive options. Identifying and fixing the auction process are crucial as this program represents an important test case in the broader goal of utilizing market methods for the provision of Medicare supplies and services. Failure of this implementation might well discourage the further application of market methods and prevent future cost savings in other areas.

To better understand the implications of our findings, it is important to understand the CMS auction process. CMS began auction pilots as a means of setting reimbursement prices for Durable Medical Equipment in 1999 in Demonstration Projects in Florida and Texas. In those pilots, and still today, firms place individual bids on a multitude of products within specified categories in an attempt to be named a Medicare provider (nonwinning bidders cannot receive Medicare reimbursements). While reimbursement prices on individual products are set using winning bids on those products, winners are actually chosen based on a "composite" bid that is a weighted average of their individual bids on the different products in a category where the weights indicate the relative importance of the product to the category.[4]

CMS selects winners beginning with the lowest composite bid and works upward until the total capacity of winners is sufficient to satisfy estimated demand in the category. However, it is important to recognize that merely selecting enough winners to satisfy demand does not guarantee quantity efficiency in the Medicare auction. This is because winning the auction does not mean that a firm becomes a Medicare supplier. Rather, winning the auction simply earns a firm the option of signing a supply contract—which the winner is free to decline since bids are not binding. This is another danger of median pricing. By setting the reimbursement price on an individual good equal to the median winning bid on the good, Medicare risks that some winning bidders' costs may exceed the reimbursement price and thus, they will be unwilling to supply and some demand will go unfulfilled.

Interestingly, median pricing has not always been the policy. In the original implementation of the Medicare auctions, reimbursement prices were set using an upwardly adjusted average of the winning bids. After the Demonstration Projects, this rule was replaced by median pricing at the same time a "bona fide" bid rule was instituted (Federal Register 2007). Bona fide bidding is simply an imposition of bid ceilings and floors by Medicare. Bid ceilings act like reserve prices and are risky since reserve prices discourage bidder participation and can cause quantity inefficiency. However, they can also lead to allocation inefficiency here because, as we show, they result in nonexistence of monotone equilibria when combined with the median-pricing rule.

Although not a direct source of inefficiency, bid floors were an interesting addition since the goal of the auctions was to lower reimbursement prices. Medicare explains in the Federal Register (2007) that floors were put in place to prevent "irrational, infeasible" lowball bids. We show that the lowball bidding phenomenon is a perfectly justifiable concern when the IPV auction model is adapted to allow for costless bidder default (i.e., nonbinding bids). The basic idea is that bidding below cost is not a dominated strategy if bids are

2. It should be noted that under-supply of goods is particularly problematic in the Medicare setting as demand for life preserving/saving medical equipment may go unfulfilled.

3. Nonexistence of equilibrium can, at times, be as much an indictment of a model as it is of the mechanism being studied. However, the model used here has survived many decades of scrutiny and been successfully employed in auction markets around the world; we examine it in light of a never before used median-pricing rule. The source of nonexistence here is the median-price rule.

4. Katzman and McGeary (2008) show that the composite bid rule itself can lead to allocation inefficiencies as it provides strong incentives for firms to skew bids away from costs.

not binding (it is dominated when bids are binding) and thus lowball bidding is an economically rational strategy. It is important however to stress that while our results are applicable to Medicare's auction due to the absence of default costs, they should not be extended to markets where default costs are positive.

The theoretical properties of the median-price auction that we present are supported by the experimental work of Merlob, Plott, and Zhang (2012, MPZ hereafter). MPZ demonstrate that, in a laboratory setting, the median-price auction with binding bids results in large allocation and quantity inefficiencies while the standard clearing-price auction with binding bids is highly efficient. In addition, they are able to experimentally produce the lowball bidding phenomenon that led Medicare to adopt bid floors and that we justify theoretically. Specifically, they show that nonbinding bids encourage lowball bidding no matter what the auction form (clearing or median price) and that these lowball bids result in significant undersupply.

Not only does the costless default associated with nonbinding bids open the door for lowball bidding, there are additional institutional details outside the realm of our model that make lowball bids even more attractive. The first is that there are complementarities in supplying multiple categories—demanders value being able to get supplies from one provider. Lowball bidding enables the provider to select a complementary and profitable set of categories to supply once the prices are announced. Second, the negative price impact of any single lowball bid is minor given the large number of winning suppliers. And finally, lowball bidding allows a supplier to postpone a difficult assessment of costs until after CMS announces prices. Lowball bidding is a simple strategy that gives the supplier maximum flexibility.

In the end, it seems evident that the adjustments to the Medicare auction rules over the past decade, from average pricing to bona fide bids, were in response to unexpected outcomes that occurred in the actual auctions. Cramton and Katzman (2010) argue that, rather than making ongoing adjustments to the current auction rules, a better strategy would be to replace this system with a proven procedure such as the clearing-price auction. Cramton (2011) specifically suggests a dynamic clearing-price auction as a preferable alternative that can be easily implemented. Within our setting, the dynamic clearing-price auction is isomorphic to Vickrey's (1962) sealed bid clearing-price auction.[5] The clearing-price auction is widely used and studied with desirable properties that are well established both in the field and in the experimental laboratory.[6] Further, dynamic implementations of the clearing-price auction have performed especially well in the field (Ausubel and Cramton 2004) and in the lab (Cramton et al. 2012).

Our article proceeds with a review of the clearing-price auction, followed by detailed analysis of the median-price auction with binding bids. Once the inefficiencies in the latter are presented and discussed, we conclude the article with a section on the median-price auction with nonbinding bids that helps explain the lowball bidding phenomenon observed in the laboratory and in the field.

II. THE MODEL

We consider an IPV model where only one product is to be supplied and multiple winners are chosen. N risk-neutral firms have unit capacities and an odd number, $W(<N)$, of winning bidders is necessary to fulfill demand.[7] Firm i's cost of providing a unit of the product is $c_i \in [L, H]$ which is drawn from the cumulative distribution function $F(c)$ with corresponding density $f(c) > 0$. It is assumed that $f(c) > 0$ for all $c \in [L, H]$ and that f has derivatives of all orders on $[L,H]$. We denote the minimum and maximum values that $f(c)$ obtains on $[L,H]$ by $f_{\min}$ and $f_{\max}$, respectively.

The reimbursement price on units supplied equals the $M = (W + 1)/2$ lowest bid under the median-pricing rule and equals the $(W + 1)$ lowest bid in the clearing-price auction. When bid ceilings and floors are imposed, bids are restricted to be no lower than $\underline{b}$ $(\leq L)$ and no higher than $\overline{b}$ $(\geq H)$. For any $n > 0$ and

5. The dynamic clearing-price auction differs from Vickrey's auction in more complex environments.

6. For experimental results, see for example, Coppinger, Smith, and Titus (1980), Cox, Roberson, and Smith (1982), Kagel (1995), and Kagel and Levin (1993, 2001, 2008).

7. The environment faced by Medicare bidders is much more complex than is our model, likely including common value components as well as multiunit capacities. We focus on the case where costs are independently distributed and bidders have unit capacities because it admits equilibrium solutions, the properties of which are sufficient to conclude that the median price auction is inefficient. It is unlikely that the median price auction would somehow become better in a more complex setting.

$1 \leq i \leq n$ we let the random variable $c_{(i:n)}$ be the ith lowest of n costs drawn from F. Thus, $c_{(1:N-1)} < c_{(2:N-1)} < \cdots < c_{(N-1:N-1)}$ are the ordered costs of a particular bidder's opponents and $c_{(1:N)} < c_{(2:N)} < \cdots < c_{(N:N)}$ are the ordered costs of all N bidders. For $1 \leq k \leq n$, the cumulative distribution of the kth lowest of n-order statistics is denoted by $F_{(k:n)}(x)$ with corresponding density function

$$f_{(k:n)}(x) = k \binom{n}{k} (1 - F(x))^{n-k} F(x)^{k-1} f(x)$$

and, for $1 \leq j \leq k \leq n$, the cumulative joint distribution of the jth and kth lowest of n-order statistics is denoted by $F_{(j,k:n)}(x, y)$ with corresponding density function

$$\begin{aligned} f_{(j,k:n)}(x, y) &= n! / ((j-1)! (k-j-1)! (n-k)!) \\ &\times F(x)^{j-1} (F(y) - F(x))^{k-j-1} (1 - F(y))^{n-k} \\ &\times f(x) f(y). \end{aligned}$$

To best represent the CMS rules, it is assumed that all firms must bid and that when bids at auction are binding, a winning bidder must supply the good even if the price reached at auction is below that bidder's cost.[8] Alternatively, when bids are not binding, winning bidders observe the auction price and then decide whether to supply the good. In the latter scenario, we use subgame perfection when examining optimal bids at auction.

III. THE CLEARING-PRICE AUCTION

Under the assumptions of our model the dynamic clearing-price auction proposed in Cramton (2011) is analogous to that studied in Vickrey (1962) where the reimbursement price paid to winners equals the lowest-losing bid. It is well known that in our environment this payment rule gives firms a dominant strategy of bidding their cost if bids are binding. The resulting market clearing price-equals the $(W+1)$th lowest cost. Because this price is above each winner's cost, all winners are willing to supply the product. The result is a fully efficient outcome in which the W lowest-cost firms end up supplying the product.

