Dynamics of a ring network of phase-only oscillators

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ARTICLE INFO

ABSTRACT

We investigate the existence, stability and bifurcation of phase-locked motions in a ring network consisting of phase-only oscillators arranged in multiple simple rings (*sub-rings*) which are themselves arranged in a single large ring. In the case of networks with three or four sub-rings, we give approximate expressions for critical coupling coefficients which must be exceeded in order for phase-locking to occur.

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1. Introduction

Recent research has dealt with the dynamics of a simple ring of oscillators, variously modeled as phase-only oscillators, and as van der Pol oscillators in both sinusoidal and relaxation limits [1,5,3,8,7,4,2]. In this work we consider the dynamics of a system composed of multiple simple rings of phase-only oscillators which themselves are arranged in a single large ring, resulting in a structure which we refer to as a *ring network*, see Fig. 1. Each individual ring, called a *sub-ring*, is composed of a number of identical phase-only oscillators which are nearest-neighbor coupled by sine functions of phase differences, all with identical coupling coefficients. The ith sub-ring is composed of n_i oscillators with uncoupled frequencies w_i and coupling coefficients α_i. We consider a system composed of m sub-rings which are connected by a larger ring called the *communication ring*. One oscillator on each sub-ring is identified as the *communication oscillator*, and it is these m communication oscillators which compose the communication ring, which has coupling coefficient α.

We label the phase of the n_i oscillators which comprise the ith sub-ring by $\theta_{i,j}$, where $j = 1,\ldots,n_i$. In particular the communication oscillator is labelled $\theta_{i,1}$. Using this notation the equations of motion become:

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta_{ij} = w_i + \alpha_i(\sin(\theta_{i,j+1} - \theta_{ij}) + \sin(\theta_{i,j-1} - \theta_{ij})) + \delta_{j,1}\alpha(\sin(\theta_{i+1,1} - \theta_{i,1}) + \sin(\theta_{i-1,1} - \theta_{i,1})),$$
$$\text{where } i = 1,\ldots,m, \; j = 1,\ldots,n_i \tag{1}$$

where $\delta_{j,i}$ is the Kronecker delta, and where we use the convention that

$$\theta_{i,n_i+1} = \theta_{i,1}, \theta_{i,0} = \theta_{i,n_i} \quad \text{and} \quad \theta_{m+1,1} = \theta_{1,1}, \quad \theta_{0,1} = \theta_{m,1} \tag{2}$$

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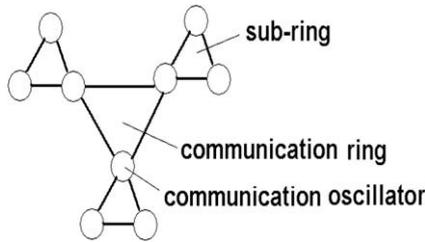
Fig. 1. A ring network.

It is convenient to define the phase differences

$$\phi_{i,j} = \theta_{i,j+1} - \theta_{i,j} \tag{3}$$

$$\psi_i = \theta_{i+1,1} - \theta_{i,1} \tag{4}$$

where $\phi_{i,0} = \theta_{i,1} - \theta_{i,0} = \theta_{i,n_i+1} - \theta_{i,n_i} = \phi_{i,n_i}$ and $\psi_0 = \theta_{1,1} - \theta_{0,1} = \theta_{m+1,1} - \theta_{m,1} = \psi_m$

whereupon the equations of motion become:

$$\frac{d}{dt}\theta_{i,j} = w_i + \alpha_i(\sin\phi_{i,j} - \sin\phi_{i,j-1}) + \delta_{j,1}\alpha(\sin\psi_i - \sin\psi_{i-1}), \quad \text{where } i = 1,\ldots,m, \; j = 1,\ldots,n_i \tag{5}$$

For example, in the case of a network consisting of three sub-rings, each of size 3, we have the following nine equations of motion (Fig. 2):

$$\frac{d}{dt}\theta_{1,1} = w_1 + \alpha_1(\sin\phi_{1,1} - \sin\phi_{1,3}) + \alpha(\sin\psi_1 - \sin\psi_3), \tag{6}$$

$$\frac{d}{dt}\theta_{1,2} = w_1 + \alpha_1(\sin\phi_{1,2} - \sin\phi_{1,1}), \tag{7}$$

$$\frac{d}{dt}\theta_{1,3} = w_1 + \alpha_1(\sin\phi_{1,3} - \sin\phi_{1,2}), \tag{8}$$

$$\frac{d}{dt}\theta_{2,1} = w_2 + \alpha_2(\sin\phi_{2,1} - \sin\phi_{2,3}) + \alpha(\sin\psi_2 - \sin\psi_1), \tag{9}$$

$$\frac{d}{dt}\theta_{2,2} = w_2 + \alpha_2(\sin\phi_{2,2} - \sin\phi_{2,1}), \tag{10}$$

$$\frac{d}{dt}\theta_{2,3} = w_2 + \alpha_2(\sin\phi_{2,3} - \sin\phi_{2,2}), \tag{11}$$

$$\frac{d}{dt}\theta_{3,1} = w_3 + \alpha_3(\sin\phi_{3,1} - \sin\phi_{3,3}) + \alpha(\sin\psi_3 - \sin\psi_2), \tag{12}$$

$$\frac{d}{dt}\theta_{3,2} = w_3 + \alpha_3(\sin\phi_{3,2} - \sin\phi_{3,1}), \tag{13}$$

$$\frac{d}{dt}\theta_{3,3} = w_3 + \alpha_3(\sin\phi_{3,3} - \sin\phi_{3,2}) \tag{14}$$
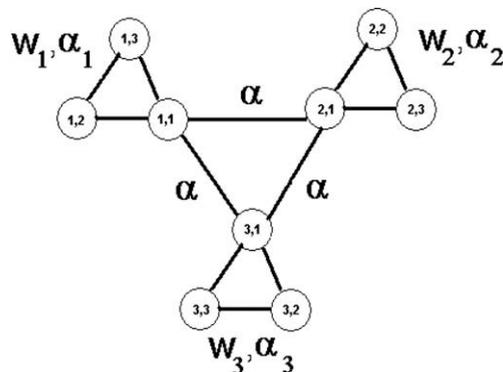
Fig. 2. A network consisting of three sub-rings, each of size 3.

2. Phase-locked motions

The dynamics of a system like that in Fig. 2 may be expected to be complicated, exhibiting various steady state modes of behavior in response to parameter changes. Amongst all such steady state motions perhaps the simplest is phase-locked motion in which each of the oscillators has the same constant frequency Ω:

$$\theta_{i,j} = \Omega t + k_{i,j} \tag{15}$$

where $k_{i,j}$ is a phase angle, constant in time. Substituting (15) into (5) we get

$$\Omega = w_i + \alpha_i(\sin\phi_{i,j} - \sin\phi_{i,j-1}) + \delta_{j,1}\alpha(\sin\psi_i - \sin\psi_{i-1}), \quad \text{where } i = 1,\ldots, m, \; j = 1,\ldots, n_i \tag{16}$$

where now

$$\phi_{i,j} = \theta_{i,j+1} - \theta_{i,j} = k_{i,j+1} - k_{i,j} \tag{17}$$
$$\psi_i = \theta_{i+1,1} - \theta_{i,1} = k_{i+1,1} - k_{i,1} \tag{18}$$

Thus $\phi_{i,j}$ and ψ_i are constants. Examination of the example of Fig. 2, Eqs. (6)–(14), shows that by adding up all the equations we obtain (in the general case)

$$N\Omega = \sum n_i w_i \tag{19}$$

where $N = \sum n_i$ is the total number of oscillators. This yields the interesting result that the phase-locked frequency Ω equals the average frequency of all N oscillators.

A natural question to ask is what are the conditions on the parameters of the system such that phase-locked motion is possible?

As an example we take the three sub-ring network of Fig. 2, Eqs. (6)–(14).

$$\Omega = w_1 + \alpha_1(\sin\phi_{1,1} - \sin\phi_{1,3}) + \alpha(\sin\psi_1 - \sin\psi_3), \tag{20}$$
$$\Omega = w_1 + \alpha_1(\sin\phi_{1,2} - \sin\phi_{1,1}), \tag{21}$$
$$\Omega = w_1 + \alpha_1(\sin\phi_{1,3} - \sin\phi_{1,2}), \tag{22}$$
$$\Omega = w_2 + \alpha_2(\sin\phi_{2,1} - \sin\phi_{2,3}) + \alpha(\sin\psi_2 - \sin\psi_1), \tag{23}$$
$$\Omega = w_2 + \alpha_2(\sin\phi_{2,2} - \sin\phi_{2,1}), \tag{24}$$
$$\Omega = w_2 + \alpha_2(\sin\phi_{2,3} - \sin\phi_{2,2}), \tag{25}$$
$$\Omega = w_3 + \alpha_3(\sin\phi_{3,1} - \sin\phi_{3,3}) + \alpha(\sin\psi_3 - \sin\psi_2), \tag{26}$$
$$\Omega = w_3 + \alpha_3(\sin\phi_{3,2} - \sin\phi_{3,1}), \tag{27}$$
$$\Omega = w_3 + \alpha_3(\sin\phi_{3,3} - \sin\phi_{3,2}) \tag{28}$$

Adding the first three, the second three and third three of these equations, we obtain

$$w_1 + w_2 + w_3 = 3w_1 + \alpha(\sin\psi_1 - \sin\psi_3), \tag{29}$$
$$w_1 + w_2 + w_3 = 3w_2 + \alpha(\sin\psi_2 - \sin\psi_1), \tag{30}$$
$$w_1 + w_2 + w_3 = 3w_3 + \alpha(\sin\psi_3 - \sin\psi_2) \tag{31}$$

where we have used $3\Omega=(w_1 + w_2 + w_3)$ from Eq. (19). We may think of Eqs. (29)–(31) as replacing Eqs. (20),(23) and (26), respectively. Note that (31) is equivalent to the sum of (29) and (30). Thus we have:

$$w_1 + w_2 + w_3 = 3w_1 + \alpha(\sin\psi_1 - \sin(-\psi_2 - \psi_1)), \tag{32}$$
$$w_1 + w_2 + w_3 = 3w_2 + \alpha(\sin\psi_2 - \sin\psi_1) \tag{33}$$

where we have used $\psi_3 = -\psi_2 - \psi_1$. Trig-expanding, we obtain

$$-2w_1 + w_2 + w_3 = \alpha(\sin\psi_1 + \sin\psi_2\cos\psi_1 + \cos\psi_2\sin\psi_1), \tag{34}$$
$$w_1 - 2w_2 + w_3 = \alpha(\sin\psi_2 - \sin\psi_1) \tag{35}$$

For given values of w_1, w_2, w_3 and α, we may or may not be able to solve these equations for ψ_1 and ψ_2. In order to find conditions on the parameters such that these equations are solvable, i.e., for a phase-locked solution to exist, we need to do some algebraic manipulations which we describe next. We used the computer algebra system Macsyma to accomplish these calculations. First we solve Eq. (34) for $\cos\psi_2$, square the result and use the identity $(\cos\psi_2)^2 = 1-(\sin\psi_2)^2$ to eliminate $\cos\psi_2$. Call the result equation A. Next we solve (35) for $\sin\psi_2$ and substitute it into equation A. The resulting equation is free of ψ_2 but contains ψ_1 in the form $\sin\psi_1$ and $\cos\psi_1$. Using the identity $(\cos\psi_1)^2 = 1-(\sin\psi_1)^2$, we eliminate all powers of $\cos\psi_1$ except the first. Then we solve for $\cos\psi_1$, square the result, and use the foregoing identity to eliminate $\cos\psi_1$. Call the result equation B, which turns out to be a 6th degree polynomial on $\sin\psi_1$. The general form of the equation is listed in the Appendix. We continue with a specific example defined by the frequencies:

$$w_1 = 1, \quad w_2 = 2.5, \quad w_3 = 3.1 \tag{36}$$

For these values, equation B takes the form:

$$-4\alpha^4 s^6 + 36\alpha^3 s^5 + 3\alpha^4 s^4 - 106.92\alpha^2 s^4 - 18\alpha^3 s^3 + 116.64\alpha s^3 - 1.62\alpha^2 s^2 - 41.9904 s^2 + 131.22\alpha s - 147.622 = 0 \tag{37}$$

where we have used the abbreviation $s = \sin\psi_1$. For a given value of α, Eq. (37) may or may not have a solution for ψ_1. By inspection, there is no solution for $\alpha = 0$, and by continuity there is no solution for sufficiently small α. The critical case will correspond to a double root, which may be computed by requiring both Eq. (37) and its derivative with respect to s to simultaneously vanish. Eliminating s between these two equations gives the following equation on α_{cr}:

$$\alpha_{cr}^{12} - 112.32\alpha_{cr}^{10} + 3145.73\alpha_{cr}^8 - 19851.3\alpha_{cr}^6 + 90477.9\alpha_{cr}^4 - 188704.9\alpha_{cr}^2 - 3470.49 = 0 \tag{38}$$