It is well established that there are other equilibria in the clearing-price auction where some firms bid less than their cost. For example, W firms bidding $\underline{b}$ and all other firms bidding $\overline{b}$ is a Nash equilibrium. However, equilibria such as this are commonly discounted by using the dominant strategy refinement and noting that bidding below cost is a dominated strategy. However, if bids are not binding, many new lowball bidding equilibria arise that cannot be refined away. Bidding below cost is not dominated when winning bidders can simply walk away from the contract. As mentioned in Section I, it is important to note that the lack of dominance follows only if there are no transaction costs associated with "walking away." Should any transaction costs from forfeiture of one's bid be incurred, bidding below costs remains dominated as in the standard IPV model.

Bidding below cost in the clearing-price auction is dominated when bids are binding because by bidding less than cost, a firm runs the risk of winning the auction but obtaining a negative payoff. But by bidding costs, that firm will never receive a negative payoff when winning, and is assured of winning whenever there would be a positive payoff. Once bids are not binding however, firms need not worry about receiving a negative payoff from a below-cost bid because they are free to walk away from the contract without penalty in the Medicare auctions. Because below-cost bids do not earn negative payoffs, bidding below cost is not dominated. It also means that in the extreme, everyone bidding the lowest allowed bid $\underline{b}$ and simply walking away is an undominated equilibrium when bids are not binding. Thus, it is not surprising that MPZ find that bidders in clearing-price auctions do in fact lowball bid in the experimental lab when bids are not binding and there are no forfeiture costs.

IV. THE MEDIAN-PRICE AUCTION

The median-price auction used by CMS sets the price equal to the median of the W winners' bids (i.e., the $M = (W+1)/2$ lowest bid). It is assumed that ties (which end up occurring with positive probability in some equilibria) are broken by choosing winners randomly (with equal probability) from those whose bids tied. If bids are not binding, winners may decline to sign a contract with CMS. When this occurs, CMS turns to the lowest losing bidder and offers that bidder a contract at the original median price. In a strictly increasing equilibrium those bidders to whom offers are subsequently made have

8. This also matches the experimental rules set forth in MPZ (2012) and allows for direct comparisons of our results to theirs.

higher costs than the firm that first declined the contract and will therefore decline the contract as well.

A. Full Information

When studying auctions it is often useful to examine a full information environment before proceeding to the more realistic environment of incomplete information. Often the full information equilibrium is the limiting case of the incomplete information case and can provide much needed intuition about a problem. For example, this is the case in the clearing-price auction with binding bids where bidding cost is a dominant strategy in both full and incomplete information environments.

We begin by assuming that all firms know that the costs are $c_1 < c_2 < \cdots < c_N$. When bids are binding in the median-price auction there are a number of equilibria that generate inefficient outcomes. For example, if $W \geq 5$ it is an equilibrium for the $M+1$ lowest-cost firms to bid identical amounts somewhere between c_{M+1} and c_{M+2} while all others bid $\overline{b}$. This sets the equilibrium price between c_{M+1} and c_{M+2} which results in at least one firm with a bid of $\overline{b}$ being chosen randomly as a winner, but being reimbursed an amount less than their cost—leading to quantity inefficiencies. Further, because some winners are chosen randomly, it is possible that not all of the W lowest-cost firms are awarded contracts—causing potential allocation inefficiencies.

When bids are not binding even more equilibria arise in the full information median-price auction. Since winning bidders are now able to decline contracts, even the extreme case where all firms bid $\underline{b}$ become an equilibrium. More importantly, these equilibria are not dominated if there are no forfeiture costs. To see that bidding below cost is not dominated (and is in fact payoff equivalent to bidding cost) in the median-price auction when bids are not binding we must consider two cases: $c \geq m$ and $c < m$, where m is the median bid. When $c \geq m$, bidding c leads to zero profit since the firm either wins at a price equal to or less than their cost (and declines the contract), or loses. By bidding less than c the firm increases the chances of winning and may lower the price by changing the median winning bid. But, whenever the firm wins it is at a price below cost, the firm declines the supply contract, and continues to earn zero profit. Alternatively, when $c < m$, bidding c and bidding below c both give profit of $m - c$ since both strategies assure the firm of winning and the median bids are the same under each strategy.

In addition to the inefficient equilibrium mentioned above, there is also an efficient equilibrium in the full information setting where the W low-cost firms bid c_{W+1}, the firm with cost c_{W+1} bids $c_{W+1} + \epsilon$,[9] and all other firms with cost greater than c_{W+1} bid c_{W+2} or greater. Interestingly, this is an equilibrium whether bids are binding or not. We will see in the next section that this efficient equilibrium is an artifact of the full information assumption and does not carry over into most incomplete information cases.

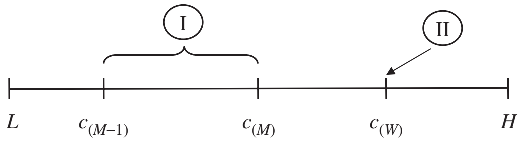
In addition to the incomplete information experiments mentioned above, MPZ also ran the auctions in a full information environment. Despite the existence of an efficient equilibrium in this setting, the full information, median-price auction was the *worst* performing of all formats tested in terms of efficiency. We conjecture two causes for this poor experimental performance. First, it appears that experimental bidders adopted strategies from different equilibria, some of which consist of lowball bidding. Second, the theoretically efficient equilibrium has firms with costs above the market clearing price simply bid above the market clearing price and accept losing. It could be that the experimental participants were not willing to bid above the market clearing price simply to support the equilibrium when doing so would result in a zero payoff, particularly since lowball bidding guaranteed them the same payoff while also giving them the option of signing the contract if the price was favorable.

B. Incomplete Information with Binding Bids

We begin by examining a potential bidder's maximization problem when there is no ceiling placed on bids to show that the bid ceiling will almost surely bind. In doing so we assume that each of bidder i's opponents is using the strictly increasing bid function $\beta(c)$ with inverse $\phi(\beta)$. Bidder i's problem (suppressing the bidder subscript i) is to choose the bid, b, in order to

9. ϵ represents the fact that the bidder with cost c_{W+1} is aggressively mixing just above c_{W+1} such that none of the bidders with lower cost want to raise their bids. See Hirshleifer and Riley (1992, Chapter 10) for a detailed discussion of the role of mixing in full information auctions.

FIGURE 1
Intuition for Equation (2)



maximize

(1)
$$P(c,b) = \int_{\phi(b)}^{H} (\beta(x) - c) f_{(M-1:N-1)}(x)\,\mathrm{d}x + (b-c)\left[F_{(M-1:N-1)}(\phi(b)) - F_{(M:N-1)}(\phi(b))\right] + \int_{L}^{\phi(b)} \int_{\phi(b)}^{H} (\beta(x) - c) f_{(M,W:N-1)}(x,y)\,\mathrm{d}y\mathrm{d}x$$

where the first term represents the case in which i's bid wins and is below the price-setting bid, the second term is when i's bid wins and sets the price, and the third term is when i's bid wins and is above the price-setting bid.

Taking the derivative of Equation (1) with respect to b, imposing symmetry, and rearranging gives the following equilibrium condition:

(2)
$$\beta'(c)\left[F_{(M-1:N-1)}(c) - F_{(M:N-1)}(c)\right] = \int_{L}^{c} (\beta(x) - c) f_{(M,W:N-1)}(x,c)\,\mathrm{d}x.$$

Figure 1 presents the intuition for Equation (2). Region I denotes the case where firm i's bid lies between $c_{(M-1)}$ and $c_{(M)}$ and thus sets the price. The LHS of Equation (2) is simply the probability that firm i's bid falls in region I times the incremental change in the price brought about by a change in his bid. The RHS of Equation (2) represents the instantaneous probability that lowering firm i's bid makes him a winner (this happens if he is at point II where $c = c_{(W)}$) times the expected payoff i receives from becoming a winner (the value of $(\beta(x) - c)$, integrated over all possible values of the equilibrium price-setting bidder's cost, $c_{(M)}$). This equilibrium is dictated by events where either the firm sets the price, or the price-setting bid is below his own. Events where the price is set by a bid above his own do not influence the equilibrium.

Unfortunately, there is no known closed-form solution to the integro-differential equation given by Equation (2). We are however able to obtain solutions in power series form once a cost distribution is specified. With that in mind, we will show that a unique monotone increasing, bounded equilibrium bid function exists in the case when $W = 3$ and costs are uniformly distributed. But, we also provide evidence that no such solution exists when $W > 3$ and costs are uniformly distributed. Thus indicating that there are relevant settings in which a bounded equilibrium bid function does not exist in the median-price auction.

Despite the lack of a general solution to Equation (2), we are able to obtain general necessary conditions that must be satisfied by any equilibrium solution to that equation. These conditions are given below in Theorem 1. But first, to provide the setting for Theorem 1, it is convenient to define $B = M - 1$ and $A = N - 1 - W$ whenever bidder i submits a winning bid that is above the price-setting bid. B is simply the number of bids submitted by bidder i's $N - 1$ opponents that are below the price-setting bid and A is the number of opponents' bids that are above the lowest-losing bid. The remaining $B - 1$ opponents (along with bidder i) submit winning bids that are above the price-setting bid. For example, if bidder i submits a winning bid that is above the price-setting bid when $W = 7$ and $N = 12$ as in MPZ, then of bidder i's $N - 1 = 11$ opponents, one will be the price setter, one will be the lowest losing bidder, $B = 3$ will have bid below the price-setting bid, $A = 4$ will have bid above the lowest-losing bid, and $B - 1 = 2$ will have bid between the price-setting bid and the lowest-losing bid.

An essential ingredient of our analysis is the operator $D_c = (1/f(c))(\mathrm{d}/\mathrm{d}c)$ (differentiation followed by division by $f(c)$). For any $k \geq 1$, the k-fold iterate of D_c will be denoted by D_c^k. Namely, D_c^k allows us to define $\gamma_B(c) = B!/(2B)!\left(D_c^B\left(cF(c)^{2B}\right)/F(c)^B\right)$ which plays a crucial role in our conclusions regarding nonexistence. By our assumption that $f(c)$ is positive and has derivatives of all orders throughout (L, H), γ_B is also defined throughout that interval. In addition, it is shown in the Appendix that γ_B can be continuously extended to the entire interval $[L, H]$ and that $\gamma_B(L) = L$.

Theorem 1. *If* β *is a solution of Equation (2) on* (L, H) *such that* $\beta'(c) > 0$ *for all* $c \in (L, H)$ *and such that* β *is continuous (and hence bounded) on* [L, H], *then*

(i) $\beta^{(n)}(L) = 0$ *for all* $n = 1, 2, \ldots, B$.
(ii) $\beta(H) = \gamma_B(H)$

(iii) $\beta(c) > c$ *for all* $c \in [L, H]$ *and* $L < \beta(L) < L + W/f_{\min}N$

(iv) $P(c, \beta(c)) > 0$ *for all* $c \in [L, H)$ *and* $Pr[\beta(c_{(M:N)}) < c_{(M+1:N)}] > 0$.

The proof of Theorem 1 is given in the Appendix. We now make some useful observations about its implications. First, we note that repeated application of the operator D_c to Equation (2) yields a linear differential equation of order $B+1$ in β for which a unique solution can be determined if values of $\beta(L)$ and $\beta^{(n)}(L)$ for $n = 1, 2, \ldots, B$ are specified. Therefore, by establishing that Equation (2) pins down the first B derivatives of β at L, Part i of Theorem 1 guarantees that Equation (2) can have at most one solution for any fixed initial value, $\beta(L)$.

The implication of Part iii of Theorem 1 is intuitively appealing. A small ratio of winners to bidders (W/N), which indicates a high level of competition, forces the lowest-cost firms to bid aggressively (just above cost) whereas low-cost firms can bid fairly high when this ratio is close to one. Part ii of the theorem provides the less intuitive conclusion that the equilibrium bid of the highest-cost firm depends only on the number of winners and *not* the total number of bidders. This is because the function γ_B is defined in terms of B (and hence W) but does not depend on N. Parts i and ii together show that at most one bounded solution exists and that if it does, then Part ii gives the exact value to which the solution converges as $c \to H$.

Part iv establishes that the median-price auction is interim individually rational but not always ex post individually rational. Interim individual rationality follows from the fact that, in equilibrium, all firms expect a non-negative expected payoff conditional from winning the auction ($P(c, \beta(c)) > 0$).[10] Ex post individual rationality fails because there is a positive probability that the bid of the price setter will be less than the costs of some or potentially all other higher winning bidders.

As stated above, solutions to Equation (2) can be expressed as power series, but doing so requires specifying the distribution of costs. The two examples that follow use the uniform distribution of costs where $F(c) = (c - L)/(H - L)$ and $f(c) = f_{\min} = 1/(H - L)$. In this setting, Parts ii and iii of Theorem 1 reduce to $\beta(H) = H + ((W-1)/(W+1))(H-L)$ and $L < \beta(L) < L + (W/N)(H-L)$.

10. Of course this expected payoff will be lessened when a binding bid ceiling is imposed, and interim Individual Rationality may be lost. We discuss this issue at the end of this section.

C. Example 1: The Case of $W = 3$ *Winners,* $U[0,1]$ *Cost Distribution*

Although it is unlikely that a CMS auction would have only three winners, our first example considers just such a case because it admits a complete mathematical analysis of the solutions to Equation (2) and sets a comparative foundation for our second example where $W = 7$.

Assuming that costs are uniformly distributed on the [0, 1] interval, Equation (2) becomes

$$(3) \quad c\,(1-c)^2\,\beta'(c) = (N-2)(N-3)\int_0^c x\,(\beta(x) - c)\,\mathrm{d}x$$

and by Theorem 1 we know that any bounded, monotone increasing solution must satisfy $\beta'(0) = 0$, $0 < \beta(0) < 3/N$, and $\beta(1) = 1.5$.

It is useful to note that any solution of Equation (3) is also a solution of the second-order differential equation

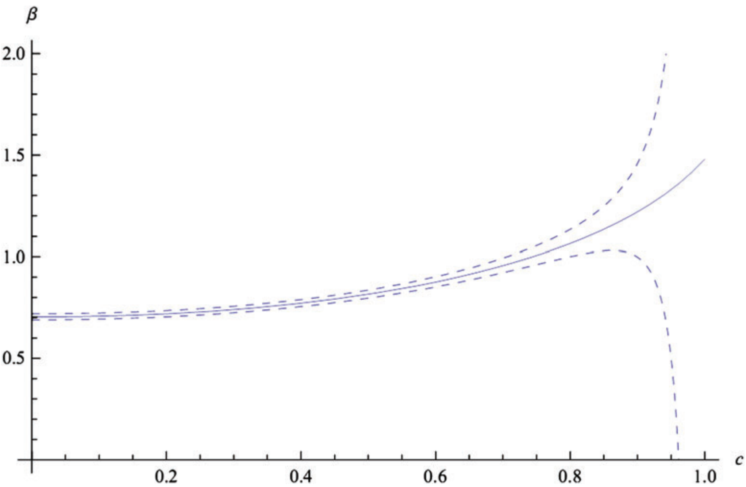
$$c\,(1-c)^2\,\beta''(c) + (1-c)(1-3c)\,\beta'(c) - (N-2)(N-3)\,c\beta(c) = -1.5\,(N-2)(N-3)\,c^2$$

obtained by differentiating Equation (3) with respect to c. Because Part i of Theorem 1 specifies that that $\beta'(0) = 0$, this second-order differential equation has a unique solution (on some open interval containing $c = 0$) for any given initial value $\beta(0) = b_0$. Furthermore, each of these solutions can be expressed as a power series $\beta(c) \equiv \beta(c, b_0) = b_0 + \sum_{n=1}^{\infty} b_n c^n$ where the sequence of coefficients b_n is defined by $b_1 = 0$, $b_2 = b_0/2$, and the three term recurrence relation

$$b_n = \left(2n(n-1)\,b_{n-1} - \left(n^2 - 2n - 2\right) b_{n-2}\right)/n^2, \quad n \geq 3.$$

It can be shown using the Ratio Test that for any choice of b_0, the above power series has radius of convergence equal to 1 and that $\beta(c, b_0)$ is a solution of Equation (3) on the interval (0, 1). It can also be shown that there is a unique $b_0 \equiv b^*$ such that $c = 1$ is also included in the interval of convergence of $\beta(c, b^*)$ and it then follows from Abel's Theorem that $\beta(c, b^*)$ is continuous throughout the interval [0, 1] (see Ahlfors

FIGURE 2
Solutions to Equation (3), When $W = 3$, $N = 4$



1979, p. 41 for details on Abel's Theorem). In addition, we discover that, for $b_0 = b*$, we have $b_n > 0$ for all $n \geq 2$ and this allows us to conclude that $\beta'(c, b*) > 0$ and $\beta''(c, b*) > 0$ for all $c \in (0, 1)$. Further investigation of the power series reveals that the solution becomes infinitely steep as $c \to 1^-$.