The smallest real positive root of Eq. (38) is

$$\alpha_{cr} = 1.8579 \tag{39}$$

When this value of α is substituted back into Eq. (37) we find the double root occurs at approximately $s = \sin\psi_1 = 0.96$. This observation suggests a method for efficiently obtaining an approximation for α_{cr}. Returning to Eqs. (34) and (35), we assert that the critical case will occur when $|\sin\psi_1| \approx 1$. So we set $\sin\psi_1 = 1$, in which case $\cos\psi_1 = 0$. Eqs. (34) and (35) become:

$$-2w_1 + w_2 + w_3 = \alpha(1 + \cos\psi_2), \tag{40}$$
$$w_1 - 2w_2 + w_3 = \alpha(\sin\psi_2 - 1) \tag{41}$$

For given values of w_1, w_2, w_3, we may obtain an expression for α_{cr}, the critical value of α, by eliminating ψ_2 from these equations via the identity $\sin^2 + \cos^2 = 1$. This results in a quadratic equation on α_{cr}:

$$\alpha_{cr}^2 + 6(w_1 - w_2)\alpha_{cr} + 2w_3^2 - 2w_2 w_3 - 2w_1 w_3 + 5w_2^2 - 8w_1 w_2 + 5w_1^2 = 0 \tag{42}$$

which gives

$$\alpha_{cr} = \pm\sqrt{2}\sqrt{-(w_3 - 2w_2 + w_1)(w_3 + w_2 - 2w_1)} + 3(w_2 - w_1) \tag{43}$$

In a similar way, the case for which $\sin\psi_1 = -1$ gives the following candidates for α_{cr}:

$$\alpha_{cr} = \pm\sqrt{2}\sqrt{-(w_3 - 2w_2 + w_1)(w_3 + w_2 - 2w_1)} - 3(w_2 - w_1) \tag{44}$$

Eqs. (43) and (44) may be written in the following form:

$$\alpha_{cr} = \pm\sqrt{2}\sqrt{-(3\Omega - 3w_2)(3\Omega - 3w_1)} \pm 3(w_2 - w_1) \tag{45}$$

where the two $\pm$ signs are to be taken independently, giving four candidate values. Note that Eqs. (29)–(31) are invariant under permutation of subscripts. Thus we may obtain eight more candidate values for α_{cr} by replacing the subscripts 1,2 in Eq. (45) by 2,3, and again by 3,1. Of these 12 candidate values for α_{cr}, our approximate method says that the value for α_{cr} is the minimum of those values which are both real and positive. (Note that the cases for which $|\sin\psi_2| = 1$ turn out to be the same as that obtained by replacing subscripts 1,2 in (45) by 2,3.)

Returning to the example of Eq. (36), we find $\Omega = (w_1 + w_2 + w_3)/3 = 2.2$. Eq. (45) gives the following four candidate values for α_{cr}:

$$\pm\frac{9}{5}\sqrt{2} \pm \frac{9}{2} = [1.95442, 7.04558, -1.95442, -7.04558] \tag{46}$$

Permuting subscripts gives eight additional candidate values:

$$\pm\frac{9}{5}\sqrt{6} \pm \frac{63}{10} = [1.89092, 10.7091, -1.89092, -10.7091] \tag{47}$$
$$\pm\frac{9}{10}\sqrt{6}i \pm \frac{9}{5} = \pm 2.20454i \pm 1.8 \tag{48}$$

Choosing the minimum of those real and positive values given in Eqs. (46)–(48), we see that

$$\alpha_{cr} = -\frac{9}{5}\sqrt{6} + \frac{63}{10} = 1.89092 \tag{49}$$

which is close to the exact value (39).

So far we have obtained a method for determining α_{cr}. Now we present a similar method for determining the critical values of the other coupling coefficients, $\alpha_{i_{cr}}$. In order to obtain $\alpha_{1_{cr}}$, we rewrite Eqs. (21) and (22) in the form

$$\Omega = w_1 + \alpha_1(\sin\phi_{1,2} - \sin\phi_{1,1}), \tag{50}$$
$$\Omega = w_1 + \alpha_1(\sin(-\phi_{1,2} - \phi_{1,1}) - \sin\phi_{1,2}) \tag{51}$$

where we have used $\phi_{1,3} = -\phi_{1,2} - \phi_{1,1}$. Trig-expanding, we obtain

$$\Omega = w_1 + \alpha_1(\sin\phi_{1,2} - \sin\phi_{1,1}), \tag{52}$$

$$\Omega = w_1 + \alpha_1(-\sin\phi_{1,2}\cos\phi_{1,1} - \cos\phi_{1,2}\sin\phi_{1,1} - \sin\phi_{1,2}) \tag{53}$$

For given values of Ω, w_1 and α_1, we may or may not be able to solve these equations for $\phi_{1,1}$ and $\phi_{1,2}$. Note that Eqs. (52) and (53) are similar in form to Eqs. (34) and (35), except here the left hand sides are the same in both equations, whereas they are in general different in Eqs. (34) and (35). We solve Eqs. (52) and (53) for $\sin\phi_{1,2}$ and $\cos\phi_{1,2}$, and use the identity $\sin^2 + \cos^2 = 1$ and some trig-simplification to obtain the following equation on $\phi_{1,1}$:

$$(\sin\phi_{1,1} + k)(2\cos\phi_{1,1}\sin\phi_{1,1} + \sin\phi_{1,1} + 4k\cos\phi_{1,1} + 5k) = 0 \tag{54}$$

where we have used k to abbreviate $k=(\Omega - w_1)/\alpha_1$. The first factor has the root

$$\sin\phi_{1,1} = -k = -\frac{\Omega - w_1}{\alpha_1} \tag{55}$$

This may or may not yield a value for $\phi_{1,1}$. The critical case corresponds to $|\sin\phi_{1,1}| = 1$ which generates the following candidates for $\alpha_{1_{cr}}$: $\pm(\Omega - w_1)$.

Now let us consider the second factor in Eq. (54). Solving this for $\cos\phi_{1,1}$ and using the identity $\sin^2 + \cos^2 = 1$ gives the following polynomial on $\sin\phi_{1,1}$ which we abbreviate as s:

$$4s^4 + 16ks^3 + (16k^2 - 3)s^2 - 6ks + 9k^2 = 0 \tag{56}$$

In order to have a real solution $\phi_{1,1}$ of Eq. (56), the root $s = \sin\phi_{1,1}$ must be (a) real and (b) its absolute value must be less than unity. The critical case for (a) is that (56) have a double root, while the critical case for (b) is that $s = 1$. For a double root we differentiate (56) with respect to s and eliminate s, giving a polynomial on k:

$$k^2(4k - 1)^2(4k + 1)^2(k^2 + 3) = 0 \tag{57}$$

which has the real solution $k = \pm 1/4$, giving the following candidates for $\alpha_{1_{cr}}$: $\pm 4(\Omega - w_1)$.