We are now prepared to discuss why $W = 3$ is more amenable to complete mathematical analysis than are cases where $W > 3$. In particular, when $W = 3$, $\beta(c, b*)$ can be expressed in the form $\beta(c, b*) = (p(c) + H(r, s, 1; c))/(1 - c)^2$ where $H(r, s, 1; c)$ is Gauss' hyper-geometric function with numerator parameters r and s that depend on W and N (see Brand 1966, 439–40 for a thorough discussion) whereas this is impossible for $W > 3$. Well-known properties of H (in particular Gauss' Theorem) can then be used in the $W = 3$ case to determine the exact value of $\beta(0, b*) = b*$ for any N. For example, when $N = 4$, Gauss' Theorem gives

$$b^* = 8 - \left(\Gamma\left(2 + \sqrt{3}\right)\Gamma\left(2 - \sqrt{3}\right) / \Gamma(1)\Gamma(3)\right) \approx 0.704.$$

Figure 2 shows the behavior of the power series solutions for the specific case where $N = 4$ (solutions for $N > 4$ are of the same basic form). The lower dashed curve begins at $b_0 = 0.69$ and diverges to negative infinity which is representative of all solutions emanating from initial values $\beta(0) = b_0 < b*$. Similarly, the upper dashed curve begins at $b_0 = 0.72$ and diverges to positive infinity which is representative of all solutions for which $\beta(0) = b_0 > b*$. Only for $\beta(0) = b* \approx 0.704$ does the solution converge on the entire interval [0, 1] and by Part ii of Theorem 1 it converges to $2W/(W + 1) = 1.5$. That solution is represented by the solid curve in Figure 2.

It is worth noting that since the solid bid function equilibrium is monotone increasing and bounded, the median-price auction is "revenue equivalent" to the clearing-price auction under that equilibrium, generating an expected reimbursement price of 0.8. Further, although the two auctions generate equivalent expected revenue and are both interim individually rational, only the clearing-price auction is ex post individually rational since the median-price auction sometimes sets the reimbursement price below the highest winning bidder's cost. The power series solution described above can be used to calculate that there is an 11.227% chance of quantity inefficiency in this example. The practical implication of these results is that equilibrium either results in bids that approach infinity or a convergent bid function where the highest bids are at least one-and-one-half times the highest costs. Clearly in either case the CMS bid ceiling will bind and outcomes will be inefficient. Unfortunately as the next example shows, this suboptimal result is probably a best-case scenario since it appears that when $W > 3$, a bounded, monotone increasing solution does not even exist.

D. Example 2: The Case with W = 7 *Winners,* U*[100,1000] Cost Distribution*

Our second example examines the case of seven winners. Here we assume that costs are uniformly distributed on [100, 1000] and set the number of bidders to $N = 12$. We choose this scenario because it corresponds to the recent experimental work of MPZ and allows us to shed light on their findings.

In general for the case of seven winners, Equation (2) becomes

$$(4)\quad (c-L)^3 (H-c)^4 \beta'(c) = K \int_L^c (x-L)^3 (c-x)^2 (\beta(x)-c)\,\mathrm{d}x,$$

where $K = (N-7)(N-6)(N-5)(N-4)/2$. For the case where $N = 12$ and $c \sim U[100, 1000]$, Theorem 1 indicates that

$$\beta'(100) = \beta''(100) = \beta'''(100) = 0,$$
$$\beta(100) < 100 + 7/12(900) = 625,$$
$$\beta(1000) = 1000 + 6/8(900) = 1,675.$$

Similar to the $W = 3$ case, integro-differential Equation (4) can be converted into a linear differential equation (of fourth order in this case). Unfortunately, Equation (4) cannot be reformulated as a hyper-geometric equation as in the $W = 3$ case (since this is only possible with linear differential equations of second order) and little is known about analytic solutions to this form of equation other than the fact that it can be expressed as a power series $\beta(c, b_0) = \sum_{n=0}^{\infty} b_n (c-L)^n$, with coefficients defined by $b_1 = b_2 = b_3 = 0$, $b_4 = K(b_0 - L)/480(H-L)^4$, $b_5 = (K(b_0-L)/150(H-L)^5) - (K/600(H-L))^4$, and the five-term recurrence relation

$$\begin{aligned} b_n = {} & (K/n(n+1)(n+2)) - (n-4)\, b_{n-4} \\ & + 4(n-3)(H-L)\, b_{n-3} \\ & - 6(n-2)(H-L)^2\, b_{n-2} \\ & + 4(n-1)(H-L)^3\, b_{n-1}/n(H-L)^4 \end{aligned}$$

for $n \geq 6$.

Once again, Part i of Theorem 1 tells us that Equation (4) pins down the first three derivatives and therefore the solutions to the equation are unique for any fixed initial condition $b_0 = \beta(L)$. Figure 3 graphs the power series solutions for four different choices of b_0 (550, 588.353, 588.354, and 640) when $L = 100$ and $H = 1000$. It shows that the qualitative nature of the family of solutions of Equation (4) for $W = 7$, $N = 12$ is similar to that in the example where $W = 3$ in that there appears to be a critical value, b^*, that separates solutions into classes that diverge to positive infinity when $\beta(L) > b^*$ and negative infinity when $\beta(L) < b^*$. For b_0 far enough away from b^* (represented by the upper and lower dashed curves), solutions quickly diverge to positive or negative infinity just as in Figure 2. However, for $\beta(L) \neq b^*$, but close to b^*, solutions do not diverge simply to positive or negative infinity like in the $W = 3$ case. Rather, they diverge to $\pm\infty$ nonmonotonically.

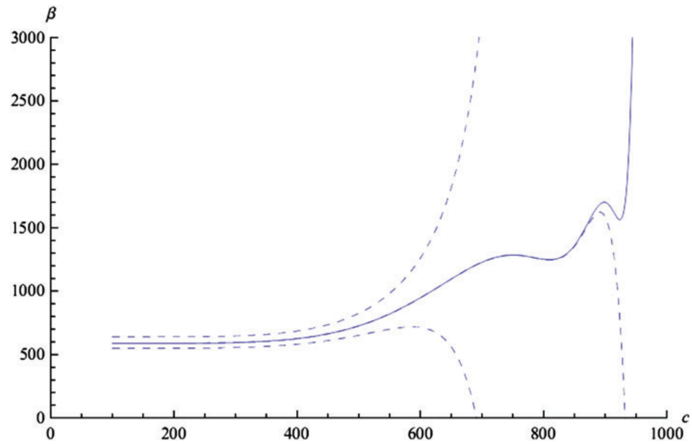
The nonmonotonic behavior of the middle curves in Figure 3 indicate that no bounded, monotone increasing equilibrium bid function exists in this example. To see this notice that any solutions starting above or below the middle two curves in Figure 3 cannot approach a finite value. It is only a solution that starts between the two middle curves that can converge, and Part ii of Theorem 1 tells us that the solution with $\beta(L) = b^*$ must converge to $\gamma_3(1000) = 1,675$. However, by uniqueness, a curve lying between the middle two curves must approach 1,675 nonmonotonically and we therefore conclude that no bounded, monotone increasing equilibrium exists.

This section has supplied convincing mathematical evidence that the median-price auction with binding bids does not admit a bounded, monotone increasing equilibrium under realistic parameter values. We have performed similar analysis for the $W = 5$ and $W = 9$ cases and found similar nonmonotonic behavior for values near the critical b^* in those cases as well. We conjecture that this nonmonotonic behavior is due to the term $(c-x)^{B-1}$ that appears in the integro-differential equation that governs the dynamics. If correct, this would explain the existence of a bounded equilibrium when $W = 3$ as the $(c-x)^{B-1}$ term vanishes since $B - 1 = 0$.

Before proceeding to the next section, we comment on how a bid ceiling might actually worsen the inefficiency results just presented. The upper dotted curve in Figure 3 represents an unbounded equilibrium where bids of the highest cost firm must approach infinity for the auction to remain interim individually rational.[11] Therefore, if an equilibrium exists, it must consist of a binding bid ceiling and take the form of the kinked,

11. It is also worth noting that if we do consider unbounded bid functions, then the upper (dotted) curve is not the only equilibrium as infinitely many, with higher initial values, exist.

FIGURE 3
Solutions to Equation (4), $W = 7$, $N = 12$



solid function in Figure 4 where low-cost firms bid according to the monotonic increasing portion of the function and high-cost firms pool their bids at $\overline{b}$.

Even if the kinked curve in Figure 4 were an equilibrium, two facts are clear. First, by not allowing high-cost firms to bid as high as the unbounded equilibrium calls for, the bid ceiling leads to a negative expected profit and thus the bid ceiling (even though set above the highest possible cost draw) will discourage participation and potentially leave too few suppliers to fulfill demand—a quantity inefficiency. Second, those high-cost firms who find it individually rational to participate will pool their bids at $\overline{b}$ and thus Medicare will have to decide which of these suppliers to select as Medicare suppliers without knowing their costs—an allocation problem.

Unfortunately, the two potential inefficiencies above are best case scenarios. We have numerically confirmed that no equilibrium of the type in Figure 4 even exists in our examples if any bid ceiling is imposed. It follows that in our examples, the only equilibrium in the binding bids median-price auction with a bid ceiling calls for low-cost firms to use mixed strategies, high-cost firms either to pool bids at $\overline{b}$ or not participate at all, and inefficiency is rampant. This matches well with MPZ who find that many bids bump up against the bid ceiling when bids are binding, bids below the bid ceiling are nonmonotonic, and the auction is highly inefficient.