For (b) the critical case corresponds to $s = \pm 1$, which together with (56) gives $k = \pm 1/5$, giving the following candidates for $\alpha_{1_{cr}}$: $\pm 5(\Omega - w_1)$.

Thus Eq. (54) has yielded six candidates for $\alpha_{1_{cr}}$, namely $\pm(\Omega - w_1)$, $\pm 4(\Omega - w_1)$ and $\pm 5(\Omega - w_1)$. Of these six candidate values, the true value of $\alpha_{1_{cr}}$ is the minimum of those values which are both real and positive. Thus we conclude that

$$\alpha_{1_{cr}} = |\Omega - w_1| \tag{58}$$

Similarly,

$$\alpha_{2_{cr}} = |\Omega - w_2|, \quad \alpha_{3_{cr}} = |\Omega - w_3| \tag{59}$$

Continuing with the previous example, we again suppose that $w_1 = 1$, $w_2 = 2.5$ and $w_3 = 3.1$, giving $\Omega = 2.2$. We previously found that (see Eq. (49)) $\alpha_{cr} = 1.89092$. We now obtain

$$\alpha_{1_{cr}} = 1.2, \quad \alpha_{2_{cr}} = 0.3, \quad \alpha_{3_{cr}} = 0.9 \tag{60}$$

Numerically simulating the network in Fig. 2 for $w_1 = 1$, $w_2 = 2.5$ and $w_3 = 3.1$ by numerically integrating Eqs. (6)–(14), we find that phase-locking occurs as long as each of α, α_1, α_2, α_3 is larger than the respective critical value obtained above.

3. Generalization

The formulas for α_{cr}, $\alpha_{1_{cr}}$, $\alpha_{2_{cr}}$, $\alpha_{3_{cr}}$ derived above apply to the example of Fig. 2 which consists of $m=3$ sub-rings, each having $n_1 = n_2 = n_3 = 3$ oscillators. These formulas have been extended to a general system with $m = 3$ sub-rings, each sub-ring with n_i oscillators. We find that for each sub-ring,

$$\alpha_{i_{cr}} = \frac{n_i - 1}{2}|\Omega - w_i|, \quad i = 1, 2, 3 \tag{61}$$

And for the communication ring, an approximate expression for α_{cr} is obtained by taking the minimum of those values which are both real and positive from amongst the following 12:

$$\pm\sqrt{-2C_iC_j} \pm (C_i - C_j), (i,j) = (1,2), (2,3), (3,1) \tag{62}$$

where the two $\pm$ signs are to be taken independently, giving four candidate values, and where

$$C_i = n_i(\Omega - w_i) \tag{63}$$

As an example, we take a system in which $n_1 = 3$, $n_2 = 4$ and $n_3 = 5$, with frequencies $w_1 = 1$, $w_2 = 2.5$ and $w_3 = 3.1$. From Eq. (19) we obtain the phase-locked frequency

$$\Omega = (3w_1 + 4w_2 + 5w_3)/12 = 2.375 \tag{64}$$

Eq. (61) then gives us the following values for the critical coupling coefficients of the sub-rings:

$$\alpha_{1_{cr}} = 1.375, \quad \alpha_{2_{cr}} = 0.1875, \quad \alpha_{3_{cr}} = 1.45 \tag{65}$$

To obtain an approximate value for the critical coupling coefficient of the communication ring, α_{cr}, we start by computing the C_i's from Eq. (63):

$$C_1 = 4.125, \quad C_2 = -0.5, \quad C_3 = -3.625 \tag{66}$$

Then from Eq. (62), α_{cr} is the minimum of those values which are both real and positive from amongst the following list:

$$[\pm 6.65602, \pm 2.59399, \pm 1.90395i \pm 3.125, \pm 2.28134, \pm 13.2187] \tag{67}$$

Therefore we obtain

$$\alpha_{cr} = 2.28134 \tag{68}$$

Numerical simulation of the 12 ODE's which describe the system of Fig. 3 gives a value of α_{cr} = 2.22 which is close to the value (68) given by our approximate method.

4. Systems with four Sub-rings

In this section we consider conditions for phase-locking in systems which have m = 4 sub-rings, each with n_i oscillators. As in the previous section, we find that for each sub-ring,

$$\alpha_{i_{cr}} = \frac{n_i - 1}{2} |\Omega - w_i|, \quad i = 1, 2, 3, 4 \tag{69}$$

where $\Omega = (w_1 + w_2 + w_3 + w_4)/4$. For the communication ring we offer an approximate formula for α_{cr} which involves taking the minimum of those values which are positive and which are greater than $\max\{\alpha_{i_{cr}}\}$, i.e., the largest of the $\alpha_{i_{cr}}$, from amongst the following four permutations:

$$\pm \frac{2C_i C_j (C_j + C_k)}{4 C_i C_j + (C_i + C_k)^2}, \quad (i, j, k) = (1, 2, 3), (2, 3, 4), (3, 4, 1), (4, 1, 2) \tag{70}$$

where

$$C_i = n_i(\Omega - w_i) \tag{71}$$

As an example, we take the system in Fig. 4, in which each sub-ring has n_i=4 oscillators with frequencies

$$w_1 = 1, \quad w_2 = 2.5, \quad w_3 = 3.1, \quad w_4 = 1.7 \tag{72}$$

We find:

$$C_1 = 4.3, \quad C_2 = -1.7, \quad C_3 = -4.1, \quad C_4 = 1.5 \tag{73}$$
$$\alpha_{1_{cr}} = 1.6125, \quad \alpha_{2_{cr}} = 0.6375, \quad \alpha_{3_{cr}} = 1.5375, \quad \alpha_{4_{cr}} = 0.5625 \tag{74}$$

An approximate value for the critical coupling coefficient of the communication ring, α_{cr}, is given by the minimum of those values which are positive and which are greater than $\max\{\alpha_{i_{cr}}\} = 1.6125$ from the following list (Eq. (70)):

$$[\pm 2.904, \pm 1.298, \pm 2.904, \pm 1.298] \tag{75}$$

Therefore we obtain

$$\alpha_{cr} = 2.904 \tag{76}$$
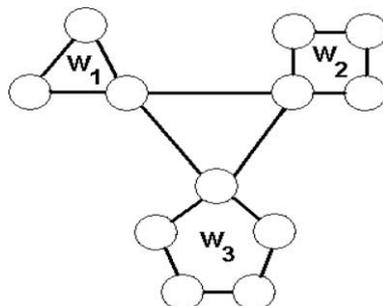
Fig. 3. A system with three sub-rings with frequencies w_1 = 1, w_2 = 2.5 and w_3 = 3.1.
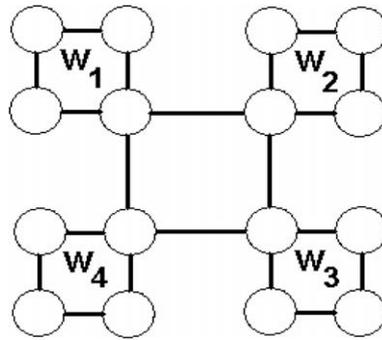
Fig. 4. A system with four sub-rings, each with the same number of oscillators, $n_i = 4$, $i = 1, 2, 3, 4$.