V. INCOMPLETE INFORMATION WITH NONBINDING BIDS

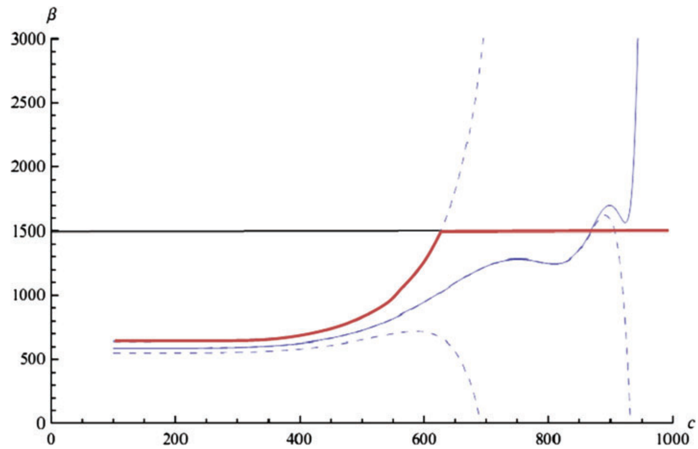
In this section, we show that a multitude of equilibria emerge in the median-price auction when bids are not binding. However, as we showed in the section on full information, in the absence of default costs, bidding below cost is not dominated when bids are not binding and these equilibria cannot be refined away. The result is that, absent explicit coordination, individual bidders may adopt strategies from any of these equilibria and the result of the auction is therefore highly unpredictable and likely inefficient.

We construct equilibria by considering situations where bidder i's opponents are using the strictly increasing bid function $\beta(c)$ (with inverse $\phi(b)$) if $c < c^*$ and are bidding $\beta(c^*)$ if $c \geq c^*$ where c^* is uniquely and implicitly defined by $\beta(c^*) = c^*$ for $c^* \in [L, H]$.[12] In essence, this means that firms with costs below c^* are bidding according to a strictly increasing bid function that eventually crosses the 45° line where $\beta(c^*) = c^*$ and that firms with costs higher than c^* all bid c^*.

If bidders are following the strategy described above, then the expected price will always be less than or equal to $\beta(c^*) = c^*$ and firms with $c \geq c^*$ earn zero expected payoff since they always

12. There may be more equilibria than those derived here. However, identifying them is unnecessary as the multiplicity that we identify is sufficient to conclude that the median-price auction performs poorly when bids are not binding.

FIGURE 4
Effect of a Binding Bid Ceiling



decline the contract to supply. These firms cannot profitably deviate by bidding higher than $\beta(c^*)$ since they would win with probability zero and thus still earn zero payoff. Similarly, they cannot profitably deviate by bidding less than $\beta(c^*)$ since the reimbursement price would be less than their cost if they won and hence they would always decline the contract and continue to earn zero payoff. Therefore, bidding $\beta(c^*)$ is equilibrium behavior for firms with $c \geq c^*$.

For firms with $c < c^*$, equilibrium bids are derived by examining the following maximization problem for $b < \beta(c^*)$:

$$(5)\quad \int_{\phi(b)}^{H} (\beta(x) - c) f_{(M-1:N-1)}(x)\,\mathrm{d}x + (b - c) \times \left[F_{(M-1:N-1)}(\phi(b)) - F_{(M:N-1)}(\phi(b))\right] + \int_{\max\{L,\phi(c)\}}^{\phi(b)} \int_{\phi(b)}^{H} (\beta(x) - c) \times f_{(M,W:N-1)}(x, y)\,\mathrm{d}y\mathrm{d}x.$$

This maximization problem is similar to the binding bids case except that in the last term x is only integrated over the interval $[\max[L, \phi(c)], \phi(b)]$ rather than $[L, \phi(b)]$. The change in the lower limit of integration is a consequence of subgame perfection which specifies that a firm will only accept the contract to supply if the auction price ends up being above his cost. Since the last term in Equation (5) is conditioned on the firm's bid being higher than the price-setting median bid, $\beta(x)$, the maximizer will only accept the contract if that price is greater than their cost, c.

Imposing symmetry, the first-order condition condition can be written as

$$(6)\quad \beta'(c)\left[F_{(M-1:N-1)}(c) - F_{(M:N-1)}(c)\right] = \int_{\max\{L,\phi(c)\}}^{c} (\beta(x) - c) f_{(M,W:N-1)}(x, c)\,\mathrm{d}x.$$

For firms with cost in the interval $[L, \beta(L)]$, Equation (6) is

$$(7)\quad \beta'(c)\left[F_{(M-1:N-1)}(c) - F_{(M:N-1)}(c)\right] = \int_{L}^{c} (\beta(x) - c) f_{(M,W:N-1)}(x, c)\,\mathrm{d}x,$$

which is exactly the same as Equation (2) that was derived above for the case of binding bids. This means that Part i of Theorem 1 applies here as well and the nonbinding bids equilibrium bid function must begin with slope of zero and the resulting solution will be unique for a given choice of $\beta(L)$.

Equilibrium bids for players with cost in the interval $[L, \beta(L)]$ are easily obtained using the appropriate power series solution (similar to those given in Examples 1 and 2). Equilibrium bids by players with costs in the interval $[\beta(L), H]$ are determined by

$$(8)\quad \beta'(c)\left[F_{(M-1:N-1)}(c) - F_{(M:N-1)}(c)\right] = \int_{\phi(c)}^{c} (\beta(x) - c) f_{(M,W:N-1)}(x, c)\,\mathrm{d}x.$$

However, Equation (8) cannot be solved analytically since it requires inverting the power series solution for bids on the $[L, \beta(L)]$ interval to obtain $\phi(c)$. Fortunately it is straightforward to numerically solve Equation (8) using a forward Euler method by numerically inverting β with Mathematica.[13]

Using the forward Euler method on Equation (8) proceeds as follows. First we obtain the power series solution to Equation (7) which gives all bids on the interval $[L, \beta(L)]$. Then, beginning with $c = \beta(L)$, we calculate $\beta(c + \delta)$ (where δ is the numerical step size) by computing $\beta(\beta(L))$ using the power series and then adding on the incremental change required by Equation (8). This incremental change is obtained by rearranging Equation (8) as

(9)
$$\beta'(c) = \int_{\phi(c)}^{c} (\beta(x) - c) f_{(M,W:N-1)}(x, c)\,\mathrm{d}x \Big/ \times \left[F_{(M-1:N-1)}(c) - F_{(M:N-1)}(c)\right]$$

and multiplying $\beta'(c)$ by δ. Thus, $\beta(c+\delta) = \beta(c) + \delta \cdot \beta'(c)$.

We applied this methodology in the setting studied by MPZ where $c \sim U[100, 1000]$ and $N = 16$ (the results are similar when using their assumption that $N = 12$). Figure 5 displays eight representative solutions to Equation (6) based on the initial values of $\beta(L)$ found in the first column of Table 1.

As in the binding bids model, there is a critical initial value $\beta(100) = b^* \approx 425.563$ in this case. The increasing solid curve in Figure 5 emanating from that initial value represents the only bounded equilibrium bid function that does not consist of any below-cost bids. The highest dashed curve is representative of all solutions to Equation (6) that start at some $\beta(100) > b^*$ as each is monotone increasing and diverges to positive infinity. The other dashed curves show equilibria with initial conditions where $\beta(100) < b^*$. Each of these dashed curves is strictly increasing until it hits the 45° line where $\beta(c) = c$ and is flat from there onward. Note that the slope of these dashed curves is zero at c^* as is required by Equation (9).

The second column of Table 1 lists the different values of c^* that the various curves obtain which provides insight into the relative slopes of the different equilibrium bid functions. When $\beta(100)$ is close to 100, even the increasing portions of these equilibrium bid functions are very flat. For instance, when the firm with $c = 100$ bids $\beta(100) = 200$, a firm with $c = 200.103$ bids only 0.103 more than the $c = 100$ firm. But, when $\beta(100) = 410$, the equilibrium function becomes much steeper with $\beta(c^*) = 508.6898$, nearly 100 units higher than $\beta(100)$.

TABLE 1
NonBinding Bids—Initial and Terminal Bids

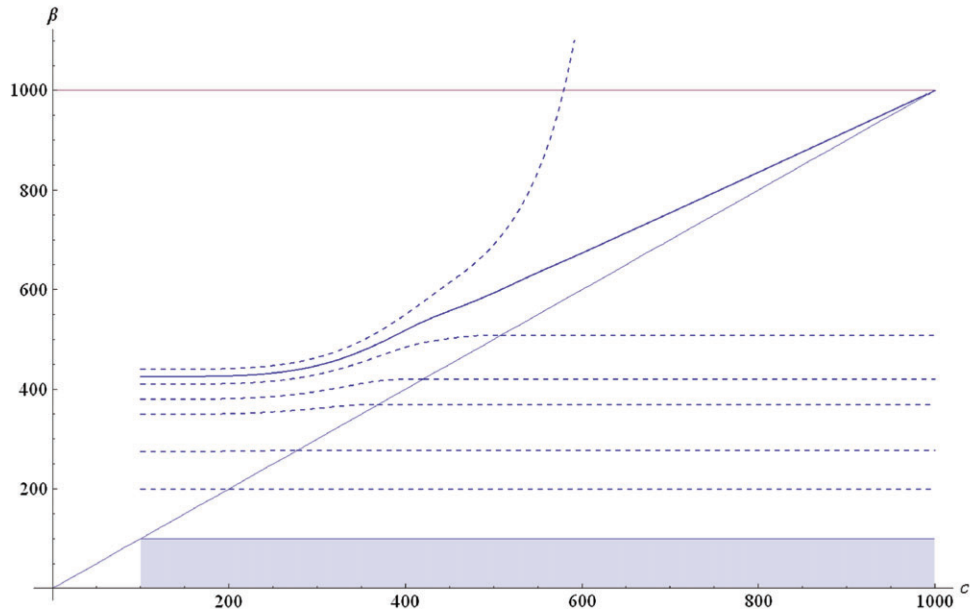
$\beta(L)$	$c^* = \beta(c^*)$
440.0	NA
425.56299908203551	1000.0
410.0	508.689782000833
380.0	420.737028973323
350.0	368.775566695461
275.0	277.219780623315
200.0	200.103249544203
100.0	100.0

The final equilibrium bid function shown in Figure 5 is the straight line at $\beta(c) = 100$. This is one of many "lowball" bid equilibria that exist. The idea is that if everyone else is bidding 100, then a firm cannot win by bidding more than 100 and thus bidding 100 and declining the contract is an equilibrium strategy for that firm. In fact, any situation where all bids are in the shaded region bounded by $\beta = 0$ and $\beta = 100$ constitutes an equilibrium for the same reason. There are many other equilibria as well, such as any situation where at least five firms bid less than 100 (no matter what the others bid) and, as discussed above, strategies in all of these equilibria are undominated when there are zero default costs. The only thing that mitigates how low bids can fall is the bid floor $\underline{b}$, which MPZ show was often binding at $\underline{b} = 50$.

It is worth noting that even if the only nonbinding bids equilibrium bid function were the solid, increasing curve in Figure 5, the median-price auction would still suffer allocation inefficiencies. Although the W lowest-cost firms would be selected as winners (since the equilibrium bid function is monotone increasing), the median winning bid sets a price such that some winning bidders decline the supply contract with positive probability. The existence of many other equilibria only compounds the case against the median-price auction with nonbinding bids. Coupled with our previous nonexistence results from the median-price binding bids model and the overwhelmingly consistent evidence from MPZ, the conclusion is clear: the median-price auction is inefficient.

13. Marshall et al. (1994) provide an excellent discussion of forward and backward Euler methods as they apply to auction problems.

FIGURE 5
NonBinding Bids Equilibrium Bid Functions



VI. CONCLUSION

Our analysis identifies two main types of inefficiencies generated by the median-price auction. By setting the auction price equal to the median winning bid, Medicare creates potential *quantity inefficiencies* as some winning bidders face a price less than their cost and therefore leave demand unfulfilled. Further, the incentives created by the median-pricing rule lead to nonexistence of equilibrium in many cases (especially when a bid ceiling is in place), thus creating *allocation inefficiencies* as high-cost firms sometimes displace low-cost firms as auction winners. These inefficiencies are unfortunate given that alternative auction formats such as the clearing-price auction have proven to perform well and are easily implemented.

The theoretical results that we present in this article are supported by the recent experimental findings of Merlob, Plott, and Zhang (2012). Their experiments show that the level of allocation and quantity inefficiencies we predict is significant in the median-price auction when bids are binding and that lowball bidding only worsens these inefficiencies when bids in their experiments are made nonbinding. Our model is easily adapted to allow for nonbinding bids and we are able to generate the lowball bid phenomenon theoretically.

It is worth noting that our theoretical model of the median-price auction, as well as the Merlob et al. experimental setting, is one where firms bid to supply a single unit of an item whereas the actual auctions involve firms bidding on multiple units of a variety of items. While a theoretical multiunit supply model of the median-price auction is analytically intractable, we feel that the failure of the median-price auction in our single unit supply model suggests that it will likely fail in more complex environments. But even under the extreme possibility that the median-price auction would perform better in a more complicated setting, why take the risk? Dynamic versions of the clearing-price auction prove highly efficient in complex theoretical and experimental settings and have been successfully implemented in the real-world.

We conclude on a positive note. Instituting auctions as a means of reducing Medicare costs was a wise move by Congress. Switching from the median-price auction to a more established procedure can eliminate the inefficiencies we have identified, guarantee health care to seniors and the disabled, and save taxpayers money. The clearing-price auction is a simple, fully efficient alternative that harnesses market forces by encouraging firms to bid their costs. Dynamic clock implementations of the clearing-price auction offer further benefits from price and assignment discovery, especially in the context of auctions for many products.

APPENDIX

Preliminaries for the Proof of Theorem 1

The operator D_c is defined on the set of all functions g that are continuously differentiable on (L, H) by $D_c g = g'/f$ where $g' = \mathrm{d}g/\mathrm{d}c$. For functions, g, that are k times continuously differentiable on (L, H), we define $D_c^k g$ to be the k-fold iterate of D_c applied to g. If p and q are functions that are continuous on $[L, H]$ with $p(L) = q(L)$ and $D_c p(c) = D_c q(c)$ for all $c \in (L, H)$, then it follows that $p(c) = q(c)$ for all $c \in [L, H]$. Furthermore, since $D_c p(c)/D_c q(c) = p'(c)/q'(c)$ (assuming that $q'(c) \neq 0$), we can use D_c in place of d/dc when working with L'Hopital's Rule (LR). We will do this frequently in giving the proof of Theorem 1.

For each $B \geq 1$, the function γ_B is defined on $[L, H]$ by

$$\gamma_B(c) = (B!/(2B)!)\left(D_c^B\left(cF(c)^{2B}\right)/F(c)^B\right), \quad c \in (L, H).$$

By the following proposition, γ_B can be extended continuously to the interval $[L, H]$ with $\gamma_B(L) = L$.

PROPOSITION 1. *For any* $B \geq 1$ *and* $0 \leq n \leq 2B$, *the limits of* $D_c^n\left(cF(c)^{2B}\right)$ *as* $c \to L^+$ *and* $c \to H^-$ *both exist and are finite. Furthermore,*

$$\lim_{c\to L^+} D_c^n\left(cF(c)^{2B}\right) = 0, \quad 0 \leq n \leq 2B-1,$$

$$\lim_{c\to L^+} D_c^n\left(cF(c)^{2B}\right)/F(c)^{2B-n} = ((2B)!/(2B-n)!)L, \quad 0 \leq n \leq 2B,$$

and

$$\lim_{c\to L^+} \gamma_B(c) = L.$$

Proof. The assertions of the proposition are clearly true for $n = 0$ so we assume that $1 \leq n \leq 2B$. By expanding $D_c^n\left(cF(c)^{2B}\right)$, we observe that

$$\text{(A1)} \quad D_c^n\left(cF(c)^{2B}\right) = ((2B)!/(2B-n)!)cF(c)^{2B-n} + \sum_{j=1}^{n}\binom{n}{j}((2B)!/(2B-n+j)!)D_c^j(c)\,F(c)^{2B-n+j}.$$

In addition, by expanding $D_c^j(c)$ we observe that $D_c(c) = 1/f(c)$ and $D_c^j(c) = \sum_{k=j+1}^{2j-1} r_{(j,k)}(c)/f(c)^k$, $j \geq 2$ where each function $r_{(j,k)}(c)$ is a sum of terms whose factors are derivatives of f. Since f is assumed to be positive-valued and to have derivatives of all orders throughout $[L, H]$, then it is clear that the limits of $D_c^j(c)$ as $c \to L^+$ and $c \to H^-$ both exist and are finite. Thus by Equation (A1), the limits of $D_c^n\left(cF(c)^{2B}\right)$ as $c \to L^+$ and $c \to H^-$ both exist and are finite. The remaining claims of the proposition then follow immediately from Equation (A1). ■

We now present two more propositions that will be used in the proof of Theorem 1.

PROPOSITION 2. *If* g *is a function such that* $D_c^k g(c) \to 0$ *as* $c \to L^+$ *for* $1 \leq k \leq n$, *then* $g^{(k)}(c) \to 0$ *as* $c \to L^+$ *for* $1 \leq k \leq n$.

Proof. The proof will proceed by induction on n. For $n = 1$, if $D_c g(c) \to 0$ as $c \to L^+$, then since $g'(c) = f(c)D_c g(c)$, it follows that $g'(c) \to 0$ as $c \to L^+$.

For any $n \geq 1$, expansion of $D_c^{n+1} g(c)$ yields

$$D_c^{n+1} g(c) = g^{(n+1)}(c)/f(c)^{n+1} + \sum_{j=n+2}^{2n+1} r_{(n+1,j)}(c)/f(c)^j$$

where each $r_{(n+1,j)}(c)$ is a sum of terms whose factors are derivatives of f and derivatives of g of order less than $n+1$. Furthermore, derivatives of g are present in each of these terms.