Numerical integration of the governing differential equations for the system of Fig. 4 with frequencies (72) in the case that all coupling parameters are taken to be equal, $\alpha = \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$, gives that $\alpha_{cr} = 2.903$, in close agreement with the foregoing. In the case that four of the five coupling parameters are fixed at values greater than their critical values, e.g., $\alpha = 3$, $\alpha_1 = 1.7$, $\alpha_3 = 1.6$, $\alpha_4 = 0.6$, and α_2 is allowed to vary, we find that locking occurs at $\alpha_2 = 0.6375$, in agreement with the foregoing.

As another example, we take the system in Fig. 5, in which $n_1 = 3$, $n_2 = 4$, $n_3 = 5$ and $n_4 = 6$, with frequencies Eq. (72). The phase-locked frequency is

$$\Omega = (3w_1 + 4w_2 + 5w_3 + 6w_4)/18 = 2.15 \tag{77}$$

And from Eq. (69) we obtain the following values for the critical coupling coefficients of sub-rings:

$$\alpha_{1_{cr}} = 1.15, \quad \alpha_{2_{cr}} = 0.525, \quad \alpha_{3_{cr}} = 1.9, \quad \alpha_{4_{cr}} = 1.125 \tag{78}$$

In order to find an approximate value for the critical coupling coefficient of the communication ring, α_{cr}, we first compute the C_i's:

$$C_1 = 3.45, \quad C_2 = -1.4, \quad C_3 = -4.75, \quad C_4 = 2.7 \tag{79}$$

Then from Eq. (70), α_{cr} is the minimum of those values which are positive and which are greater than max $\{\alpha_{i_{cr}}\} = 1.9$ from the following list:

$$[\pm 3.370, \pm 0.964, \pm 3.180, \pm 0.981] \tag{80}$$

Therefore we obtain

$$\alpha_{cr} = 3.180 \tag{81}$$

Numerical integration of the 18 ODE's which describe the system with frequencies Eq. (72) in the case that all sub-ring coupling parameters are larger than their respective critical values, e.g., $\alpha_1 = 1.2$, $\alpha_2 = 0.6$, $\alpha_3 = 2$, $\alpha_4 = 1.2$, gives that $\alpha_{cr} = 3.15$ which is close to the above approximate value.

For a third example, we again take the system in Fig. 5 but with frequencies:

$$w_1 = 3.3, \quad w_2 = 1.7, \quad w_3 = 2.1, \quad w_4 = 1 \tag{82}$$

We find

$$\alpha_{1_{cr}} = 1.456, \quad \alpha_{2_{cr}} = 0.217, \quad \alpha_{3_{cr}} = 0.511, \quad \alpha_{4_{cr}} = 2.111 \tag{83}$$
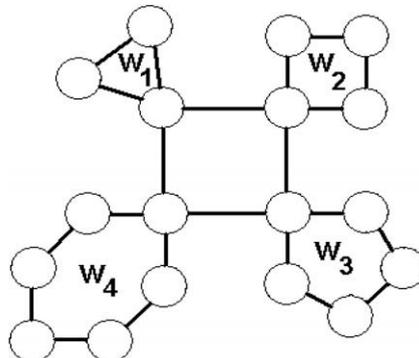
Fig. 5. A system with four sub-rings, with $n_1 = 3$, $n_2 = 4$, $n_3 = 5$ and $n_4 = 6$.

The list of candidates for max $\{\alpha_{i_{cr}}\}$ is

$$[\pm 0.162, \pm 0.194, \pm 1.520, \pm 2.960] \tag{84}$$

giving the approximate result:

$$\alpha_{cr} = 2.960 \tag{85}$$

Numerical integration with all sub-ring coupling parameters taken larger than their respective critical values gives $\alpha_{cr} = 2.90$.

In the first example, which involves Fig. 4 with frequencies (72), there are only 2 positive candidates for α_{cr}. One of these is less than max$\{\alpha_{i_{cr}}\}$. In the second example, which involves Fig. 5 with frequencies (72), there are four positive candidates for α_{cr}. Two of these are less than max$\{\alpha_{i_{cr}}\}$. In the third example, which involves Fig. 5 with frequencies (82), there are 4 positive candidates for α_{cr}. Three of these are less than max$\{\alpha_{i_{cr}}\}$. In all three cases the rule of choosing the minimum of those positive candidate values which are greater than max$\{\alpha_{i_{cr}}\}$ gives a reasonable approximation of the numerically computed value for α_{cr}.

5. Symmetric solutions

We have thus far focused on conditions for phase-locking, but we have not considered the phases themselves in a phase-locked motion. So let us return to Eq. (55) which we used to obtain a condition on α_1 for locking. Now however we will use it to study the phases. Substituting Eq. (55) into (52), we see that

$$\sin \phi_{1,2} = 0 \tag{86}$$

Subtracting (22) from (21) and using (86), we obtain

$$-\sin \phi_{1,3} = \sin \phi_{1,1} \Rightarrow \sin(\theta_{1,3} - \theta_{1,1}) = \sin(\theta_{1,2} - \theta_{1,1}) \tag{87}$$

One solution of this equation is $\theta_{1,2} = \theta_{1,3}$, in which case the sub-ring is said to be *symmetric*. A phase-locked solution will be said to be symmetric if all its sub-rings are symmetric. Eq. (55) gives a value for $\sin \phi_{1,1}$:

$$\sin \phi_{1,1} = \sin(\theta_{1,2} - \theta_{1,1}) = -\frac{\Omega - w_1}{\alpha_1} \tag{88}$$

Since arcsine is a multivalued function, Eq. (88) will give two values for $\phi_{1,1}$. One of these will have $\cos \phi_{1,1} > 0$, and the other will have $\cos \phi_{1,1} < 0$.

As an example we consider the system of Fig. 2 with Eqs. (6)–(14) and frequencies (36) for parameters $\alpha = \alpha_1 = \alpha_2 = \alpha_3 = 2$. Note that these values lie above the previously computed critical values, cf. Eqs. (49) and (60), so a phase-locked motion is expected to exist. Numerical integration of Eqs. (6)–(14) gives the following values for the phase-locked steady state:

$$\theta_{1,1} = 0 \tag{89}$$
$$\theta_{1,2} = 5.6397 \tag{90}$$
$$\theta_{1,3} = 5.6397 \tag{91}$$
$$\theta_{2,1} = 0.9683 \tag{92}$$
$$\theta_{2,2} = 1.1189 \tag{93}$$
$$\theta_{2,3} = 1.1189 \tag{94}$$
$$\theta_{3,1} = 1.3516 \tag{95}$$
$$\theta_{3,2} = 1.8183 \tag{96}$$
$$\theta_{3,3} = 1.8183 \tag{97}$$

Since the phase of this periodic motion is arbitrary, we have chosen $\theta_{1,1}$ as zero in the above list. These give the following values for $\phi_{i,j}$ and ψ_i:

$$\phi_{1,1} = 5.6397, \quad \sin \phi_{1,1} = -0.6, \quad \cos \phi_{1,1} = 0.8 \tag{98}$$
$$\phi_{1,2} = 0, \quad \sin \phi_{1,2} = 0, \quad \cos \phi_{1,2} = 1 \tag{99}$$
$$\phi_{1,3} = -5.6397, \quad \sin \phi_{1,3} = 0.6, \quad \cos \phi_{1,3} = 0.8 \tag{100}$$
$$\phi_{2,1} = 0.1506, \quad \sin \phi_{2,1} = 0.15, \quad \cos \phi_{2,1} = 0.9887 \tag{101}$$
$$\phi_{2,2} = 0, \quad \sin \phi_{2,2} = 0, \quad \cos \phi_{2,2} = 1 \tag{102}$$
$$\phi_{2,3} = -0.1506, \quad \sin \phi_{2,3} = -0.15, \quad \cos \phi_{2,3} = 0.9887 \tag{103}$$
$$\phi_{3,1} = 0.4667, \quad \sin \phi_{3,1} = 0.45, \quad \cos \phi_{3,1} = 0.8930 \tag{104}$$
$$\phi_{3,2} = 0, \quad \sin \phi_{3,2} = 0, \quad \cos \phi_{3,2} = 1 \tag{105}$$
$$\phi_{3,3} = -0.4667, \quad \sin \phi_{3,3} = -0.45, \quad \cos \phi_{3,3} = 0.8930 \tag{106}$$
$$\psi_1 = 0.9683, \quad \sin \psi_1 = 0.8239, \quad \cos \psi_1 = 0.5667 \tag{107}$$

$$\psi_2 = 0.3833, \quad \sin\psi_2 = 0.3740, \quad \cos\psi_2 = 0.9274 \tag{108}$$

$$\psi_3 = -1.3516, \sin\psi_3 = -0.9761, \cos\psi_3 = 0.2174 \tag{109}$$

All of these agree with the derived formula for symmetric solutions, cf. Eq. (88). Note that all have positive values for the cosines of $\phi_{i,j}$ and ψ_i. This raises the following question: since we have seen in Eq. (87) that there are other solutions besides symmetric solutions, and since we have seen in Eq. (88) that in the case of symmetric solutions, they exist with both positive and negative $\cos\phi_{i,j}$, we may ask why the numerical result showed only symmetric solutions, and those with only positive $\cos\phi_{i,j}$? The answer lies in the stability of motion.

6. Stability considerations

So far we have studied the *existence* of phase-locked motions in ring networks. We have identified critical values of the coupling coefficients α and α_i which must be exceeded for phase-locked motions to exist. In this section we consider the question of the *stability* of phase-locked motions [6].

We have defined phase-locked motions by Eq. (15):

$$\theta_{i,j} = \Omega t + k_{i,j} \tag{110}$$

In order to study the stability of such a motion, we set

$$\theta_{i,j} = \Omega t + k_{i,j} + u_{i,j} \tag{111}$$

and linearize in $u_{i,j}$. This will result in a linear system with constant coefficients, the eigenvalues of which will determine stability.

As an example we consider the system of Fig. 2 with Eqs. (6)–(14). Substituting (111) in (6)–(14) and linearizing, we obtain:

$$\frac{d}{dt}u_{1,1} = \alpha_1[\cos\phi_{1,1}(u_{1,2} - u_{1,1}) - \cos\phi_{1,3}(u_{1,1} - u_{1,3})] + \alpha[\cos\psi_1(u_{2,1} - u_{1,1}) - \cos\psi_3(u_{1,1} - u_{3,1})], \tag{112}$$

$$\frac{d}{dt}u_{1,2} = \alpha_1[\cos\phi_{1,2}(u_{1,3} - u_{1,2}) - \cos\phi_{1,1}(u_{1,2} - u_{1,1})], \tag{113}$$

$$\frac{d}{dt}u_{1,3} = \alpha_1[\cos\phi_{1,3}(u_{1,1} - u_{1,3}) - \cos\phi_{1,2}(u_{1,3} - u_{1,2})], \tag{114}$$

$$\frac{d}{dt}u_{2,1} = \alpha_2[\cos\phi_{2,1}(u_{2,2} - u_{2,1}) - \cos\phi_{2,3}(u_{2,1} - u_{2,3})] + \alpha[\cos\psi_2(u_{3,1} - u_{2,1}) - \cos\psi_1(u_{2,1} - u_{1,1})], \tag{115}$$

$$\frac{d}{dt}u_{2,2} = \alpha_2[\cos\phi_{2,2}(u_{2,3} - u_{2,2}) - \cos\phi_{2,1}(u_{2,2} - u_{2,1})], \tag{116}$$

$$\frac{d}{dt}u_{2,3} = \alpha_2[\cos\phi_{2,3}(u_{2,1} - u_{2,3}) - \cos\phi_{2,2}(u_{2,3} - u_{2,2})], \tag{117}$$

$$\frac{d}{dt}u_{3,1} = \alpha_3[\cos\phi_{3,1}(u_{3,2} - u_{3,1}) - \cos\phi_{3,3}(u_{3,1} - u_{3,3})] + \alpha(\cos\psi_3(u_{1,1} - u_{3,1}) - \cos\psi_2(u_{3,1} - u_{2,1})), \tag{118}$$

$$\frac{d}{dt}u_{3,2} = \alpha_3[\cos\phi_{3,2}(u_{3,3} - u_{3,2}) - \cos\phi_{3,1}(u_{3,2} - u_{3,1})], \tag{119}$$

$$\frac{d}{dt}u_{3,3} = \alpha_3[\cos\phi_{3,3}(u_{3,1} - u_{3,3}) - \cos\phi_{3,2}(u_{3,3} - u_{3,2})] \tag{120}$$

where the arguments of the trig terms are to be evaluated on the solution whose stability is being sought.