Now assume the proposition (the induction hypothesis) to hold for n and suppose that $D_c^k g(c) \to 0$ as $c \to L^+$ for $1 \leq k \leq n+1$. Then by the induction hypothesis we have $g^{(k)}(c) \to 0$ as $c \to L^+$ for $1 \leq k \leq n$ and hence

$$\lim_{c\to L^+} \sum_{j=n+2}^{2n+1} \frac{r_{(n+1,j)}(c)}{f(c)^j} = 0.$$

Since $D_c^{n+1} g(c) \to 0$ as $c \to L^+$, then $g^{(n+1)}(c) \to 0$ as $c \to L^+$ and the induction argument is complete. ■

PROPOSITION 3. *For any* $B \geq 1$ *and any* $c \in [L, H]$,

$$\int_L^c F(u)^B (F(c) - F(u))^{B-1} f(u)\,\mathrm{d}u = (B!(B-1)!/(2B)!)F(c)^{2B}.$$

Proof. The assertion of the proposition is clearly true when $B = 1$ so we assume $B > 1$. Let $p(c)$ and $q(c)$ be, respectively, the expressions on the left- and right-hand sides of the identity that is to be proved. Clearly $p(L) = q(L) = 0$. Also,

$$D_c p(c) = \int_L^c F(u)^B (B-1)(F(c) - F(u))^{B-2} f(u)\,\mathrm{d}u$$

and

$$D_c q(c) = (B!(B-1)!/(2B-1)!)F(c)^{2B-1}$$

and hence, $D_c p(L) = D_c q(L) = 0$. By continuing to apply D_c we find that $D_c^n p(L) = D_c^n q(L) = 0$ for $0 \leq n \leq B-1$ and $D_c^B p(c) = D_c^B q(c) = (B-1)!F(c)^B$ for all $c \in [L, H]$. Since $D_c^{B-1} p(L) = D_c^{B-1} q(L)$, then $D_c^{B-1} p(c) = D_c^{B-1} q(c)$ for all $c \in [L, H]$. By continuing this reasoning, we conclude that $p(c) = q(c)$ for all $c \in [L, H]$. ■

Proof of Theorem 1, Part i

By Proposition 3, Equation (2) can be written as

$$\text{(A2)} \qquad p(c)\,D_c\beta(c) = R(c)$$

where

$$K = A!/(N-1-B)!, \qquad \text{and}$$

$$p(c) = KF(c)^B(1 - F(c))^{B+1},$$

(A3) $$R(c) = (1/(B-1)!)\int_L^c F(u)^B (F(c) - F(u))^{B-1} \times \beta(u) f(u)\,\mathrm{d}u - (B!/(2B)!)cF(c)^{2B}.$$

Direct computation gives

$$D_c^n R(c) = (1/(B-1-n)!)\int_L^c F(u)^B \times (F(c)-F(u))^{B-n-1}\beta(u) f(u)\,\mathrm{d}u - (B!/(2B)!)D_c^n(cF(c)^{2B}), \quad 0 \le n \le B-1$$

and $D_c^B R(c) = F(c)^B (\beta(c) - \gamma_B(c))$.

Since $\lim_{c\to L^+} D_c^n (cF(c)^{2B}) = 0$ for $0 \le n \le 2B-1$ by Proposition 1, then $\lim_{c\to L^+} D_c^n R(c) = 0$ for $0 \le n \le B$ and we can apply L'Hopital's Rule (using the operator D_c in place of d/dc) to obtain

$$\begin{aligned}
&\lim_{c\to L^+}((2B)!R(c))/(B!F(c)^{2B}) \\
&= \lim_{c\to L^+}(2B-1)!D_c R(c)/(B!F(c)^{2B-1}) \\
&= \lim_{c\to L^+}(2B-2)!D_c^2 R(c)/(B!F(c)^{2B-2}) \\
&\vdots \\
&= \lim_{c\to L^+}(B+1)!D_c^{B-1}R(c)/(B!F(c)^{2B+1}) \\
&= \lim_{c\to L^+} D_c^B R(c)/F(c)^B \\
&= \lim_{c\to L^+}(\beta(c) - \gamma_B(c)) \\
&= \beta(L) - L
\end{aligned}$$

by Proposition 1 and the assumption that β is continuous at $c = L$. We conclude that

(A4) $$\lim_{c\to L^+} D_c^n R(c)/F(c)^{2B-n} = (B!/(2B-n)!)(\beta(L) - L) \quad \text{for } 0 \le n \le B.$$

In addition, expansion of $D_c^n p(c)$ gives

$$D_c^n p(c) = (KB!/(B-n)!)F(c)^{B-n} + K\sum_{j=1}^{B+1}(-1)^j \binom{B+1}{j}((B+j)!/(B+j-n)!)F(c)^{B+j-n}$$

for $0 \le n \le B$ and thus

(A5) $$\lim_{c\to L^+} D_c^n p(c)/F(c)^{B-n} = KB!/(B-n)! \quad \text{for } 0 \le n \le B.$$

We will now show by induction on n that if $B \ge n$ and $1 \le k \le n$, then

(A6) $$\lim_{c\to L^+} D_c^k \beta(c)/F(c)^{B-k+1} \text{ exists and is finite}$$

and

(A7) $$\beta^{(k)}(L) = \lim_{c\to L^+}\beta^{(k)}(c) = 0.$$

The base case in our inductive proof is $B \ge 1$ and $k = 1$. In this case, we divide both sides of Equation (A2) by $F(c)^{2B}$ to obtain

$$(p(c)/F(c)^B)(D_c\beta(c)/F(c)^B) = R(c)/F(c)^{2B}.$$

Since

$$\lim_{c\to L^+} p(c)/F(c)^B = K$$

by Equation (A5) and

$$\lim_{c\to L^+} R(c)/F(c)^{2B} = (B!/(2B)!)(\beta(L) - L)$$

by Equation (A4), then

$$\lim_{c\to L^+} D_c\beta(c)/F(c)^B = (B!/K(2B)!)(\beta(L) - L)$$

which establishes Equation (A6) in the case $n = 1$. Since the above limit is finite, we also have $D_c\beta(c) \to 0$ as $c \to L^+$ and hence $\beta'(c) \to 0$ as $c \to L^+$ by Proposition 2. In addition, since β is assumed to be continuous at $c = L$, then

$$\lim_{c\to L^+}(\beta(c) - \beta(L))/(c - L) = \lim_{c\to L^+}\beta'(c)/1 = 0$$

which shows both that $\beta'(L) = 0$ and that β' is continuous at $c = L$.

Now assume that the induction hypothesis $B \ge n$ and $1 \le k \le n \Rightarrow$ Equations (A6) and (A7) hold for n and suppose that $B \ge n+1$ and $1 \le k \le n+1$. Then both Equations (A6) and (A7) hold for $1 \le k \le n$ by the induction hypothesis. Since

$$p(c)D_c^{n+1}\beta(c) + \sum_{j=1}^{n}\binom{n}{j}D_c^j p(c)D_c^{n+1-j}\beta(c) = D_c^n R(c),$$

we have

$$\begin{aligned}
&(p(c)/F(c)^B)(D_c^{n+1}\beta(c)/F(c)^{B-n}) \\
&+ \sum_{j=1}^{n}\binom{n}{j}(D_c^j p(c)/F(c)^{B-j})(D_c^{n+1-j}\beta(c)/F(c)^{B-(n-j)}) \\
&= D_c^n R(c)/F(c)^{2B-n}.
\end{aligned}$$

Also, since

$$\lim_{c\to L^+} p(c)/F(c)^B = K,$$

$$\lim_{c\to L^+} D_c^n R(c)/F(c)^{2B-n} = (B!/(2B-n)!)(\beta(L) - L)$$

by Equation (A4),

$$\lim_{c\to L^+} D_c^j p(c)/F(c)^{B-j} = KB!/(B-j)!, \quad 1 \le j \le n$$

by Equation (A5), and

$$\lim_{c\to L^+} D_c^{n+1-j}\beta(c)/F(c)^{B-(n-j)} \text{ exists and is finite for } 1 \le j \le n$$

by the induction hypothesis, we obtain

$$\lim_{c\to L^+} D_c^{n+1}\beta(c)/F(c)^{B-n} \quad \text{exists and is finite.}$$

This shows that Equation (A6) holds for $n+1$ and also shows that $D_c^{n+1}\beta(c) \to 0$ as $c \to L^+$. We also have that $\beta^{(k)}(L) = \lim_{c\to L^+}\beta^{(k)}(c) = 0$ for $1 \le k \le n$ by the induction hypothesis. By Proposition 2, we conclude that $\beta^{(k)}(c) \to 0$ as $c \to L^+$ for $1 \le k \le n+1$. Finally,

$$\lim_{c\to L^+}(\beta^{(n)}(c) - \beta^{(n)}(L))/(c - L) = \lim_{c\to L^+}\beta^{(n+1)}(c)/1 = 0$$

shows both that $\beta^{(n+1)}(L) = 0$ and that $\beta^{(n+1)}$ is continuous at $c = L$.