As in the previous section, we take as an example this system with frequencies (36) and parameters $\alpha = \alpha_1 = \alpha_2 = \alpha_3 = 2$. We substitute the values for $\cos\phi_{i,j}$ and $\cos\psi_i$ derived in Eqs. (98) thru (109) into Eqs. (112) thru (120), and obtain the following 9×9 matrix:

$$\begin{pmatrix}
-4.76 & 1.60 & 1.60 & 1.13 & 0 & 0 & 0.43 & 0 & 0 \\
1.60 & -3.60 & 2.0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1.60 & 2.0 & -3.60 & 0 & 0 & 0 & 0 & 0 & 0 \\
1.13 & 0 & 0 & -6.94 & 1.97 & 1.97 & 1.85 & 0 & 0 \\
0 & 0 & 0 & 1.97 & -3.97 & 2.0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.97 & 2.0 & -3.97 & 0 & 0 & 0 \\
0.43 & 0 & 0 & 1.85 & 0 & 0 & -5.86 & 1.79 & 1.79 \\
0 & 0 & 0 & 0 & 0 & 0 & 1.79 & -3.79 & 2.0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1.79 & 2.0 & -3.79
\end{pmatrix} \tag{121}$$

which turns out to have the following nine eigenvalues:

$$\lambda = 0, -0.55, -0.94, -4.66, -5.42 \pm 1.00i, -6.93 \pm 0.73i, -9.45 \tag{122}$$

The presence of the zero eigenvalue is associated with the non-uniqueness of $\theta_{1,1}$, corresponding to a one dimensional continuum of equilibria. The other eight eigenvalues all have negative real parts, confirming the stability of the phase-locked motion.

This example suggests the result that if coupling parameters α and α_i are larger than their critical values, then there will exist a phase-locked solution which is symmetric and stable.

This may be proved by following an approach taken by Rogge and Aeyels [8] (for simple rings) based on Gershgorin's theorem: Let A be a square matrix. Imagine a disk in the complex plane centered at a point corresponding to a_{ii}, being an element on the main diagonal. Now let the radius of this disk be the sum of the absolute value of all the other terms in that row $\sum_{j \neq i} |a_{ij}|$. Then every eigenvalue of A lies in one of these disks.

In order to use this theorem we need to have an expression for the general stability matrix, this being a generalization of the example matrix (121). The row corresponding to a generic non-communication oscillator θ_{ij} contains three non-zero terms:

$$\alpha_i \cos \phi_{ij-1} \quad - \alpha_i(\cos \phi_{ij} + \cos \phi_{ij-1}) \quad \alpha_i \cos \phi_{ij} \tag{123}$$

where the middle term lies on the main diagonal and the other two terms lie on other diagonals. In the case of a communication oscillator, the corresponding row contains three additional terms:

$$\alpha \cos \psi_i \ldots - \alpha(\cos \psi_i + \cos \psi_{i-1}) \ldots \alpha \cos \psi_{i-1} \tag{124}$$

where again the middle term lies on the main diagonal and the other two terms lie on other diagonals. By choosing the coupling coefficients α_i and α larger than their critical values we guarantee that $\sin \phi_{ij}$ and $\sin \psi_i$ will give real values for ϕ_{ij} and ψ_i. However since sine is multivalued, each of ϕ_{ij} and ψ_i can take on two values mod 2π, leading to each of the cosine terms in (121) being either positive or negative. If we choose the solution for which the cosine terms are positive, then we may use Gershgorin's theorem to prove that the phase-locked solution is stable.

Note that the row sum of each row is zero. Also note that the terms on the main diagonal are negative and all other terms are positive. Thus the Gershgorin disks lie in the left half-plane and are tangent to the imaginary axis at the origin. Then Gershgorin's theorem tells us that all eigenvalues lie in the left half-plane or are 0, which proves stability of the phase-locked motion.

7. Bifurcations

In this section, the number of phase-locked solutions is investigated for varying coupling coefficients α_i and α. We again use the example of the system in Fig. 2 with parameters $(\omega_1, \omega_2, \omega_3) = (1, 2.5, 3.1)$ and $\alpha = \alpha_1 = \alpha_2 = \alpha_3 = 2$. We saw previously that for this case (a) a stable symmetric solution exists, (b) that there are other unstable solutions which exist, and (c) the critical coupling coefficient within each sub-ring ($\alpha_{i,cr}$) is independent of the corresponding values in the other sub-rings and the communication ring. Consequently, we may independently investigate the effect of varying each coupling coefficient.

7.1. The communication ring

The Appendix gives the generalized formula for the existence of phase-locked solutions in the communication ring. Note that if $\alpha = 0$, there are no phase-locked solutions but as $\alpha \to \infty$, there are six solutions for the phase-locked equation. We therefore expect the appearance of new roots at critical values of α, i.e. where there is a double root.

Thus, for the ω parameters discussed above, Eq. (38) yields critical communication ring coupling coefficients at $\alpha = \pm 1.8579, \pm 5.7564, \pm 8.5119$, with the corresponding number of real solutions to $\sin \psi_1$ increasing from 0 to 2, from 2 to 4 and from 4 to 6, respectively. Each of these is accompanied by an appropriate $\sin \psi_2$ and $\sin \psi_3$ which satisfies Eq. (126). We note that there are two possible candidates for ψ_i for each term; however the appropriate phase-locked solution must further satisfy the consistency condition,

$$\psi_1 + \psi_2 + \psi_3 = 0 \mod(2\pi).$$

Imposing this condition yields a single possible solution (ψ_1, ψ_2, ψ_3) per root of Eq. (127). Fig. 6 shows the corresponding bifurcation diagram.

7.2. The sub-rings

From Eqs. (54) and (57), the critical coupling coefficients for the sub-rings are (a) $\alpha_i = (\Omega - \omega_i)$ and (b) $\alpha_i = 4(\Omega - \omega_i)$. Condition (a) yields

$$\sin \phi_{1,2} = 0$$

from which

$$\phi_{1,2} = 0, \pi.$$

Eq. (88) gives the actual phase difference.
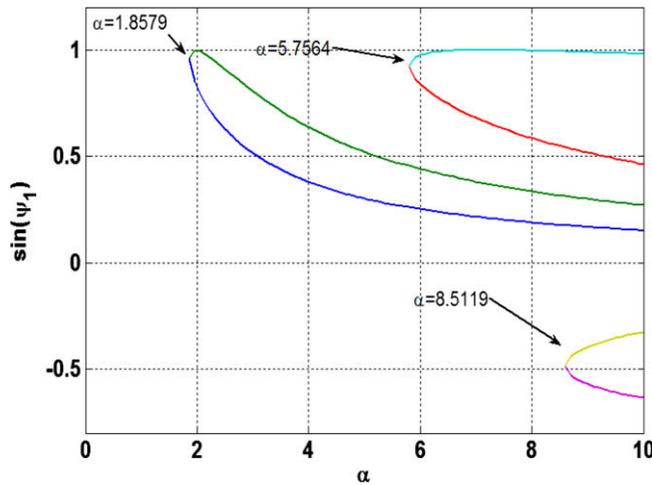
Fig. 6. Bifurcation diagram for the communication ring dynamics.
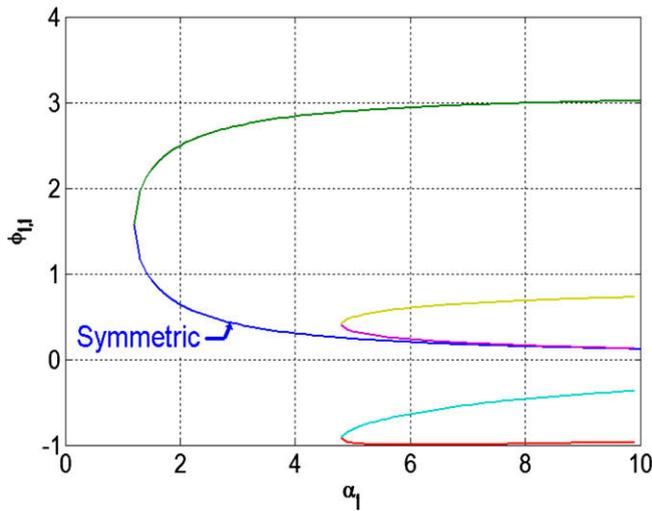
Fig. 7. Bifurcation diagram for the sub-ring dynamics.

Combining this with the consistency condition,

$$\phi_{i,1} + \phi_{i,2} + \phi_{i,3} = 0 \mod 2\pi$$

the solutions corresponding to $\phi_{1,2} = 0$ are equivalent to $\theta_{1,2} = \theta_{1,3}$, i.e. the solution is symmetric. For one of these symmetric solutions all the $\cos \phi_{i,j}$ terms are positive while the other has two negative $\cos \phi_{i,j}$ terms. The other combination $\phi_{1,2} = \pi$ and the consistency condition, yields two solutions of the form $\theta_{1,2} = \pi - \theta_{1,3}$. These solutions also have two negative $\cos \phi_{i,j}$ terms. Thus on exceeding the lower critical coupling coefficient, four phase-locked solution candidates appear.

Now consider (b). Eq. (57) shows that there are two double roots of the sub-ring coupling coefficient equation at $\alpha = \pm 4$ $(\Omega - \omega_i)$. Thus *four* new phase-locked solutions emerge at the second critical coefficient, two corresponding to $\sin \phi_{1,1} = 1$ and two with $\sin \phi_{1,1} = -1$. Imposing the consistency condition yields Fig. 7, the bifurcation diagram for sub-ring 1; similar curves are obtained for the other sub-rings.

8. Conclusions

We have studied the existence, stability and bifurcation of phase-locked motions in ring networks which consist of a large communication ring, each node of which contains an oscillator attached to a sub-ring. In the case of systems with three or four sub-rings we have given approximate expressions for critical coupling coefficients which must be exceeded for phase-

locking to occur. We have identified a type of phase-locked motion called symmetric, and we have shown that if the coupling coefficients are all greater than critical, a stable symmetric phase-locked motion will exist.

An important conclusion of this work is that phase-locking in each sub-ring can occur independently of the motion in the other sub-rings, cf. Eq. (61). On the other hand phase-locking in each sub-ring is not sufficient for phase-locking of the entire structure. For the latter an additional condition must be satisfied, cf. Eqs. (62) and (70).

This work represents a first step towards understanding the dynamics of networks of oscillators that are more complicated than simple rings. Future research is expected to include comparable studies of yet more complicated networks, and of motions within such networks which are more complicated than phase-locked motions.

Acknowledgements

Appendix.

The equations for the phase differences within the communication ring in the general three sub-ring system may be written as (cf. Eqs. (33) and (32)):

$$\sin\psi_2 = \sin\psi_1 + \frac{C_2}{\alpha} \tag{125}$$

$$-\sin(\psi_1 + \psi_2) = \sin\psi_1 + \frac{C_2}{\alpha} + \frac{C_3}{\alpha} \tag{126}$$

where C_i is defined by Eq. (63).

Expanding the trigonometric term and substituting for $\sin\psi_2$ in Eq. (126)

$$-\cos\psi_2 = -1 + \cot\psi_1\left(\sin\psi_1 + \frac{C_2}{\alpha}\right) + \frac{C_2}{\alpha\sin\psi_1} + \frac{C_3}{\alpha\sin\psi_1}$$

Squaring, adding to (125) squared, and finally substituting for $\cos\psi_1$, yields the following relationship for $\sin\psi_1$ (cf. Eq. (37)):

$$F_\alpha(\sin\psi_1) = a_6\sin^6\psi_1 + a_5\sin^5\psi_1 + a_4\sin^4\psi_1 + a_3\sin^3\psi_1 + a_2\sin^2\psi_1 + a_1\sin\psi_1 + a_0 = 0 \tag{127}$$

where

$$a_6 = 4\alpha^4$$
$$a_5 = 8\alpha^3(C_2 + 2C_3)$$
$$a_4 = 4\alpha^2(C_3^2 + 6C_2C_3 + 6C_2^2) - 3\alpha^4$$
$$a_3 = 8\alpha C_2(C_2 + C_3)(2C_2 + C_3) - 4\alpha^3(C_3 + 2C_2)$$
$$a_2 = 4C_2^2(C_2 + C_3)^2 + 2\alpha^2(C_3^2 - 2C_3C_2 - 2C_2^2)$$
$$a_1 = 4\alpha C_3^2(C_3 + 2C_2)$$
$$a_0 = C_3^2(C_3 + 2C_2)^2$$

When $\alpha = 0$, Eq. (127) is reduced to

$$F_0(\sin\psi_1) = 4C_2^2(C_2 + C_3)^2\sin^2\psi_1 + C_3^2(C_3 + 2C_2)^2 = 0.$$

Since both terms are semi-positive definite, this equation has no real roots unless $C_3 = 0$ or $C_3 + 2C_2 = 0$. Meanwhile as $\alpha \to \infty$, the equation is reduced to

$$F_\infty(\sin\psi_1) = 4\alpha^4\sin^6\psi_1 - 3\alpha^4\sin^4\psi_1 = 0$$

which yields six real roots (two simple roots at $\pm\sqrt{3}/2$ and a quadruple root at 0) for $\sin\psi_1$. Therefore, as α is increased the number of real roots of the equation increases from 0 to 6. There is therefore a minimum value of α for which a real solution will exist. If we let $\sin\psi_1 = s$, then the number of roots change when both $F(s)$ and $\mathrm{d}F/\mathrm{d}s = 0$, i.e., when there is a double root.

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