Proof of Theorem 1, Part ii

To prove Part ii of Theorem 1, we first observe that because β is continuous at $c = H$ we have

$$\begin{aligned}\infty &> \beta(H)\\ &= \lim_{c\to H^-} \beta(c)\\ &= \lim_{c\to H^-} (1 - F(c))\,\beta(c)/(1 - F(c))\\ &= \lim_{c\to H^-} (1 - F(c))\,(D_c\beta(c) - \beta(c))/(-1)\\ &= \lim_{c\to H^-} \big(\beta(c) - (1 - F(c))\,D_c\beta(c)\big)\end{aligned}$$

if this limit exists. Clearly it cannot be the case that $\lim_{c\to H^-} (1 - F(c))\,D_c\beta(c)$ is ∞ or any finite number other than zero because this would contradict L'Hopital's Rule. Therefore, either $\lim_{c\to H^-} (1 - F(c))\,D_c\beta(c) = 0$ or this limit does not exist. To determine which is the case, we use Equation (A2) to obtain

$$\lim_{c\to H^-} KF(c)^B (1 - F(c))\,D_c\beta(c) = \lim_{c\to H^-} R(c)\,/\,(1 - F(c))^B.$$

Since $\lim_{c\to H^-} R(c)$ exists and is finite by Proposition 1 and the assumption that β is continuous throughout $[L, H]$, then the limit on the right of the above equation must be equal to zero (for otherwise it would be ∞ which would contradict what was stated above). This implies that both $R(c) \to 0$ and $(1 - F(c))D_c\beta(c) \to 0$ as $c \to H^-$. Hence we can apply L'Hopital's Rule to obtain

$$\begin{aligned}0 &= \lim_{c\to H^-} KF(c)^B (1 - F(c))\,D_c\beta(c)\\ &= \lim_{c\to H^-} R(c)\,/\,(1 - F(c))^B\\ &= \lim_{c\to H^-} D_cR(c)\,/\big(-B\,(1 - F(c))^{B-1}\big)\end{aligned}$$

if the latter limit exists. However, $\lim_{c\to H^-} D_cR(c)$ exists and is finite (by Proposition 1 and the assumption that β is bounded throughout $[L, H]$) and hence it must be the case that $\lim_{c\to H^-} D_cR(c) = 0$ so as not to contradict L'Hopital's Rule. By continuing along this line of reasoning, we obtain

$$\begin{aligned}0 &= \lim_{c\to H^-} KF(c)^B (1 - F(c))\,D_c\beta(c)\\ &= \lim_{c\to H^-} R(c)\,/\,(1 - F(c))^B\\ &\;\;\vdots\\ &= \lim_{c\to H^-} (-1)^B D_c^B R(c)/B!\\ &= \lim_{c\to H^-} F(u)^B \big(\beta(u) - \gamma_B(u)\big)/B!\\ &= \big(\beta(H) - \gamma_B(H)\big)/B!\end{aligned}$$

and conclude that $\beta(H) = \gamma_B(H)$.

It is also easily seen by direct computation that $\gamma_B(H) = H + ((W - 1)/(W + 1))(H - L)$ when F is the uniform distribution.

Proof of Theorem 1, Part iii

If β is a bounded and monotone increasing equilibrium for the median-price auction, then a firm whose cost is c and who bids $\beta(c)$ has expected payoff

$$\begin{aligned}\pi(c) = P(c, \beta(c)) &= \int_c^H f_{(B:N-1)}(u)\,(\beta(u) - c)\,du\\ &+ \binom{N-1}{B} F(c)^B (1 - F(c))^{N-1-B} (\beta(c) - c)\\ &+ \int_L^c \int_c^H ((N - 1)!/(B!\,(B - 1)!A!))F(u)^B\\ &\times (F(y) - F(u))^{B-1} (1 - F(y))^A (\beta(u) - c) f(u) f(y)\,dy\,du\end{aligned}$$

By differentiating and using the first-order condition for equilibrium, Equation (2), we obtain

$$\begin{aligned}\pi'(c) = &-\int_c^H f_{(B:N-1)}(u)\,du\\ &- \binom{N-1}{B} F(c)^B (1 - F(c))^{N-1-B}\\ &- \int_L^c \int_c^H ((N - 1)!/(B!\,(B - 1)!A!))F(u)^B\\ &(F(y) - F(u))^{B-1} (1 - F(y))^A f(u) f(y)\,dy\,du\end{aligned}$$

which shows that $\pi'(L) = -1$ and $\pi'(H) = 0$. By differentiating again we obtain $\pi''(c) = f_{(W:N-1)}(c)$.

These observations yield the following lemma, which also establishes the first assertion of Part iv of Thereom 1.

LEMMA 1. *If β is a bounded and monotone increasing equilibrium for the median-price auction and all bidders bid according to β, then the expected profit for a bidder of cost* c ∈ *[*L, H*] is*

$$\pi(c) = \int_c^H \big(1 - F_{(W:N-1)}(u)\big)\,du.$$

Proof. We have shown above that $\pi''(c) = f_{(W:N-1)}(c)$ and that $\pi'(L) = -1$. This implies that $\pi'(c) + 1 = \int_L^c f_{(W:N-1)}(u)\,du = F_{(W:N-1)}(c)$. In addition, since $\pi(H) = 0$ we obtain $(0 - \pi(c)) + (H - c) = \int_c^H F_{(W:N-1)}(u)\,du$ or $\pi(c) = \int_c^H \big(1 - F_{(W:N-1)}(u)\big)\,du$. ■

COROLLARY 1. *If β is an equilibrium for the median-price auction, then the expected profit of the lowest cost firm satisfies* $f_{min}\pi(L) \le W/N \le f_{max}\pi(L)$. *Hence, in the case of the uniform distribution* ($f(c) \equiv 1/(H - L)$)*, we have* $\pi(L) = (W/N)(H - L)$.

Proof. By Lemma 1 we have $\pi(L) = \int_L^H \big(1 - F_{(W:N-1)}(u)\big)du$ and since the integrand is positive we obtain

$$\begin{aligned}f_{\min}\pi(L) &\le \int_L^H \big(1 - F_{(W:N-1)}(u)\big) f(u)\,du\\ &= 1 - \int_L^H F_{(W:N-1)}(u) f(u)\,du.\end{aligned}$$

Also, since (in general)

$$F_{(k,n)}(u)f(u) = \sum_{j=0}^{n-k} \binom{n}{j} (1 - F(u))^j F(u)^{n-j} f(u)$$

$$= (1/(n+1)) \sum_{j=0}^{n-k} f_{(n-j+1:n+1)}(u),$$

then

$$F_{(W:N-1)}(u)f(u) = (1/N) \sum_{j=0}^{N-W-1} f_{(N-j:N)}(u)$$

and we obtain

$$\int_L^H F_{(W:N-1)}(u)f(u)\,\mathrm{d}u = \frac{1}{N} \sum_{j=0}^{N-W-1} F_{(N-j:N)}(H)$$

$$= (N - W)/N.$$

Hence, $f_{\min}\pi(L) \le 1 - (N - W)/N = W/N$. The proof of the second assertion of the corollary is similar. ∎

We now give the proof of Part iii of Theorem 1. First, to show that $\beta(c) > c$ for all $c \in (L, H]$ we let $c \in (L, H]$ be arbitrary and refer to Equation (2). Since $\beta'(c) > 0$, the integral on the right of Equation (2) is positive and hence there must exist some point $u^* \in [L, c)$ such that $\beta(u^*) - c > 0$. Since $c > u^*$ and β is monotone increasing on $[u^*, c]$, we thus have that $\beta(c) > c$. Since $G(c) = \beta(c) - c > 0$ for all $c \in (L, H]$ and $G'(L) = \beta'(L) - 1 = -1$ by Part i of Theorem 1, then it cannot be the case that $G(L) = \beta(L) - L = 0$ because this would contradict the fact that $G(c) > 0$ throughout $(L, H]$. Therefore $\beta(L) > L$.

To complete the proof, we use the assumption that β is monotone increasing throughout $[L, H]$ and Corollary 1 to obtain

$$\beta(L) - L = \int_L^H f_{(B:N-1)}(u)(\beta(L) - L)\,\mathrm{d}u$$

$$< \int_L^H f_{(B:N-1)}(u)(\beta(u) - L)\,\mathrm{d}u$$

$$= \pi(L) \le W/(f_{\min}N)$$

which shows that $B(L) < L + W/(f_{\min}N)$.

Proof of Theorem 1, Part iv

Lemma 1 establishes that $\pi(c) > 0$ for all $c \in [L, H)$ (and that $\pi(H) = 0$). The second assertion in Part iv is verified by noting that

$$\Pr[\beta(c_{(M:N)}) < c_{(M+1:N)}]$$

$$= \int_L^{\phi(H)} \int_{\beta(x)}^H f_{(M,M+1:N)}(x, y)\,\mathrm{d}y\mathrm{d}x$$

$$= \int_L^{\phi(H)} (N!/(M-1)!(N-M)!) F(x)^{M-1}$$

$$\times (1 - F(\beta(x)))^{N-M} f(x)\,\mathrm{d}x > 0.$$

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